$\mathcal{C}^\infty-$Rings: an Interplay Between Geometry and Logics

Anillos $\mathcal{C}^\infty$ - una Interacción entre la Geometría y la Lógica

Jean Cerqueira Berni[1,a], **Hugo Luiz Mariano**[1,b]

Abstract. In this work we give an overview of some logical and geometric aspects of $\mathcal{C}^\infty$-rings, presenting some results concerning its universal algebraic aspects, introducing new results in Smooth Commutative Algebra and presenting some categorial considerations about certain special types of them.

Keywords: $\mathcal{C}^\infty$-rings; Smooth Commutative Algebra; Sheaves and Logic.

Resumen. En este trabajo damos una descripción general de algunos aspectos lógicos y geométricos de los anillos $\mathcal{C}^\infty$, presentando algunos resultados sobre sus aspectos algebraicos universales, introduciendo nuevos resultados en Álgebra Conmutativa Suave y presentando algunas consideraciones categóricas sobre ciertos tipos especiales de ellos.

Palabras claves: Anillos $\mathcal{C}^\infty$; Álgebra Conmutativa Suave; Haces y Lógica.

Mathematics Subject Classification: 13Nxx, 18Fxx.

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Introduction

This work is an overview of the main themes studied in the doctoral research of the first author (see [3]), talking about the theory of $\mathcal{C}^\infty-$rings and pointing some interplays with Logic and Geometry. This is an extended abstract of a talk given by the second author as part of the activities of the "Seminário de Geometría y Lógica", which took place at the Universidad Nacional de Colombia, Bogotá on February 27th, 2019.

We have studied a general class of rings of smooth functions, the class of $\mathcal{C}^\infty-$rings, frequently used in virtue of their applications to Singularity Theory (rings of germs, Weil algebras, jets of smooth functions etc.) and in order to construct topos-models for Synthetic Differential Geometry (cf. [10])[1]. Such

[1]Department of Mathematics of the Institute of Mathematics and Statistics, University of São Paulo, São Paulo, SP, Brazil

[a]jeancb@ime.usp.br

[b]hugomar@ime.usp.br

[1]A first step in this direction was given by F. W. Lawvere, in a series of lectures given in 1967 (cf. Lawvere's seminal work, "Categorical Dynamics")

topoi must have an internal language that is capable to provide room for some very useful (specially for heuristic and practical purposes) "infinitesimal entities"[2], such as nilpotent elements, for instance, which are not compatible with classical logic, but rather with a "weaker" one, the intuitionistic logic, which allows fewer deductions.

The category of $\mathcal{C}^\infty-$rings presents many advantages as a setting in which one can give a more intuitive treatment of Differential Geometry over the category of all finite-dimensional smooth manifolds and smooth functions, **Man**. This occurs partly because the category of $\mathcal{C}^\infty-$rings is a *variety* (*i.e.*, it is the class of models of an equational theory) and partly because it includes, as a full subcategory, the (dual of the) category **Man**.

We follow a complementary trail to the ones one usually finds nowadays - which seem to be more "algebro-geometric" - deepening the understanding about $\mathcal{C}^\infty-$rings analyzing them *per se*, using tools of Category Theory and describing them from an algebraic and from a logical-categorial viewpoint.

Algebraic Geometry and Commutative Algebra hold strong connections, to the extent that one passes from algebraic varieties to commutative rings via some well-known functors. Thus, properties of a commutative ring (algebraic variety) can be translated into properties of algebraic varieties (commutative rings). There is an analogous way to connect Differential Geometry and a "smooth" version of Commutative Algebra - the commutative algebra of $\mathcal{C}^\infty-$rings, that we call "Smooth Commutative Algebra".

We give some new definitions and generalize results on "Smooth Commutative Algebra" (see [6]), presented in [17] and [18]: we analyze and study "smooth" notions of "radical ideals" ($\mathcal{C}^\infty-$radical ideal, presented by the first time in [17]), "saturation of a multiplicative set" (smooth saturation), "ring of fractions" ($\mathcal{C}^\infty-$ring of fractions with respect to some subset of a $\mathcal{C}^\infty-$ring, generalizing the definition given by I. Moerdijk and G. Reyes in Theorem 1.4 of [17]), "reduced rings" ($\mathcal{C}^\infty-$reduced $\mathcal{C}^\infty-$rings), "fields" ($\mathcal{C}^\infty-$fields), "von Neumann regular rings" (von Neumann regular $\mathcal{C}^\infty-$rings), "prime spectrum" ($\mathcal{C}^\infty$-Zariski spectrum), "real spectrum" ($\mathcal{C}^\infty$-real spectrum), among others. We present some results regarding preservation properties of some features of $\mathcal{C}^\infty-$rings under limits and colimits. We present, apart from the results of Smooth Commutative Algebra, an interesting result that establishes a spectral bijection from the $\mathcal{C}^\infty-$spectrum to the $\mathcal{C}^\infty-$analog of the real spectrum of a $\mathcal{C}^\infty-$ring. We also present another order-theoretical analysis of $\mathcal{C}^\infty$-rings based on [4].

The central notion of $\mathcal{C}^\infty-$spectrum of a $\mathcal{C}^\infty-$ring, introduced by I. Moerdijk and G. Reyes in [17], about which they make a more detailed exposition in later papers (see [18], for instance) is addressed carefully. We present some contributions to the study of this spectrum, giving an explicit description of the "smooth" Zariski topology, and we present a result that asserts that the $\mathcal{C}^\infty-$spectrum of a $\mathcal{C}^\infty-$ring is a spectral space.

[2]Even Alexander Grothendieck, as observed by R. Lavendhomme in [14], insisted on not excluding nilpotent elements in Algebraic Geometry

We introduce the notion of von Neumann regular $\mathcal{C}^{\infty}-$ring and some of their properties, together with a categorial and a logical treatment for them (see [8]). In particular, the subcategory of $\mathcal{C}^{\infty}\mathbf{Rng}$ consisting of all von Neumann regular $\mathcal{C}^{\infty}-$rings is characterized as the closure under small limits of the category of $\mathcal{C}^{\infty}-$fields, *i.e.*, it is the smallest subcategory of $\mathcal{C}^{\infty}\mathbf{Rng}$ which contains all $\mathcal{C}^{\infty}-$fields and is closed under small limits. We establish, by two different methods, that the subcategory of von Neumann regular $\mathcal{C}^{\infty}$-rings is reflective in the category of all $\mathcal{C}^{\infty}$-rings.

We present a result that says that "$\mathcal{C}^{\infty}-$spectra of von Neumann regular $\mathcal{C}^{\infty}-$rings classify Boolean spaces": a precise and even stronger result holds, which has no analog in the purely algebraic setting of von Neumann regular rings (see for instance [1]).

We develop the first results on the categorial-logical aspects of the theory of $\mathcal{C}^{\infty}$-rings: we describe in details the classifying toposes for the theories of $\mathcal{C}^{\infty}-$rings, adapting a proof of the analogous construction for commutative rings given in [15]), local $\mathcal{C}^{\infty}-$rings and von Neumann regular $\mathcal{C}^{\infty}-$rings (see [5]).

Finally, this work also lists some possible developments - mainly on model-theoretic and order-theoretic aspects of the theory of $\mathcal{C}^{\infty}$-rings - that, due to the lack of time, were not developed in the first author's thesis - but that we intend to address in the future.

1. The Universal Algebra of $\mathcal{C}^{\infty}$-Rings

In this section we present $\mathcal{C}^{\infty}$-rings in two different settings, and describe explicitly some of the constructions one can perform with them. The detailed proofs of the results of this sections are given in [7].

There are many ways of defining a $\mathcal{C}^{\infty}$-ring, depending on the the framework one wants to use them. Here we give two (equivalent) definitions of this mathematical object. We begin by making a parallel between $\mathcal{C}^{\infty}-$rings and $\mathbb{R}-$algebras (which should be more familiar to the reader) by regarding them both as Lawvere theories. Recall the following (functorial) definition of an $\mathbb{R}-$algebra:

Definition 1.1. An $\mathbb{R}-$algebra A can be regarded as a finite product preserving functor from the category **Pol** to the category of sets, $A : \mathbf{Pol} \to \mathbf{Set}$, where $\mathrm{Obj}(\mathbf{Pol}) = \{\mathbb{R}^n | n \in \mathbb{N}\}$ and

$$\mathbf{Pol}\,(\mathbb{R}^m, \mathbb{R}^n) = \{p : \mathbb{R}^m \to \mathbb{R}^n | p \text{ is a polynomial function}\}$$

Thus, every polynomial map, $p : \mathbb{R}^m \to \mathbb{R}^n$, given by an $n-$tuple of polynomials with real coefficients, $(p_1, \cdots, p_n)$, can be "interpreted" in **Set**, as a map $A(p) : A^m \to A^n$.

Analogously, we have the following:

Definition 1.2. A $\mathcal{C}^{\infty}-$ring A is a finite product preserving functor from $\mathcal{C}^{\infty}$, the category whose objects are the Euclidean spaces, that is, $\mathrm{Obj}(\mathcal{C}^{\infty}) = \{\mathbb{R}^n | n \in \mathbb{N}\}$, and whose morphisms are all smooth functions - that is, $\mathrm{Hom}_{\mathcal{C}^{\infty}}(\mathbb{R}^m, \mathbb{R}^n) = \mathcal{C}^{\infty}(\mathbb{R}^m, \mathbb{R}^n)$, to **Set**, $A : \mathcal{C}^{\infty} \to \mathbf{Set}$.

In order to explicitly give constructions involving $\mathcal{C}^{\infty}$-rings, we give a more "universal-algebraic" definition, describing them in a first-order language using only functional symbols.

Definition 1.3. A $\mathcal{C}^{\infty}-$**structure** on a set A is a pair $\mathfrak{A} := (A, \Phi)$, where:

$$\begin{array}{rccc} \Phi : & \bigcup_{n\in\mathbb{N}} \mathcal{C}^{\infty}(\mathbb{R}^n, \mathbb{R}) & \to & \bigcup_{n\in\mathbb{N}} \mathrm{Func}\,(A^n; A) \\ & (f : \mathbb{R}^n \stackrel{\mathcal{C}^{\infty}}{\to} \mathbb{R}) & \mapsto & \Phi(f) := (f^A : A^n \to A) \end{array}$$

that is, Φ interprets the **symbols** of all smooth functions of n variables as $n-$ary function symbols on A.

Definition 1.4. Given two $\mathcal{C}^{\infty}-$structures $\mathfrak{A} = (A, \Phi)$ and $\mathfrak{B} = (B, \Psi)$, a **morphism of** $\mathcal{C}^{\infty}-$**structures** is a function $\varphi : A \to B$ such that for every $n \in \mathbb{N}$ and every $f \in \mathcal{C}^{\infty}(\mathbb{R}^n, \mathbb{R})$ the following diagram commutes:

$$\begin{array}{ccc} A^n & \xrightarrow{\varphi^{(n)}} & B^n \\ {\scriptstyle \Phi(f)}\downarrow & & \downarrow{\scriptstyle \Psi(f)} \\ A & \xrightarrow[\varphi]{} & B \end{array}$$

that is, such that $\Psi(f) \circ \varphi^{(n)} = \varphi \circ \Phi(f)$.

We are going to denote by $\mathcal{C}^{\infty}\mathbf{Str}$ the category of all $\mathcal{C}^{\infty}-$structures and $\mathcal{C}^{\infty}-$homomorphisms.

The theory of $\mathcal{C}^{\infty}-$rings can be described within a first-order language, $\mathbb{L}$, with a denumerable set of variables, $\mathrm{Var}\,(\mathbb{L})$ whose non-logical symbols are the $n-$ary function symbols $\mathcal{F}_{(n)} = \{f^{(n)} \mid f \in \mathcal{C}^{\infty}(\mathbb{R}^n)\}$ for each $n \geq 0$.

Now, $\mathcal{C}^{\infty}$-rings are exactly the $\mathcal{C}^{\infty}$-structures that preserve all equations between smooth functions. More precisely, consider the following classes of equations:

(E1) For every $n \in \mathbb{N}$ and for every $k \leq n$, denoting the projection on the $k-$th coordinate by $p_k : \mathbb{R}^n \to \mathbb{R}$, the equations:

$$\mathbf{Eq}^{n,k}_{(1)} = \{p_k(x_1, \cdots, x_n) = x_k | x_1, \cdots, x_n \in \mathrm{Var}\,(\mathbb{L})\}$$

(E2) For every $n, k \in \mathbb{N}$, and for every $(n+2)-$tuple of **symbols** of smooth functions, $(f, g_1, \cdots, g_n, h)$ such that $f \in \mathcal{F}_{(n)}$, $g_1, \cdots, g_n, h \in \mathcal{F}_{(k)}$ e $h = f \circ (g_1, \cdots, g_n)$, the equations:

$$\mathbf{Eq}^{n,k}_{(2)} = \{h(x_1, \cdots, x_k) = f(g_1(x_1, \cdots, x_k), \cdots, g_n(x_1, \cdots, x_k)) | \\ x_1, \cdots, x_k \in \mathrm{Var}\,(\mathbb{L})\}$$

Now we are able to give the following:

Definition 1.5. A $\mathcal{C}^\infty-$**ring** is a $\mathcal{C}^\infty-$structure, $\mathfrak{A} = (A, \Phi)$ such that:

- For every $n \in \mathbb{N}$, $k \leq n$, denoting the projection on the $k-$th coordinate by $p_k : \mathbb{R}^n \to \mathbb{R}$:

$$\mathfrak{A} \models (\forall x_1) \cdots (\forall x_n)(p_k(x_1, \cdots, x_n) = x_k)$$

  that is, $\Phi(p_k) = \pi_k : A^n \to A$;

- For every $n, k \in \mathbb{N}$, $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$, $h, g_1, \cdots, g_n \in \mathcal{C}^\infty(\mathbb{R}^k, \mathbb{R})$ such that $h = f(g_1, \cdots, g_n)$:

$$\mathfrak{A} \models (\forall \vec{x})(h(\vec{x}) = f(g_1(\vec{x}), \cdots, g_n(\vec{x})))$$

  that is, $\Phi(h) = \Phi(f)(\Phi(g_1), \cdots, \Phi(g_n))$.

The theory of $\mathcal{C}^\infty-$rings is an equational theory, so many classical results hold, such as:

- **Birkhoff's HSP theorem**: the subclass $\mathcal{C}^\infty$**Rng** of $\mathcal{C}^\infty$**Str** is closed under products, substructures and homomorphic images;
- For each $n \in \mathbb{N}$, $\mathcal{C}^\infty(\mathbb{R}^n)$ is the free $\mathcal{C}^\infty-$ring on n generators;
- Whenever $\{(A_i, \Phi_i) | i \in I\}$ is a *directed family* of $\mathcal{C}^\infty-$subrings of (A, Φ) then $\left(\bigcup_{i \in I} A_i, \bigcup_{i \in I} \Phi_i\right)$ is a $\mathcal{C}^\infty-$subring of (A, Φ)
- Given a family of $\mathcal{C}^\infty-$subrings $\{(A_\alpha, \Phi_\alpha) | \alpha \in \Lambda\}$ of (A, Φ), $\left(\bigcap_{\alpha \in \Lambda} A_\alpha, \Phi'\right)$ is a $\mathcal{C}^\infty-$subring of (A, Φ)
- The *ad hoc* **Theorem of Homomorphism** and its consequences hold.
- The congruences of any $\mathcal{C}^\infty-$ring are classified by their ring-theoretic ideals.

Definition 1.6. A $\mathcal{C}^\infty-$ring is finitely generated whenever there is some $n \in \mathbb{N}$ and some ideal $I \subseteq \mathcal{C}^\infty(\mathbb{R}^n)$ such that $A \cong \mathcal{C}^\infty(\mathbb{R}^n)/I$. The category of all finitely generated $\mathcal{C}^\infty-$rings is denoted by $\mathcal{C}^\infty\mathbf{Rng}_{\text{fg}}$.

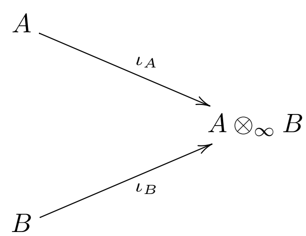
Definition 1.7. A $\mathcal{C}^\infty-$ring is finitely presented whenever there is some $n \in \mathbb{N}$ and some finitely generated ideal $I \subseteq \mathcal{C}^\infty(\mathbb{R}^n)$ such that $A \cong \mathcal{C}^\infty(\mathbb{R}^n)/I$.

Whenever A is a finitely presented $\mathcal{C}^\infty-$ring, there is some $n \in \mathbb{N}$ and some $f_1, \cdots, f_k \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that:

$$A = \frac{\mathcal{C}^\infty(\mathbb{R}^n)}{\langle f_1, \cdots, f_k \rangle}$$

The category of all finitely presented $\mathcal{C}^\infty-$rings is denoted by $\mathcal{C}^\infty\mathbf{Rng}_{\text{fp}}$

Remark 1.8. As a consequence of the fact that $\mathcal{C}^\infty\mathbf{Rng}$ is a variety of of algebras, the category of $\mathcal{C}^\infty-$rings has all coproducts. Given two $\mathcal{C}^\infty-$rings, we denote their coproduct by:



In order to describe concretely the coproducts in $\mathcal{C}^\infty\mathbf{Rng}$, we first compute the coproduct of two finitely generated $\mathcal{C}^\infty-$rings: given two finitely generated $\mathcal{C}^\infty-$rings, say $A = \mathcal{C}^\infty(\mathbb{R}^n)/I$ and $B = \mathcal{C}^\infty(\mathbb{R}^m)/J$, their **coproduct** is given by:

$$A \otimes_\infty B = \frac{\mathcal{C}^\infty(\mathbb{R}^n)}{I} \otimes_\infty \frac{\mathcal{C}^\infty(\mathbb{R}^m)}{J} \cong \frac{\mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^m)}{\langle I, J\rangle}$$

where $\langle I, J\rangle = \langle\{f \circ \pi_1, g \circ \pi_2 | (f \in I)\&(g \in J)\}\rangle$

Since every $\mathcal{C}^\infty-$ring is the directed colimit of finitely generated $\mathcal{C}^\infty-$rings, given any $A \cong \varinjlim_{i \in I} A_i$ and $B \cong \varinjlim_{j \in J} B_j$, we have:

$$A \otimes_\infty B \cong \varinjlim_{(i,j) \in I \times J} A_i \otimes_\infty B_j$$

Remark 1.9. Both categories $\mathcal{C}^\infty\mathbf{Rng}_{\mathrm{fg}}$ and $\mathcal{C}^\infty\mathbf{Rng}_{\mathrm{fp}}$ are closed under finite coproducts.

The category $\mathcal{C}^\infty\mathbf{Rng}$ is such that:

- For every set X there is a free $\mathcal{C}^\infty-$ring determined by X
- any $\mathcal{C}^\infty-$ring is a homomorphic image of some free $\mathcal{C}^\infty-$ring;
- a $\mathcal{C}^\infty-$homomorphism is monic if, and only if it is an injective map;
- any indexed set of $\mathcal{C}^\infty-$rings, $\{(A_\alpha, \Phi_\alpha) | \alpha \in I\}$ has a coproduct in $\mathcal{C}^\infty\mathbf{Rng}$;

The category of all $\mathcal{C}^\infty-$rings is a concrete category, and the forgetful functor $U : \mathcal{C}^\infty\mathbf{Rng} \to \mathbf{Set}$ has a left adjoint, $L : \mathbf{Set} \to \mathcal{C}^\infty\mathbf{Rng}$. Moreover, the forgetful functor creates all limits and filtered colimits, and using a general argument, it can be shown that the category $\mathcal{C}^\infty\mathbf{Rng}$ has all small colimits.

Definition 1.10. Given any $\mathcal{C}^\infty-$ring A and any set S, one has the $\mathcal{C}^\infty-$ring of $\mathcal{C}^\infty-$polynomials with variables in S, $A\{S\}$, given by the coproduct $A\{S\} := A \otimes_\infty L(S) = A \otimes_\infty \mathcal{C}^\infty(\mathbb{R}^S)$.

This construction (together with quotients) enables us to prove that $\mathcal{C}^\infty-$rings of fractions exist in $\mathcal{C}^\infty\mathbf{Rng}$.

2. Topics in Smooth Commutative Algebra

As mentioned in the introduction, $\mathcal{C}^{\infty}$-rings have a "rich Commutative Algebra", which can be related to Differential Geometry in a similar way that Commutative Algebra relates to Algebraic Geometry. In this section we present the main results of this theory, omitting their proofs (which can be found in [6]). We begin with the central notion of "universally" inverting elements of a $\mathcal{C}^{\infty}$-ring:

Definition 2.1. The **smooth ring of fractions** of A with respect to $S \subseteq A$ is a pair $(A\{S^{-1}\}, \eta_S)$ which satisfies the following conditions:

(1) $(\forall s \in S)(\eta_S(s) \in (A\{S^{-1}\})^{\times})$;

(2) If $f : A \to B$ is such that $(\forall s \in S)(f(s) \in B^{\times})$, then there is a unique $\widetilde{f} : A\{S^{-1}\} \to B$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\eta_S} & A\{S^{-1}\} \\ & \searrow^{f} & \downarrow{\scriptstyle \widetilde{f}} \\ & & B \end{array}$$

After the definition, we explicitly construct the $\mathcal{C}^{\infty}$-ring of fractions as follows.

Let A be a $\mathcal{C}^{\infty}-$ring and let $S \subseteq A$ be any subset. Consider the coproduct:

$$A \xrightarrow{\iota_A} A \otimes_{\infty} L(S) \xleftarrow{\iota_{L(S)}} L(S)$$

And take $A\{S^{-1}\} \cong A\{S\}/\langle\{x_s \cdot \iota_A(s) - 1 | s \in S\}\rangle$, where $x_s := \iota_{L(S)}(s)$, together with the composition $\mathcal{C}^{\infty}-$homomorphism:

$$\begin{array}{ccc} A & \xrightarrow{\iota_A} & A\{S\} \\ & \searrow_{\eta_S} & \downarrow{\scriptstyle q} \\ & & \dfrac{A\{S\}}{\langle\{x_s \cdot \iota_A(s) - 1 | s \in S\}\rangle} \end{array}$$

It can be proved that such a construction is (up to $\mathcal{C}^{\infty}$-isomorphisms) *the* $\mathcal{C}^{\infty}-$ring of fractions of A with respect to $S \subseteq A$.

The notion of $\mathcal{C}^{\infty}-$**saturation** of a subset S makes the study of $\mathcal{C}^{\infty}-$rings of fractions more clear:

Definition 2.2. Let (A, Φ) be a $\mathcal{C}^{\infty}-$ring and let $S \subseteq A$ be any subset. The $\mathcal{C}^{\infty}-$**saturation** of S is:

$$S^{\infty-\mathrm{sat}} := \{a \in A : \eta_S(a) \in (A\{S^{-1}\})^{\times}\} = \eta_S^{\dashv}[A\{S^{-1}\}^{\times}] =$$

$$= \bigcap \{f^{\dashv}[B^{\times}] | f : A \to B \,\text{is}\, \mathcal{C}^{\infty} - \text{homomorphism s.t. } f[S] \subseteq B^{\times}\},$$

where $\eta_S^{\dashv}[A\{S^{-1}\}^{\times}]$ denotes the preimage of $A\{S^{-1}\}^{\times}$ by η_S.

The ring-theoretic saturation and the $\mathcal{C}^{\infty}-$saturation are connected by the following result:

Proposition 2.3. *Given a $\mathcal{C}^{\infty}-$ring A and a subset $S \subseteq A$, let $\mathcal{U}(A)$ be its underlying $\mathbb{R}-$algebra and $\eta_S : \mathcal{U}(A) \to \mathcal{U}(A)[S^{-1}]$ be the (ordinary) ring of fractions. We always have:*

- $S^{\text{sat}} \subseteq S^{\infty-\text{sat}}$
- *Whenever* $S^{\infty-\text{sat}} \subseteq S^{\text{sat}}$, *we have* $\mathcal{U}(A)[S^{-1}] \cong \mathcal{U}(A\{S^{-1}\})$.

Moreover, we have:

(i) $A^{\times} \subseteq S^{\infty-\text{sat}}$

(ii) $S \subseteq S^{\infty-\text{sat}}$

(iii) $S \subseteq T \Rightarrow S^{\infty-\text{sat}} \subseteq T^{\infty-\text{sat}}$

(iv) $S^{\infty-\text{sat}} = \langle S \rangle^{\infty-\text{sat}}$

Also, $(S^{\infty-\text{sat}})^{\infty-\text{sat}} = S^{\infty-\text{sat}}$. *Since it is inflationary, monotonic and idempotent, we can say that* $(\cdot)^{\infty-\text{sat}}$ *is a* ***closure operator***.

Moreover, whenever $\{S_i | i \in I\}$ *is a directed family of subsets of a* $\mathcal{C}^{\infty}-$*ring* A *we have:*

$$(\cup_{i\in I} S_i)^{\infty-\text{sat}} = \cup_{i\in I} S_i^{\infty-\text{sat}}$$

Remark 2.4. In general, the canonical ring homomorphism from $\mathcal{U}(A)[S^{-1}]$ to $\mathcal{U}(A\{S^{-1}\})$ is not an isomorphism, so in general, $S^{sat} \subsetneq S^{\infty-\text{sat}}$.

For instance, if $f \in \mathcal{C}^{\infty}(\mathbb{R}^n)$, then $\mathcal{C}^{\infty}(\mathbb{R}^n)\{\{f\}^{-1}\} \cong \mathcal{C}^{\infty}(U_f)$, where $U_f = \{x \in \mathbb{R}^n \mid f(x) \neq 0\}$, then $\{f\}^{\infty-\text{sat}} = \{g \in \mathcal{C}^{\infty}(\mathbb{R}^n) : Z(g) \subseteq Z(f)\}$, where $Z(g) = \mathbb{R}^n \setminus U_g = \{x \in \mathbb{R}^n \mid g(x) = 0\}$, but $\{f\}^{\text{sat}} = \{g \in \mathcal{C}^{\infty}(\mathbb{R}^n) : \exists h \in \mathcal{C}^{\infty}(\mathbb{R}^n), h \cdot g = f^k \text{ for some } k \geq 1\}$. To be specific, consider $A = \mathcal{C}^{\infty}(\mathbb{R})$ and $f(x) = \sin(x), x \in \mathbb{R}$. We have $\{f\}^{\infty-\text{sat}} = \{g \in \mathcal{C}^{\infty}(\mathbb{R}) : Z(g) \subseteq \{n \cdot \pi \mid n \in \mathbb{Z}\}) \supsetneq \{g \in \mathcal{C}^{\infty}(\mathbb{R}) : \exists h \in \mathcal{C}^{\infty}(\mathbb{R}), h \cdot g = f^k \text{ for some } k \geq 1\} = \{f\}^{\text{sat}}$, since $x^3 \in \{\sin(x)\}^{\infty-\text{sat}} \setminus \{\sin(x)\}^{\text{sat}}$.

On the other hand, as we will see in **Section 3**, whenever A is a Von Neumann regular $\mathcal{C}^{\infty}$-ring we have $\mathcal{U}(A)[S^{-1}] \cong \mathcal{U}(A\{S^{-1}\})$, so these two saturations coincide, that is, $S^{\infty-\text{sat}} = S^{\text{sat}}$.

The notion of $\mathcal{C}^{\infty}-$saturation enables us to articulate the following extension of **Theorem 1.4** of [17], and to present an alternative description of $A\{S^{-1}\}$:

Theorem 2.5. *Let A be any $\mathcal{C}^{\infty}-$ring, $S \subseteq A$ and $\eta_S : A \to A\{S^{-1}\}$ the universal $\mathcal{C}^{\infty}-$homomorphism. We have:*

(i) $(\forall b \in A\{S^{-1}\})(\exists c \in S^{\infty-\text{sat}})(\exists d \in A)(b \cdot \eta_S(c) = \eta_S(d))$

(ii) $(\forall b \in A)(\eta_S(b) = 0 \rightarrow (\exists c \in S^{\infty-\text{sat}})(c \cdot b = 0))$

Remark 2.6. Any $f : A \to B$ which satisfies (i) and (ii) is (up to isomorphism) the $\mathcal{C}^{\infty}-$ring of fractions.

Remark 2.7. Using the notation:

$$c = \frac{\eta_S(a)}{\eta_S(b)} \doteq c \cdot \eta_S(b) = \eta_S(a)$$

we write $A\{S^{-1}\} = \{\eta_S(a)/\eta_S(b) | (a \in A) \& (b \in S^{\infty-\text{sat}})\}$.

It can be proved that the forgetful functor:

$$\widetilde{U} : \mathcal{C}^{\infty}\mathbf{Rng} \to \mathbf{CRing}$$

has a left adjoint:

$$\mathcal{C}^{\infty}\mathbf{Rng} \underset{\widetilde{L}}{\overset{\widetilde{U}}{\rightleftarrows}} \mathbf{CRing}$$

and since any $\mathcal{C}^{\infty}-$ring is an $\mathbb{R}-$algebra, the forgetful functor $\mathcal{U} : \mathcal{C}^{\infty}\mathbf{Rng} \to \mathbb{R} - \mathbf{Alg}$ has a left adjoint:

$$\mathcal{C}^{\infty}\mathbf{Rng} \underset{\mathcal{L}}{\overset{\mathcal{U}}{\rightleftarrows}} \mathbb{R} - \mathbf{Alg}$$

3. Smooth Commutative Algebra versus Commutative Algebra

The forgetful functors $\mathcal{U} : \mathcal{C}^{\infty}\mathbf{Rng} \to \mathbb{R} - \mathbf{Alg}$ and $\widetilde{U} : \mathcal{C}^{\infty}\mathbf{Rng} \to \mathbf{CRing}$ allow us to define some notions like:

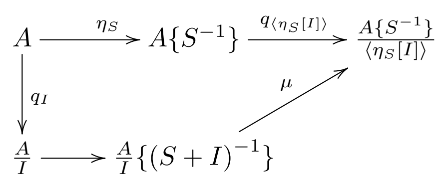
- A $\mathcal{C}^{\infty}-$**field** is a $\mathcal{C}^{\infty}-$ring A such that $\widetilde{U}(A)$ is a field;
- A $\mathcal{C}^{\infty}-$**domain** is a $\mathcal{C}^{\infty}-$ring A such that $\widetilde{U}(A)$ is a domain;
- A $\mathcal{C}^{\infty}-$**local** $\mathcal{C}^{\infty}-$**ring** is a $\mathcal{C}^{\infty}-$ring A such that $\widetilde{U}(A)$ is a local ring.

- A **von Neumann regular** $\mathcal{C}^\infty-$**ring** is a $\mathcal{C}^\infty-$ring A such that $\widetilde{U}(A)$ is a von Neumman regular ring, *i*.e..

$$(\forall a \in A)(\exists! b \in A)((a^2 b = a)\&(b^2 a = b))$$

Just as in **CRing**, in $\mathcal{C}^\infty$**Rng**, one has the "commutation between taking the ring of fractions and taking quotients", as we prove in the following:

Theorem 3.1. *Let A be a $\mathcal{C}^\infty-$ring, I be an ideal of A and $S \subseteq A$. There is a unique $\mathcal{C}^\infty-$isomorphism $\mu : \left(\frac{A}{I}\right)\{(S+I)^{-1}\} \to \frac{A\{S^{-1}\}}{\langle \eta_S[I]\rangle}$ such that the following diagram commutes:*



In **Commutative Algebra**, whenever $\mathfrak{p}$ is a prime ideal of A, we have that

$$A_{(\mathfrak{p})} = \varinjlim_{a \notin \mathfrak{p}} A[a^{-1}]$$

is a local ring. The same is not true in $\mathcal{C}^\infty$**Rng**. in order to obtain a local $\mathcal{C}^\infty$-ring when "localizing" it by some prime ideal, one must require an extra property, which - differently from the ordinary Commutative Algebra, is not a consequence of being prime. This property is given in the following:

Definition 3.2. Let A be a $\mathcal{C}^\infty-$ring and let $I \subseteq A$ be an ideal. The $\mathcal{C}^\infty-$**radical of** I is:

$$\sqrt[\infty]{I} = \left\{a \in A |\, (A/I)\,\{a + I^{-1}\} \cong 0\right\}$$

Using filters and ideals, and then passing the arguments "to the colimit", one proves that the $\mathcal{C}^\infty-$radical of any ideal is an ideal. One proves, similarly, that whenever $\mathfrak{p}$ is a prime ideal of A, $\sqrt[\infty]{\mathfrak{p}}$ is a prime ideal of A.

Remark 3.3. It should be noted that there exists a $\mathcal{C}^\infty$-ring A and a prime ideal $\mathfrak{p} \subseteq A$ that *is not* $\mathcal{C}^\infty$-radical. For instance, if $A = \mathcal{C}^\infty(\mathbb{R})$ and $\mathfrak{p}_0 = \{f \in \mathcal{C}^\infty(\mathbb{R}) \mid \frac{d^n f}{d^n x}(0) = 0, \forall n \in \mathbb{N}\}$, then $\mathfrak{p}_0$ is a prime ideal in A that *is not* $\mathcal{C}^\infty$-radical (cf. **Example 1.2** of [18]).

Proposition 3.4. *Given a $\mathcal{C}^\infty-$ring A and a prime $\mathcal{C}^\infty$-radical ideal $\mathfrak{p} \subseteq A$, the $\mathcal{C}^\infty$-ring:*

$$A_{\{\mathfrak{p}\}} = A\{(A \setminus \mathfrak{p})^{-1}\} \cong \varinjlim_{a \notin \mathfrak{p}} A\{a^{-1}\}$$

is a local $\mathcal{C}^\infty-$ring.

It is convenient to describe the $\mathcal{C}^\infty$-radical of and ideal in terms of the smooth saturation of their elements. In fact, we have the following equality:

$$\sqrt[\infty]{I} = \{a \in A | (\exists b \in I)\&(\eta_a(b) \in (A\{a^{-1}\})^\times)\} = \{a \in A | I \cap \{a\}^{\infty-\text{sat}} \neq \varnothing\}$$

Remark 3.5. Let A be a $\mathcal{C}^\infty-$ring and denote by $\mathfrak{I}(A)$ the lattice of its ideals. The map:

$$\begin{array}{rccc} \sqrt[\infty]{(\cdot)}: & \mathfrak{I}(A) & \rightarrow & \mathfrak{I}(A) \\ & I & \mapsto & \sqrt[\infty]{I} \end{array}$$

is a closure operator.

The following result shows that the smooth saturation is "preserved" by passing to the quotient, in the sense of the following:

Proposition 3.6. *Let A be a $\mathcal{C}^\infty-$ring, let I be any ideal and $q_I : A \to \dfrac{A}{I}$ be the quotient $\mathcal{C}^\infty-$homomorphism. We have:*

$$(\forall b \in A)(b \in \{a\}^{\infty-\text{sat}} \to q_I(b) \in \{a + I\}^{\infty-\text{sat}})$$

The converse of the above implication, however, is false.

Definition 3.7. Given a $\mathcal{C}^\infty-$ring A, we say that an ideal $I \subseteq A$ is a $\mathcal{C}^\infty-$**radical ideal** if, and only if:

$$I = \sqrt[\infty]{I}$$

Denoting by $\mathfrak{I}^\infty(A)$ the set of all $\mathcal{C}^\infty-$radical ideals of A, one has the following properties:

(i) $(\forall \alpha \in \Lambda)(I_\alpha \in \mathfrak{I}^\infty(A)) \Rightarrow \bigcap_{\alpha \in \Lambda} I_\alpha \in \mathfrak{I}^\infty(A)$;

(ii) If $\{I_\alpha | \alpha \in \Lambda\}$ is an upwards directed system of elements of $\mathfrak{I}^\infty(A)$, then $\bigcup_{\alpha \in \Lambda} I_\alpha \in \mathfrak{I}^\infty(A)$;

(iii) Given two $\mathcal{C}^\infty-$rings A, B and a $\mathcal{C}^\infty-$homomorphism $f : A \to B$, whenever $J \in \mathfrak{I}^\infty(B)$ we have $f^{\dashv}[J] \in \mathfrak{I}^\infty(A)$.

3.1. $\mathcal{C}^\infty-$reduced $\mathcal{C}^\infty-$rings

Now we give the notion of "reducedness" which is proper to our notion of $\mathcal{C}^\infty$-radical ideal. One can think of a $\mathcal{C}^\infty$-reduced $\mathcal{C}^\infty$-ring as a $\mathcal{C}^\infty$-ring that is free of $\infty-$nilpotents (see [9])

Definition 3.8. A $\mathcal{C}^\infty-$ring A is $\mathcal{C}^\infty-$**reduced** if, and only if $\sqrt[\infty]{(0)} = (0)$. In the case that $A = \mathcal{C}^\infty(\mathbb{R}^E)/I$ for some set E and some ideal $I \subseteq \mathcal{C}^\infty(\mathbb{R}^E)$, A is $\mathcal{C}^\infty-$reduced if, and only if $\sqrt[\infty]{I} = I$

Concerning $\mathcal{C}^\infty$-reduced $\mathcal{C}^\infty$-rings, we have:

(i) Every $\mathcal{C}^\infty-$field is a $\mathcal{C}^\infty-$reduced $\mathcal{C}^\infty-$domain;

(ii) Any free $\mathcal{C}^\infty-$ring is $\mathcal{C}^\infty-$reduced;

(iii) Given two $\mathcal{C}^\infty-$rings, A, B and a $\mathcal{C}^\infty-$monomorphism $\jmath : A \to B$, if B is $\mathcal{C}^\infty-$reduced then A is $\mathcal{C}^\infty-$reduced;

(iv) Every $\mathcal{C}^\infty-$subring of a $\mathcal{C}^\infty-$field is $\mathcal{C}^\infty-$reduced ;

(v) The directed colimit of $\mathcal{C}^\infty-$reduced $\mathcal{C}^\infty-$rings is $\mathcal{C}^\infty-$reduced.

(vi) If D is a $\mathcal{C}^\infty-$reduced $\mathcal{C}^\infty-$domain and $a \in D$ is such that $D\{a^{-1}\} \cong 0$, then $a = 0$

3.2. The Smooth Zariski Spectrum

In Commutative Algebra, the Zariski (or "prime") spectrum of a ring R consists of the set of all prime ideals of R together with a spectral topology. Since in this context every prime ideal is also radical, no requirement is made as for the "radicalness". As we have already commented, in order to $A_{\{\mathfrak{p}\}}$ to be a local $\mathcal{C}^\infty$-ring, one must require $\mathfrak{p}$ to be a $\mathcal{C}^\infty$-radical ideal. Motivated by this, we have the following:

Definition 3.9. The **Smooth Zariski Spectrum** of a $\mathcal{C}^\infty-$ring A is defined as the set:

$$\mathrm{Spec}^\infty(A) := \{\mathfrak{p} \in \mathrm{Spec}\,(\widetilde{U}(A)) | \sqrt[\infty]{\mathfrak{p}} = \mathfrak{p}\}$$

together with the **smooth Zariski topology**, Zar^∞ , generated by:

$$\{D^\infty(a) | a \in A\},$$

where $D^\infty(a) = \{\mathfrak{p} \in \mathrm{Spec}^\infty(A) | a \notin \mathfrak{p}\}$. Such a topology is **spectral**.

With the concepts given so far, we can formulate the following result, which is analogous to the separation theorems one finds in Commutative Algebra (see, for example, [2]):

Theorem 3.10 (**Separation Theorems**). *Let A be a $\mathcal{C}^\infty-$ring, $S \subseteq A$ and I be some ideal of A.*

(a) If I is a $\mathcal{C}^\infty-$radical ideal, then: $I \cap \langle S \rangle = \varnothing \iff I \cap S^{\infty-\mathrm{sat}} = \varnothing$

(b) If $S \subseteq A$ is $\mathcal{C}^\infty-$saturated , then: $I \cap S = \varnothing \iff \sqrt[\infty]{I} \cap S = \varnothing$

(c) If $S \subseteq A$ is $\mathcal{C}^\infty-$saturated, then:

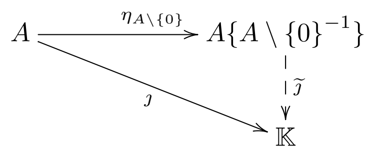
$$I \cap S = \varnothing \iff (\exists \mathfrak{p} \in \mathrm{Spec}^\infty(A))((I \subseteq \mathfrak{p}) \& (\mathfrak{p} \cap S = \varnothing)).$$

(d) For any $\mathfrak{p} \in \mathrm{Spec}^\infty(A)$, we have $A \setminus \mathfrak{p} = A \setminus \mathfrak{p}^{\infty-\mathrm{sat}}$

(e) $\sqrt[\infty]{I} = \bigcap\{\mathfrak{p} \in \mathrm{Spec}^\infty(A) | I \subseteq \mathfrak{p}\}$

The following result combines the notions of "$\mathcal{C}^\infty$-domain" and "$\mathcal{C}^\infty$-reducedness":

Proposition 3.11. *Whenever A is a $\mathcal{C}^\infty-$reduced $\mathcal{C}^\infty-$domain then:*
- $A\{A \setminus \{0\}^{-1}\}$ *is a $\mathcal{C}^\infty-$field;*
- $\eta_{A\setminus\{0\}} : A \to A\{A \setminus \{0\}^{-1}\}$ *is a $\mathcal{C}^\infty-$monomorphism;*
- *Given any $\mathcal{C}^\infty-$field $\mathbb{K}$ and any $\mathcal{C}^\infty-$monomorphism $\jmath : A \to \mathbb{K}$, there is a unique $\mathcal{C}^\infty-$homomorphism $\widetilde{\jmath} : A\{A \setminus \{0\}^{-1}\} \to \mathbb{K}$ such that the following diagram commutes:*



Thus, $\mathrm{Frac}\,(A) = A\{A \setminus \{0\}^{-1}\}$.

Theorem 3.12. *Given a $\mathcal{C}^\infty-$ring A and $\mathfrak{p} \in \mathrm{Spec}^\infty(A)$, we have:*

$$A\{A \setminus \mathfrak{p}^{-1}\}/\mathfrak{m}_\mathfrak{p} \cong (A/\mathfrak{p}) \left\{A/\mathfrak{p} \setminus \{0 + \mathfrak{p}\}^{-1}\right\}$$

We denote any of these constructions, when there is no danger of confusion, by $k_\mathfrak{p}(A)$.

3.3. Von Neumann Regular $\mathcal{C}^\infty$-Rings

Recall that a (commutative) von Neumann regular ring is a ring R in which the following sentence holds:

$$(\forall a \in R)(\exists! b \in R)((a^2 b = a)\&(b^2 a = b)).$$

A von Neumann regular $\mathcal{C}^\infty-$ring is, analogously, a $\mathcal{C}^\infty$-ring A such that $\widetilde{U}(A)$ is a von Neumann regular ring. The category of all von Neumann regular $\mathcal{C}^\infty-$rings is denoted by $\mathcal{C}^\infty$**vNRng**. This category is closed under quotients, so the congruences of any von Neumann regular $\mathcal{C}^\infty-$ring are classified by ideals (for detailed proofs of the results of this section, see [8]).

Whenever A is a von Neumann regular $\mathcal{C}^\infty-$ring, we have:

- For every $a \in A$ there is some idempotent $e \in A$ such that $A\{a^{-1}\} = A/\langle 1 - e\rangle$.
- Given any $S \subseteq A$, we have $\widetilde{U}(A\{S^{-1}\}) = \widetilde{U}(A)[S^{-1}]$. As a consequence:
- $\mathcal{C}^\infty$**vNRng** is closed in $\mathcal{C}^\infty$**Rng** under localizations;
- A is a $\mathcal{C}^\infty-$reduced $\mathcal{C}^\infty-$ring;

- $\mathrm{Spec}^{\infty}(A) = \mathrm{Spec}(A) = \mathrm{Specm}(A)$;
- For every $\mathfrak{p} \in \mathrm{Spec}^{\infty}(A)$, $A_{\{\mathfrak{p}\}}$ is a $\mathcal{C}^{\infty}-$field;

Apart from the facts given above, we have the following results:

- The inclusion functor $\iota : \mathcal{C}^{\infty}\mathbf{vNRng} \hookrightarrow \mathcal{C}^{\infty}\mathbf{Rng}$ creates filtered colimits
- The limit in $\mathcal{C}^{\infty}\mathbf{Rng}$ of a diagram of von Neumann-regular $\mathcal{C}^{\infty}-$rings is a von Neumann-regular $\mathcal{C}^{\infty}-$ring. In particular, $\mathcal{C}^{\infty}\mathbf{vNRng}$ is a complete category and the inclusion functor preserves all limits.
- The category $\mathcal{C}^{\infty}\mathbf{vNRng}$ is the smallest subcategory of $\mathcal{C}^{\infty}\mathbf{Rng}$ which contains every $\mathcal{C}^{\infty}-$field and is closed under limits.

Von Neumann regular $\mathcal{C}^{\infty}$-rings have a $\mathcal{C}^{\infty}$-spectrum with some "nice" properties. In fact, one characterizes a von Neumann regular $\mathcal{C}^{\infty}$-ring by means of the topology of its $\mathcal{C}^{\infty}$-spectrum. This is the content of the following:

Theorem 3.13. *A is a von Neumann regular $\mathcal{C}^{\infty}-$ring if, and only if A is $\mathcal{C}^{\infty}-$reduced and $\mathrm{Spec}^{\infty}(A)$ is a Boolean space.*

4. Order Theory for $\mathcal{C}^{\infty}$-Rings

There are at least two natural (complementary) approaches to the order theory of $\mathcal{C}^{\infty}$-ring. In one hand we have a *canonical binary relation* on each $\mathcal{C}^{\infty}$-ring, that, under mild conditions, is a partial order. On the other hand, we can develop a natural notion of *$\mathcal{C}^{\infty}$-real spectrum* of a general $\mathcal{C}^{\infty}$-ring.

Definition 4.1. Let A be a $\mathcal{C}^{\infty}-$ring. The **canonical binary relation** on A is

$$\prec_A = \{(a,b) \in A \times A \mid (\exists u \in A^{\times})(b - a = u^2)\}$$

The relation $\prec$ is preserved under $\mathcal{C}^{\infty}$-morphisms and is compatible with the operations $+$ and $\cdot$:

- $a \prec b, \Rightarrow a + x \prec b + x$;
- $0 \prec x, a \prec b \Rightarrow a.x \prec b.x$

Proposition 4.2. *Consider the $\mathcal{C}^{\infty}$-reduced $\mathcal{C}^{\infty}-$ring $A = \dfrac{\mathcal{C}^{\infty}(\mathbb{R}^E)}{I}$ (i.e. $I = \sqrt[\infty]{I}$), the its canonical relation can be characterized by:*

$$f + I \prec g + I \iff (\exists \varphi \in I)(\forall x \in Z(\varphi))(f(x) < g(x)),$$

where $Z(\varphi) = \varphi^{\dashv}[\{0\}] = \{x \in \mathbb{R}^E \mid \varphi(x) = 0\}$.

Proposition 4.3. *For any $\mathcal{C}^{\infty}$-reduced $\mathcal{C}^{\infty}-$ring A, we have:*

1. *The relation $\prec_A$ is a strict pre-order relation, whenever A is non-trivial.*

2. $$1 + \sum A^2 \subseteq A^\times,$$

   so every non trivial $\mathcal{C}^\infty-$ring is semi-real.

Note that, by the above result, -1 can't have a square root in any $\mathcal{C}^\infty$-field. Thus a quadratically closed field *cannot* be the underlying field of any $\mathcal{C}^\infty$-field. In particular, any algebraically closed field (like the field of complex numbers, $\mathbb{C}$) cannot be the underlying field of any $\mathcal{C}^\infty$-field.

The order theory of $\mathcal{C}^\infty$-fields is very interesting:

Proposition 4.4. *If A is a $\mathcal{C}^\infty-$field, then:*

1. *The canonical relation $\prec_A$ is a strict linear order, thus it holds the tricotomy law, or, equivalently:*

   $$a \neq 0 \Rightarrow (0 \prec a) \text{ or } (a \prec 0)$$

2. *Any $\mathcal{C}^\infty$-field F together with its canonical preorder $\prec$ is such that $\mathcal{U}(F)$ is a* **real closed field** *([17], Theorem 2.10).*

The above result on $\mathcal{C}^\infty$-fields, combined with the separation theorems, is essential to get remarkable results on other approach to the order theory of $\mathcal{C}^\infty$-ring based on the notion of:

Definition 4.5. Given a $\mathcal{C}^\infty$-ring A, a **$\mathcal{C}^\infty$-ordering** on A is a subset $P \subseteq A$ such that:

- $P + P \subseteq P$;
- $P \cdot P \subseteq P$;
- $P \cup (-P) = A$;
- $P \cap (-P) \in \mathrm{Spec}^\infty(A)$.

Definition 4.6. The $\mathcal{C}^\infty-$**real spectrum** of a $\mathcal{C}^\infty-$ring A is the set:

$$\mathrm{Sper}^\infty(A) = \{P \subseteq A | P \text{ is a } \mathcal{C}^\infty - \text{ordering on } A\}$$

together with the topology Har^∞, generated by:

$$\{H^\infty(a) | a \in A\},$$

where $H^\infty(a) = \{P \in \mathrm{Sper}^\infty(A) | a \in P \setminus (P \cap (-P))\}$.

Note that if A is a $\mathcal{C}^\infty$-field, then its $\mathcal{C}^\infty$-real spectrum is a singleton space, where its unique ordering is $P_A = \{a \in A : 0 \prec a\} \cup \{0\}$.

In general, we have the following remarkable result:

Theorem 4.7. *Given any $\mathcal{C}^\infty-$ring A, the function:*

$$\begin{array}{cccc} \mathrm{supp}^\infty : & \mathrm{Sper}^\infty(A) & \to & \mathrm{Spec}^\infty(A) \\ & P & \mapsto & P \cap (-P) \end{array}$$

is a **continuous bijection**.

5. Special topics concerning Von Neumann regular $\mathcal{C}^\infty$-rings

5.1. The von Neumann regular hull of a $\mathcal{C}^\infty$-ring

We can establish, by two different methods, that the subcategory of von Neumann regular $\mathcal{C}^\infty$-rings is reflective in the category of all $\mathcal{C}^\infty$-rings. We emphasise here a sheaf-theoretic construction, thus we start this subsection with the following:

Definition 5.1. A $\mathcal{C}^\infty-$space is a pair, $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X : \mathrm{Open}\,(X)^{\mathrm{op}} \to \mathcal{C}^\infty\mathbf{Rng}$ is a sheaf whose stalks are local $\mathcal{C}^\infty-$rings.

A morphism between two $\mathcal{C}^\infty-$spaces, $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is a pair:

$$(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

where $f : X \to Y$ is a continuous function and $f^\sharp : f^{\dashv}[\mathcal{O}_Y] \to \mathcal{O}_X$ is a morphism of sheaves such that for every $x \in X$,

$$f^\sharp_x : f^{\dashv}[\mathcal{O}_{Y,f(x)}] \to \mathcal{O}_{X,x}$$

is a local $\mathcal{C}^\infty-$homomorphism.

The category of $\mathcal{C}^\infty-$spaces and their morphisms is denoted by $\mathcal{C}^\infty\mathbf{Sp}$.

Proposition 5.2. *Given a $\mathcal{C}^\infty-$ring A, we have a (essentially) unique sheaf*

$$\Sigma_A : \mathrm{Open}\,(\mathrm{Spec}^\infty(A))^{\mathrm{op}} \to \mathcal{C}^\infty\mathbf{Rng}$$

such that for every $a \in A$ one has:

$$\Sigma_A(D^\infty(a)) \cong A\{a^{-1}\}$$

Moreover, this sheaf is such that all its stalks, $A_{\{\mathfrak{p}\}}$, are local $\mathcal{C}^\infty-$rings. We have a left-adjoint full and faithful functor:

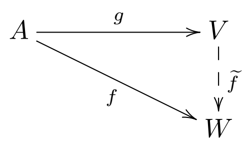
$$\Sigma : \mathcal{C}^\infty\mathbf{Rng} \to \mathcal{C}^\infty\mathbf{Sp}$$

Thus $\Gamma(\Sigma(A)) \cong A$.

We have already seen that whenever A is a von Neumann regular $\mathcal{C}^{\infty}-$ring,

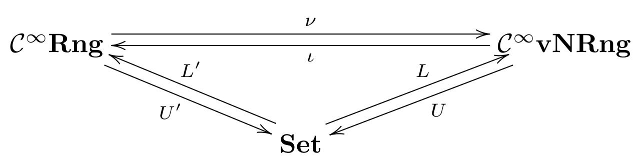
- $\mathrm{Spec}^{\infty}(A)$ is a Boolean space;
- For every $\mathfrak{p} \in \mathrm{Spec}^{\infty}(A)$, $A_{\{\mathfrak{p}\}}$ is a $\mathcal{C}^{\infty}-$field.

Definition 5.3. Given a $\mathcal{C}^{\infty}-$ring A, a von Neumann regular $\mathcal{C}^{\infty}-$ring V together with a $\mathcal{C}^{\infty}-$homomorphism $g : A \to V$ is a **von Neumann regular hull of** A if, and only if given any von Neumann regular $\mathcal{C}^{\infty}-$ring W and any $\mathcal{C}^{\infty}-$homomorphism $f : A \to W$, there is a unique $\mathcal{C}^{\infty}-$homomorphism $\widetilde{f} : V \to W$ such that the following diagram commutes:



Remark 5.4.
- Using free $\mathcal{C}^{\infty}-$rings, coproducts, quotients and colimits, one proves that **every** $\mathcal{C}^{\infty}-$**ring** A **has a von Neumann regular hull,** $(\mathrm{vN}(A), \nu : A \to \mathrm{vN}(A))$.
- It follows that $\iota : \mathcal{C}^{\infty}\mathbf{vNRng} \hookrightarrow \mathcal{C}^{\infty}\mathbf{Rng}$ has a left adjoint (say, ν), *i.e.*, $\mathcal{C}^{\infty}\mathbf{vNRng}$ is a reflective subcategory of $\mathcal{C}^{\infty}\mathbf{Rng}$.
- The forgetful functor $U : \mathcal{C}^{\infty}\mathbf{vNRng} \to \mathbf{Set}$ has a left-adjoint, $L : \mathbf{Set} \to \mathcal{C}^{\infty}\mathbf{vNRng}$.

We have the following diagram:



Given any $\mathcal{C}^{\infty}-$ring A, we know that $\mathrm{Spec}^{\infty}(A)$ is a spectral space. We can refine this topology in order to obtain a Boolean space. Such a topology is called **the constructible topology**, and it is generated by $\{D^{\infty}(a) \cap Z^{\infty}(b) | a, b \in A\}$, where $Z^{\infty}(b) = \mathrm{Spec}^{\infty}(A) \setminus D^{\infty}(b)$. The smooth spectrum together with the constructible topology will be denoted by $\mathrm{Spec}^{\infty-\mathrm{const}}(A)$.

Next we give some remarks that allows us, with a sheaf-theoretic method, to obtain the von Neumann-regular hull of a $\mathcal{C}^{\infty}$-ring.

- For each $\mathcal{C}^{\infty}-$ring, A, we build a presheaf P_A on the basis of the topology of $\mathrm{Spec}^{\infty-\mathrm{const}}(A)$ such that its stalks at each $\mathfrak{p} \in \mathrm{Spec}^{\infty}(A)$ are the $\mathcal{C}^{\infty}-$fields $k_{\mathfrak{p}}(A)$.

- Given any basic open set V of $\mathrm{Spec}^{\infty-\mathrm{const}}(A)$, there are $a, b \in A$ such that $V = D^{\infty}(a) \cap Z^{\infty}(b)$, so we define:

$$P_A(V) = \varinjlim_{V = D^{\infty}(a) \cap Z^{\infty}(b)} \left(A/\sqrt[\infty]{(b)} \right) \{ a + \sqrt[\infty]{(b)}^{-1} \}$$

- The stalk of this pre-sheaf at each $\mathfrak{p} \in \mathrm{Spec}^{\infty}(A)$ is the $\mathcal{C}^{\infty}-$field:

$$(P_A)_{\mathfrak{p}} = (A/\mathfrak{p}) \left\{ A/\mathfrak{p} \setminus \{0 + \mathfrak{p}\}^{-1} \right\} \cong k_{\mathfrak{p}}(A)$$

Now we proceed as follows:

- For each $\mathcal{C}^{\infty}$-ring A, $P_A : (\mathcal{B}(\mathrm{Spec}^{\infty-\mathrm{const}}(A)), \subseteq)^{\mathrm{op}} \to \mathcal{C}^{\infty}\mathbf{Rng}$[3] is a presheaf on the (canonical) basis of the constructible topology on $\mathrm{Spec}^{\infty}(A)$, whose stalks are isomorphic to $k_{\mathfrak{p}}(A)$, $\mathfrak{p} \in \mathrm{Spec}^{\infty}(A)$. Moreover there is a canonical $\mathcal{C}^{\infty}$-homomorphism:

$$\phi_A : A \to P_A(\mathrm{Spec}^{\infty-\mathrm{const}}(A))) \cong A/\sqrt[\infty]{(0)}$$

in such a way that we obtain a functor $A \overset{\mathbb{F}}{\mapsto} P_A(\mathrm{Spec}^{\infty-\mathrm{const}}(A)))$ and a natural transformation $\phi : \mathrm{Id} \Rightarrow \mathbb{F}$.

- For each $\mathcal{C}^{\infty}$-ring A, let G_A be the sheaf functor associated with the presheaf P_A - both defined on the basis of the constructible topology of the $\mathcal{C}^{\infty}$-spectrum of A, and both share the same (up to isomorphisms) stalks. Moreover, since there is a natural transformation $P_A \Rightarrow G_A$, there is a canonical $\mathcal{C}^{\infty}$-homomorphism $\gamma_A : A \to G_A(\mathrm{Spec}^{\infty-\mathrm{const}}(A)))$, in such a way that we obtain a functor $A \overset{\mathbb{G}}{\mapsto} G_A(\mathrm{Spec}^{\infty-\mathrm{const}}(A)))$ and a natural transformation $\gamma : \mathrm{Id} \Rightarrow \mathbb{G}$.

- For each $\mathcal{C}^{\infty}$-ring A, take H_A the (unique, up to isomorphism) sheaf extension G_A from the basis of the constructible topology to all the opens sets of this topology, so $H_A \restriction \cong G_A$, and both functors keep the same stalks (up to isomorphism).

Moreover, since there is a natural transformation (isomorphism) $G_A \Rightarrow H_A \restriction$, there is a canonical $\mathcal{C}^{\infty}$-homomorphism $\mu_A : A \to H_A(\mathrm{Spec}^{\infty-\mathrm{const}}(A)))$, in such a way that we obtain a functor:

$$A \overset{\mathbb{H}}{\mapsto} H_A(\mathrm{Spec}^{\infty-\mathrm{const}}(A)))$$

and a natural transformation $\mu : \mathrm{Id} \Rightarrow \mathbb{H}$.

- Since H_A is a sheaf of $\mathcal{C}^{\infty}$-rings, defined over all the open subsets of a Boolean space, and whose stalks are $\mathcal{C}^{\infty}$-fields, $\mathbb{H}(A) = H_A(\mathrm{Spec}^{\infty-\mathrm{const}}(A)))$ is a Von Neumann regular $\mathcal{C}^{\infty}$-ring. Thus, $A \overset{\mathbb{H}}{\mapsto} H_A(\mathrm{Spec}^{\infty-\mathrm{const}}(A)))$ determines a functor $\mathcal{C}^{\infty}\mathbf{Rng} \to \mathcal{C}^{\infty}\mathbf{vNRng}$ and we have a natural transformation $\mu : \mathrm{Id} \Rightarrow i \circ \mathbb{H}$.

[3] Here, $\mathcal{B}(\mathrm{Spec}^{\infty-\mathrm{const}}(A))$ denotes the basic open sets of $\mathrm{Spec}^{\infty-\mathrm{const}}(A)$

Theorem 5.5. (**bis**) *The inclusion functor* $\mathcal{C}^\infty\mathbf{vNRng} \hookrightarrow \mathcal{C}^\infty\mathbf{Rng}$ *has a left adjoint functor* $\mathrm{vN} : \mathcal{C}^\infty\mathbf{Rng} \to \mathcal{C}^\infty\mathbf{vNRng}$. *In more details: let A be any $\mathcal{C}^\infty-$ring. The pair $(\mathbb{H}(A), \mu_A : A \to \mathbb{H}(A))$ is the $\mathcal{C}^\infty-$von Neumann regular hull of A, that is, for every von Neumann-regular $\mathcal{C}^\infty-$ring V and for every $\mathcal{C}^\infty-$homomorphism $f : A \to V$ there is a unique $\mathcal{C}^\infty-$homomorphism $\widetilde{f} : \mathbb{H}(A) \to V$ such that the following diagram commutes:*

$$\begin{array}{ccc} A & \xrightarrow{\mu_A} & \mathbb{H}(A) \\ & \searrow^{f} & \downarrow{\scriptstyle \exists! \widetilde{f}} \\ & & V \end{array}$$

Proposition 5.6. *The functor* $\mathrm{vN} : \mathcal{C}^\infty\mathbf{Rng} \to \mathcal{C}^\infty\mathbf{vNRng}$ *preserves all colimits. In particular it preserves:*

- *directed inductive limits;*
- *coproducts (= $\mathcal{C}^\infty$-tensor products in $\mathcal{C}^\infty\mathbf{Rng}$);*
- *coequalizers/quotients.*

The following results are specific to the functor vN, i.e., they are not general consequences of it being a left adjoint functor[4]:

- vN preserves localizations;
- vN preserves *finite* products.

5.2. Von Neumann regular $\mathcal{C}^\infty-$rings and Boolean Algebras

We apply von Neumann regular $\mathcal{C}^\infty$-rings to naturally represent Boolean Algebras in a strong sense: i.e., not only all Boolean algebras are isomorphic to the Boolean algebra of idempotents of a von Neumann regular $\mathcal{C}^\infty$-ring, also every homomorphism between such boolean algebras of idempotents is induced by a $\mathcal{C}^\infty$-homomorphism.

Given any $\mathcal{C}^\infty-$ring A, let $\widetilde{B}(A) = \{e \in \widetilde{U}(A) | e^2 = e\}$,

$$\begin{array}{rccc} \widetilde{B}: & \mathcal{C}^\infty\mathbf{Rng} & \to & \mathbf{Bool} \\ & A & \mapsto & \widetilde{B}(A) \\ & A \xrightarrow{f} A' & \mapsto & \widetilde{B}(A) \xrightarrow{\widetilde{B}(f)} \widetilde{B}(A') \end{array}$$

is a (covariant) functor.

Now, if A is any $\mathcal{C}^\infty-$ring, we define the following homomorphism of Boolean algebras:

[4]The last one can be established by a analysis of the sheaf-theoretic construction of the Von Neumann regular hull.

$$\begin{array}{rccc} \jmath_A : & \widetilde{B}(A) & \to & \mathrm{Clopen}\,(\mathrm{Spec}^{\infty}(A)) \\ & e & \mapsto & D^{\infty}(e) = \{\mathfrak{p} \in \mathrm{Spec}^{\infty}\,(A) | e \notin \mathfrak{p}\} \end{array}$$

The map $\jmath_A : \widetilde{B}(A) \to \mathrm{Clopen}(\mathrm{Spec}^{\infty}\,(A))$ suggests that *the idempotent elements of the Boolean algebra* $\widetilde{B}(A)$ *associated with a* $\mathcal{C}^{\infty}-$*ring* A *hold a strong relationship with the canonical basis of the Zariski topology* of $\mathrm{Spec}^{\infty}(A)$.

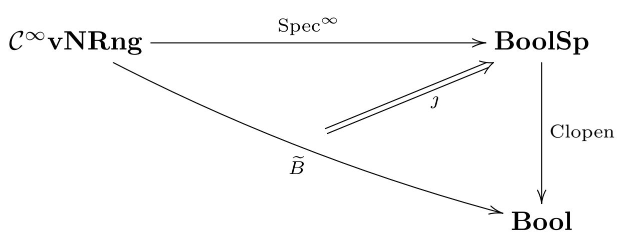
We have the following:

Theorem 5.7. *Let* A *be a von Neumann regular* $\mathcal{C}^{\infty}-$*ring. The map:*

$$\begin{array}{rccc} \jmath_A : & \widetilde{B}(A) & \to & \mathrm{Clopen}\,(\mathrm{Spec}^{\infty}(A)) \\ & e & \mapsto & D^{\infty}(e) = \{\mathfrak{p} \in \mathrm{Spec}^{\infty}\,(A) | e \notin \mathfrak{p}\} \end{array}$$

is an isomorphism of Boolean algebras.

Theorem 5.8. *We have the following diagram of categories, functors and natural isomorphisms:*



where **BoolSp** *denotes the category of all Boolean spaces and their morphisms.*

Lemma 5.9. *Let* (X, τ) *be a Boolean topological space. Define* $\mathcal{R}_X :=$

$$\Big\{R \subseteq X \times X | (R \text{ is an equivalence relation on } X) \ \& \ (\frac{X}{R} \text{ is a } \textbf{discrete}(= \textbf{finite}) \text{ quotient space})\Big\},$$

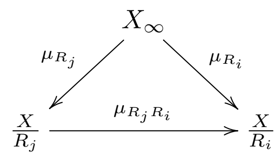
which is partially ordered by inclusion. Whenever $R_i, R_j \in \mathcal{R}$ *are such that* $R_j \subseteq R_i$, *we have the continuous surjective map:*

$$\begin{array}{rccc} \mu_{R_j R_i} : & \dfrac{X}{R_j} & \twoheadrightarrow & \dfrac{X}{R_i} \\ & [x]_{R_j} & \mapsto & [x]_{R_j} \end{array}$$

so we have the inverse system $\{\frac{X}{R_i}; \mu_{R_j R_i} : \frac{X}{R_j} \to \frac{X}{R_i}\}$. *By definition*

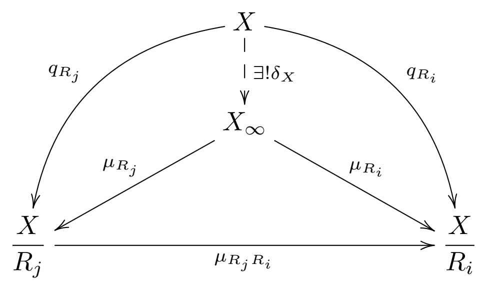
$$\lim_{\substack{\longleftarrow \\ R \in \mathcal{R}}} \frac{X}{R} = \Bigg\{([x]_{R_i})_{R_i \in \mathcal{R}} \in \prod_{R \in \mathcal{R}} \frac{X}{R} | (R_j \subseteq R_i \to (p_{R_i} (([x]_{R_i})_{R_i \in \mathcal{R}}) = \mu_{R_j R_i} \circ p_{R_j}(([x]_{R_i})_{R_i \in \mathcal{R}})))\Bigg\}$$

Let X_∞ *denote* $\varprojlim_{R\in\mathcal{R}} \frac{X}{R}$*, so we have the following cone:*



We consider X_∞ *together with the induced subspace topology of* $\prod_{R\in\mathcal{R}} \frac{X}{R}$.

By the universal property of X_∞*, there is a unique continuous map* $\delta_X : X \to X_\infty$ *such that the following diagram commutes:*



Then $\delta_X : X \to X_\infty$ *is a homeomorphism, so:*

$$\delta_X : X \stackrel{\approx}{\to} \varprojlim_{R\in\mathcal{R}} \frac{X}{R}$$

that is, X *a profinite space.*

Theorem 5.10. *Let* $\mathbb{K}$ *be a* $\mathcal{C}^\infty-$*field and define the contravariant functor:*

$$\begin{array}{rccc} \widehat{k}: & \mathbf{BoolSp} & \to & \mathcal{C}^\infty\mathbf{vNRng} \\ & (X,\tau) & \mapsto & R_X := \varinjlim_{R\in\mathcal{R}} \mathbb{K}^{U\left(\frac{X}{R}\right)} \end{array},$$

where $U(X/R)$ *denotes the underlying set of* X/R*. Then there is a natural isomorphism:*

$$\varepsilon : \mathrm{Id}_{\mathbf{BoolSp}} \stackrel{\cong}{\Rightarrow} \mathrm{Spec}^\infty \circ \widehat{k}.$$

Thus, the functor $\mathrm{Spec}^\infty : \mathcal{C}^\infty\mathbf{vNRng} \to \mathbf{BoolSp}$ *is full and* *<u>isomorphism-dense</u>, that is, for each Boolean space* (X,τ) *there is a von Neumann-regular* $\mathcal{C}^\infty-$*ring,* R_X*, such that:*

$$\mathrm{Spec}^\infty(R_X) \approx (X,\tau).$$

Theorem 5.11. *Let* $\mathbb{K}$ *be a* $\mathcal{C}^\infty-$*field. Define the covariant functor (composition of contravariant functors):*

$$\check{K} = \widehat{k} \circ \text{Stone} : \mathbf{Bool} \to \mathcal{C}^{\infty}\mathbf{vNRng}.$$

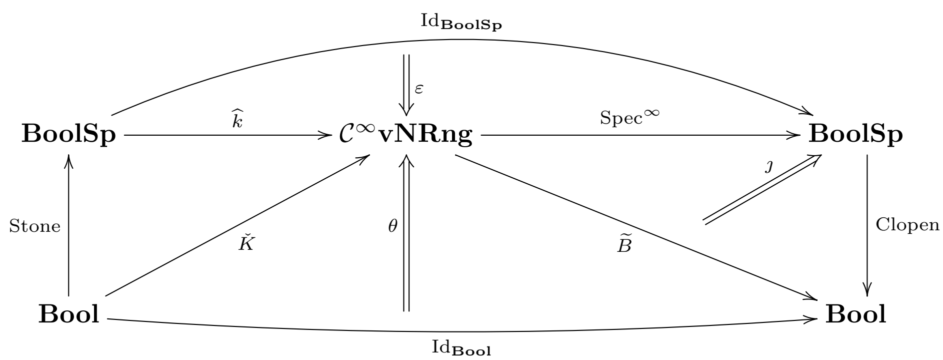
There is a natural isomorphism:

$$\theta : \text{Id}_{\mathbf{Bool}} \overset{\cong}{\Rightarrow} \widetilde{B} \circ \check{K}.$$

Thus, the functor $\widetilde{B} : \mathcal{C}^{\infty}\mathbf{vNRng} \to \mathbf{Bool}$ is full and isomorphism-dense.
In particular: given any $\mathcal{C}^{\infty}-$field $\mathbb{K}$ and any Boolean algebra B, there is a von Neumann regular $\mathcal{C}^{\infty}-$ring which is a $\mathbb{K}-$algebra, $\check{K}(B)$, such that

$$\widetilde{B}(\check{K}(B)) \cong B.$$

The following diagram summarizes the main functorial connections stablished in this section:



6. Some Logical Aspects of $\mathcal{C}^{\infty}-$rings

It is a well-known fact that any first-order geometric mathematical theory admits a (unique up to equivalence) *classifying topos*, which contains a *universal model* of the theory (cf. [16] and [15], for example). Since $\mathcal{C}^{\infty}$-rings can be described as models of geometric axioms, such a theory has a classifying topos. The next result describes the classifying topos of the theory of $\mathcal{C}^{\infty}$-rings.

Theorem 6.1. *The category*

$$\mathcal{C}^{\infty}\mathbf{Rng}_{\text{fp}}^{\text{op}}$$

is a category with finite limits freely generated by the $\mathcal{C}^{\infty}-$ring-object $\mathcal{C}^{\infty}(\mathbb{R})$, i.e., for any category with finite limits $\mathcal{C}$, the evaluation of a left-exact functor $F : \mathcal{C}^{\infty}\mathbf{Rng}_{\text{fp}}^{\text{op}} \to \mathcal{C}$ at $\mathcal{C}^{\infty}(\mathbb{R})$ yields the following equivalence of categories:

$$\begin{array}{rccc} \text{ev}_{\mathcal{C}^{\infty}(\mathbb{R})} : & \mathbf{Lex}\,(\mathcal{C}^{\infty}\mathbf{Rng}_{\text{fp}}^{\text{op}}, \mathcal{C}) & \to & \underline{\mathcal{C}^{\infty} - \text{Rings}}\,(\mathcal{C}) \\ & F & \mapsto & F(\mathcal{C}^{\infty}(\mathbb{R})) \end{array}$$

Theorem 6.2. *The presheaf topos* $\mathbf{Sets}^{\mathcal{C}^\infty-\mathbf{Rng}_{\rm fp}}$ *is a classifying topos for* $\mathcal{C}^\infty-$*rings, and the universal* $\mathcal{C}^\infty-$*ring* R *is the* $\mathcal{C}^\infty-$*ring object in* $\mathbf{Sets}^{\mathcal{C}^\infty-\mathbf{Rng}_{\rm fp}}$ *given by the forgetful functor from* $\mathcal{C}^\infty\mathbf{Rng}_{\rm fp}$ *to* $\mathbf{Set}$*. Thus, for any Grothendieck topos* $\mathcal{E}$ *there is an equivalence of categories, natural in* $\mathcal{E}$*:*

$$\begin{array}{ccc} {\rm Geom}\,(\mathcal{E}, \mathbf{Sets}^{\mathcal{C}^\infty-\mathbf{Rng}_{\rm fp}}) & \to & \underline{\mathcal{C}^\infty{\rm Ring}\,(\mathcal{E})} \\ f & \mapsto & f^*(R) \end{array}$$

6.1. The Smooth Zariski Site

Let $\mathcal{C} = \mathcal{C}^\infty\mathbf{Rng}_{\rm fp}{}^{\rm op}$ be the (skeleton of the) opposite category of the finitely presented $\mathcal{C}^\infty-$rings. We are going to describe the **smooth Grothendieck-Zariski pretopology** on $\mathcal{C}^\infty\mathbf{Rng}_{\rm fp}{}^{\rm op}$.

Given a finitely presentable $\mathcal{C}^\infty$-ring A, we will define the collection of all cocovering families of A, denoted by coCov (A). Together this will yield a map:

$$\begin{array}{cccc} {\rm coCov}: & {\rm Obj}\,(\mathcal{C}^\infty\mathbf{Rng}_{\rm fp}) & \to & \wp(\wp({\rm Mor}\,(\mathcal{C}^\infty\mathbf{Rng}_{\rm fp}))) \\ & A & \mapsto & {\rm coCov}\,(A) \end{array}$$

For every $n-$tuple of elements of A, $(a_1, \cdots, a_n) \in A \times A \times \cdots, \times A$, $n \in \mathbb{N}$, such that $\langle a_1, a_2, \cdots, a_n\rangle = A$, a family of $\mathcal{C}^\infty-$homomorphisms $k_i : A \to B_i$ such that:

(i) For every $i \in \{1, \cdots, n\}$, $k_i(a_i) \in {B_i}^\times$;

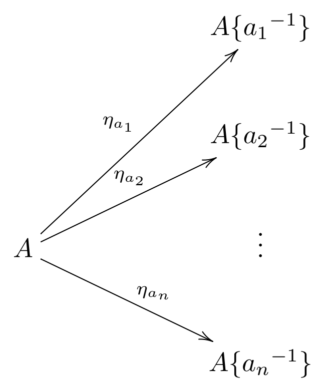
(ii) For every $i \in \{1, \cdots, n\}$, if $k_i(a) = 0$ for some $a \in A$, there is some $s_i \in \{a_i\}^{\infty-{\rm sat}}$ such that $a \cdot s_i = 0$;

(iii) For every $b \in B_i$ there are $c \in \{a_i\}^{\infty-{\rm sat}}$ and $d \in A$ such that $b \cdot k_i(c) = k_i(d)$.

will be a co-covering family of the finitely presentable $\mathcal{C}^\infty-$ring A.

In terms of diagrams, the "generators" of the co-covering families are:

$$\begin{array}{c} \langle a_1, \cdots, a_n\rangle = A \\ \Big\Updownarrow \\ \bigcup_{i=1}^n D^\infty(a_i) = {\rm Spec}^\infty(A) \end{array}$$



Given a finitely presented $\mathcal{C}^\infty-$ring A, a **covering family** for A in $\mathcal{C}^\infty\mathbf{Rng}_{\rm fp}^{\rm op}$ is given by:

$$\mathrm{Cov}\,(A) = \{f_i^{\rm op} : B_i \to A | (f_i : A \to B_i)_{i \in I} \in \mathrm{coCov}\,(A)\}.$$

We prove that Cov is a Grothendieck pretopology on the category[5] $\mathcal{C}^\infty\mathbf{Rng}_{\rm fp}^{\rm op}$. We have the Grothendieck topology on $\mathcal{C}^\infty\mathbf{Rng}_{\rm fp}^{\rm op}$:

$$J_{\rm Cov} : \mathrm{Obj}\,(\mathcal{C}^\infty\mathbf{Rng}_{\rm fp}) \to \wp(\wp(\mathrm{Mor}\,(\mathcal{C}^\infty\mathbf{Rng}_{\rm fp})))$$

given by:

$$J_{\rm Cov}\,(A) := \{\overleftarrow{S} \subseteq \bigcup_{B \in \mathrm{Obj}\,(\mathcal{C}^\infty\mathbf{Rng}_{\rm fp})} \mathrm{Hom}_{\mathcal{C}^\infty\mathbf{Rng}_{\rm fp}}\,(B, A) | S \in \mathrm{Cov}\,(A)\},$$

where $\overleftarrow{S}$ denotes the sieve completion of S, turning $(\mathcal{C}^\infty\mathbf{Rng}_{\rm fp}^{\rm op}, J_{\rm Cov})$ into a small site, that we denote by $\mathcal{Z}^\infty$.

The forgetful functor $\mathcal{O} : \mathcal{C}^\infty\mathbf{Rng}_{\rm fp} \to \mathbf{Sets}$ is called the structure sheaf of the Grothendieck-Zariski smooth topos. The smooth Grothendieck-Zariski topology $J_{\rm Cov}$ constructed above is subcanonical.

The Mitchell-Bénabou language motivates us to define a local $\mathcal{C}^\infty-$ring object in a topos $\mathcal{E}$: it is a $\mathcal{C}^\infty-$ring object R in $\mathcal{E}$ for which the (geometric) formula:

$$(\forall a \in R)((\exists b \in R)(a \cdot b = 1) \vee (\exists b \in R)((1 - a) \cdot b = 1))$$

is valid. By definition, this means that the union of the subobjects:

$$\{a \in R | \exists b \in R(a \cdot b = 1)\} \rightarrowtail R,$$
$$\{a \in R | \exists b \in R((1 - a) \cdot b = 1)\} \rightarrowtail R$$

of R is all of R. Equivalently, consider the two subobjects of the product $R \times R$ defined by:

$$\begin{cases} U = \{(a, b) \in R \times R | a \cdot b = 1\} \rightarrowtail R \times R \\ V = \{(a, b) \in R \times R | (1 - a) \cdot b = 1\} \rightarrowtail R \times R \end{cases} \tag{1}$$

The $\mathcal{C}^\infty-$ring object R is local if, and only if, the two composites $U \rightarrowtail R \times R \stackrel{\pi_1}{\to} R$ and $V \rightarrowtail R \times R \stackrel{\pi_1}{\to} R$ form an epimorphic family in $\mathcal{E}$.

Theorem 6.3. *The smooth Grothendieck-Zariski topos, $\mathcal{Z}^\infty$ is a classifying topos for local $\mathcal{C}^\infty-$rings, i.e., for any Grothendieck topos $\mathcal{E}$, there is an equivalence of categories:*

$$\mathrm{Geom}\,(\mathcal{E}, \mathcal{Z}^\infty) \simeq \underline{\mathcal{C}^\infty\mathrm{LocRng}}\,(\mathcal{E}) \tag{2}$$

[5]In fact, a locally small category that is equivalent to a small category.

where $\mathcal{C}^\infty\mathrm{LocRng}\,(\mathcal{E})$ *is the category of local* $\mathcal{C}^\infty-$*ring-objects in* $\mathcal{E}$*.*

The universal local $\mathcal{C}^\infty-$*ring is the structure sheaf* $\mathcal{O}$ *of the Grothendieck-Zariski smooth topos*

Theorem 6.4. *The category*

$$\mathcal{C}^\infty\mathbf{vNRng}_{\mathrm{fp}}^{\mathrm{op}}$$

is a category with finite limits freely generated by the von Neumann regular $\mathcal{C}^\infty-$*ring* $\mathrm{vN}(\mathcal{C}^\infty(\mathbb{R}))$*, i.e., for any category with finite limits* $\mathcal{C}$*, the evaluation of a left-exact functor* $F:\mathcal{C}^\infty\mathbf{vNRng}_{\mathrm{fp}}^{\mathrm{op}}\to\mathcal{C}$ *at* $\mathrm{vN}\,(\mathcal{C}^\infty(\mathbb{R}))$ *yields the following equivalence of categories:*

$$\begin{array}{rccc} \mathrm{ev}_{\mathrm{vN}(\mathcal{C}^\infty(\mathbb{R}))}: & \mathbf{Lex}\,(\mathcal{C}^\infty\mathbf{vNRng}_{\mathrm{fp}}^{\mathrm{op}},\mathcal{C}) & \to & \underline{\mathcal{C}^\infty-\mathbf{vNRng}}\,(\mathcal{C}) \\ & F & \mapsto & F(\mathrm{vN}\,(\mathcal{C}^\infty(\mathbb{R}))) \end{array}$$

Theorem 6.5. *The presheaf topos* $\mathbf{Sets}^{\mathcal{C}^\infty\mathbf{vNRng}_{\mathrm{fp}}}$ *is a classifying topos for von Neumann regular* $\mathcal{C}^\infty-$*rings, and the universal von Neumann regular* $\mathcal{C}^\infty-$*ring* R *is the von Neumann regular* $\mathcal{C}^\infty-$*ring object in*

$$\mathbf{Sets}^{\mathcal{C}^\infty-\mathbf{vNRng}_{\mathrm{fp}}}$$

given by the forgetful functor from $\mathcal{C}^\infty\mathbf{vNRng}_{\mathrm{fp}}$ *to* $\mathbf{Set}$*. Thus, for any Grothendieck topos* $\mathcal{E}$ *there is an equivalence of categories, natural in* $\mathcal{E}$*:*

$$\begin{array}{ccc} \mathrm{Geom}\,(\mathcal{E},\mathbf{Sets}^{\mathcal{C}^\infty\mathbf{vNRng}_{\mathrm{fp}}}) & \to & \mathcal{C}^\infty\mathbf{vNRng}\,(\mathcal{E}) \\ f & \mapsto & f^*(R) \end{array}$$

7. Future Works

As a continuation of the former research topics, we describe some promising topics on which we are focusing next. The quest for answering the following questions and researching the following topics is the next step of our research.

(I) Analyze special classes of $\mathcal{C}^\infty$-rings.

(II) Develop Real Algebraic Geometry and Quadratic Form Theory of $\mathcal{C}^\infty$-rings.

(III) Is a Gelfand $\mathcal{C}^\infty-$ring the global section of locally ringed space over a compact Hausdorff space?

(IV) If so, can we obtain from that, by sheaf theoretic methods, a ($\mathcal{C}^\infty-$Gelfand hull for each ($\mathcal{C}^\infty-$)ring?

(V) Can we apply these ideas and methods to the case of sheaf of $\mathcal{C}^\infty$-rings whose stalks are archimedian local $\mathcal{C}^\infty-$rings (see [18], [12])?

(VI) And in the vein of Grothendieck topologies, are there natural and useful versions of étale ([11]) and/or real-étale ([20]) sites in the setting of $\mathcal{C}^\infty$-rings?

(VII) Are there other sensible descriptions of classifying toposes for other distinguished classes of $\mathcal{C}^\infty-$rings?

(VIII) In particular, is there a nice description of the theory of von Neumann regular $\mathcal{C}^\infty-$rings in the language of $\mathcal{C}^\infty$-rings?

(IX) It is natural to ask if the class of $\mathcal{C}^\infty$-fields is model-complete in the language of $\mathcal{C}^\infty$-rings or even admits elimination of quantifiers (possibly in the language expanded by a unary predicate for the positive cone of a ordering). If the former holds, then the relation $\mathcal{R}$ between pairs of morphism with the same source and targets $\mathcal{C}^\infty$-fields, considered in **Section 5** of **Chapter 2** of [3], that encodes Sper^∞, is already a transitive relation (as occurs in the algebraic case). If the latter holds, then it is possible to adapt the definition and results provided in [19] on "Model-theoretic Spectra" and describe "logically" the spectral topological spaces $\mathrm{Spec}^\infty(A)$ and/or $\mathrm{Sper}^\infty(A)$ as certain classes of equivalence of homomorphisms of A into models of a nice theory T. Moreover, since the techniques in this paper provide structural sheaves of "definable functions", we could compare them with others previously defined and determine other new natural model-theoretic spectra in $\mathcal{C}^\infty$-structures.

(X) After a good understanding of the model-theory of distinguished classes of $\mathcal{C}^\infty$-rings, it is natural consider model-theoretic aspects of simple constructions/structures related to the $\mathcal{C}^\infty$. In particular, should be better studied the class of modules over $\mathcal{C}^\infty$-rings, that has been considered by D. Joyce in the recent development of the Algebraic Geometry of $\mathcal{C}^\infty$-rings ([12]). It will be interesting start the model-theoretic analysis of Modules over Finitely Generated $\mathcal{C}^\infty$-rings in the vein of [13]: by defining a 2 sorted language that contains axioms for a $\mathcal{C}^\infty$-ring part and a module of it and (at least) symbols for the Grassmanians to express linear (in)dependence without the need of quantifiers

(XI) Finally there is also a mathematical aspect of $\mathcal{C}^\infty$-rings that seems to be only laterally considered: since each free $\mathcal{C}^\infty$-ring $\mathcal{C}^\infty(\mathbb{R}^X)$ encodes many possible $\mathbb{R}$-derivations, every "smooth polynomial ring" $A\{X\}$ in the set of variables X with coefficients over a $\mathcal{C}^\infty$- ring A admits many A-derivations. Thus classes of $\mathcal{C}^\infty$-rings endowed with derivations should be interesting and deserve a systematic study under many aspects, including the model-theoretic one.

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