Sheaf Categories on semicartesian monoidal categories: logical and cohomological aspects

Categorías de haces en categorías monoidales semicartesianas: aspectos lógicos y cohomológicos

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Let $(\mathcal{O}(X), \subseteq)$ denote the category whose objects are the elements of the set $\mathcal{O}(X)$ of all open sets of a topological space X, and the morphisms are given by the inclusion $\subseteq$. A presheaf on a topological space X with values in Set is a functor $F : (\mathcal{O}(X), \subseteq)^{op} \to Set$. Sheaves are presheaves that satisfy a certain gluing property and, categorically, they can be described as follows: for all U an open of X and all $U = \bigcup_{i \in I} U_i$ an open cover of U a presheaf is a sheaf if the following diagram is an equalizer in Set

$$F(U) \xrightarrow{e} \prod_{i \in I} F(U_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{(i,j) \in I \times I} F(U_i \cap U_j)$$

where:

1. $e(t) = \{t_{|_{U_i}} \mid i \in I\}, \;\; t \in F(U)$
2. $p((t_k)_{k \in I}) = (t_{i_{|_{U_i \cap U_j}}})_{(i,j) \in I \times I}$
   $q((t_k)_{k \in I}) = (t_{j_{|_{U_i \cap U_j}}})_{(i,j) \in I \times I}, \;\; (t_k)_{k \in I} \in \prod_{k \in I} F(U_k)$

In other words, F is sheaf if, and only if, given U an open of X and an open cover $U = \bigcup_{i \in I} U_i$ of U, for any $s_i \in F(U_i)$ a *compatible family* (i.e., such that $s_{i_{|_{U_i \cap U_j}}} = s_{j_{|_{U_i \cap U_j}}}$ for all $i, j \in I$) there is a unique $s \in F(U)$ such that $s_{|_{U_i}} = s_i, i \in I$. We say s is the *gluing* of the compatible family.

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Two concepts are essential on sheaf's definition: the covering for a open subset U of X, as a union of smaller open subsets, and the intersection between the open subsets that are part of the covering. It is well known that we can replace the union with the join and the intersection by the meet to obtain sheaves on locales (complete lattices in which finite meets distribute over arbitrary joins). Equivalently, locales are complete Heyting algebras. More generally, Grothendieck pretopologies provide the notion of coverings for any small category $\mathcal{C}$ with pullbacks, and a sheaf on $\mathcal{C}$ is defined by replacing the intersection of open subsets with pullbacks of objects of $\mathcal{C}$. In other words, a sheaf on $\mathcal{C}$ is a presheaf $F : \mathcal{C}^{op} \to Set$ such that for each object C of $\mathcal{C}$ and all coverings $\{f_i : C_i \to C\}$ of C we have an equalizer diagram of the form:

$$F(C) \longrightarrow \prod_i F(C_i) \rightrightarrows \prod_{i,j} F(C_i \times_C C_j)$$

Further, given a Grothendieck pretopology, there is a *generated Grothendieck topology*. A presheaf $F : \mathcal{C}^{op} \to Set$ is a sheaf for the Grothendieck pretopology iff it is a sheaf for the generated Grothendieck topology generated by the pretopology [9, Chapter III. 4, Proposition 1]. Then we obtain *Grothendieck toposes* – any category equivalent to the category of sheaves $Sh(\mathcal{C}, J)$, where $\mathcal{C}$ is a small category and J is a Grothendieck topology. The observation, by W. Lawvere and M. Tierney in the 1970s, that a Grothendieck topos has categorical properties that make it close to the category Set of all sets and functions gave rise to *elementary toposes* — cartesian closed categories with pullbacks, a terminal object, and a subobject classifier. Currently, elementary toposes are used in Foundations, having an internal language (of Mitchell-Bénabou) and correspondent semantic (of Kripke-Joyal), allowing variations of Cohen's forcing techniques in higher-order logic in terms of category theory.

In this work, we expand topos theory investigating what happens if we replace the meet operation $\wedge$ with a more general operation $\odot$ of a (commutative) *quantale* $(Q, \leq, \odot)$, i.e, a complete lattice $(Q, \leq)$ with a (commutative) monoid $(Q, \odot)$ such that $a \odot \bigvee_{i \in I} b_i = \bigvee_{i \in I} a \odot b_i$. There are other conceptions of sheaves on quantales in the literature ([4, 5, 7, 11, 12]), but we propose an original one: a sheaf on a quantale is defined as sheaf on a locale, we maintain that the cover of an element U is given by joins $U = \bigvee_{i \in I} U_i$ but we replace the meet $\wedge$ with the operation $\odot$. Note that maps of the form $F(U_i) \to F(U_i \odot U_j)$ are obtained if $U_i \odot U_j \leq U_i$, which is guaranteed for *semicartesian* quantales. So, actually, we are studying sheaves on semicartesian quantales.

Moreover, observe that while $\wedge$ is a pullback in the category $(H, \leq)$ given by a locale H, the operation $\odot$ may not be a pullback in $(Q, \leq)$, which means our notion of a sheaf on quantales may not be a sheaf in the usual sense and then we wonder: is it possible to define sheaves for categories that, instead of having pullbacks, have some kind of generalization of a pullback? How to define such a generalization? We answer this by considering *pseudo-pullbacks*: Let $(\mathcal{C}, \otimes, 1)$ be a semicartesian monoidal category with equalizers. The pseudo-pullback of morphisms $f : A \to C$ and $g : B \to C$ is the equalizer of the parallel arrows

$$A \otimes B \underset{g \circ \pi_2}{\overset{f \circ \pi_1}{\rightrightarrows}} C \quad \text{where } (A \otimes B \overset{\pi_1}{\to} A) \;=\; (A \otimes B \overset{id_A \otimes !_B}{\to} A \otimes 1 \underset{\cong}{\overset{\rho_A}{\to}} A) \text{ and}$$
$$(A \otimes B \overset{\pi_2}{\to} B) \;=\; (A \otimes B \overset{!_A \otimes id_B}{\to} 1 \otimes B \underset{\cong}{\overset{\lambda_B}{\to}} B)$$

then by weakening the hypothesis that the category in question has pullbacks, i.e. equalizers and non-empty finite products, to only ask it to have equalizers and some monoidal structure. Thereafter, we substitute the pullback in the Grothendieck pretopology with the pseudo-pullback and define a *Grothendieck prelopology*, with additional modifications. Thus, we expand the notion of sheaves using the description with an equalizer diagram but with a more general view of covering. Warning: to encompass our notion of sheaves of quantales, it is not enough to just replace the pullback with the pseudo-pullback in the *stability axiom* in the definition of a Grothendieck pretopology. We provide the complete definition of a Grothendieck prelopology and the respective notion of a sheaf for semicartesian categories with pullbacks in the PhD thesis [15].

Since the monoidal operation $\odot$ is a pseudo-pullback in the semicartesian monoidal category $(Q, \leq)$, our sheaves on quantales are a particular case of $Sh(\mathcal{C}, L)$, where L is a Grothendieck prelopology. Moreover, when the monoidal structure is given by the cartesian product, the pseudo-pullback is the pullback, and the Grothendieck prelopology is the Grothendieck pretopology. In fact, we are generalizing the sheaf definition, and we are studying ways of expanding topos theory and its applications.

The sheafification functor and the monoidal closed structure: In [15, Proposition 4.2.13], we proved that $Sh(\mathcal{C}, L)$ is λ-orthogonality class in $PSh(\mathcal{C})$. It is known that:

Theorem 0.1. *[1, Theorem 1.39] Let $\mathcal{K}$ be a locally λ-presentable category. The following conditions on a full subcategory $\mathcal{A}$ of $\mathcal{K}$ are equivalent:*

(i) $\mathcal{A}$ is a λ-orthogonality class in $\mathcal{K}$;

(ii) $\mathcal{A}$ is a reflective subcategory of $\mathcal{K}$ closed under λ-directed colimits.

Furthermore, $\mathcal{A}$ is locally λ-presentable.

Therefore, the inclusion $Sh(\mathcal{C}, L) \hookrightarrow Set^{\mathcal{C}^{op}}$ has a left adjoint that we call *sheafification functor*. In [15, Proposition 3.2.5] we proved that if F is a sheaf on Q and u is a fixed element of Q then $F(u \odot -)$ is a sheaf on Q. Since

Theorem 0.2. *[6, Proposition 1.1] Let $C = (C, \otimes, 1)$ be a small monoidal category. A reflective embedding $a \dashv i : D \to PSh(C)$ admits normal enrichment if and only if the functor $F(U \otimes -)$ is isomorphic to some object in D whenever F is an object of D and U is an object of C.*

Then we conclude that our reflective embedding – with a being the sheafification functor and $i : PSh(Q) \to Sh(Q)$ the inclusion – admits normal enrichment. This, in particular, means that $Sh(Q)$ is monoidal closed.

Expanded sheaf cohomology: We want to expand the sheaf cohomology theory available in the literature (as exposed in [8]) and investigate how the cohomology of $Sh(\mathcal{C}, L)$ is related with other (co)homologies and K-theories. In particular, we hope to relate sheaf cohomology with Hochschild (co)homology, and apply the theory in contexts whose cohomological constructions seem to be in stage of development, as it happens in functional analysis for C^*-algebras in [10] and for Banach spaces in [13]. At the moment, we replicated the definition of Čech cohomology for topological spaces but for commutative ring with unity. Using an adequate quantale morphism τ between the locale $\mathcal{O}(X)$ of open subsets of a topological space X and the quantale $\mathcal{I}(C(X))$ of ideals of the ring of continuous real-valued functions $C(X)$, we proved that the Čech cohomology group of X with coefficients in any sheaf F on $\mathcal{O}(X)$ is isomorphic to the Čech cohomology group of $C(X)$ with coefficients in the sheaf $F \circ \tau$ on $\mathcal{I}(C(X))$. Such construction is avaible at [15, Theorem 4.5.8].

$Sh(\mathcal{C}, L)$ **is not an elementary topos:** Given an elementary topos $\mathcal{E}$ and E any object of $\mathcal{E}$ the poset of subobjects of E is a Heyting algebra. On the other hand, the monoidal structure of $Sh(Q)$ induces a binary operation on the lattice of subobjects of the terminal sheaf on Q. Concerning these points, we have established the following:

Theorem 0.3. *[15, Theorem 3.4.6] Let Q be a commutative semicartesian quantale. Then the lattice of subobjetcs of the terminal sheaf in $Sh(Q)$ is quantalic isomorphic to Q.*

Where the terminal sheaf is $\mathbf{1} = Hom_Q(-, 1)$, with $1 = \top$ being the top element of Q, and the isomorphism is the composition of isomorphisms $h_Q = j_Q \circ i_Q \circ h_Q$ such that

$$\begin{aligned} h_Q \colon Q &\to Represented(Sh(Q)) \\ q &\mapsto Q(-, q) \end{aligned}$$

$$\begin{aligned} i_Q \colon Represented(Sh(Q)) &\to Representable(Sh(Q))/isos \\ Q(-, q) &\mapsto [Q(-, q)]_{iso} \end{aligned}$$

$$\begin{aligned} j_Q \colon Representable(Sh(Q))/isos &\to Sub(\mathbf{1}) \\ [R]_{iso} &\mapsto [R \cong Q(-, q) \rightarrowtail Q(-, 1) \cong \mathbf{1}]_{iso}. \end{aligned}$$

So, if Q does not have a localic structure in addition to the quantalic one, that is, if the meet does not distribute over the arbitrary joins, we conclude that $Sh(Q)$ does not have subobject classifier. This suggests that $Sh(\mathcal{C}, L)$ is a monoidal closed category instead of a cartesian closed category and the corresponding internal logic would go from intuitionistic logic to some version of affine logic (a special case of linear logic [14]) because of its relation with the quantalic structure. In the future, we plan to generalize elementary toposes from an axiomatic description of $Sh(\mathcal{C}, L)$.

Candidates for subobject classifier: Note that if $\mathcal{C}$ is a category with all finite limits with subobject classifier, then the poset $Sub(C)$ of subobjects of C, for every object C in $\mathcal{C}$, meets distribute over any existing join. So, in general, we do not expect that $Sh(Q)$ has subobject classifier. In [16], we defined two "best idempotent approximation functors" $(-)^-$ and $(-)^+$ from the commutative semicartesian quantale Q to the locale $Idem(Q)$. Then, inspired by the subobject classifier of the category of sheaves on a locale, we defined sheaves Ω^- and Ω^+. The sheaf Ω^- is not a subobject classifier but is "essentially" classified a certain class of monomorphisms (which we called *dense*), thus providing a possible alternative for what should be subobject classifier for those generalized toposes we envision. The functor Ω^+ actually is a subobject classifier, however, we had to impose extra conditions over the quantale Q. Therefore, such extra conditions are implying that Q has an underlying localic structure.

Monoidal toposes: There are some clues that even the analysis of presheaf categories over a monoidal semicartesian small category will be of mathematical interest to particular applications, since they immediately provide examples of monoidal toposes.

This is part of a bigger project about monoidal generalizations of topos theory, with Caio Mendes, and José Alvim from the University of São Paulo [2], [3].

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