DOI: https://doi.org/10.15446/bol.mat.v31n1.122437

Boletín de Matemáticas **31**(1) (2024)

Informative gluing and collapsing of pre-orders

Pegamiento y colapso informativo de preordenes

Alfredo Roque Freire[1,a], **Hugo Luiz Mariano**[2,b]

Abstract. Motivated by some mathematical instances of comparisons between structures or theories, we introduce notions of compatibility and informative morphisms between pre-orders. We analyze gluings and collapses between pre-orders, concerning information.[3]

Keywords: compatibility, information, gluing, collapse.

Resumen. Motivados por algunas instancias matemáticas de comparaciones entre estructuras o teorías, introducimos nociones de compatibilidad y morfismos informativos entre preordenes. Analizamos pegamientos y colapsos entre preordenes, a nivel de información.

Palabras claves: compatibilidad, información, pegamiento, colapso.

Mathematics Subject Classification:

Recibido: julio de 2023 Aceptado: diciembre de 2023

1. Introduction

The phenomena we would like to consider is that of a order intuitively perceived as linear, even though we might not have a mathematical criteria for its linearity. One may, for instance, idealize the linear order of 'theory strength' or 'consistency strength'. One often hears set theorist talk in this terms when discussing large cardinal axioms. But any attempt (under reasonable conditions) of constructing an actual comparisons between theories is doomed to be incomplete, and any order relation over the universe of first order theories invariably produces a pre/partial order.

Having in mind a **linear concept**, we may nevertheless describe comparison strategies, knowing that all of them must in some way lose information. Our goal is to investigate methods of overcoming/dealing-with this information loss

[1]Departament of Mathematics, University of Aveiro, Aveiro, Portugal
[2]Institute of Mathematics and Statistics, University of São Paulo, São Paulo, Brazil
[a]alfrfreire@gmail.com
[b]hugomar@ime.usp.br
[3]This research was supported by FCT through CIDMA and projects UIDB/04106/2020 and UIDP/04106/2020.

by articulating collections of methods instead of considering each of them in isolation. This creates however a methodological difficulty. For one should have trouble connecting mathematically comparison strategies with the ‘intended’ linear order – never present to be treated as an object. In this section, we suggest starting from a **known linear order**. Families of partial orders over this linear order can be used to represent incomplete representations of a linear concept. This will allow us to picture the broad scenario of the phenomena. Ultimately, we claim that this picture is a good representation of the unreachable linear orders.

There are plenty occurrences of distinct hierarchies in various fields of Mathematics, but the emphasize:

- In Logic: logics stronger or more expressible than others via ”inclusions”: of infinitary languages ($L_{\kappa,\lambda} \subseteq L_{\kappa',\lambda'}$ whenever $\kappa \leq \kappa', \lambda \leq \lambda'$, (classical) first order logic is contained in a higher order logic, classical logic is stronger than intuitionistic logic

- In Set Theory: there are (at least) two hierarchies of ”size” of large cardinals: (i) $type(1) \leq type(2)$ if the consistence of type 2 entails the consistence of type 1; (ii) $type(1) \leq' type(2)$ if the first cardinal of type 2 occurs before the first cardinal of type 1.

2. Gluing bad partial orders over $\mathbb{N}$

For a given rational number $0 < r < 1$, let us consider the order relation $\leq_r$ such that for $a, b \in \mathbb{N}$

$$a \leq_r b \Longleftrightarrow a = b \vee a \leq r.b$$

Each rational r provides us with a ‘imperfect’ partial order over the natural numbers. Notably, for any two partial orders $\leq_r$ and $\leq_q$ with $r < q$ ($r = r_1/r_2$ and $q = q_1/q_2$), we have that $x \leq_r y$ implies $x \leq_q y$ for all x and y. There are however cases for which $x \leq_q y$ even if x and y are not comparable in $\leq_r$ (e.g. $x = r_2.q_1$ and $y = r_2.q_2$). As a result, the union of both order is precisely $\leq_q$.

Orders like $\leq_r$ are but one way of representing bad orders of $\mathbb{N}$. Let A be an infinite and coinfinite subset of $\mathbb{N}$, we define the order of A:

$$x \leq^A y \Leftrightarrow x = y \text{ or } \exists a \in A(x < a \leq y \ or \ x \leq a < y)$$

It is clear that $A \subseteq B$, $x \leq^A y \Rightarrow x \leq^B y$. Furthermore, the union of all $\leq^A$ with A infinite and coinfinite subset of $\mathbb{N}$ is the original order of the natural numbers.

3. Translations between first-order theories

If our chosen method is *interpretation*[1], we should obtain a pre-order $\leq^I$ over the universe of theories U. Different than what we had in the previous section where

[1] Definition. as in first section of [2]

relations are antisymmetric, theories can mutually interpret each other without being the same theory. This is for instance the case between Peano arithmetic (PA) and Zermelo Fraenkel set theory with the negation of the axiom of infinity (ZF_{fin}). We should note however that some important information is lost in this mutual relation of interpretation. If A is the Ackermann interpretation of set theory in arithmetic, then PA 'sees' a stronger set theory than simply ZF_{fin}, i.e. PA proves formulas φ^A such that $ZF_{fin} \nvdash \varphi$.

Using the method of interpretation, we may equate theories that, under a more nuanced inspection, can be shown to be different in strength. With the addition in ZF_{fin} of the axiom stating that every set is hereditarily finite, we obtain the theory ZF_{hfin} which is bi-interpretable[2] with PA (see. [3]). As a consequence, the interpreted version of set theory in PA is precisely the theory ZF_{hfin} and the interpreted version of arithmetic in ZF_{hfin} is precisely PA. Hence, if bi-interpretation is a better sign of 'equal expressive strength', then ZF_{fin} is short of axioms to reach the expressiveness of PA.

Bi-interpretations are mutual relations between theories, they do not produce an order over the universe of theories. In this scenario, we may obtain an order relation with the following notion: let T_1 and T_2 be first order theories, we say T_1 is stronger than T_2 (notation: $T_1 \geq^S T_2$) when there is a subtheory of T_1 that is bi-interpretable with T_2. Notably, many other notions are possible[3] and, as we shall see, this is indeed invited. But for now let us limit ourselves to this version of strong interpretation.

Let ZF_{-infin} be ZF without the axiom of infinity, then we have the following representation of the pre-orders obtained from interpretation and strong interpretation (arrows are assumed to be transitive and circles represent equivalent classes of indistinguishable theories):

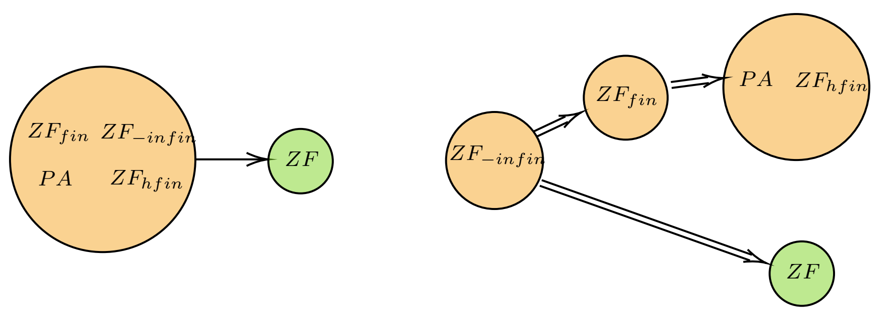

Figure 1: Theories under interpretation (left) and strong interpretation (right).

The important aspects to notice in this figure is that (i) though we distinguish PA and ZF_{hfin} from ZF_{fin} and ZF_{-infin} using strong interpretation, (ii) we

[2]definition

[3]For instance, T_1 is stronger than T_2 (notation: $T_1 \geq^{S'} T_2$) when there is a extension of T_2 that is bi-interpretable with T_1

loose the comparison from PA and ZF_{hfin} and ZF. This results from the fact that no subtheory of ZF is bi-interpretable with any extension of PA (see. [1, p. 151 - 152]). In short, we gain some distinctions but we loose some others. Nonetheless, we may reason that ZF cannot be compared to PA using strong interpretation because this method requires that the theories are too much connected. We may thus say:

1. Interpretation is too weak to distinguish some theories that should be distinguished.

2. Strong interpretation requires too much to establish a connection between theories and thus some theories distinguished by interpretations become undistinguished.

4. Gluing and compatibility of pre-order under information

With the previously described in mind, we should use information from both methods to obtain a more complete picture of the strength relation between first order theories. But how would it work and what are the conditions for it to work? The strategy we will adopt is to fix a hierarchy of the kind of information available in each order relation. The hierarchy, in turn, is determined by what we will call **facilitating-interpretation**:

> By weakening a method of comparison, we facilitate that comparison relations are established.

We illustrate by what we call the **paradigmatic case** in the following figure:

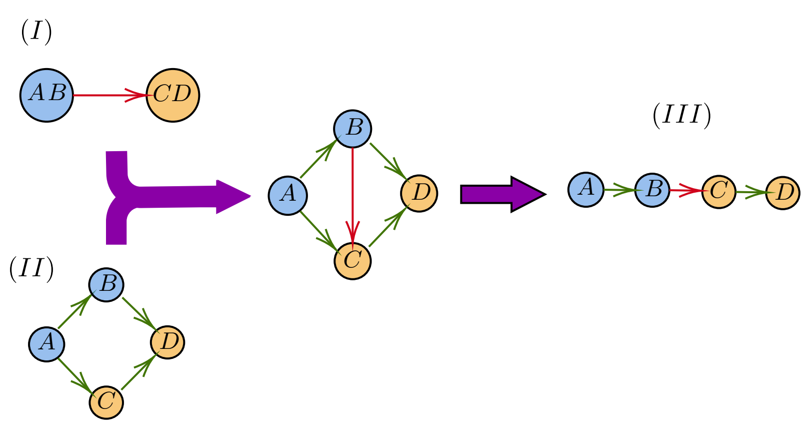

Figure 2: Paradigmatic case for gluing.

The operation to obtain (III) from (I) and (II) is that of finding the minimally informative order that has better information than both (I) and (II). Formally, we define:

Definition 4.1. Let P be a binary relation over a domain D. The level of information $info(x, y, P)$ about $x, y \in D$ in P is defined as follows:

$$info(x, y, P) = \begin{cases} 0; x \not{P} y \wedge y \not{P} x \\ 1; xPy \wedge yPx \\ 2; xPy \wedge y \not{P} x \text{ (or } yPx \wedge x \not{P} y) \end{cases}$$

Then we can define what makes an order more informative than another:

Definition 4 .2. For two binary relations P_1 and P_2 over the same domain D, we say that P_2 is more informative than P_1 when

$$\forall x, y \in D \ (info(x, y, P_1) \leq info(x, y, P_2))$$

More generally, we consider pairs (D_i, P_i), $i = 1, 2$ where $P_i \subseteq D_i \times D_i$. A function $f : D_1 \to D_2$ is a **informative morphism**[4] whenever

$$\forall x, y \in D_1 \ (info(x, y, P_1) \leq info(f(x), f(y), P_2))$$

From the definition, we can now observe that (III) in [2] is more informative than (I) and (II). Specifically, (III) is more informative than (I) for the pairs $\langle A, B\rangle$ and $\langle C, D\rangle$; and it is more informative than (II) for the pair $\langle B, C\rangle$. Now, regarding interpretation and strong interpretation in [1], we can apply the same idea, obtaining the more informative comparison for the selected FOT:

Concerning this notion of morphisms, we can consider two problems of "optimization" for convenient families $\{(D, P_i) : i \in I\}$. The minimal optimization problem – or **collapsing problem** – and the maximal optimization problem – or **gluing problem**.

Before moving on, we now consider the concrete construction of the more informative order. When possible, the informative pre-order for a collection of pre-orders Σ is obtained by making an the following operation:

Definition 4 .3. Let P be a pre-order, call $P^{inv} = \{\langle a, b\rangle \mid \langle b, a\rangle \in P\}$.

Definition 4 .4. Let Σ be a family of pre-orders, $I = \bigcap \Sigma$ and $U = \bigcup \Sigma$. We call $\mathcal{P}^{\Sigma}$ the transitive closure of $I \cup (U \setminus (I \cup I^{inv}))$. We call $\mathcal{P}^{\Sigma}$ the informative gluing of Σ.

The important property we get in with this definition is thus:

[4]Note that: (i) the identity function $id_D : D \to D$ is a informative morphism iff P_2 is more informative than P_1; (ii) the composition of informative morphisms is an informative morphism. Thus we can consider a category of pairs (D, P) and informative morphisms between them. The analysis of the full subcategory of pairs (D, P) where P is a pre-order on D is the goal of this work in progress.

Proposition 4.5. *Let P be a pre-order in the family Σ of pre-orders over a domain D and fix any $a, b \in D$. If aPb, then $a\mathcal{P}^{\Sigma}b$ or $b\mathcal{P}^{\Sigma}a$.*

Proof. Let $I = \bigcap \Sigma$ and $U = \bigcup \Sigma$ and suppose that, while aPb, neither $a\mathcal{P}^{\Sigma}b$ nor $b\mathcal{P}^{\Sigma}a$. Then $\langle a, b\rangle \notin I$, $\langle b, a\rangle \notin I$ and so $\langle b, a\rangle \notin I^{inv}$ and $\langle a, b\rangle \notin I^{inv}$. But, from aPb, we have that $\langle a, b\rangle \in U$. Therefore $\langle a, b\rangle \in U \setminus (I \cup I^{inv})$. $\square$

The gluing operation requires nothing from Σ, so the fact that we may obtain a more informative pre-order can only be due to an intrinsic characteristic of the collection Σ. The condition for $\mathcal{P}^{\Sigma}$ to be more informative than every pre-order in Σ is that every pre-order in Σ is compatible with each other.

Definition 4.6. Let P be a pre-order and V a set of information values (i.e. $V \subseteq \{0, 1, 2\}$). Define $P \restriction V$ as the restriction of P to the pairs that are of information level V:

$$P \restriction V = \{\langle x, y\rangle \mid info(x, y, P) \in V\}$$

In particular, we use P_{max} when $V = \{2\}$.

Definition 4.7. Let Σ be a family of pre-orders. We say that Σ is compatible w.r.t. maximal information when $S = \bigcup\limits_{P \in \Sigma} P_{max}$ do not form cycles of any size (or equivalently S is contained in a partial order).

Theorem 4.8. *Suppose that P_1, P_2 are pre-orders defined over the same domain D. Then the following statements are equivalent:*

1. *P_1 and P_2 are compatible w.r.t. maximal information.*
2. *$\mathcal{P}^{\{P_1,P_2\}}$ is a pre-order in D that is more informative than P_1 and P_2.*
3. *There is a pre-ordered set (D', P') and informative morphisms $f : (D, P_i) \to (D', P')$, $i = 1, 2$.*

Proof. Suppose $P = \mathcal{P}^{\{P_1,P_2\}}$ is not more informative than P_1. Then there are $a, b \in D$ such that $Info(a, b, P)$ is smaller than $Info(a, b, P_1)$. Thus either (i) $Info(a, b, P) = 0$ and $Info(a, b, P_1) \geq 1$ or (ii) $Info(a, b, P) = 1$ and $Info(a, b, P_1) = 2$. If $Info(a, b, P_1) \geq 1$, either aP_1b or bP_1a; so, from Property 4.5, $Info(a, b, P)$ cannot be 0. Thence (i) is not possible. We conclude that $Info(a, b, P) = 1$ and that aPb and bPa. Because aPb is obtained with the transitive closure of connections established with P_1 or P_2, there is a sequence $a, c_1, c_2, \ldots, b$ connected point by point through P_1 or P_2. Analogously, we obtain a sequence $b, d_1, d_2, \ldots, a$. The connection of these two sequences form the cycle $s = \langle a, c_1, c_2, \ldots, b, d_1, d_2, \ldots, a\rangle$.

Let us then suppose that P_1 and P_2 are compatible w.r.t. maximal information. Then s cannot be a cycle of maximal information, i.e. it is not present in $P_1^{max} \cup P_2^{max}$. This means that at least one of the connections $\langle k, q\rangle$ in s cannot be made with P_1^{max} nor with P_2^{max}. $\square$

Two pre-orders P_1 and P_2 are said to be compatible for $a, b \in D$ if they **agree** whenever they **assert with maximal information**.

Let P^m is the subset of P containing only maximal information of P.

Moreover, P_1 and P_2 are compatible when $P_1^m \cup P_2^m$ forms no cycle.

5. A categorial encoding of generalized safe collapses and safe gluings

Lets start with the simplest one: the safe collapse. Recall that if $(D, P_i)_{i \in I}$ is a family of pre-orders over the same domain D, then the safe collapse of this family is just the pre-ordered set $(D, \bigcap_{i \in I} P_i)$.

If P is a binary relation on D, then this data can be described alternatively by the characteristic function $\check{P} : D \times D \to \{0, 1\}$.

Thus, if P' is a binary relation on D', then a map $f : D' \to D$ is a relation preserving morphism $f : (D', P') \to (D, P)$ iff

$$\check{P}' \leq \check{P} \circ (f \times f),$$

where the consider the coordinatewise ordering between functions $g, h : D \times D \to \{0, 1\}$ into the poset $(\{0, 1\}, \leq)$.

Note that:
(i) P is a reflexive relation iff $(D \stackrel{diag}{\to} D \times D \stackrel{\check{P}}{\to} \{0, 1\}) = (D \stackrel{1}{\to} \{0, 1\})$;
(ii) P is a transitive relation iff $(\check{P} \circ \pi^3_{12}) \wedge (\check{P} \circ \pi^3_{23}) \leq (\check{P} \circ \pi^3_{13}) : D \times D \times D \to \{0, 1\}$;
(iii) P is a symmetric relation iff $\check{P} = (\check{P} \circ \pi^2_{21}) : D \times D \to \{0, 1\}$.

The safe collapse of a family of pre-ordered sets $(D, P_i)_{i \in I}$, is represented by the function $p : D \times D \to \{0, 1\}$ given by $p(x, y) = \bigwedge_{i \in I} p_i(x, y)$, where $p_i = \check{P}_i$. The family of order preserving morphisms $(id : (D, P) \to (D, P_i))_{i \in I}$ satisfies the universal property: For each pre-ordered set (D', P') and order preserving morphisms $(f : (D', P') \to (D, P_i))_{i \in I}$ there is a unique order preserving morphism $f : (D', P') \to (D, P)$ such that $((D', P') \stackrel{f}{\to} (D, P) \stackrel{id}{\to} (D, P_i)) = ((D', P') \stackrel{f}{\to} (D, P_i))$, $i \in I$.

Another construction can be considered for general diagrams of pre-ordered sets $\Gamma : \mathcal{I} \to Poset$, $\Gamma(i) = (D_i, P_i)$, $lim\Gamma = (C, P)$, where

$$C = \{\vec{x} = (x_i)_{i \in I} \in \prod_{i \in I} D_i : \forall i, j \in I, \forall h : i \to j, \Gamma(h)(x_i) = x_j\}$$

$$p(\vec{x}, \vec{y}) = \bigwedge_{i \in I} p_i(x_i, y_i)$$

On the other hand, the safe gluing of a family of pre-ordered sets $(D, P_i)_{i \in I}$, is represented by the function $p : D \times D \to \{0, 1\}$ given by $p(x, y) = (\bigvee_{i \in I} p_i(x, y))^{(t)}$, where $p_i = \check{P}_i$ and $\check{P}$ is the least pre-order relation o D containing $\bigcup_{i\in I} P_i$, i.e. $\check{P} =$ transitive closure$(\bigcup_{i\in I} P_i) \cup \Delta_D$, where $\Delta_D = \{(x,x) : x \in D\}$ (if $I \neq \emptyset$, then $\Delta_D \subseteq$ transitive closure$(\bigcup_{i\in I} P_i)$).

Another construction can be considered for general diagrams of pre-orded sets $\Gamma : \mathcal{I} \to Poset$, $\Gamma(i) = (D_i, P_i)$, $colim\Gamma = (G, P)$, where

$$G = (\bigsqcup_{i\in ob(\mathcal{I})} D_i)/\equiv$$

and $\equiv$ is an "optimal" equivalence relation on $\bigsqcup_{i\in ob(\mathcal{I})})D_i)$ and P is the transitive closure of $\bigcup_{i\in I}(\alpha_i \times \alpha_i)[P_i]$ and $\alpha_i : D_i \to G$ is such that $\alpha_i(x_i) = [(x_i, i)]$, $x_i \in D_i$.

Naturally, the previous encoding allows us to generalize the previous considerations just by replacing the finite poset $(\{0,1\}, \leq)$ by any other complete lattice $(L, \leq)$. Thus:

1. A L-pre-ordered set is just a pair (D, p) where $p : D \times D \to L$ is a function such that:
   (i) p is a L-reflexive L-relation, i.e.: $p(x,x) = 1, \forall x \in D$ or $(D \stackrel{diag}{\to} D \times D \stackrel{p}{\to} L) = (D \stackrel{1}{\to} L)$;
   (ii) p is a L-transitive L-relation, i.e.: $p(x,y) \wedge p(y,z) \leq p(x,z), \forall x,y,z \in D$ or $(\check{P} \circ \pi^3_{12}) \wedge (\check{P} \circ \pi^3_{23}) \leq (\check{P} \circ \pi^3_{13}) : D \times D \times D \to L$.

2. If (D', p') is a L-preordered set, then a map $f : D' \to D$ is a L-monotonous morphism $f : (D', p') \to (D, p)$ iff

$$p' \leq p \circ (f \times f).$$

3. The safe collapse of a family of L-preordered sets $(D, p_i)_{i\in I}$, is the L-preordered set (D, p) represented by the function $p : D \times D \to L$ given by $p(x,y) = \bigwedge_{i\in I} p_i(x,y)$. The family of L-monotonous morphisms $(id : (D,p) \to (D, p_i))_{i\in I}$ satisfies an universal property: for each L-pre-ordered set (D', p') such that there is a function $f : D' \to D$ such that, for each $i \in I$, $f : (D', p') \to (D, p_i)$ is a L-monotonous morphism, then $f : (D', p') \to (D, p)$ is an well-defined L-monotonous morphism and it the unique morphism such that the family $(f : (D', p') \to (D, p_i))_{i\in I}$ factors through $(id : (D, p) \to (D, p_i))_{i\in I}$.

4. Limits $lim\Gamma = (C, p)$

   where

$$C = \{\vec{x} = (x_i)_{i\in I} \in \prod_{i\in I} D_i : \forall i,j \in I, \forall h : i \to j, \Gamma(h)(x_i) = x_j\}$$

$$p(\vec{x}, \vec{y}) = \bigwedge_{i\in I} p_i(x_i, y_i).$$

$p(\vec{x},\vec{y}) \wedge p(\vec{y},\vec{z}) \leq p_i(x_i,y_i) \wedge p_i(y_i,z_i) \leq p_i(x_i,z_i)$, thus $p(\vec{x},\vec{y}) \wedge p(\vec{y},\vec{z}) \leq \bigwedge_{i\in I} p_i(x_i,z_i) = p(\vec{x},\vec{z})$

5. gluing: works if L is a locale (= complete Heyting algebra)

   then (D,p) is $p = (\bigvee_{i\in I} p_i)^{(t)} \bigvee \delta_D$, where

   $\delta_D : X \times X \to L$, $\delta_D(x,x) = \top$, $\delta_D(x,y) = \bot$, if $x \neq y$

   $r^{(t)}(x,z) = \bigvee_{n\in\mathbb{N},\gamma\in Path_n(D)(x,z)} \bigwedge_{i<n} r(\gamma(i),\gamma(i+1))$, where $Path_n(D)(x,z) = \{\gamma \in Func(n+1,D) : \gamma(0) = x, \gamma(n) = z\}$

   if $I \neq \emptyset$, then $\delta_D \leq (\bigvee_{i\in I} p_i)^{(t)}$

   Then: $r \leq r^{(t)}$; if r' is a transitive L-relation on D such that $r \leq r'$ then $r^{(t)} \leq r'$ and, if L is a locale (= complete Heyting algebra),then $r^{(t)}$ is a transitive L-relation

   let $n,m \in \mathbb{N}$, $\gamma \in Path_n(D)(x,y), \tau \in Path_m(D)(y,z)$. Consider the "juxtaposition" $\tau * \gamma \in Path_{n+m}(D)(x,z)$, then

   $\bigwedge_{i<n} r(\gamma(i),\gamma(i+1)) \wedge \bigwedge_{j<m} r(\tau(j),\tau(j+1)) = \bigwedge_{k<n+m} r(\tau*\gamma(k),\tau*\gamma(k+1)) \leq r^{(t)}(x,z)$

   Thus $\bigvee_{n,m\in\mathbb{N},\gamma\in Path_n(D)(x,z)} \tau \in Path_m(D)(y,z) \bigwedge_{i<n} r(\gamma(i),\gamma(i+1)) \wedge \bigwedge_{j<m} r(\tau(j),\tau(j+1)) \leq r^{(t)}(x,z)$

   If we assume that L is a locale, then $r^{(t)}(x,y) \wedge r^{(t)}(y,z) = \bigvee_{n,m\in\mathbb{N},\gamma\in Path_n(D)(x,z)} \tau \in Path_m(D)(y,z) \bigwedge_{i<n} r(\gamma(i),\gamma(i+1)) \wedge \bigwedge_{j<m} r(\tau(j),\tau(j+1))$ then $r^{(t)}$ is the least transitive L-relation above r.

With the obvious notion of identities and compositions, the L-pre-ordered sets and the L-monotonous morphisms determines a category.

The former category can be enlarged by adding more general morphisms, just considering maps $F : D' \times D' \to D \times D$ as a morphism $F : (D',p') \to (D,p)$ iff

$$p' \leq p \circ F.$$

Let us call $L - gPoset$ the category of all L-pre-ordered sets and general morphisms between them (with obvious composition and identities). We can easily perform the analogous of the previous collapse constructions in this general category $L - gPoset$. For instance, the safe collapse of a family of L-preorded set is the same L-preodered set considered above.

If $t : L \to K$ is a complete morphism of complete lattices, we have a induced functor

$$P(t) : L - gPoset \to K - gPoset$$

such that

$$P(t)(F : (D',p') \to (D,p)) = F : (D', t\circ p') \to (D, t\circ p)$$

This functor preserves safes collapses.

Moreover, clearly the map $(t : L \to K) \mapsto (P(t) : L-gPoset \to K-gPoset)$ determines a functor.

6. A categorial encoding of generalized informative collapses and informative gluings

After the previous warm up section, we turn our attention to categories directly related with the main subject of this work: informative morphisms.

Now, it is not surprising that we can represent the information of a binary relation $P \subseteq X \times X$ as a certain "symmetric" function:

$i(P) : X \times X \to \{0, 1, 2\}$, i.e., such that $i(P) \circ t = i(P)$, where $t = (\pi_2, \pi_1)$.

More precisely: $i(P)^{-1}[\{0\}] = (X \times X \setminus P) \cup (X \times X \setminus t[P])$; $i(P)^{-1}[\{1\}] = P \cap t[P]$; $i(P)^{-1}[\{2\}] = (P \cap (X \times X \setminus t[P])) \cup (t[P] \cap (X \times X \setminus P))$.

Thus, if P' is a binary relation on X', then a map $f : X' \to X$ is an informative morphism $f : (X', P') \to (X, P)$ iff

$$i(P') \leq i(P) \circ (f \times f),$$

where the consider the coordinatewise ordering between functions $g, h : X' \times X' \to \{0, 1, 2\}$ into the poset $(\{0, 1, 2\}, \leq)$.

We can expand this category in many natural ways:

1. replacing the finite lattice $\{0, 1\} \cup \{2\}$ by a complete lattice $L \cup \{\infty\}$, where L is a complete lattice.

2. considering an information concerning a L-relation on X (a function $p : X \times X \to L$, where L is a complete lattice) as a function $i(p) : X \times X \to L \cup \{\infty\}$ such that:

   - $i(p) \circ t = i(p)$;

   - $i(p)^{-1}[\{\bot\}] = \{(x, y) : p(x, y) = \bot = p(y, x)\}$;

   - $i(p)^{-1}[\{\infty\}] = \{(x, y) : (p(x, y) = \top, p(y, x) = \bot)$ *or* $(p(y, x) = \top, p(x, y) = \bot)\}$

3. considering a morphism as a function $F : X' \times X' \to X \times X$ such that $i(p') \leq i(p) \circ F$

Remark 6.1. In forthcoming work, we will develop a categorical approach to informative collapses and informative gluings.

References

[1] Alfredo Roque Freire, *Estudo comparado do comprometimento ontológico das teorias de classes e conjuntos*, Ph.D. thesis, University of Campinas, 2019.

[2] Alfredo Roque Freire and Joel David Hamkins, *Bi-interpretation in weak set theories*, Journal of Symbolic Logic **86** (2021), no. 2, 609–634.

[3] Richard Kaye and Tin Lok Wong, *On interpretations of arithmetic and set theory*, Notre Dame Journal of Formal Logic **48** (2007), no. 4, 497–510.