Tripos models of Internal Set Theory

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Abstract. We introduce a categorical approach to nonstandard methods, loosely inspired on Internal Set Theory (IST) and reliant on the rich notions of hyperdoctrine and tripos from categorical logic. A different point of view is adopted: the axioms of IST should be expressed in terms of interactions between doctrines, which can be thought of as saying that instead of adding a new predicate we only add new quantifiers. Moreover, it differs from the typical approaches in that the foremost axiom is Standardisation rather than Transfer, all of which can be expressed in a point-free way. This opens the way to the usage of nonstandard proofs methods in the internal language of a doctrine.

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1. Introduction

Categorical approaches to nonstandard analysis have historically been attempted from the semantics side, being closer in spirit to Robinson's own work — this can be seen in the papers by I. Moerdijk and E. Palmgren [5] [9], [8], [10], for example. There is, however, a syntactic form of NSA that provides an useful alternative and has not been as explored in the literature[2]: Internal Set Theory. We'll briefly describe a categorification of it developed in [7].

E. Nelson introduced Internal Set Theory (IST) [6] in an attempt to make the methods of nonstandard analysis more accessible to mathematicians (and physicists) not acquainted with Logic, particularly model theory. His approach was to extend ZFC by adding a new unary 'standardness' predicate $\mathrm{st}(x)$ and three axiom schemata (chiefly Transfer) to govern the behaviour of this new notion, providing a reasonably contained set of rules one can use to make new proofs. This turns out to be a conservative extension of ZFC, so that it raises no new foundational issues and can be used to prove classical results: any theorem of IST that can be stated in the language of ZFC is provable in ZFC,

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[2]A notable exception is [2].

even if the IST proof cannot. While Nelson's approach is perfectly reasonable, there is room for improvement on a few grounds: the underlying logic of the theory is unnecessarily classical, tied to Set Theory, and predicated on the idea of points/elements.

Addressing these matters is important if the theory is to be made constructive, and also in order to facilitate looking for models that fit new settings (e.g., algebraic geometry). In order to do so, we change the perspective: instead of having a new predicate and axioms to govern it, we simply assume we have *new quantifiers* that interact with the usual ones in certain prescribed ways, expressing the idea of *Transfer*. This can be formalised via usage of *tripos theory* [1], a tool from categorical logic that allows one to express a topos as an indexed partial order of 'predicates'. It has the advantage that the underlying universe for the resulting theory is no longer necessarily a model of set theory, but rather just a topos, which is strictly more general [12], and we simultaneously abandon the reliance on elements when doing so.

2. Doctrinal models of IST

Nelson's IST expands ZFC by adding a new unary 'standardness' predicate st, along with three axiom schemata to establish its behaviour. While it is slightly confusing, a general formula in the new language is called 'external', whereas an *internal formula* is simply a formula in the language of ZFC. The new axioms are as follows [3]:

- **Scheme of Transfer**: If $A(x, t_1, ..., t_k)$ is *internal*, then

$$\forall^{st} t_1 ... \forall^{st} t_k \; (\forall^{st} x \; A(x, t_1, ..., t_k) \Rightarrow \forall x \; A(x, t_1, ..., t_k))$$

- **Scheme of Standardisation**: If $C(z)$ is *any* formula, then

$$\forall^{st} x \; \exists^{st} y \; \forall^{st} z \; (z \in y \iff z \in x \wedge C(z))$$

- **Scheme of Idealisation**: If $R(x, y)$ is *internal*, then

$$\forall^{st}_{\text{fin}} z \; \exists x \; \forall y \in z \; R(x, y) \iff \exists x \; \forall^{st} y \; R(x, y)$$

Transfer is the main tool for proving facts about sets in IST, as it serves as a compactness principle: any internal feature of standard sets (perhaps expressed by using standard parameters) is actually true of all sets.

Standardisation serves as the next best thing to Separation — note that while we still have a Separation axiom, it only applies to internal formulae (as ZFC has nothing to say about the predicate st). In particular, there is no 'subset of all standard elements' of A, but if A is standard then there is an

[3]Here the abbreviations $\forall^{\mathrm{st}}$, $\forall^{\mathrm{st}}_{\mathrm{fin}}$, and $\exists^{\mathrm{st}}$ have the expected meaning.

unique standard subset of A whose *standard elements* are precisely those of A (albeit nothing can be said about its nonstandard elements).

The strangest one is Idealisation, which is inspired by Nelson's ultrafinitist beliefs; it nonetheless plays the practical role of proving the existence of nonstandard elements and some special sets, including infinitesimals. Nelson's original paper [6] includes several examples of how these axioms can be used to prove results in different areas of mathematics in a perhaps conceptually simpler manner.

We will employ the tools of categorical logic (mainly the notions of hyperdoctrine and tripos) to implement the key ideas that the axioms of Internal Set Theory should be construed abstractly as interactions between two kinds of quantifiers, that Transfer and Standardisation ought to be formulated without reference to elements, and that a set should be standard if, and only if, it coincides with its standardisation. Moreover, due to the type-theoretic nature of toposes, the abstract versions of the IST axioms should capture the Δ_0-versions of their original counterparts.

Our starting point is noting that the standard sets and maps of any model of IST form a category. Moreover, this category has enough structure to interpret intuitionistic higher order logic, and the inclusion functor into **Set** preserves it. We briefly recall the relevant notions:

Definition 2.1. A category $\mathcal{C}$ is (properly) cartesian-closed if it has finite limits and each product functor $(-) \times A \colon \mathcal{C} \to \mathcal{C}$ has a right adjoint $(-)^A$. A functor $F \colon \mathcal{C} \to \mathcal{D}$ is a cartesian-closed functor if it preserves finite products and the canonical morphisms $F(B^A) \to F(B)^{F(A)}$ are isomorphisms for all objects $A, B \in \mathcal{C}$.

It is well-known that cartesian-closed categories provide a categorical semantics for the simply typed λ-calculus [3]. The concept is useful here since a cartesian-closed category admits an 'object of maps from A to B', namely B^A, enabling us to reason about such things internally. With a little extra structure, we can interpret higher order intuitionistic logic:

Definition 2.2. A properly cartesian-closed category $\mathcal{C}$ is said to be an (elementary) topos if it admits a subobject classifier: an object Ω together with a morphism $1 \xrightarrow{\top} \Omega$ such that any monomorphism $S \to A$ in $\mathcal{C}$ is the pullback of $\top$ along some unique morphism $\chi_S \colon A \to \Omega$. We say that χ_S *classifies* the monomorphism S.

In any topos, the objects of the form Ω^A are internal Heyting algebras that play the role of 'power sets'. In fact, there is enough structure available to replicate the usual constructions in Set Theory: the evaluation map $\Omega^A \times A \to \Omega$ classifies a monomorphism $\in_A \to \Omega^A \times A$, which has a universal property making it behave as a typed membership relation. In particular, elements of Ω^A may be seen as predicates over A. These are preserved by *logical functors*: functors that preserve finite limits and power objects.

An interested reader may consult [12] for details on the connection between topos theory and set theory.

Another approach is to consider an family of Heyting algebras indexed by a category with finite limits, and posit that their elements are 'predicates' on the objects of the indexing category $\mathcal{C}$. Pulling back a predicate along a morphism in $\mathcal{C}$ plays the role of substitution in a logical formula, and adjoints to such map play the role of quantification. This is a key insight due to Lawvere [4], which is captured by the notion of hyperdoctrine:

Definition 2.3. Let $\mathcal{C}$ be a category with finite limits. A first-order hyperdoctrine P is a $\mathcal{C}$-indexed Heyting pre-algebra [4] such that:

- For each $f\colon A \to B$ in $\mathcal{C}$, the functor $P(B) \xrightarrow{P(f)} P(A)$ of preorders preserves implication and has both a left adjoint $\exists(f)$ and a right adjoint $\forall(f)$;
- Such adjoints satisfy the Beck-Chevalley condition: given a pullback square $\begin{tikzcd} X \arrow[r, "k"] \arrow[d, "h"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & Y \end{tikzcd}$ in $\mathcal{C}$, there's an iso $P(f) \circ \exists(g) \cong \exists(k) \circ P(h)$, and an iso $P(f) \circ \forall(g) \cong \forall(k) \circ P(h)$.

We can compare hyperdoctrines P and Q over $\mathcal{C}$ by considering *Heyting transformations* $\alpha\colon P \to Q$ between them: those are the indexed functors whose components are morphisms of Heyting algebras, and that commute with the respective quantification functors: for every morphism $f\colon A \to B$ in $\mathcal{C}$, the diagrams

$$\begin{tikzcd} P(A) \arrow[r, "\exists_f^P"] \arrow[d, "\alpha_A"'] & P(B) \arrow[d, "\alpha_B"] \\ Q(A) \arrow[r, "\exists_f^Q"'] & Q(B) \end{tikzcd} \qquad \begin{tikzcd} P(A) \arrow[r, "\forall_f^P"] \arrow[d, "\alpha_A"'] & P(B) \arrow[d, "\alpha_B"] \\ Q(A) \arrow[r, "\forall_f^Q"'] & Q(B) \end{tikzcd}$$

commute. In particular, we have a notion of embedding for hyperdoctrines (the componentwise monic Heyting transformations).

This is enough to capture first-order intuitionistic logic, but we can upgrade it to a valid semantics for higher-order logic by adding a 'generic predicate':

Definition 2.4. Let $\mathcal{C}$ be a category with finite limits. A tripos over $\mathcal{C}$ is a first-order hyperdoctrine with a generic predicate, that is, there is a predicate $\sigma \in P(\Sigma)$ for some $\Sigma \in \mathcal{C}$ such that for each $\phi \in P(A)$ there is a map $\chi_\phi : A \to \Sigma$ in $\mathcal{C}$ with $\phi \cong P(\chi_\phi)(\sigma)$.

[4]By a Heyting pre-algebra we just mean the pre-ordered counterpart to Heyting algebras.

A tripos can be seen as a logical presentation of a topos: every tripos induces a topos via the tripos-to-topos construction and, conversely, every topos $\mathcal{I}$ has a canonical tripos $\mathbf{Sub}_{\mathcal{I}}$ (the tripos whose Heyting pre-algebras are given the the global elements of Ω^A for each object A of the topos, i.e., the subobjects of A). The advantage of working with a tripos is that you can reason about the predicates directly. More details can be found in [1, 11, 14].

The first step is to study the triposes that arise from the classical set-theoretic setting in [6]:

Lemma 2.5 (ZFC+Transfer)**.** *Fix a model of (set-theoretic) IST, and denote by* **StSet** *and* **Set** *the categories of all standard sets and standard functions, and the category of all sets and functions of the model, respectively. Then* **StSet** *is a topos and the inclusion functor* $\iota\colon \mathbf{StSet} \hookrightarrow \mathbf{Set}$ *is logical.*

We abstract away from this by considering the following data:

- A topos $\mathcal{I}$ (of 'internal sets');
- A hyperdoctrine $\mathfrak{X}$ over $\mathcal{I}$ (of 'external predicates');
- An embedding $[\cdot]\colon \mathbf{Sub}_{\mathcal{I}} \hookrightarrow \mathfrak{X}$;
- A cartesian-closed category $\mathbb{S}$, and a cartesian-closed faithful functor $\iota\colon \mathbb{S} \to \mathcal{I}$.

The topos $\mathcal{I}$ plays the role of the 'internal world'. The predicates of the hyperdoctrine $\mathfrak{X}$ represent the 'external formulae' with a free variable of sort given by an 'internal set', and the transformation $[\cdot]\colon \mathbf{Sub}_{\mathcal{I}} \to \mathfrak{X}$ describes how internal predicates can be viewed externally. The subcategory $\iota\colon \mathbb{S} \to \mathcal{I}$ provides the sorts for which the Axiom of Standardisation will hold (in Nelson, that's **StSet**). Note that we can pull $\mathfrak{X}$ along ι to obtain a hyperdoctrine $\mathfrak{X}^*$ over $\mathbb{S}$, where $\mathfrak{X}^*(A) = \mathfrak{X}(\iota(A))$. We can thus speak of internal and external predicates about standard sets.

Definition 2.6. A *standardisation transformation* is an $\mathbb{S}$-indexed transformation $\diamond\colon \mathfrak{X}^* \to \mathfrak{X}^*$ such that:

- $\diamond$ is idempotent, i.e., $\diamond_A^2 \cong \diamond_A$ for each $A \in \mathrm{ob}(\mathbb{S})$;
- For all $A \in \mathbb{S}$, $\diamond_A$ preserves finite meets;
- Each $\diamond_A$ satisfies the Frobenius reciprocity property with respect to implication:

  $$\diamond_A(p \Rightarrow \diamond_A(p')) = (\diamond_A(p) \Rightarrow \diamond_A(p')) = \diamond_A(\diamond_A(p) \Rightarrow p') \qquad \text{in } \mathfrak{X}^*(A)$$

  for all $p, p' \in \mathfrak{X}^*(A)$.

With the right semantics, these conditions ensure that Standardisation holds *according to* $\mathfrak{X}$. The fixed points $\mathbb{S}^\diamond(A) = \{p \in \mathfrak{X}^*(A) \colon \diamond_A(p) \cong p\}$ of a standardisation transformation serve as predicates for a hyperdoctrine over $\mathbb{S}$, which we *define* as the standard ones. As an example, if $\iota \colon \mathcal{E} \to \mathcal{F}$ is the inverse image part of an open surjection of toposes, then the comonad induced by the geometric morphism at the level of subobjects is a standardisation transformation $\mathbf{Sub}^*_{\mathcal{F}} \to \mathbf{Sub}^*_{\mathcal{F}}$, and the doctrine $\mathbb{S}^\diamond$ is equivalent to $\mathbf{Sub}_{\mathcal{E}}$. Note that the inclusion $\mathbb{S}^\diamond \hookrightarrow \mathfrak{X}^*$ factors through $[\cdot] \colon \mathbf{Sub}^*_{\mathcal{I}} \to \mathfrak{X}^*$. When that happens, we say we have a *doctrinal IST context*.

Given such a context, we can view Transfer as property of the inclusion $\mathbb{S}^\diamond \hookrightarrow \mathfrak{X}^*$ of hyperdoctrines. The intuition is that applying an internal quantifier to an internal formula with standard parameters should still yield an internal formula with standard parameters. This leads us to the realisation that Δ_0-Transfer holds internally to $\mathfrak{X}^*$ whenever $\mathbb{S}^\diamond \hookrightarrow \mathfrak{X}^*$ is a conservative Heyting transformation. A similar (but more complicated) treatment can be applied to a bounded version of Idealisation, and a doctrinal IST context for which the above hold and for which $\mathfrak{X}$ is a tripos over $\mathcal{I}$ is called a *tripos model of IST*. We can then verify the following:

Theorem 2.7 (Soundess for tripos models). *Let $\mathcal{L}$ be an $\mathcal{I}$-typed relational language, and ϕ be a sentence in $\mathcal{L}$. If ϕ is provable in using intuitionistic logic without equality, Δ_0-Transfer, Standardisation, and Bounded Idealisation, then $\mathfrak{X} \models \phi$ for any tripos model of IST and choice of interpretation of $\mathcal{L}$ in $\mathfrak{X}$.*

Models arise from the usual ultrapowers of sets, but also from situations as in [2]. Transfer and Standardisation can also be verified to hold in doctrinal models arising from open continuous surjections of locales (in particular, topological spaces).

3. Future work

A paper under review [13] includes a proof that every topos satisfying the internal axiom of choice (e.g., toposes of G-sets and Boolean étendues, including sheaves over complete Boolean algebras) occurs as a universe of standard objects and maps. However, the conditions under which Bounded Idealisation holds for localic models in general is still unknown, and techniques to build new models from old ones need to be developed.

Explicit constructions of models out of toposes ought to be considered in the future, and there is room for serious consideration to applications (in the guise of nonstandard proof of results in fields like algebraic geometry, say).

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