Horn filter pairs and Craig interpolation in Propositional Logic

Pares filtro de Horn y interpolación de Craig en Lógica Proposional

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A filter pair ([1]) can be seen as a presentation of a logic, different in style from the usual syntactic presentations in terms of axioms and rules, or semantic presentations in terms of matrices and also a tool for creating and analyzing logics. The notion of filter pair can be developed into several different directions and applications. In this work we will introduce the κ-Horn filter pairs and apply this notion to obtain Craig interpolation results on certain classes of propositional logics, including some without algebraic semantics. The proofs of all assertions will be presented in full details in [5].

From now on, κ will designate a infinite regular cardinal.

Definition 1. A κ-filter pair ([4]) for the signature Σ is a contravariant functor G from Σ-algebras to κ-algebraic lattices together with a natural transformation $i\colon G \to \wp(-)$ from G to the functor taking an algebra to the power set of its underlying set, which preserves arbitrary infima and κ-directed suprema.

We will consider Horn theories defined on expansions Σ^+ of an algebraic language Σ by relational symbols $\Sigma \hookrightarrow \Sigma^+$, i.e. Σ^+ is a first-order language.

An *atomic embedding* is a morphism of Σ-structures that (preserves and) reflects the satisfaction of atomic formulas. For instance:

1. Let $\Sigma^+ = \Sigma$ be any algebraic signature. Then atomic embeddings are precisely the injective homomorphisms, since injectivity means precisely that all equalities are reflected.

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2. Let Σ be any algebraic signature, let $\Sigma^+ = \Sigma \cup \{\leq\}$ possess an additional binary relation symbol; this expanded language is considered in [7]. Let $\mathbf{K}$ be a class of structures for which the symbol $\leq$ is interpreted as a partial order. Then an atomic embedding is precisely an order preserving and order reflecting injective homomorphism.

3. Let Σ be any algebraic signature, let $\Sigma^+ = \Sigma \cup \{D\}$ possess an additional unary relation symbol; this expanded language is considered in [6]. Let $\mathbf{K}$ be the class of matrices for some logic where the filter is the interpretation of the unary relation D. Then an atomic embedding is precisely a matrix embedding.

Definition 2. Let L be a first-order language. All concepts below refers to this language.

1. A *strict basic Horn formula* is a formula of the form $\bigwedge_{i\in I} P_i(\vec{x}) \to P(\vec{x})$, with I some set (possibly $I = \emptyset$, in this case $\bigwedge_{i\in I} P_i(\vec{x}) = \top$) and (positive) atomic formulas P_i, P which are not equivalent to $\bot$.

2. A *universal strict basic Horn sentence* is a sentence of the form

   $\forall \vec{x}\colon \bigwedge_{i\in I} P_i(\vec{x}) \to P(\vec{x})$, with P_i, P as above.

3. A *H-theory* is a theory $\mathbb{T}$ which can be axiomatized by strict universal basic Horn sentences.

4. Let κ be a regular cardinal. A *κ-H-theory* is a theory which can be axiomatized by universal basic Horn sentences from $L_{\kappa\kappa}$, i.e. by a set of basic Horn sentences which contains less than κ distinct variables and in which the occurring conjunctions have less than κ constituents.

5. A *H-class* (respect. κ-H-class) $\mathbf{K}$ is the class of models of a H-theory (respect. κ-H-theory) $\mathbb{T}$, i.e. $\mathbf{K} = Mod(\mathbb{T})$.

Definition 3. Let Σ be an algebraic language and consider a first-order expansion $\Sigma^+ \supseteq \Sigma$. All concepts below refer to this language $L = \Sigma^+$.

- For a Σ-structure A we define the set $G^{\mathbb{T}}(A)$ as follows:

  $G^{\mathbb{T}}(A) := \{(\theta, S) \mid \theta$ is a Σ-congruence on A and S an interpretation of $\mathfrak{R}$ on A/θ s.t. the resulting Σ^+-structure on A/θ is a $\mathbb{T}$-model$\}$

- We define a binary relation on $G^{\mathbb{T}}(A)$ as follows:

  $(\theta, S) \leq (\theta', S') \;:\Leftrightarrow\; \theta \subseteq \theta'$ and the induced quotient map $q_{\theta\theta'} : A/\theta \twoheadrightarrow A/\theta'$ is a homomorphism of Σ^+-structures for the interpretations S, S' i.e., if $R \in \mathfrak{R}_n$, then ${q_{\theta\theta'}}_*(R^S) := q^n_{\theta\theta'}[R^S] \subseteq R^{S'}$ (or, equivalently, $R^S \subseteq q^*_{\theta\theta'}(R^{S'}) := (q^n_{\theta\theta'})^{-1}[R^{S'}]$).

If a universal basic Horn sentence is true in each A/θ_i for a collection $(\theta_i, S_i)_{i\in I}$, then it is true in the product structure $\Pi_{i\in I} A/\theta_i$. The intersection structure can be described as the substructure of the product structure given by the diagonal map $\delta\colon A/\cap_i \theta_i \hookrightarrow \Pi_{i\in I} A/\theta_i$. By definition of the intersection structure, δ preserves and reflects the validity of atomic formulas. It is immediate that it then also preserves and reflects the validity of basic Horn formulas and therefore of universal basic Horn sentences. Thus if $\mathbb{T}$ is axiomatized by universal basic Horn sentences, and these sentences are true in every A/θ_i, then they are true in the intersection structure.

The following (more specialized) result holds:

Proposition 4. If $\mathbb{T}$ is a universal basic κ-Horn theory and A is a Σ-structure, then $G^{\mathbb{T}}(A)$ is a κ-algebraic lattice.

The κ-directed suprema in $G^{\mathbb{T}}(A)$ are given by the set-theoretic union of the congruences, and the interpretations of the relation symbols.

Building on the previous result, we obtain the:

Theorem 5. Let τ be a set of *atomic* Σ^+-formulas with at most one free variable, such that $|\tau| < \kappa$. The collection of maps $i^\tau = (i^\tau_A)_{A\in\Sigma-Str}$, defined by

$$\begin{array}{rcl} i^\tau_A : G^{\mathbb{T}}(A) & \to & (\mathcal{P}(A), \subseteq) \\ (\theta, S) & \mapsto & \{a \in A \mid \forall\varphi(x) \in \tau\colon\ A/(\theta, S) \vDash \varphi(a)\} \end{array}$$

is a natural transformation and for any $A \in \Sigma - Str$, i^τ_A preserves arbitrary infima and κ-directed suprema. In other words, $(G^{\mathbb{T}}, i^\tau)$ is a κ-filter pair.

In general, if (G, i) is a κ-filter pair and $A \in \Sigma$-Str, then $i_A : G(A) \to \mathcal{P}(A)$ has a left adjoint $\Xi_A : \mathcal{P}(A) \to G(A)$ and $\Xi_A \circ i_A$ is a closure operator that is structural and κ-ary. Thus we have an abstract κ-logic on A. We present a correspondence between the Craig interpolation property for a logic associated to a Horn filter pair $(G^{\mathbb{T}}, i^\tau)$, satisfying an extra condition, and the τ-amalgamation property in the Horn class $\mathbb{T}$-Mod.

We introduce the central notions and results for this work.

Definition 6. Let Σ be an algebraic signature and $\Sigma^+ \supseteq \Sigma$ a first order signature with the same function symbols.

1. A logic l has the *Craig entailment interpolation property* if for every set of formulas Γ, with variables $var(\Gamma)$ and every formula φ with variables $var(\varphi)$, if $\Gamma \vdash \varphi$ then there is a set of formulas Γ' with the variables in $var(\Gamma) \cap var(\varphi)$ such that $\Gamma \vdash \Gamma'$ and $\Gamma' \vdash \varphi$.

2. We shall say that a class $\mathbf{K}$ of Σ^+-structures has the *atomic amalgamation property* if given $A, B, C \in \mathbf{K}$ and atomic embeddings $i_B\colon A \to B$, $i_C\colon A \to C$, there exist a Σ^+-structure $D \in \mathbf{K}$ and atomic embeddings $e_B : B \to D$, $e_C : C \to D$ such that $e_B \circ i_B = e_C \circ i_C$.

3. Let (G, i) be a filter pair. Given sets X, Y such that $Z := X \cap Y \neq \emptyset$, define i_X, resp. i_Y, to be the inclusions $Z \hookrightarrow X$, resp. $Z \hookrightarrow Y$. Given a theory $T \subseteq Fm(X)$ of the logic associated to (G, i), define $T'' := T \cap Fm(Z)$ and $T' \subseteq Fm(Y)$ to be the smallest $Fm(Y)$-theory containing T''.

4. The filter pair (G, i) has the *theory lifting property*, if the following holds:

   For every pair of sets X, Y and theory $T \subseteq Fm(X)$ as above there exist $\theta_T \in G(Fm(X))$, $\theta_{T''} \in G(Fm(Z))$ and $\theta_{T'} \in G(Fm(Y))$ such that

   (a) $T = i(\theta_T)$, $T' = i(\theta_{T'})$ and $T'' = i(\theta_{T''})$

   (b) $G(i_X)(\theta_T) = \theta_{T''} = G(i_Y)(\theta_{T'})$

As an application of the previously presented results and concepts, we obtain the:

Theorem 7. Let Σ be a signature, Σ^+ an extension by relation symbols, $\mathbb{T}$ a κ-Horn theory over Σ^+, and τ a set of fewer than κ atomic Σ^+-formulas in at most one variable. Suppose that the filter pair $(G^{\mathbb{T}}, i^{\tau})$ has the theory lifting property. If $\mathbf{K} := \mathrm{Mod}(\mathbb{T})$ has the atomic amalgamation property, then the logic L associated to $(G^{\mathbb{T}}, i^{\tau})$ has the Craig entailment property.

The following result provides a natural class of examples:

Proposition 8. Let $(Co_{\mathbf{K}}, i)$ be a congruence filter pair (see [2], [3]). Under each of the conditions below we have that $(Co_{\mathbf{K}}, i)$ has the theory lifting property.

1. If $i_A : G(A) \to \mathcal{P}(A)$ is injective, for each $A \in \Sigma - Str$.

2. If the collection of left adjoints $(\Xi_A)_{A \in \Sigma\text{-Str}}$ is a natural transformation with respect to variable inclusions, i.e. such that for every inclusion of sets $Z \subseteq X$ the following diagram (in which $j_X \colon Fm(Z) \hookrightarrow Fm(X)$ denotes the induced map of formula algebras) commutes:

$$
\begin{array}{ccccc}
Fm(Z) & & Co_{\mathbf{K}}(Fm(Z)) & \xleftarrow{\ \Xi_{Fm(Z)}\ } & Fi_l(Fm(Z)) \\
\downarrow{\scriptstyle j_X} & & {\scriptstyle Co_{\mathbf{K}}(j_X)}\uparrow & & \uparrow{\scriptstyle j_X^{-1}} \\
Fm(X) & & Co_{\mathbf{K}}(Fm(X)) & \xleftarrow{\ \Xi_{Fm(X)}\ } & Fi_l(Fm(X))
\end{array}
$$

Examples 9.

1. Let l be an algebraizable logic with equivalent semantics in a quasivariety $\mathbf{K}$. If $\mathbf{K}$ has the amalgamation property, then the l has the Craig entailment property.

2. An equation of formulas $\langle \varphi, \psi \rangle \in Fm(X)$ is called *balanced* if φ, ψ contain exactly the same variables. A variety is called balanced if it can be defined by balanced equations. Let $\mathbf{K}$ be a balanced variety satisfying the amalgamation property. Then any logic l with an algebraic semantics in $\mathbf{K}$ given by a balanced equation satisfies the Craig interpolation property.

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References

[1] P. Arndt, H. L. Mariano, and D. C. Pinto, *Finitary filter pairs and propositional logics*, South American Journal of Logic **4** (2018), no. 2, 257–280.

[2] ______, *Congruence filter pairs, adjoints and Leibniz hierarchy*, arXiv:2109.01065 (2021).

[3] ______, *Congruence filter pairs, equational filter pairs and adjoints*, submitted (2023).

[4] ______, *Filter pairs and natural extensions of logics*, Archive for Mathematical Logic **62** (2023), 113–145.

[5] ______, *Horn filter pairs and applications to Craig interpolation*, in preparation (2023).

[6] P. Dellunde and R. Jansana, *Some characterization theorems for infinitary universal horn logic without equality*, The Journal of Symbolic Logic **61** (1996), no. 4, 1242–1260.

[7] J. G. Raftery, *Order algebraizable logics*, Annals of Pure and Applied Logic **164** (2013), 251–283.