# On categories of quantale valued sets

**Sobre Categorías de Conjuntos Quantale-Valuados**

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**Abstract.** We introduce and study some categories of quantale valued sets defined over commutative semicartesian quantales (quantales "with projections"). This extends known results on locale valued sets and establishes a preliminary categorial step to deal with internal logic of such categories.

**Keywords:** quantales, categories, fuzzy-sets.

**Resumen.** Introducimos y estudiamos algunas categorías de conjuntos valuados por quantales semicartesianos conmutativos (quantales "con proyecciones"). Esto amplía los resultados conocidos sobre conjuntos valuados localmente y establece un paso categorial preliminar para tratar con la lógica interna de tales categorías.

**Palabras claves:** quantales, categorias, conjuntos difusos.

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The notion of sheaf (of sets, abelian groups, etc.) over a topological space produces a kind of parametrized and coherent family of structures and is present in many areas of Mathematics. This depends only on the lattice of the open subsets of the topological space, that forms a locale (or complete Heyting algebra).

An $\mathbb{H}$-set is a set $X$ endowed with a similarity (or co-distance) function with values in $\mathbb{H}$, $\delta : X \times X \to \mathbb{H}$, satisfying axioms called of symmetry ($\delta(x, y) = \delta(y, x)$) and transitivity ($\delta(x, y) \wedge \delta(y, z) \leq \delta(x, z)$). In the 1970s, the topos of sheaves over a locale $\mathbb{H}$ was described, alternatively, as the category of $\mathbb{H}$-sets ([7], [4]), endowed with a convenient notion of relational morphism (or, equivalently, as the category of Scott-complete $\mathbb{H}$-sets endowed with functional morphisms).

**Definition 0.1** (Unital Quantale)**.** A (unital) quantale is a structure $(\mathscr{Q}, \otimes, 1, \leq)$ such that:

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(i) $(\mathscr{Q}, \leq)$ is a complete lattice;

(ii) $(\mathscr{Q}, \otimes, 1)$ is a monoid;

(iii) the general distributive laws $a \otimes \bigvee_{i\in I} b_i = \bigvee_{i\in I} a \otimes b_i$ ; $(\bigvee_{i\in I} b_i) \otimes a = \bigvee_{i\in I}(b_i \otimes a)$ holds.

Moreover, the quantale $(\mathscr{Q}, \otimes, 1, \leq)$ is:

- commutative whenever $(\mathscr{Q}, \otimes, 1)$ is a commutative monoid;
- semicartesian whenever $a \otimes b \leq a \wedge b$;
- integral whenever $1 = \top$.

*Remark* 0.2. A locale is a commutative semicartesian quantale where $\otimes = \wedge$. A unital quantale is semicartesian iff it is integral (i.e., $1 = \top = max\mathscr{Q}$).

There is an important technical "strength" condition considered by Höhle in [8]. With this, Scott-completeness makes sense for commutative semicartesian quantales. $\mathscr{Q}$ is "strong" if:

- $e \otimes e = e$ and $e \leq \bigvee A$, then $e \leq \bigvee\{a \otimes a : a \in A\}$.

The main examples of strong quantales are complete Heyting Algebras and $([0,1], \leq, \cdot)$. Some MV-Algebras, like the "standard MV-algebra" $([0,1], \wedge, \vee, \oplus, \otimes, 0, 1)$ ([5]), are not strong.

Right-sided idempotent quantales, (i.e. $a \otimes \top \leq a$) have long been studied as candidates for substituting the locales in the standard of $\mathbb{H}$-Sets ([9], [6]). It should be emphasised that the quantales that enjoy *both* idempotency and integrality are automatically commutative, in fact they are locales with $\wedge$ as their monoidal product, and all locales enjoy those properties.

In this work we consider a class quantales – that of the semicartesian (and sometimes integral) quantales (which are also commutative), this means quantales with "projections" $(a \otimes b \leq a, b)$; naturally this includes the class of all locales. While the choice of idempotency and right-sidedness is justified by a large class of examples of such quantales of interest – namely, the set of closed right-ideals of a given $C^*$ algebra – our choice of integrality is orthogonal to idempotency (having both implies $\otimes = \wedge$ and $\mathscr{Q}$ is a locale). Our choice could be seen as embodying both the quantales of the ideals of commutative unital rings and $([0,1], \leq, \cdot)$ – which is isomorphic to $([0,\infty], \geq, +)$, thus we generalize metric spaces in a sense.

**Definition 0.3** ($\mathscr{Q}$-set)**.** Let $\mathscr{Q}$ be a commutative semicartesian quantale. A $\mathscr{Q}$-set is a pair $(X, \delta)$ where $X$ is a set and $\delta : X \times X \to \mathscr{Q}$ is a function satisfying:

$$\delta(x,y) = \delta(y,x)$$
$$\delta(x,y) \otimes \delta(y,z) \leq \delta(x,z)$$
$$\delta(x,y) \otimes \delta(y,y) = \delta(x,y)$$

We abbreviate $\delta(x,x)$ by $\mathrm{E}\,x$, read as the "extent" of $x$. It is immediate that $\mathrm{E}\,x$ is idempotent and $\delta(x,y) \leq \mathrm{E}\,x \otimes \mathrm{E}\,y$. We adopt the notation $\mathrm{E}\,\mathscr{Q}$ to denote the subset of all idempotent members of the quantale $\mathscr{Q}$.

The definition of $\mathscr{Q}$-set for one such quantale is a direct generalization of the one for complete Heyting algebras when one realizes that, for complete Heyting algebras, $\otimes = \wedge$ which makes the last axiom redundant in the presence of the second.

When $\mathscr{Q} = ([0,\infty], \geq, +)$, the category of $\mathscr{Q}$-sets is equivalent to the extended pseudometric spaces.

We consider functional morphisms between $\mathscr{Q}$-sets, $f : (X,\delta) \to (X',\delta')$, to be ordinary set-theoretical functions between the underlying sets that: (i) preserve extents; (ii) increase (possibly not strictly) $\delta$'s. *i.e.* $\mathrm{E} = \mathrm{E}' \circ f$ and $\delta \leq \delta' \circ (f \times f)$.

The resulting category $\mathscr{Q}$-$\mathbf{Set}_f$ has rather good properties:

**Theorem 0.4.** *([2]) $\mathscr{Q}$-$\mathbf{Set}_f$ is a complete and cocomplete category.*

We have described explicit constructions of the limits and colimits of this category. Equalizers and colimits descriptions are similar with the category of sets, more precisely, the forgetful functor to **Set** create them. The terminal object is the subset of idempotent elements of the quantale ($1 = \mathrm{E}\,\mathscr{Q}$) and the product is defined as a "fibered" cartesian product for each extent. In this category, epis are the surjective morphisms, monos are the injective morphisms and the regular monos are the delta preserving functions: $\delta \circ (f \times f) = f \circ \delta$.

Note that the category $\mathscr{Q}$-$\mathbf{Set}_f$ is not balanced: If $(X,\delta)$ is a $\mathscr{Q}$-set, then $(X,\delta^m)$ is a $\mathscr{Q}$-set, where $\delta^m(x,y) := \delta(x,x) \otimes \delta(y,y)$, and identity function $id_X : (X,\delta) \to (X,\delta^m)$ determine a morphism that is epi and mono but is not an isomorphism, in general.

**Definition 0.5.** (cf. [1]) Let $\kappa$ a regular cardinal. An object $C$ in a category $\mathcal{C}$ is called *$\kappa$-compact* if, and only if, for all $\kappa$-filtered diagram $F : \mathcal{D} \to \mathcal{C}$, $\mathcal{C}(c, \operatorname{colim}_{d\in\mathcal{D}} F(d)) \cong \operatorname{colim}_{d\in\mathcal{D}} \mathcal{C}(c, F(d))$. This category $\mathcal{C}$ is *$\kappa$-locally presentable* if, and only if, $\mathcal{C}$ has small colimits, is locally small and there is a small set $S$ of $\kappa$-compact objects such that every object of $\mathcal{C}$ is a $\kappa$-filtered colimit of elements of $S$.

**Theorem 0.6.** *([2]) $\mathscr{Q}$-$\mathbf{Set}_f$ is $\kappa$-locally presentable, where $\kappa = max\{\aleph_0, card(\mathscr{Q})^+\}$.*

We emphasize that in [10], a work where is developed a theory of (functorial) sheaves over semicartesian quantales, it is established that the corresponding category of sheaves over $\mathscr{Q}$ is $\kappa$-locally presentable, where $\kappa$ is the same regular cardinal described above.

The following result indicates that the category $\mathscr{Q}$-$\mathbf{Set}_f$ supports a rich class of additional structures.

**Theorem 0.7.** *([2]) $\mathscr{Q}$-$\mathbf{Set}_f$ is a monoidal closed category (for many natural monoidal structures).*

The proof of the above result relies, strongly, on the previous theorem and on deep results in the theory of locally presentable categories.

In our research, we have met multiple ways of defining a monoidal structure on the category with the same "spirit": the categorical *product* is given by a subset of the product of the underlying sets with $\delta$ defined as the meet of the $\delta$'s of the coordinates. The obvious way to perturb this definition is to swap the $\wedge$ for a $\otimes$, which works. However, we found plenty of ways to alter the construction.

**Theorem 0.8.** *([2]) $\mathscr{Q}$-$\mathbf{Set}_f$ admits a classifier for regular subobjects.*

Let the $\mathscr{Q}$-set $\Omega = (\mathrm{E}\,\mathscr{Q} \dot{\sqcup} \mathrm{E}\,\mathscr{Q}, \delta_\Omega)$, where $\delta_\Omega((e,i),(e',i')) = e \otimes e'$. Then $\top : \mathrm{E}\,\mathscr{Q} \to \Omega$, $e \mapsto (e,1)$ is a regular monomorphism and is "the" classifier for all regular subobjects (i.e., the class of all classes of equivalence of regular monomorphisms). For $f : A \to B$, its characteristic morphism $\chi_f : B \to \Omega$ is defined as $\chi_f(f(a)) = (\mathrm{E}\,a, 1)$, and $\chi_f(b) = (\mathrm{E}\,b, 0)$, if $b \notin A$. Note that this is the best result possible since the pullback of $\top : 1 \to \Omega$ must be a regular monomorphism.

We emphasize that the category $\mathscr{Q}$-$\mathbf{Set}_f$ – even when $\mathscr{Q}$ is a locale – is not equivalent to the category of (functorial) sheaves of this locale, $Sh(\mathscr{Q})$, since the latter is a balanced category and the former is not. However, we can present to all commutative semi-cartesian quantale: (i) an alternative notion of morphism (the notion of relational morphism); (ii) a subclass of the class of all $\mathscr{Q}$-set (the class of extensional and Scott complete $\mathscr{Q}$-set) that, combined, determine categories that are equivalent to each other, and that are equivalent to $Sh(\mathscr{Q})$ whenever $\mathscr{Q}$ is a locale:

$$Sh(\mathscr{Q}) \simeq \mathscr{Q}\text{-}\mathbf{Set}_r \simeq \mathscr{Q}\text{-}\mathbf{Set}_r^{compl} \simeq \mathscr{Q}\text{-}\mathbf{Set}_f^{compl}$$

In the sequel, we will provide definitions, constructions and results that encompasses the last three equivalences of categories to the more general setting of commutative and semicartesian quantales.

**Definition 0.9.** A *relational* morphism of $\mathscr{Q}$-sets $\varphi : (X,\delta) \to (Y,\delta')$ is a function $\varphi : X \times Y \to \mathscr{Q}$ such that:

$$\begin{aligned}
\varphi(x,y) \otimes \delta(x,x') &\leq \varphi(x',y)\\
\varphi(x,y) \otimes \delta'(y,y') &\leq \varphi(x,y')\\
\varphi(x,y) \otimes \varphi(x,y') &\leq \delta'(y,y')\\
\varphi(x,y) \otimes \mathrm{E}\,x \otimes \mathrm{E}'\,y &= \varphi(x,y)\\
\bigvee_{y \in Y} \varphi(x,y) &= \mathrm{E}\,x
\end{aligned}$$

This class of objects and morphisms with the canonical relation composition defines a category with identities being $\mathbf{id}_X(x,x') = \delta(x,x')$.

**Definition 0.10.** There is a functor $\mathcal{G}_{\mathcal{R}} : \mathscr{Q}\text{-}\mathbf{Set}_f \to \mathscr{Q}\text{-}\mathbf{Set}_r$ which is the underlying *graph* functor, because it takes functional morphisms to their graph relation. More precisely, $\mathcal{G}_{\mathcal{R}}(X) = X$ and given $f : X \to Y$

$$(\mathcal{G}_{\mathcal{R}}\, f)(x, y) = \delta(f(x), y)$$

**Definition 0.11.** A *singleton* of $(X, \delta)$ is a function $\sigma : X \to \mathscr{Q}$ such that:

$$\sigma(x) \otimes \mathrm{E}\, x = \sigma(x)$$
$$\sigma(x) \otimes \delta(x, y) \leq \sigma(y)$$
$$\sigma(x) \otimes \sigma(y) \leq \delta(x, y)$$
$$\sigma(x) \otimes \bigvee_{x' \in X} \sigma(x') = \sigma(x) \qquad \text{(strictness condition)}$$

And it is *representable* iff:

$$\exists x_\sigma \in X(\forall y \in X(\sigma(y) = \delta(x_\sigma, y))$$

When $\mathscr{Q}$ is a locale (more precisely, $\otimes = \wedge$), then the above notion of singleton coincides with the notion presented in [4].

**Definition 0.12.** A $\mathscr{Q}$-Set $X$ is *separable* or *extensional* iff any of these equivalent conditions hold:

(i) $\mathrm{E}\, x = \delta(x, y) = \mathrm{E}\, y \Rightarrow x = y$

(ii) $\mathrm{E}\, x \vee \mathrm{E}\, y = \delta(x, y) \Rightarrow x = y$

(iii) $\forall z \in X(\delta(x, z) = \delta(y, z)) \Rightarrow x = y$

**Definition 0.13.** Let $\mathbf{s}(X)$ the set of singletons of $X$. We call $X$ *Scott-separable* if, and only if, the function $\eta : X \to \mathbf{s}(X), \eta(x) = \sigma_x : X \to \mathscr{Q}, \sigma_x(y) = \delta(x, y)$ is injective. In addition, $X$ is *Scott-complete* if, and only if, $\eta$ is a bijection.

We examined the notion of completeness (and separability) via (unique) representability of singletons (Scott-complete $\mathscr{Q}$-sets). For strong quantales, it makes sense to speak of "the Scott completion" of a given $\mathscr{Q}$-set: this provides a "completion" functor $c_f : \mathscr{Q}\text{-}\mathbf{Set}_f \to \mathscr{Q}\text{-}\mathbf{Set}_f^{compl}$.

**Definition 0.14.** When $\mathscr{Q}$ is strong, we can define also a notion of completion of a $\mathscr{Q}$-set. Let $(X, \delta)$ be a $\mathscr{Q}$-set. Then the Scott-completion of $(X, \delta)$ is $(\mathbf{s}(X), \delta_{\mathbf{s}})$, where $\delta_{\mathbf{s}}(\sigma, \sigma') := \bigvee_{x \in X} \sigma(x) \otimes \sigma'(x)$.

The strictness condition on $\sigma$ guarantees that $\bigvee_{x \in X} \sigma(x)$ is idempotent, and the strength condition on $\mathscr{Q}$ guarantees that $(\mathbf{s}(X), \delta_{\mathbf{s}})$ is an object in $\mathscr{Q}\text{-}\mathbf{Set}_f$.

We have obtained the:

**Theorem 0.15.** *([3]) The full subcategory of Scott-complete $\mathscr{Q}$-sets with functional morphisms is reflective. More precisely, the full inclusion functor $i_f$ : $\mathscr{Q}\text{-}\mathbf{Set}_f^{compl} \hookrightarrow \mathscr{Q}\text{-}\mathbf{Set}_f$ is right adjoint to the completion functor $c_f : \mathscr{Q}\text{-}\mathbf{Set}_f \to \mathscr{Q}\text{-}\mathbf{Set}_f^{compl}$.*

Concerning categories of $\mathscr{Q}$-sets endowed with relational morphisms we have obtained:

**Theorem 0.16.** *([3])*

1. *Every $\mathscr{Q}$-set is relationally isomorphic to its own Scott completion.*

2. *The full inclusion functor $i_r$ : $\mathscr{Q}\text{-}\mathbf{Set}_r^{compl} \hookrightarrow \mathscr{Q}\text{-}\mathbf{Set}_r$ establishes an equivalence of categories.*

More explicitly, the relational isomorphism between $X$ and $\mathbf{s}(X)$ is defined by the evaluation map $ev_X : \mathbf{s}(X) \times X \to \mathscr{Q}$, $ev_X(\sigma, x) := \sigma(x)$.

Finally, we have established:

**Theorem 0.17.** *([3]) For strong quantales , the categories of Scott-complete $\mathscr{Q}$-sets and relational morphisms and Scott complete $\mathscr{Q}$-sets functional morphisms are isomorphic: $\mathscr{Q}\text{-}\mathbf{Set}_f^{compl} \cong \mathscr{Q}\text{-}\mathbf{Set}_r^{compl}$*

An isomorphism can be obtained by the restriction of the "graph functor" to the full subcategories of complete $\mathscr{Q}$-sets.

Some good categorial properties have been proved about these functional and relational categories of Scott-complete $\mathscr{Q}$-sets. Despite the broad category of $\mathscr{Q}$-sets and functional morphisms not being equivalent to the Scott-complete one, all of them are complete and cocomplete, and the limits looks the same, since the Scott-complete ones determine a reflective full subcategory of the category $\mathscr{Q}\text{-}\mathbf{Set}_f$. Additional analyses are needed concerning the properties of locally presentable and monoidal (or monoidal closed) structures of these full subcategories of Scott-complete $\mathscr{Q}$-sets.

Further research is also needed to describe a precise relationship between the categories of $\mathscr{Q}$-sets and the category $Sh(\mathscr{Q})$, of all functorial sheaves over $\mathscr{Q}$ and natural transformations (see [10]), where $\mathscr{Q}$ is a commutative and semicartesian quantale.

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