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On geometric forms for Cauchy's and Flett's mean value theorems

Sobre formas geométricas de los teorema del valor medio de Cauchy y de Flett

Sergio A. Carrillo[a]

Abstract. We give two mean value type theorems for differentiable curves in the Euclidean space. These unify several versions of Cauchy's and Flett's mean value theorems and provide equivalent geometric statements to their classical counterparts. We also discuss some particular cases for curves in two and three dimensions, and some related results.

Keywords: Cauchy's mean value theorem; Flett's theorem; Wachnicki's theorem; differentiable curves.

Resumen. Presentamos dos teoremas del valor medio para curvas diferenciables en el espacio euclidiano. Estos unifican varias versiones de los teoremas del valor medio de Cauchy y Flett y proporcionan enunciados geométricos equivalentes a sus contrapartes clásicas. También analizamos algunos casos particulares para curvas en dos y tres dimensiones, y algunos resultados relacionados.

Palabras claves: Teorema del valor medio de Cauchy, teorema de Flett, teorema de Wachnicki, curvas diferenciables.

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1. Introduction

One of the central theorems in Calculus of one variable is Lagrange's mean value theorem (MVT) for derivatives. This is often explained by means of the apparently more general statement known as:

Cauchy's MVT. Given continuous functions $f, g : [a, b] \to \mathbb{R}$ which are differentiable in (a, b), there is $c \in (a, b)$ such that

$$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a)). \tag{1}$$

[1]Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá, Colombia
[a]sacarrillot@unal.edu.co

Among its applications we highlight the proof of L'Hôpital's rule [15]. Lagrange's MVT corresponds to the particular case $g(t) = t$, having a clear geometric insight suitable for students to quickly understand its meaning. More precisely, it assures the existence of a point $(c, f(c))$ for which its tangent line to the graph of f is parallel to the line passing through $(a, f(a))$ and $(b, f(b))$. It is often mentioned in class that these results are equivalent since both are tantamount to Rolle's theorem. For an alternative recent proof of the equivalence between Cauchy's and Lagrange's theorems when $g'(t) \neq 0$ for all t, see [3].

Cauchy's MVT admits several generalizations including higher order derivatives [1] or funny looking forms [13]. The reader may consult [11], [12, Chapter 7] and the references therein for more information. Some other generalizations have also been studied, see, e.g., [11]. The theorem also allows variants involving auxiliary functions with appealing geometric interpretations. Two of them are Flett's and Wachnicki's theorems, see [6, 18]. The former can be stated as follows:

Wachnicki's theorem. If f and g are differentiable in $[a, b]$, $g'(t) \neq 0$, for all $t \in [a, b]$, and $f'(a)/g'(a) = f'(b)/g'(b)$, then there is $\eta \in (a, b)$ such that

$$\frac{f(\eta) - f(a)}{g(\eta) - g(a)} = \frac{f'(\eta)}{g'(\eta)}. \tag{2}$$

The case $g(t) = t$ is precisely Flett's original theorem. In geometric terms, Flett's result assumes that the tangent lines to the graph of f at the boundary points are parallel. Then, it proves the existence of a point $(\eta, f(\eta))$ for which the tangent line of the graph of f passes through the point $(a, f(a))$. These two results are equivalent in the same way that Cauchy's and Lagrange's theorems are. In fact, the argument in [3] follows the same lines as the proof in [9], which is older.

Since graphs of real-valued functions are nothing but particular cases of curves in Euclidean space, it is natural to wonder if there are mean value type theorems in this context. After all, the geometry of Cauchy's MVT theorem is usually explained by using the curve $t \in [a, b] \mapsto (f(t), g(t)) \in \mathbb{R}^2$. It is well-known that the MVT theorem does not extend immediately to vector-valued maps nor with values in a Banach space. In this case, the correct and valuable extension is called the mean value inequality. Quoting Dieudonné: ' '*the real nature of the mean value theorem is exhibited by writing it as an inequality, and not as an equality*" [4, p. 148]. For a quick proof of this result the reader may consult [8]. In contrast, there are some interesting related results for vector-valued differentiable maps, such as a complex Rolle's theorem, see [5]. There are also some related results in the setting of topological vector spaces, see, e.g., [7, 10, 14] Thus it is valid to wonder for MVTs in higher dimensions.

The goal of this note is to contribute to MVTs for curves by giving two extra equivalent statements for both Cauchy's and Flett's theorems involving curves in Euclidean space. They have the advantage of having a geometric flavor that might help to give students a broader understanding of the original theorems.

The results are suitable to Calculus classes in one variable with a bare knowledge of analytic geometry, namely, the inner product $\langle v, w\rangle = v_1w_1+\cdots+v_nw_n = \|v\|\,\|w\|\cos\theta$ to measure the oriented angle θ between $v = (v_1,\ldots,v_n)$ and $w = (w_1,\ldots,w_n) \in \mathbb{R}^n$. We will write $v \perp w$ when the vectors are orthogonal, i.e., $\langle v, w\rangle = 0$. Under these notations, our results are as follows.

Theorem 1.1. *Let $\gamma : [a, b] \to \mathbb{R}^n$ be a continuous curve which is differentiable in (a, b). If $v \in \mathbb{R}^n$ is such that $\langle \gamma(b) - \gamma(a), v\rangle = 0$, then there is a $c \in (a, b)$ such that $\gamma'(c) \perp v$.*

Theorem 1.2. *Let $\gamma : [a, b] \to \mathbb{R}^n$ be a differentiable curve. If $v \in \mathbb{R}^n$ satisfies $\langle \gamma'(b) - \gamma'(a), v\rangle = 0$, then there is a $\eta \in (a, b)$ such that*

$$\langle \gamma'(\eta), v\rangle = \left\langle \frac{\gamma(\eta) - \gamma(a)}{\eta - a}, v \right\rangle.$$

Theorem 1.1 was inspired in the extension of Cauchy's MVT involving $n \geq 2$ functions presented in [17, Theorem 3], see Corollary 5.1. In fact, we arrived to our statement by searching for a geometric interpretation of the aforementioned result. Its formulation is not new and it can be found in [16] for higher order derivatives. Nonetheless, we have no precise reference for Theorem 1.2.

2. Proof of the results

We start by the analogous result to Cauchy's MVT.

Proof of Theorem 1.1. Consider the function

$$A(t) = \langle \gamma(t) - \gamma(a), v\rangle\,,$$

which is differentiable in (a, b) and continuous in $[a, b]$. Since $A(a) = 0$ and by hypothesis $A(b) = 0$, by Rolle's theorem, there is $c \in (a, b)$ such that $A'(c) = \langle \gamma'(c), v\rangle = 0$, as required. □

Remark 2.1. It may happen that $\gamma'(c) = 0$. In fact, the conclusion $\gamma'(t_0) \perp v$ holds trivially for $v = 0$ or for points $t_0 \in (a, b)$ where $\gamma'(t_0) = 0$. However, if $\gamma : [a, b] \to \mathbb{R}^n$ is *regular*, i.e., $\gamma'(t) \neq 0$, for all $t \in (a, b)$, γ has a well-defined affine tangent line at every point $\gamma(t)$. In this case, Theorem 1.1 gives a tangent vector $\gamma'(c) \neq 0$, orthogonal to v, see Figure 1.

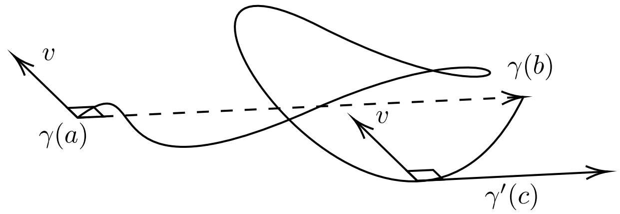

Figure 1: The vector $\gamma'(c)$ in Theorem 1.1

We prove now the analogous to Flett's theorem. To be more self-contained, we provide two proofs: one applying Flett's theorem and one following Wachnicki's argument in [18].

Proof of Theorem 1.2. Apply Flett's theorem to $A(t) = \langle \gamma(t) - \gamma(a), v \rangle$ which is differentiable on $[a, b]$ and satisfies $A'(a) = \langle \gamma'(a), v \rangle = \langle \gamma'(b), v \rangle = A'(b)$.

For a direct proof, let $B : [a, b] \to \mathbb{R}$ be given by $B(a) = \langle \gamma'(a), v \rangle$ and

$$B(t) = \left\langle \frac{\gamma(t) - \gamma(a)}{t - a}, v \right\rangle = \frac{A(t)}{t - a}, \qquad t \in (a, b].$$

Then B is continuous on $[a, b]$ and differentiable on $(a, b]$. We need to show the existence of $\eta \in (a, b)$ such that $B'(\eta) = 0$. By contradiction assume that $B'(t) \neq 0$, for all $t \in (a, b)$. By Darboux's theorem for derivatives, the sign of B' is constant. Assume that $B'(t) > 0$, for all $t \in (a, b)$. Thus, B is strictly increasing, and in particular $B(a) < B(b)$. However, computing $B'(t)$ we find that $(t - a)B'(t) = \langle \gamma'(t), v \rangle - B(t)$. Using the hypothesis we conclude that

$$(b - a)B'(b) = \langle \gamma'(b), v \rangle - B(b) = B(a) - B(b) < 0.$$

Again by Darboux's theorem, the inequality $B'(b) < 0$ is impossible since $B'(t) > 0$ for all $t \in (a, b)$. The case $B'(t) < 0$ follows the same lines, thus completing the proof. $\square$

Remark 2.2. In both theorems we can assume $v \neq 0$ to avoid a trivial statement. Moreover, the previous proofs are meaningful only when $A(t) \not\equiv 0$ and $B(t) \not\equiv 0$. Otherwise, the trace of γ is contained in the hyperplane P orthogonal to v and passing through $\gamma(a)$. Thus $A'(t) = \langle \gamma'(t), v \rangle \equiv 0$, i.e., the tangent vectors to γ are also in P, so Theorems 1.1 and 1.2 vacuously hold for any $c \in (a, b)$.

We conclude this section with the case of closed curves. What happens when $\gamma(a) = \gamma(b)$? Assuming also that γ is not contained in a hyperplane, i.e., $A(t) \not\equiv 0$, we see that $v \in \mathbb{R}^n$ can be chosen arbitrarily in Theorem 1.1. Therefore, we have the following result.

Corollary 2.3. *Let $\gamma : [a, b] \to \mathbb{R}^n$ be continuous and differentiable in (a, b). If γ is closed, for every $v \in \mathbb{R}^n$, there is $c \in (a, b)$ such that $\gamma'(c) \perp v$.*

This is particularly interesting when $n = 2$. For instance, assume that $\gamma : [a, b] \to \mathbb{R}^2$ is differentiable in (a, b) and that the right derivative $\gamma'_+(a)$ exists and it is non-zero. Then, we can choose $v \in \mathbb{R}^2$ such that $\gamma'_+(a) \perp v$. The previous result asserts the existence of $c \in (a, b)$ such that $\gamma'(c) \perp v$. If $\gamma'(c) \neq (0, 0)$, this vector is parallel to $\gamma'_+(a)$. The same result is true if $\gamma'_-(b)$ exists and it is non-zero. In particular, if $\gamma'(a) = \gamma'(b)$ and γ is regular, there is $c \in (a, b)$ such that the tangent line to γ at $t = c$ is parallel to the tangent line to γ at $t = a$.

3. The case of plane curves

Theorem 1.1 is equivalent to Rolle's theorem. In fact, let $f : [a, b] \to \mathbb{R}$ be continuous, differentiable in (a, b), and such that $f(a) = f(b)$. Then the curve $\gamma(t) = (t, f(t))$ is regular and satisfies $\gamma(b) - \gamma(a) = (b - a, 0) \perp (0, 1)$. Our result provides a $c \in (a, b)$ such that $\gamma'(c) = (1, f'(c)) \perp (0, 1)$, i.e., $f'(c) = 0$, as needed. In conclusion, we obtain.

Corollary 3.1. *Rolle's theorem, Lagrange's MVT, Cauchy's MVT and Theorem 1.1 are equivalent statements.*

In the same way, Theorem 1.2 is equivalent to Flett's result. In fact, if $f : [a, b] \to \mathbb{R}$ is differentiable and $f'(a) = f'(b)$, then curve $\gamma(t) = (t, f(t))$ is also differentiable and $\gamma'(b) = \gamma'(a)$. Thus we can choose $v \in \mathbb{R}^2$ arbitrary. By taking $v = (0, 1)$, we see that there is $\eta \in (a, b)$ such that $\gamma'(\eta) - \frac{\gamma(\eta)-\gamma(a)}{\eta-a} \perp (0, 1)$, i.e., $f'(\eta) = \frac{f(\eta)-f(a)}{\eta-a}$, as wanted.

Corollary 3.2. *Flett's theorem, Wachnicki's theorem and Theorem 1.2 are equivalent statements.*

Let us now apply Theorems 1.1 and 1.2 to a plane curve

$$\gamma(t) = (f(t), g(t)) \in \mathbb{R}^2.$$

Recall first that given any vector $w = (w_1, w_2) \neq (0, 0)$, there is a natural orthogonal vector $w^\perp := (-w_2, w_1)$ to it such that $\{w, w^\perp\}$ is oriented positively, i.e., $\det(w, w^\perp) = \|w\|^2 > 0$.

Let us assume first that f and g are as in Cauchy's MVT. If $\gamma(b) \neq \gamma(a)$ (the curve is not closed), we can choose

$$v = (\gamma(b) - \gamma(a))^\perp = (g(a) - g(b), f(b) - f(a)),$$

as a valid vector. Thus, there is $c \in (a, b)$ such that $\gamma'(c) \perp v$, i.e., (1) holds. Since we are in $\mathbb{R}^2$, if $\gamma'(c) \neq (0, 0)$, then $\gamma'(c)$ is parallel to $\gamma(b) - \gamma(a)$. In this way, we recover the usual geometric interpretation of Cauchy's MVT in (1): if

$\gamma(a) \neq \gamma(b)$ and γ is regular, there is a point $\gamma(c)$ on the curve such that the tangent line to γ at $t = c$ is parallel to the segment passing through $\gamma(b)$ and $\gamma(a)$.

Assume now that $g'(t) \neq 0$, for all t. Using Darboux's theorem as in the proof of Theorem 1.2 we see that $\frac{g(b)-g(a)}{g'(c)} > 0$. Since

$$\gamma(b) - \gamma(a) = \frac{g(b) - g(a)}{g'(c)} \cdot \gamma'(c),$$

we conclude that, not only $\gamma(b) - \gamma(a)$ and $\gamma'(c)$ are parallel, but they also have the same direction, as one vector is a positive multiple of the other.

Let us assume now that f and g satisfy the conditions of Wachnicki's theorem. Here the hypothesis means that $\gamma'(a)$ and $\gamma'(b)$ have the same direction. Indeed, since

$$\gamma'(b) = \lambda \cdot \gamma'(a), \qquad \text{where} \qquad \lambda = \frac{g'(b)}{g'(a)} = \frac{f'(b)}{f'(a)},$$

and the hypothesis requires that $g'(t) \neq 0$, for all t, Darboux's theorem implies $\lambda > 0$. Then, Wachnicki's theorem gives a point $\eta \in (a, b)$ such that (2) holds, i.e.,

$$\gamma(\eta) - \gamma(a) = \frac{g(\eta) - g(a)}{g'(\eta)} \cdot \gamma'(\eta).$$

As before, this means that $\gamma(\eta) - \gamma(a)$ and $\gamma'(\eta)$ have the same direction, i. e., they are parallel.

4. The case of curves in $\mathbb{R}^3$

Another version of Cauchy's MVT for three functions can be expressed in terms of determinants. In fact, given a continuous curve $\gamma = (f, g, h) : [a, b] \to \mathbb{R}^3$ which is differentiable in (a, b), there is a point $c \in (a, b)$ such that

$$\begin{vmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f'(c) & g'(c) & h'(c) \end{vmatrix} = 0. \tag{3}$$

Cauchy's MVT corresponds to $h(t) \equiv 1$. The existence of c follows by applying Rolle's theorem to $D(t) = \det(\gamma(a), \gamma(b), \gamma(t))$, since $D(a) = D(b) = 0$ and $D'(t) = \det(\gamma(a), \gamma(b), \gamma'(t))$. Equation (3) and this argument are only meaningful when $\gamma(a)$ and $\gamma(b)$ are linearly independent —otherwise, $D(t) \equiv 0$—. Indeed, the geometric content of (3) is clear in this case: there is $c \in (a, b)$ such that $\gamma'(c)$ lies on the subspace E generated by $\gamma(a)$ and $\gamma(b)$.

We can also use Theorem 1.1 to draw these conclusions. In the first case, use the cross product $v_0 = \gamma(a) \times \gamma(b) \neq (0, 0, 0)$, which is orthogonal to E, and in particular to $\gamma(b) - \gamma(a)$. By Theorem 1.1 there is $c \in (a, b)$ such that

$$\langle \gamma(a) \times \gamma(b), \gamma'(c) \rangle = \det(\gamma(a), \gamma(b), \gamma'(c)) = 0.$$

Therefore, $\gamma'(c) \in E$, i.e., (3) holds for this c. Note also that $A(t) = D(t)$ for this v_0.

In the second case, assume that $\gamma(a)$ and $\gamma(b)$ lie on the same half-line L emerging from the origin. Then, for every vector $w \in \mathbb{R}^3$ in the orthogonal plane to L, there is $c \in (a, b)$ such that $\gamma'(c) \perp w$.

5. Other consequences

We conclude this note with a simple proof of the original motivation for Theorem 1.1, namely, a Cauchy MVT for $n \geq 2$ functions.

Corollary 5.1 (Theorem 3, [17]). *Let $\gamma = (f_1, \ldots, f_n) : [a, b] \to \mathbb{R}^n$ be a continuous curve, differentiable in (a, b). Assume that $f_j(b) \neq f_j(a)$, for all $j = 1, \ldots, n$. If $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ are such that $\alpha_1 + \cdots + \alpha_n = 0$, then there is $c \in (a, b)$ such that*

$$\sum_{j=1}^{n} \frac{\alpha_j}{f_j(b) - f_j(a)} f_j'(c) = 0.$$

Proof. It follows from the Theorem 1.1 by choosing
$v = \left(\frac{\alpha_1}{f_1(b) - f_1(a)}, \ldots, \frac{\alpha_n}{f_n(b) - f_n(a)} \right)$. □

Corollary 5.2. *Let $\gamma = (f_1, \ldots, f_n) : [a, b] \to \mathbb{R}^n$ be a differentiable curve. Assume that $f_j'(b) \neq f_j'(a)$, for all $j = 1, \ldots, n$. If $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ are such that $\alpha_1 + \cdots + \alpha_n = 0$, then there is $c \in (a, b)$ such that*

$$\sum_{j=1}^{n} \frac{\alpha_j}{f_j'(b) - f_j'(a)} \left(f_j'(\eta) - \frac{f_j(\eta) - f_j(a)}{\eta - a} \right) = 0.$$

Proof. Apply Theorem 1.1 using the vector $v = \left(\frac{\alpha_1}{f_1'(b) - f_1'(a)}, \ldots, \frac{\alpha_n}{f_n'(b) - f_n'(a)} \right)$. □

Example 5.3. A classical application of Rolle's theorem is to show that a polynomial

$$p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$

has at least one root in $(0, 1)$ provided that $c_0 + \frac{c_1}{2} + \frac{c_2}{3} \cdots + \frac{c_n}{n+1} = 0$. We can prove this using Theorem 1.1. In fact, consider the curve $\gamma(t) = (t, t^2, \ldots, t^n, t^{n+1})$ and the vector $v = (c_0, \frac{c_1}{2}, \ldots, \frac{c_n}{n+1})$ in $\mathbb{R}^{n+1}$. Since $\langle \gamma(1) - \gamma(0), v \rangle = 0$, there is $c \in (0, 1)$ such that $\langle \gamma'(c), v \rangle = p(c) = 0$, as required.

6. The case of antiderivatives of curves

As well as there are MVTs for differentiable functions, there are related results when dealing with functions given by the antiderivative of a continuous function. These are known as MVTs for integrals. We conclude this work with a brief discussion of this case.

For instance, let $f, g : [a, b] \to \mathbb{R}$ be continuous. Then, there is $c \in (a, b)$ such that

$$f(c) \cdot \int_a^b g(t)dt = g(c) \cdot \int_a^b f(t)dt.$$

This follows from Cauchy's MVT applied to $F(x) = \int_a^x f(t)dt$, $G(x) = \int_a^x g(t)dt$, since $F'(x) = f(x)$ and $G'(x) = g(x)$, by the fundamental theorem of Calculus. In particular, if $g(t) \equiv 1$, there is $c \in (a, b)$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(t)dt,$$

which is nothing but the usual MVT for integrals. Another related statement is Wayment's theorem first proved in [19], and obtained applying Flett's theorem to $F(x)$ under the assumption that $f(a) = F'(a) = F'(b) = f(b)$. Surprisingly, it was only recently that this theorem was generalized by means of Wachnicki's theorem, see [2]. More precisely, we have the following result:

Wayment's generalized theorem [2]. Let $f, g : [a, b] \to \mathbb{R}$ be continuous and $g(x) \neq 0$, for all $x \in [a, b]$. If $f(a) = f(b)$, then there exists at least one point $c \in (a, b)$ such that

$$\int_a^c f(t)g(t)dt = f(c) \int_a^c g(t)dt.$$

The result follows using Wachnicki's theorem for the functions $\widetilde{F}(x) = \int_a^x f(t)g(t)dt$ and $G(x) = \int_a^x g(t)dt$ which are differentiable in $[a, b]$. In particular, there is $c \in (a, b)$ such that

$$(c-a)f(c) = \int_a^c f(t)dt.$$

This is the original statement of Wayment, corresponding to $g(t) \equiv t$.

In our setting, we offer the following result that follows intermediately from Theorem 1.2 when applied to the curve $\psi(t) = \int_a^t \gamma(s)ds$ (the integral is taken component-wise) and also from Wayment's theorem using $f(t) = \langle \gamma(t), v \rangle$.

Corollary 6.1. *Let $\gamma : [a, b] \to \mathbb{R}^n$ be a continuous curve. If $v \in \mathbb{R}^n$ satisfies $\langle \gamma(b) - \gamma(a), v \rangle = 0$, then there is a $\eta \in (a, b)$ such that*

$$\langle \gamma(\eta), v \rangle = \frac{1}{\eta - a} \int_a^\eta \langle \gamma(t), v \rangle \, dt.$$

Additionally, note that by taking $n = 1$, $\gamma(t) = f(t)$, and $v = 1$, Corollary 6.1 implies Wayment's theorem. Thus, these are equivalent statements.

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