

Einstein metrics on cohomogeneity one manifolds

Métricas de Einstein en variedades de cohomogeneidad uno

Brian Grajales^{1,a}

Abstract. This note presents results concerning the Einstein equation for metrics that preserve the symmetries induced by a cohomogeneity one action of a compact Lie group on a compact manifold. Specifically, we discuss the existence of Einstein metrics that are invariant under a cohomogeneity one action of the special orthogonal group on the complex projective space.

Keywords: Einstein metrics, Cohomogeneity one manifold, Complex projective space.

Resumen. Esta nota presenta resultados sobre la ecuación de Einstein para métricas que preservan las simetrías inducidas por una acción de cohomogeneidad uno de un grupo de Lie compacto sobre una variedad compacta. Específicamente, se discute la existencia de métricas de Einstein que son invariantes bajo una acción de cohomogeneidad uno del grupo ortogonal especial sobre el espacio proyectivo complejo.

Palabras claves: Métricas de Einstein, Variedad de cohomogeneidad uno, Espacio proyectivo complejo.

Mathematics Subject Classification: 53C25, 22E99.

Recibido: noviembre de 2024

Aceptado: septiembre de 2025

1. Introduction

Einstein metrics are a topic of great interest in mathematics and physics, because they correspond to critical points of the Einstein-Hilbert functional, establishing a relation between these metrics and concepts related to energy in general relativity. A Riemannian metric g on a differentiable manifold M is an Einstein metric if it satisfies the equation

$$\text{Ric} = \lambda g, \tag{1}$$

¹Departamento de Matemática, Universidade Estadual de Maringá, Maringá, Brazil

^abdgtriana@uem.br

where $\lambda \in \mathbb{R}$ and Ric is the Ricci curvature tensor corresponding to g . While solving equation (1) is generally challenging, it becomes more manageable when we focus on manifolds with certain symmetry properties. Among such structures, cohomogeneity one manifolds are especially interesting: a compact Lie group acts on the manifold so that the orbit space is one-dimensional. In this context, when considering metrics that preserve the transformations induced by the action, equation (1) reduces to a system of ordinary differential equations, allowing us to employ tools from dynamical systems theory to study it.

In this note, we aim to discuss the existence of smooth Einstein metrics on a compact manifold M that admits a cohomogeneity one action of a compact Lie group G , and under the assumption that there are no exceptional orbits. In Section 2, we will discuss the foundational results in [5] regarding the description of G -invariant Riemannian metrics on such manifolds. This section will also address the associated Ricci tensor and the Einstein equation (1), which can be formulated as an initial value problem for a system of ordinary differential equations. In Section 3, we will illustrate these concepts through a specific example involving the Lie group $\text{SO}(n+1)$, which acts with cohomogeneity one on the n -dimensional complex projective space.

To emphasize the significance of the results presented here, note that in recent decades, there has been a significantly larger body of literature on Einstein metrics on homogeneous spaces (cohomogeneity zero manifolds), in contrast to the case of cohomogeneity one manifolds. This is no surprise, as the Einstein condition for homogeneous metrics reduces to a system of algebraic equations, making the problem more accessible. Consequently, much has been achieved in understanding Einstein metrics on these spaces. In contrast, for cohomogeneity one manifolds, the Einstein condition leads to a system of ordinary differential equations, which is generally more complex to solve and requires the consideration of initial conditions. This transition from algebraic to differential equations introduces new challenges and makes the study of Einstein metrics on cohomogeneity one manifolds an under-explored area. The need for sophisticated analytical tools, including methods from differential geometry and dynamical systems, highlights the novelty of the current work.

2. The Einstein equation

Let G be a compact, connected Lie group acting on a differentiable manifold M such that $M/G \approx [0, 1]$, and assume that there are no exceptional orbits. The points in the open interval $(0, 1)$ correspond to the principal orbits of the action, while the boundary points 0 and 1 correspond to the singular orbits. Consider an arbitrary G -invariant metric on M and a geodesic $c : [0, 1] \rightarrow M$ orthogonal to all orbits. Let K be the isotropy group of $c(0)$, and $Q := G \cdot c(0) \approx G/K$. By compactness of G , there exists a principal orbit whose isotropy group

$G_{c(t)}$ has the minimal number of components among all principal orbits. As a consequence of [6, Theorem 1], there exists a neighborhood V of $c(t)$ such that for all s with $c(s) \in V$, the isotropy group $G_{c(s)}$ is conjugate to a subgroup of $G_{c(t)}$, that is, there exists $a \in G$ such that

$$aG_{c(s)}a^{-1} \subseteq G_{c(t)}.$$

We can take V small enough so that the orbit of each point in V is principal, ensuring that

$$\dim aG_{c(s)}a^{-1} = \dim G_{c(s)} = \dim G_{c(t)}, \quad \text{for all } s \text{ with } c(s) \in V.$$

Since $G_{c(t)}$ was chosen to have the minimal number of components among principal isotropy groups, it follows that

$$aG_{c(s)}a^{-1} = G_{c(t)}.$$

Thus, for each $t \in (0, 1)$, there exists an open neighborhood V_t such that the isotropy groups of all points $c(s) \in V_t$ are conjugate. Now, given $0 < t_1 < t_2 < 1$, we can cover the compact set $c([t_1, t_2])$ with a finite number of such neighborhoods, concluding that the isotropy groups of $c(t_1)$ and $c(t_2)$ are conjugate. This allows us to consistently identify the isotropy groups of all points $c(t)$ for $0 < t < 1$ with a single group H . Moreover, we can assume that $H \subseteq K$ because, again by [6, Theorem 1], for a sufficiently small t , the isotropy group of $c(t)$ is conjugate to a subgroup of $G_{c(0)} = K$. In this case, K/H is diffeomorphic to a sphere \mathbb{S}^ℓ of dimension $\ell \geq 1$ (see [7, Theorem 2]).

Any G -invariant metric \hat{g} on M is determined by its restriction g to the union M_0 of all principal orbits, given by

$$g = dt^2 + g(t) \tag{2}$$

where $g(t)$ is a one-parameter family of G -invariant metrics on $P := G/H$ that extends smoothly over the interval $[0, 1]$.

Let \mathfrak{g} , \mathfrak{k} and \mathfrak{h} denote the Lie algebras of G , K and H , respectively. Consider an $\text{Ad}(G)$ -invariant inner product (\cdot, \cdot) on \mathfrak{g} and let $\mathfrak{m} := \mathfrak{h}^\perp$ be the orthogonal complement of \mathfrak{h} in \mathfrak{g} . Observe that for $X \in \mathfrak{m}$, $Y \in \mathfrak{h}$ and $h \in H$, the $\text{Ad}(G)$ -invariance of (\cdot, \cdot) implies that

$$(\text{Ad}(h)X, Y) = (X, \text{Ad}(h^{-1})Y) = 0.$$

Therefore, \mathfrak{m} is an Ad^H -invariant subspace of \mathfrak{g} and the representation

$$\text{Ad}^H \big|_{\mathfrak{m}} : H \longrightarrow \text{GL}(\mathfrak{m})$$

is well defined. It is well-known that this representation is equivalent to the isotropy representation of the homogeneous space G/H .

Proposition 2.1. *The subspace \mathfrak{m} can be decomposed into irreducible pairwise orthogonal Ad^H -invariant subspaces $\mathfrak{m}_1, \dots, \mathfrak{m}_s$ such that, for each $i \in \{1, \dots, s\}$, either $\mathfrak{m}_i \subseteq \mathfrak{k}$ or $\mathfrak{m}_i \cap \mathfrak{k} = \{0\}$.*

Proof. Let \mathfrak{m}' be the orthogonal complement of \mathfrak{h} in \mathfrak{k} , and let \mathfrak{m}'' be the orthogonal complement of \mathfrak{k} in \mathfrak{g} . Then, we have

$$\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}' \quad \text{and} \quad \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}'' = \mathfrak{h} \oplus \mathfrak{m}' \oplus \mathfrak{m}''.$$

Since \mathfrak{m}'' is orthogonal to \mathfrak{k} and $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}'$, it follows that \mathfrak{m}'' is orthogonal to \mathfrak{m}' . This implies that $\mathfrak{m}' \oplus \mathfrak{m}'' \subseteq \mathfrak{m}$, and

$$\dim(\mathfrak{m}' \oplus \mathfrak{m}'') = \dim \mathfrak{k} - \dim \mathfrak{h} + \dim \mathfrak{g} - \dim \mathfrak{k} = \dim \mathfrak{g} - \dim \mathfrak{h} = \dim \mathfrak{m},$$

so we conclude that $\mathfrak{m} = \mathfrak{m}' \oplus \mathfrak{m}''$. The fact that (\cdot, \cdot) is $\text{Ad}(G)$ -invariant implies that \mathfrak{m}' is Ad^H -invariant and \mathfrak{m}'' is Ad^K -invariant. Since $H \subseteq K$, it follows that \mathfrak{m}'' is also Ad^H -invariant. As H is a closed subgroup of the compact group G , it is itself compact. Therefore, the representations $\text{Ad}^H|_{\mathfrak{m}'}$ and $\text{Ad}^H|_{\mathfrak{m}''}$ decompose into pairwise orthogonal irreducible Ad^H -invariant subspaces (see, for instance, [2, Proposition 1.9]). These decompositions yield a decomposition of \mathfrak{m} into pairwise orthogonal Ad^H -invariant irreducible subspaces in which the summands of \mathfrak{m}' are contained in \mathfrak{k} , while the summands of \mathfrak{m}'' intersect \mathfrak{k} only trivially. \square

Let $\mathfrak{m}_1, \dots, \mathfrak{m}_s$ be as in Proposition 2.1. Define

$$\mathfrak{p} := \bigoplus_{i \in J_+} \mathfrak{m}_i \quad \text{and} \quad \mathfrak{n} := \bigoplus_{i \in J_-} \mathfrak{m}_i,$$

where J_+ is the set of indices i such that $\mathfrak{m}_i \subseteq \mathfrak{k}$, and J_- corresponds to $\mathfrak{m}_i \cap \mathfrak{k} = \{0\}$. The subspaces \mathfrak{p} and \mathfrak{n} can be identified with $T_{eH}(K/H) \cong T_{eH}\mathbb{S}^\ell$ and $T_{eK}(G/K)$, respectively. We assume that \mathfrak{n} is $\text{Ad}(K)$ -irreducible and that (\cdot, \cdot) corresponds to the standard metric of curvature one on \mathbb{S}^ℓ when restricted to \mathfrak{p} .

For each $t \in (0, 1)$, the metric $g(t)$ can be seen as an $\text{Ad}(H)$ -invariant inner product on \mathfrak{m} . As a result, there exists a positive, (\cdot, \cdot) -self-adjoint, and $\text{Ad}^H|_{\mathfrak{m}}$ -equivariant operator $P(t) : \mathfrak{m} \rightarrow \mathfrak{m}$ which satisfies

$$g(t)(X, Y) = (P(t)X, Y), \quad X, Y \in \mathfrak{m}.$$

When

$$P(t) = f_1(t)^2 I_{\mathfrak{m}_1} + \dots + f_s(t)^2 I_{\mathfrak{m}_s}$$

for some smooth functions $f_i : (0, 1) \rightarrow \mathbb{R}^+$, $i = 1, \dots, s$, we say that the metric $g(t)$ is *diagonal*. In this case, the shape operator $L(t)$ for the principal orbit through $c(t)$, $0 < t < 1$ is given by

$$L(t) = \frac{1}{2} P(t)^{-1} P'(t) = \frac{f_1'(t)}{f_1(t)} I_{\mathfrak{m}_1} + \dots + \frac{f_s'(t)}{f_s(t)} I_{\mathfrak{m}_s}.$$

If $T = \partial/\partial t$, then the tangent space $T_{c(t)}M$ of M at $c(t)$ can be decomposed into the g -orthogonal subspaces $\text{span}\{T\}$ and \mathfrak{m} , where \mathfrak{m} corresponds to $T_{c(t)}(G \cdot c(t))$. Consider an (\cdot, \cdot) -orthonormal basis $\{e_\alpha\}$ of \mathfrak{m} , adapted to the decomposition

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s.$$

If B denotes the Killing form on \mathfrak{g} , then for each $i \in \{1, \dots, s\}$, the restriction $-B|_{\mathfrak{m}_i \times \mathfrak{m}_i}$ defines an $\text{Ad}(H)$ -invariant inner product on \mathfrak{m}_i . Thus, there exists a constant b_i such that

$$-B|_{\mathfrak{m}_i \times \mathfrak{m}_i} = b_i(\cdot, \cdot)|_{\mathfrak{m}_i \times \mathfrak{m}_i}$$

(see, for instance, [4, Corollary 2.11]).

The following proposition provides the Einstein equations for the metric (2).

Proposition 2.2. *The metric*

$$g = dt^2 + f_1(t)^2(\cdot, \cdot)|_{\mathfrak{m}_1 \times \mathfrak{m}_1} + \cdots + f_s(t)^2(\cdot, \cdot)|_{\mathfrak{m}_s \times \mathfrak{m}_s} \quad (3)$$

is Einstein for the constant $\lambda \in \mathbb{R}$ if and only if the functions f_i , $i = 1, \dots, s$ satisfy the following equations:

$$d_1 \frac{f_1''}{f_1} + \cdots + d_s \frac{f_s''}{f_s} = -\lambda, \quad (4)$$

$$\frac{f_i''}{f_i} - \frac{b_i}{2f_i^2} - \frac{1}{d_i} \sum_{k, \ell=1}^s \frac{f_i^4 - 2f_k^4}{4f_i^2 f_k^2 f_\ell^2} [ik\ell] + (d_i - 1) \frac{f_i'^2}{f_i^2} + \sum_{\substack{k=1 \\ k \neq i}}^s d_k \frac{f_i' f_k'}{f_i f_k} = -\lambda, \quad (5)$$

$$i = 1, \dots, s,$$

$$\sum_{k, \ell=1}^s \frac{f_i^2 f_j^2 - 2f_k^4 + 2f_k^2 f_\ell^2}{4f_k^2 f_\ell^2} \sum_{e_\gamma \in \mathfrak{m}_k} ([e_\delta, e_\gamma]_{\mathfrak{m}_\ell}, [e_\eta, e_\gamma]_{\mathfrak{m}_\ell}) = 0, \quad (6)$$

$$i, j = 1, \dots, s$$

where $e_\delta \in \mathfrak{m}_i$, $e_\eta \in \mathfrak{m}_j$,

$$[ik\ell] := \sum_{e_\alpha \in \mathfrak{m}_i} \sum_{e_\beta \in \mathfrak{m}_k} \sum_{e_\gamma \in \mathfrak{m}_\ell} ([e_\alpha, e_\beta], e_\gamma)^2$$

and

$$d_k := \dim \mathfrak{m}_k, \quad k = 1, \dots, s.$$

Proof. The metric (3) is Einstein for the constant λ if and only if

$$\text{Ric} = \lambda g,$$

where Ric denotes the Ricci tensor associated with the metric g . According to [5, Proposition 1.14, Remark 1.16b], the components of the Ricci tensor are

$$\begin{aligned}\text{Ric}(T, T) &= -d_1 \frac{f_1''}{f_1} - d_2 \frac{f_2''}{f_2} - d_3 \frac{f_3''}{f_3}, \\ \text{Ric}(e_\delta, e_\eta) &= \sum_{k, \ell=1}^s \frac{f_i^2 f_j^2 - 2f_k^4 + 2f_k^2 f_\ell^2}{4f_k^2 f_\ell^2} \sum_{e_\gamma \in \mathfrak{m}_k} \left([e_\delta, e_\gamma]_{\mathfrak{m}_\ell}, [e_\eta, e_\gamma]_{\mathfrak{m}_\ell} \right), \\ \text{Ric}(e_\delta, T) &= 0, \\ \text{Ric}(e_\delta, e_\delta) &= \left\{ \frac{b_i}{2f_i^2} + \sum_{k, \ell=1}^s \frac{f_i^4 - 2f_k^4}{4f_i^2 f_k^2 f_\ell^2} \begin{bmatrix} \ell \\ i \ k \end{bmatrix} - \frac{f_i'}{f_i} \sum_{k=1}^s d_k \frac{f_k'}{f_k} + \frac{f_i'^2}{f_i^2} - \frac{f_i''}{f_i} \right\} g(e_\delta, e_\delta),\end{aligned}$$

where $e_\delta \in \mathfrak{m}_i$, $e_\eta \in \mathfrak{m}_j$, $i \neq j$ and the constants $\begin{bmatrix} \ell \\ i \ k \end{bmatrix}$ satisfy

$$\sum_{e_\gamma \in \mathfrak{m}_k} \left\| [X, e_\gamma]_{\mathfrak{m}_\ell} \right\|^2 = \begin{bmatrix} \ell \\ i \ k \end{bmatrix} \|X\|^2, \quad X \in \mathfrak{m}_i.$$

Moreover, we have the relation

$$\begin{aligned}[ik\ell] &= \sum_{e_\alpha \in \mathfrak{m}_i} \sum_{e_\beta \in \mathfrak{m}_k} \sum_{e_\gamma \in \mathfrak{m}_\ell} ([e_\alpha, e_\beta], e_\gamma)^2 \\ &= \sum_{e_\alpha \in \mathfrak{m}_i} \sum_{e_\beta \in \mathfrak{m}_k} \left\| [e_\alpha, e_\beta]_{\mathfrak{m}_\ell} \right\|^2 \\ &= \sum_{e_\alpha \in \mathfrak{m}_i} \begin{bmatrix} \ell \\ i \ k \end{bmatrix} \|e_\alpha\|^2 \\ &= \sum_{e_\alpha \in \mathfrak{m}_i} \begin{bmatrix} \ell \\ i \ k \end{bmatrix} \\ &= \begin{bmatrix} \ell \\ i \ k \end{bmatrix} d_i,\end{aligned}$$

which implies

$$\sum_{k, \ell=1}^s \frac{f_i^4 - 2f_k^4}{4f_i^2 f_k^2 f_\ell^2} \begin{bmatrix} \ell \\ i \ k \end{bmatrix} = \frac{1}{d_i} \sum_{k, \ell=1}^s \frac{f_i^4 - 2f_k^4}{4f_i^2 f_k^2 f_\ell^2} [ik\ell]$$

The result follows from the fact that $g(T, T) = g(e_\delta, e_\delta) = 1$ and $g(e_\delta, e_\eta) = 0$. \square

Note that the metric (3) is not defined at the singular orbits. However, according to [3, Corollary 2.6], if the functions f_1, \dots, f_s satisfy the equations (5) and can be smoothly extended to the singular orbits, then the metric \hat{g} determined

by (3) becomes an Einstein metric defined on the whole manifold M . From the results in [3, Section 3], we can deduce the following necessary and sufficient conditions for the functions f_i to extend smoothly to 0:

$$\begin{aligned} f_i(0) = f_i''(0) = 0, \quad f_i'(0) = 1, \quad i \in J_+ \\ f_i(0) = \zeta, \quad f_i'(0) = 0, \quad i \in J_-. \end{aligned} \tag{7}$$

Remark 2.3. By considering the reparametrization $\tilde{c}(t) = c(1-t)$, we can easily derive the similar conditions on the functions f_i for them to extend smoothly to $t = 1$.

The results discussed in this section reveal that the problem of finding a G -invariant Einstein metric on a cohomogeneity one manifold can be translated into solving a system of ordinary differential equations defined on the interval $(0, 1)$ with specific boundary conditions at $t = 0$ and $t = 1$.

3. $\mathrm{SO}(n+1)$ -invariant Einstein metrics on $\mathbb{C}P^n$

Consider the action $U \cdot [z] = [Uz]$, where $U \in \mathrm{SO}(n+1)$, $z \in \mathbb{C}^{n+1} - \{0\}$ and $[z]$ denotes the class of z in $\mathbb{C}P^n$. For any natural number $n \geq 2$, this is a cohomogeneity one action (see [8, 9]). Let $c : [0, 1] \rightarrow \mathbb{C}P^n$ be the curve defined by

$$c(t) := [0 : \cdots : 0 : 1 - t : \sqrt{-1}],$$

where, for each $(z_0, z_1, \dots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}$, the notation $[z_0 : \cdots : z_n]$ denotes its equivalence class in the complex projective space $\mathbb{C}P^n$. The isotropy group of $c(t)$ is

$$\begin{aligned} H &= \mathrm{SO}(n+1)_{c(t)} \approx \mathrm{SO}(n-1) \times \mathbb{Z}_2, \quad \text{if } 0 < t < 1, \\ K_1 &= \mathrm{SO}(n+1)_{c(1)} \approx \mathrm{S}(\mathrm{O}(n) \times \mathrm{O}(1)), \\ K_0 &= \mathrm{SO}(n+1)_{c(0)} \approx \mathrm{SO}(n-1) \times \mathrm{SO}(2). \end{aligned}$$

A principal orbit P is diffeomorphic to the homogeneous space $\frac{\mathrm{SO}(n+1)}{\mathrm{SO}(n-1) \times \mathbb{Z}_2}$. Fix the $\mathrm{Ad}(\mathrm{SO}(n+1))$ -invariant inner product

$$(\cdot, \cdot) := -\frac{2}{n-1}B,$$

where B is the Killing form of $\mathfrak{so}(n+1)$. With respect to this inner product, the isotropy representation of P decomposes into the following irreducible pairwise orthogonal submodules

$$\begin{aligned} \mathfrak{m}_1 &= \mathrm{span}_{\mathbb{R}} \{E_{n+1,n} - E_{n,n+1}\}, \\ \mathfrak{m}_2 &= \mathrm{span}_{\mathbb{R}} \{E_{nj} - E_{jn} : 1 \leq j \leq n-1\}, \\ \mathfrak{m}_3 &= \mathrm{span}_{\mathbb{R}} \{E_{n+1,j} - E_{j,n+1} : 1 \leq j \leq n-1\}, \end{aligned}$$

where E_{ij} denotes the $(n+1) \times (n+1)$ matrix with 1 in the (i, j) -entry and zeros elsewhere. The singular orbits are

$$Q_1 = \frac{\mathrm{SO}(n+1)}{\mathrm{S}(\mathrm{O}(n) \times \mathrm{O}(1))} \approx \mathbb{R}P^n \text{ and } Q_0 = \frac{\mathrm{SO}(n+1)}{\mathrm{SO}(n-1) \times \mathrm{SO}(2)}.$$

The quotient K_i/H , $i = 0, 1$ is diffeomorphic to the sphere \mathbb{S}^{ℓ_i} , where $\ell_1 = n-1$ and $\ell_0 = 1$. The set $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$, where

$$\begin{aligned} \mathcal{B}_1 &= \left\{ \frac{1}{2} (E_{n+1,n} - E_{n,n+1}) \right\}, \\ \mathcal{B}_2 &= \left\{ \frac{1}{2} (E_{nj} - E_{jn}) : j = 1, \dots, n-1 \right\}, \\ \mathcal{B}_3 &= \left\{ \frac{1}{2} (E_{n+1,j} - E_{j,n+1}) : j = 1, \dots, n-1 \right\}. \end{aligned}$$

is an (\cdot, \cdot) -orthonormal basis of \mathfrak{m} adapted to the decomposition $\mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$. With respect to this basis, we have

$$[ijk] = \begin{cases} \frac{n-1}{4}, & \text{if } \{i, j, k\} = \{1, 2, 3\}, \\ 0, & \text{otherwise.} \end{cases}$$

Consider an $\mathrm{SO}(n+1)$ -invariant metric

$$g = dt^2 + f_1(t)^2(\cdot, \cdot)|_{\mathfrak{m}_1 \times \mathfrak{m}_1} + f_2(t)^2(\cdot, \cdot)|_{\mathfrak{m}_2 \times \mathfrak{m}_2} + f_3(t)^2(\cdot, \cdot)|_{\mathfrak{m}_3 \times \mathfrak{m}_3} \quad (8)$$

on $\mathbb{C}P^n - \{Q_0, Q_1\}$. Using (7), we can show that g can be extended smoothly to Q_0 and Q_1 if and only if

$$\begin{aligned} f_1(0) &= f_1''(0) = 0, \quad f_1'(0) = 1, \\ f_2(0) &= f_3(0) = \zeta_0, \quad f_2'(0) = f_3'(0) = 0 \\ f_2(1) &= f_2''(1) = 0, \quad f_2'(1) = -\frac{1}{2}, \\ f_1(1) &= f_3(1) = \zeta_1, \quad f_1'(1) = f_3'(1) = 0, \end{aligned}$$

for some (arbitrary) $\zeta_0, \zeta_1 > 0$. Therefore, by Proposition 2.2, we conclude that an $\mathrm{SO}(n+1)$ -invariant metric \hat{g} on $\mathbb{C}P^n$ determined by (8) is Einstein with respect to the constant $\lambda \in \mathbb{R}$ if and only if

$$\begin{aligned} \frac{f_1''}{f_1} - \frac{n-1}{4f_1^2} - \frac{n-1}{8} \left(\frac{f_1^2}{f_2^2 f_3^2} - \frac{f_2^2}{f_1^2 f_3^2} - \frac{f_3^2}{f_1^2 f_2^2} \right) + (n-1) \frac{f_1' f_2'}{f_1 f_2} \\ + (n-1) \frac{f_1' f_3'}{f_1 f_3} = -\lambda, \end{aligned}$$

$$\begin{aligned} & \frac{f_2''}{f_2} - \frac{n-1}{4f_2^2} - \frac{1}{8} \left(\frac{f_2^2}{f_1^2 f_3^2} - \frac{f_1^2}{f_2^2 f_3^2} - \frac{f_3^2}{f_1^2 f_2^2} \right) + (n-2) \frac{f_2'^2}{f_2^2} + (n-1) \frac{f_2' f_3'}{f_2 f_3} \\ & + \frac{f_1' f_2'}{f_1 f_2} = -\lambda, \end{aligned}$$

$$\begin{aligned} & \frac{f_3''}{f_3} - \frac{n-1}{4f_3^2} - \frac{1}{8} \left(\frac{f_3^2}{f_1^2 f_2^2} - \frac{f_1^2}{f_2^2 f_3^2} - \frac{f_2^2}{f_1^2 f_3^2} \right) + (n-2) \frac{f_3'^2}{f_3^2} + (n-1) \frac{f_2' f_3'}{f_2 f_3} \\ & + \frac{f_1' f_3'}{f_1 f_3} = -\lambda, \end{aligned}$$

$$\begin{aligned} (f_1(0), f_2(0), f_3(0), f_1'(0), f_2'(0), f_3'(0)) &= (0, \zeta_0, \zeta_0, 1, 0, 0), \quad f_1''(0) = 0, \\ (f_1(1), f_2(1), f_3(1), f_1'(1), f_2'(1), f_3'(1)) &= (\zeta_1, 0, \zeta_1, 0, -1/2, 0), \quad f_2''(1) = 0, \end{aligned}$$

where $\zeta_0, \zeta_1 > 0$. Using analytic techniques and [1, Theorem 2.3], we can prove that the initial value problem above lacks a solution (f_1, f_2, f_3) defined on the whole interval $[0, 1]$. The idea of the proof consists of showing the existence and uniqueness of the system of differential equations above with the initial condition at $t = 0$, and proving that such a solution lies on an invariant surface that does not contain the final point, thus preventing it from satisfying the final condition at $t = 1$. Consequently, we have the following result:

Proposition 3.1. *There exists no smooth $\text{SO}(n+1)$ -invariant Einstein metric on $\mathbb{C}P^n$.*

References

- [1] C. Böhm, *Inhomogeneous Einstein metrics on low-dimensional spheres and other low-dimensional spaces*, Invent. math. **134** (1998), 145–176.
- [2] T. Bröcker and T. tom Dieck, *Elementary representation theory. In: Representations of compact Lie Groups*, Graduate Texts in Mathematics, vol. 98, Springer Berlin Heidelberg, 1985.
- [3] J.H. Eschenburg and M.Y. Wang, *The initial value problem for cohomogeneity one Einstein metrics*, J. Geom. Anal. **10** (2000), 109–137.
- [4] B. Grajales and L. Grama, *Equigeodesic vectors on compact homogeneous spaces with equivalent isotropy summands*, Ann. Mat. Pura Appl. **203** (2024), no. 6, 2741–2768.

- [5] K. Grove and W. Ziller, *Cohomogeneity one manifolds with positive Ricci curvature*, Invent. math. **149** (2002), 619–646.
- [6] D. Montgomery and L. Zippin, *A theorem of Lie groups*, Bull. Am. Math. Soc. **48** (1942), no. 6, 448–452.
- [7] P.S. Mostert, *On a compact lie group acting on a manifold*, Ann. Math. **65** (1957), no. 3, 447–455.
- [8] R. Takagi, *On homogeneous real hypersurfaces in a complex projective space*, Osaka J. Math. **10** (1973), no. 3, 495–506.
- [9] F. Uchida, *Classification of compact transformation groups on cohomology complex projective spaces with codimension one orbits*, Jpn. J. Math. **3** (1977), no. 1, 141–189.