The Essence of Scott's "Continuous Lattices"

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Abstract. Following Fernando Zalamea's call for *mathematical criticism* [4], we engage in an immersion and reflection on Dana Scott's seminal paper *Continuous Lattices* [3]. During that process, we turn to *making* mathematics by extracting the essence of Scott's arguments, thereby engaging in what we might call mathematical *praxis*. We reveal that Scott's construction of a model of λ-calculus actually happens in a more general setting than it first seems, which includes the use of a particular geometric context [1] to then construct a poset-enriched semicartesian monoidal category in which the inverse system constructions actually take place. We ponder the philosophical implications of this approach.

Keywords: Lambda calculus, domain theory, category theory, philosophy of mathematical practice, mathematical praxis.

Resumen. Siguiendo el llamado de Fernando Zalamea por una crítica matemática [4], nos sumergimos y reflexionamos sobre el artículo seminal de Dana Scott, "Continuous Lattices" [3]. Durante ese proceso, pasamos a *hacer* matemáticas extrayendo la esencia de los argumentos de Scott, participando así en lo que podríamos llamar *praxis* matemática. Revelamos que la construcción de Scott de un modelo de lambda-cálculo en realidad ocurre en un entorno más general de lo que parece a primera vista, que incluye el uso de un contexto geométrico particular [1] para luego construir una categoría monoidal semicartesiana enriquecida sobre conjuntos ordenados en la que se hace efectiva la construcción con el sistema inverso. Reflexionamos sobre las implicaciones filosóficas de este enfoque.

Palabras claves: Lambda cálculo, teoría de los dominios, teoría de categorías, filosofía de la práctica matemática, praxis matemática.

Domain Theory's wide applicability contrasts with its localized origin by the hands of a single person: Dana Scott. One of the seminal works in the area is his paper entitled "Continuous Lattices" [3], which introduces the titular

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structures. They are a very special kind of both topological space and complete lattice, the collective of which form a category with remarkable structure, and within which they display very useful properties. Chiefly, they are the injective objects of that category (but only in regards to open inclusions). So, specifically, continuous lattices are the T_0 spaces D for which given any *open inclusion* $i : X \to Y$ and function $f : X \to D$, there is a function $Y \to D$ extending f.

Out of continuous lattices, Scott then builds the widely recognized first model of the untyped lambda calculus, prompting then decades of work on the semantics of computational systems by way of partially specified data structures. Lambda models are spaces where there is suitable structure for interpreting the syntax of the untyped lambda calculus, such as application, abstraction, *etc* (*cf.* [2], Ch. 14).

Explicitly, the model is the limit of a sequence

$$\cdots \xrightarrow{j_{n+2}} X_{n+2} \xrightarrow{j_{n+1}} X_{n+1} \xrightarrow{j_n} \cdots \xrightarrow{j_0} X_0 \tag{1}$$

where given an X_0 each X_{i+1} is defined as $X_i^{X_i}$.

Apart from his intellectual power, part of the reason for the elegance in Scott's arguments is that they are much more general then the setting in which he puts them. Although the ostensible environment of the paper is that of a particular closed cartesian category (the injective T_0 spaces), we argue that the same techniques work, almost without modification, in a wider context. Indeed, the present work is an effort to *unravel* Scott's arguments.

"Unraveling" is not an idle nomination. To unravel an argument means to pull appart its strands; to look at its narrative structure and dissociate its objects into more essential components. Although the whole argument may apply only in a particular setting, it is often the case that it is a composition of smaller arguments and techniques, much more elegant in their own right, that speak of different kinds of structures that, in that context, offer different views of the domain of discourse.

This process leads us, in the end, to what could be called the *essence* of that argument. That is, the particular structures and properties that make that argument what it is.

In the case of *Continuous Lattices*, in the search of such essence, we can observe three disctinct parts, or *acts*: first, Scott presents a discussion about T_0 spaces and the injective objects (relative to open inclusions) in that category. Second, we are introduced to a notion of order between points of a topological space (nowadays we call it *specialization*), which in turn leads us to the characterization that a special kind of order (the continuous lattices) correspond to exactly those injective objects. Lastly, Scott constructs among the continuous lattices spaces that are their own function spaces using a well-known argument involving inverse systems.

But after all this analytical work with topological spaces, equipped with more modern conceptions of categorical thinking, we are left wondering what exactly happened here. Some passages are strikingly categorical in style, which suggests some interesting structure may be lurking beneath the whole thing.

Our reading takes us the following, more synthetic reconstruction of Scott's argument: we investigate the properties of injective objects relative to a special class of arrows in the category of T_0 spaces. From that special class, as well, we are able to enrich that category over posets[1]. We observe that the injective objects interact with the orders: they are characterized as those objects that have continuous lattices as Hom-sets; and further, the exponentials in that subcategory are related to sheaves over a particular Grothendieck topology. The special class of arrows and the Grothendieck topology interact to give what is called a *geometric context* (see [1]). Finally, we get from all of it exactly what is needed to build the λ-models using inverse systems: a semicartesian monoidal category enriched over some special posets, from which the construction is straightforward and elegant.

While this may seem a more complicated version of Scott's arguments, it actually accomplishes three things: first, it improves our understanding of his techniques in this work; second, it sheds light onto what class of structures his arguments *actually* apply (poset-enriched semicartesian monoidal categories and particular types of geometric contexts); third, it points the direction of the most fruitful generalizations and extensions (What other contexts work? How does it all relate to the toposes?).

In the end, the whole process of unravelling this already well-known argument leads us to some useful abstractions, some elegant formalisms, and to a deeper understanding of the reasons Scott's arguments work. This is a worthwhile task. Indeed, this seems to be what philosopher Fernando Zalamea calls "mathematical criticism" [4] (standing in analogy to literary criticism); that is, "to explore, know, feel, live the works under study, and only then describe, calibrate, evaluate, explain them". This is the "immersion" we seek. But we also add that such a critical approach is useful in actually *making* mathematics, as illustrated by Grothendieck's rising sea metaphor. So, in a sense, we turn Zalamea's proposal on its head, and connect its ends together to not only *immerse* ourselves in mathematics and reflect upon it, but to use that experience to then *make* mathematics. It is practice and reflection together: mathematical *praxis*.

References

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[1]This is notable: Scott's arguments work just as well with generalized points (arrows) as it does with global points.

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