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Scattering rigidity on the Lorentzian setting

Rigidez de dispersión en el ambiente Lorentziano

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Abstract. We survey some recent results regarding geometric rigidity on Lorentzian manifolds, in the context of geometric inverse problems.

Keywords: Scattering rigidity, Boundary rigidity, Stationary manifold, Lorentzian geometry.

Resumen. Revisamos algunos resultados recientes sobre la rigidez geométrica en variedades Lorentzianasn en el contexto de problemas inversos geométricos.

Palabras claves: Rigidez de dispersión, rigidez de borde, variedad estacionaria, geometría Lorentziana.

Mathematics Subject Classification: 53C24, 53C50, 53C22, 58J47.

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1. Introduction

In inverse problems, one tries to understand the "interior" of an unknown object by making indirect measurements "outside". We write interior and outside in a vague way intentionally. For example, one wants to obtain information (geometry or topology) of the interior of a domain, or the values of a coefficient of a Partial Differential Equation (PDE) on the interior of a (sub)domain.

Most of the problems in the area come from real world applications. Some of the more common examples are the following: geophysics, string theory, invisibility cloaking, relativity, medical imaging (Computed Tomography Scans, Ultrasound Scans, Electrical Impedance Tomography), Thermoacoustic Imaging, neural networks, and archaeology.

In this note we will focus on geometric inverse problems. In these kinds of problems, "some" information on the boundary of a compact Riemannian manifold (M, g) is given, and one tries to recover the metric on the inside up to some natural obstruction.

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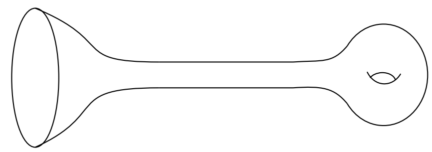

Figure 1: A non-simple manifold.

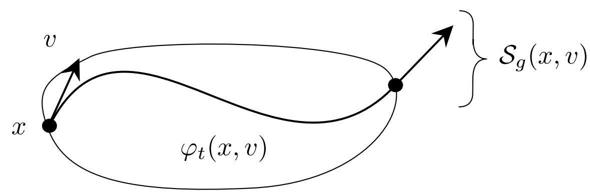

Figure 2: The scattering map.

2. A short review of results in the Riemannian setting

Questions about rigidity in the Riemannian setting were motivated in the beginning of the XX century by geophysicists: one of the main problems was to understand the structure of the Earth by measuring the travel times of seismic waves. The mathematical problem was proposed by Michel [13] in the 80's. In brief, the problem asks if the distance between boundary points determine the metric of a Riemannian manifold inside the manifold.

It is not hard to see that there is a gauge: if $f\colon M \to M$ is a diffeomorphism fixing the boundary pointwise, then $b_{f^*g} = b_g$, where (M, g) is a Riemannian manifold with boundary, and b_g denotes the boundary distance function, that is, the restriction of the distance function induced by g to boundary points. Even with the gauge in mind, there are some manifolds that are not boundary rigid: The manifold in Figure 1 is not boundary rigid because geodesics avoid the part on the right. Another geometric notion is the scattering: beginning from a point and velocity on the boundary, one follows the geodesic flow up to hit the boundary again:

The inverse problem regarding the scattering is the same as before, one wants to determine the metric by the knowledge of the scattering map.

In the last 20 years, two notions have played a role to deal with these problems.

Definition 2.1. We say that (M, g) is simple if it is non-trapping, there are no conjugate points, and ∂M is strictly convex.

Non-trapping can be thought as the property that one cannot have infinite

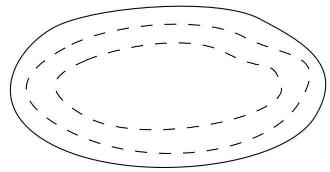

Figure 3: A convex foliation given by preimages of a convex function.

length geodesics, or, equivalently, that the geodesics that begin at the boundary, exit the manifold in finite time.

It can be shown that the scattering rigidity and the boundary rigidity problem are equivalent in simple geometries; we refer to the book by G. P. Paternain, M. Salo, and G. Uhlmann [21].

Definition 2.2. We say that (M, g) satisfies the *foliation condition* if there exists a strictly convex function on M.

The first major breakthrough was due to L. Pestov and G. Uhlmann

Theorem 2.3 ([22])**.** *Let (M, g) and $(M, \hat{g})$ be simple surfaces. If $b_g = b_{\hat{g}}$, then there is a diffeomorphism $f\colon M \to M$ with $f|_{\partial M} = id_{\partial M}$ so that $\hat{g} = f^*g$.*

One of the most important observations in [22] is that b_g determines the Dirichlet-to-Neumann (DN) map of the Dirichlet problem for the Laplace-Beltrami operator. Then, one uses a previous result by M. Lassas and G. Uhlmann [11], that shows that the DN map determines the metric (up to gauge).

In higher dimensions, a breakthrough took more than ten years, and is due to P. Stefanov, G. Uhlmann and A. Vasy:

Theorem 2.4 ([28])**.** *Assume that (M, g) is a simple manifold with $\dim M \geq 3$ that satisfies the foliation condition. Let $\hat{g}$ another metric such that ∂M is strictly convex with respect to $\hat{g}$. If $b_g = b_{\hat{g}}$, then the manifolds are isometric by an isometry fixing the boundary pointwise.*

We only mention some brief words about the proof, which is far beyond the scope of this notes. The idea of the proof is to solve the linear problem in a neighborhood of a point in the boundary, by creating an artificial boundary. It can be shown that the linearization of b_g, the geodesic X-ray transform (an integral operator that integrates tensors over geodesics) composed with a suitable backprojection (a sort of adjoint operator) is an elliptic operator in the Melrose's scattering calculus [12]. Then, one pulls this information to the non-linear problem.

At this point, there is a purely geometric question that is natural to ask. Do simple manifolds satisfy the foliation condition? The converse is false. F. Monard [14] constructed examples that can be foliated but admit conjugate points, hence, they are not simple. Note that if the answer this is affirmative, then we can remove the hypothesis about the foliation condition in Theorem 2.4.

Finally, we mention that there are generic rigidity results [26] and rigidity results for non-simple metrics [27], both works by P. Stefanov and G. Uhlmann. For more details and references about geometric inverse problems in the Riemannian setting, we refer to the recent book by G. P. Paternain, M. Salo and G. Uhlmann [21].

3. Rigidity in the Lorentzian setting

Regarding geometric rigidity questions in the Lorentzian setting, results are not as extensive nor strong in the semi-Riemannian setting.

A relation between the lens relation, a geometric object, and the Dirichlet-to-Neumann (DN) map is given by P. Stefanov and Y. Yang [29]. There, the authors studied the DN map on a cylinder-like Lorentzian manifold associated to the wave equation related to the metric, a magnetic field, and a potential. They showed that the DN map is a Fourier Integral Operator with canonical relation given by the lens relation.

Regarding rigidity, there are a few but notable results. We first present a result by Y. Kurylev, M. Lassas, and G. Uhlmann [8]. One of the questions they addressed is the determination of the space-time by passive measurements, that is, to determine the structure of space-time when by observing wavefronts produced by point sources. They obtained the following result:

Theorem 3.1 ([8]). *Let (M, g) be an open smooth globally hyperbolic Lorentzian manifold of dimension $n \geq 3$ and let $p^{\pm} \in M$ be the points of a time-like geodesic $\hat{\mu}([-1,1]) \subset M$, $p^{\pm} = \hat{\mu}(s_{\pm})$. Let $V \subset M$ be a neighborhood of $\hat{\mu}([-1,1])$ and $W \subset I^-(p^+) \setminus J^-(p^-)$ be a relatively compact set. Assume that we know the collection of earliest light observation sets*

$$\mathcal{E}_V(W) = \{\mathcal{E}_V(q) : q \in W\}.$$

Then the topological, the differential and the conformal structure of W are determined up to diffeomorphism.

Here $I^-(p^+)$ is the chronological past of p^+ and $J^-(p^-)$ is the causal past of p^-.

In physics literature, the earliest light observation set are called *light-cone cuts.*

The idea of the proof is to copy the structure of W into $\mathcal{E}_U(W)$, where U is a smaller observation set $U \subset V$, by using the lightlike geodesics that can be observed from W.

We now present a related result by P. Hintz and G. Uhlmann [7]. They use light observation sets on the boundary rather than in the interior. We need the notion of *broken null-geodesic*, which are null-geodesics which undergo reflection

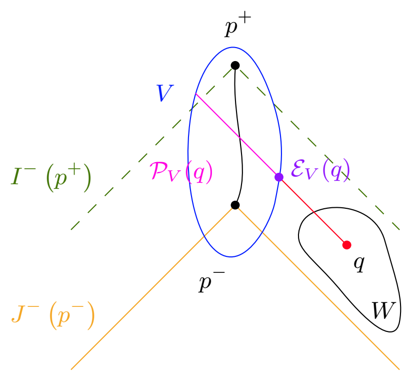

Figure 4: A pictural description of Theorem 3.1. The light observation sets are used to determine the structure of W. $\mathcal{P}_V(q)$ denotes the light observation set of q, i.e., where the light from q can be detected in V. The points in $\mathcal{E}_V(q)$ are the ones in $\mathcal{P}_V(q)$ that cannot be reached by a future pointing timelike path starting at a point in $\mathcal{P}_V(q)$.

at the boundary ∂M preserving their velocity tangent to ∂M, see Figure 5. The *future light cone* $\mathcal{L}_q^+$ from a point $q \in M$ is defined as the union of all future-directed broken null-geodesics

Theorem 3.2 ([7]). *Let (M, g) be a $(n+1)$-dimensional Lorentzian connected manifold with non-empty boundary, such that:*

- *there exists a proper, surjective function $t : M \to \mathbb{R}$ such that dt is everywhere timelike;*
- *the boundary ∂M is timelike;*
- *∂M is strictly null-convex: if ν denotes the outward pointing unit normal*

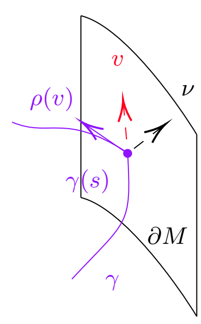

Figure 5: A broken geodesic at its reflection at $\gamma(s) \in \partial M$. Here $v = \dot{\gamma}(s)$ and $\rho(v) = v - 2g(v, \nu)$.

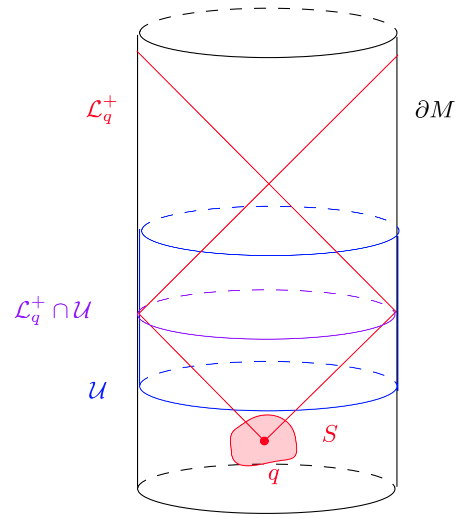

Figure 6: The setting of Theorem 3.2. Observations of S (in red) are only made from $\mathcal{U}$ (in blue). In particular, q is observable from the intersection (in purple) of its light observation set $\mathcal{L}_q^+$ (in red) with $\mathcal{U}$.

vector field on ∂M, then

$$\Pi(v,v) = g(\nabla_v \nu, v) > 0, \quad \forall v \in T_p\partial M \setminus 0.$$

Let $S \subset M^\circ$ with compact closure, and let $\mathcal{U} \subset \partial M$ be open. Consider the following additional assumptions:

1. $\mathcal{L}_{q_1}^+ \cap \mathcal{U} \neq \mathcal{L}_{q_2}^+ \cap \mathcal{U}$ *for* $q_1 \neq q_2 \in \bar{S}$*;*
2. *points in S and $\mathcal{U}$ are not null-conjugate.*

Then the smooth manifold $\mathcal{U}$ and the unlabelled collection of boundary light observation sets $\mathcal{S} = \{\mathcal{L}_q^+ \cap \mathcal{U} : q \in S\}$ determine $(S, [g|_S])$ topologically, differentiabily and conformally.

It is not hard to see that the hypothesis 1 is necessary, otherwise, we could have two different sets that are indistinguishable from $\mathcal{U}$.

The proof of Theorem 3.2 proceeds in three steps. In first place, one defines a topology on $\mathcal{S}$ by declaring collections of boundary light observation sets to be open if they intersect, respectively miss, a fixed open, respectively compact, subset of $\mathcal{U}$ (this topology turns out to be equal to the subspace topology). Secondly, they reconstruct (intrinsically within $\mathcal{S}$ and $\mathcal{U}$) a large class of functions that are smooth on S, that are used to construct a differentiable structure. To reconstruct the conformal class of g on S, they identify a large number of null-geodesics $s \mapsto q(s)$ in S in terms of the boundary light observation sets of the points $q(s)$. Note that since light cones are well-defined by just using the conformal class of a metric, one cannot hope to recover the metric itself. Finally, one can also recover the time orientation on S by analyzing the behavior of $\mathcal{L}_q^+ \cap \mathcal{U}_j$ as q moves along a timelike curve in S.

Recently, Y. Yi and Y. Zhang studied the DN map for disjoint sets, using a multiple reflections, see [32].

Now, we move on to geometric rigidity results similar to those presented in Section 2. G. Uhlmann, Y. Yang, and H. Zhou studied a version of a boundary rigidity result in the Lorentzian case for cylindrical domains in $\mathbb{R}^{1+n}$ endowed with standard stationary metrics [30].

Definition 3.3. A connected Lorentzian manifold (M, g) is called *standard stationary spacetime*, if has the form $M = \mathbb{R} \times N$, and is endowed with the following metric

$$g_{ij} = -\tilde{\lambda}(x)dt^2 + 2\tilde{\omega}_i(x)dtdx^i + \tilde{h}_{ij}(x)dx^i dx^j, \tag{1}$$

where $\tilde{\lambda} \in C^\infty(N)$ is strictly positive, $\tilde{\omega}$ is a 1-form on N, and $\tilde{h}$ a Riemannian metric on N.

Examples of this kind of manifolds include the Minkowski spacetime (which is also static, i.e., the one form is zero), and the Kerr spacetime.

We recall that the time separation function is given by

$$\tau_g(z, y) = \begin{cases} \sup\{L(\gamma) : \gamma \text{ is a causal curve from } z \text{ to } y\} & \text{if } y \in J^+(z), \\ 0 & \text{otherwise,} \end{cases}$$

where $J^+(z)$ is the causal future of z, i.e., points that can be joined to z by causal curves starting at z

$$L(\gamma) = \int_\gamma \sqrt{-g(\dot{\gamma}, \dot{\gamma})}.$$

We also need the following notions

Definition 3.4. Let g be as in (1) on $\mathbb{R}^{1+n}$, which differs from the Minkowski metric δ only on $\mathbb{R} \times \Omega$. Suppose g is close to δ.

1. We say that g satisfies the *orthogonal assumption* if there exists a hyperplane $H \subset \mathbb{R}^n \setminus \Omega$, so that $\tilde{\omega}(\dot{\sigma}) = 0$ along any geodesic σ, with respect to the Riemannian metric $\tilde{h}$, normal to H.

2. Given two metrics g and g' satisfying the orthogonal assumption with respect to the same hyperplane H, we say that g and g' satisfy the *spatial distance assumption* with respect to the hyperplane H if $d_h(x, y) = d_{h'}(x, y)$ for all $x, y \in \partial\Omega$ with $x - y$ almost normal (in Euclidean product) to H, i.e. the angle between $x - y$ and H is in some small neighborhood of $\pi/2$. Here d_h is the Riemannian distance function with respect to metric h.

The orthogonal assumption implies that there exist global coordinates with $\tilde{\omega}_1 \equiv 0$.

Theorem 3.5 ([30])**.** *Let g and g' be two standard stationary Lorentzian metrics in $\mathbb{R}^{1+n}$, which differ from the standard Minkowski metric δ only on $\overline{\Omega}$, and*

$$(g, g') \in \mathcal{G}_K := \{(g, g') : \|g - g'\|_{H^2(\overline{\Omega})} \leq K\|g - g'\|_{H^1(\overline{\Omega})}\},$$

for some constant $K \geq 1$. Suppose these metrics are ε-close to δ in the $C_0^k(\bar{\Omega})$-norm, $k \geq 2n + 6$ and satisfy the orthogonal and spatial distance assumption with respect to the same hyperplane H. Then, there exists $T_0 = T_0(\Omega, \varepsilon) > 0$, such that for any $T > T_0$, if $\Gamma = [0, T] \times \partial\Omega$ and

$$\tau_g|_{\Gamma\times\Gamma} = \tau_{g'}|_{\Gamma\times\Gamma},$$

then there exists a diffeomorphism f on $\overline{\Omega}$ fixing the boundary,

$$h' = f^*h, \quad \tilde{\omega}' = f^*\tilde{\omega}, \quad \tilde{\lambda}' = f^*\tilde{\lambda}.$$

The proof relies on a perturbative argument and follows the idea by P. Stefanov and G. Uhlmann on the Riemannian case [25]. One compare the flows with the different metrics to obtain an integral identity (known as the Stefanov–Uhlmann identity), which in this case turn to be a (weighted) ray transform involving the differences of the metrics and its derivatives. One analysis this as an FIO and uses the closeness to the Euclidean metric to obtain its invertibility.

In [4], G. Eskin investigates the rigidity of cylindrical domains in $\mathbb{R}^{1+n}$ endowed with time independent Lorentzian metrics using null geodesics. For metrics close to the Minkowski one, Y. Wang [31] proved that the scattering relation of null geodesics between two Cauchy surfaces uniquely determines the metric perturbation (up to gauge).

In [24], P. Stefanov examines scattering rigidity in standard stationary manifolds through lightlike geodesics. We proceed to describe the setting, see Figure 7. Let U, V two submanifolds either timelike or spacelike (see page 142 in [20]) of points $x_0 \in U$, $y_0 \in V$, so that they are connected by a lightlike geodesic $[0, 1] \ni t \to \gamma(t)$, see Figure 1 in [24]. Assume that x_0 and y_0 are not conjugate along γ, and that γ_0 is transversal to U and V at their only points of intersection, x and y. Assume that γ_0 is future pointing at x_0, and we choose a time orientation on V so that γ_0 is future pointing at y as well. We also fix orientation on U and V in classical sense, calling the sides containing γ_0 interior, and the other ones exterior. Set $v_0 := \dot{\gamma}(0), w_0 := \dot{\gamma}(1)$. Let v_0', w_0' be their orthogonal projections on TU and TV, respectively. They must be timelike/spacelike depending on which case we have. Let $\mathcal{U}, \mathcal{V}$ be small timelike/spacelike conic neighborhoods of (x_0, v_0') in TU, and of (y_0, w_0') in TV, respectively.

Definition 3.6. The scattering relation $\mathcal{S} : \mathcal{U} \to \mathcal{V}$ is defined by $\mathcal{S}(x, v') = (y, w')$ as follows. Let v be the lightlike vector at x with orthogonal projection v' on T_xU, pointing to the interior; then $y \in V$ is the point where the geodesic

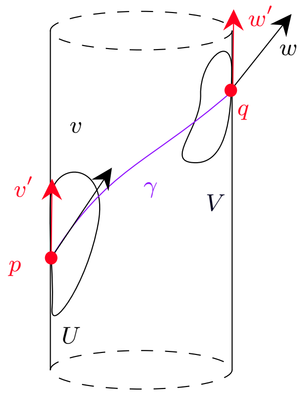

Figure 7: The scattering relation

$\gamma_{x,v}$ issued from (x, v) meets V, and w' is the orthogonal projection on T_yV of its direction there, and $\gamma_{x,v}$ is the geodesic starting at x with velocity v.

We observe that we can write (1) as

$$g_{ij} = -\lambda(x)(dt - \omega_i dx^i)^2 + h_{ij}(x)dx^i dx^j, \tag{2}$$

and we will write $g = g_{h,\omega,\lambda}$ for short.

By exploiting a relation between lightlike geodesics and magnetic geodesics on the base manifold, one gets:

Theorem 3.7. *Let $g = g_{h,\omega,\lambda}$, $g' = g_{h',\omega,\lambda'}$ standard stationary metrics on $M = \mathbb{R} \times N$. Assume that the associated magnetic systems are simple, and one of the following conditions holds:*

1. *$\hat{h}$ and h are in the same conformal class;*
2. *we are in the real analytic setting;*
3. $\dim M = 1 + 2$.

If $S = S'$ (i.e., the scattering relations corresponding to g and g' are the same), then the metrics are gauge equivalent, in the sense that there exists a diffeomorphism $f : N \to N$ fixing ∂N pointwise, and a smooth function φ vanishing on ∂N so that

$$\hat{h} = f^* h, \quad \hat{\omega} = f^*(\omega + d\varphi).$$

Remark 3.8. The simplicity assumption in Theorem 3.7 can be thought as the natural generalization of Definition 2.1 to magnetic systems. Indeed, for Riemannian manifolds, simplicity is equivalent (see [21] Section 3.8) to have strictly convex boundary and ask for the exponential map to be a diffeomorphism (in a proper domain). Hence, the definition of simplicity for magnetic systems is analogue to this last two conditions: one requires that the boundary is magnetic convex and that the exponential map defined by the magnetic geodesic flow is a diffeomorphism at every point. For further details, we refer to [3].

One also has a generic local rigidity result in higher dimensions, just as in the work on rigidity of magnetic systems by N. Dairbekov, G. P. Paternain, P. Stefanov and G. Uhlmann [3]. Note that we do not obtain information of λ. This is because λ does not play a role when working with lightlike vectors. We also mention that in this work, a scattering relation for covectors is studied, and a boundary defined function that plays the role of a boundary distance function on the Lorentzian case. This last problem is non-trivial because, if we attempt to set a distance function as a direct analog of the Riemannian one, given two points, they may not be connected by a (unique or not) lightlike geodesic. This is a property that we really want so that we can relate it to $\mathcal{S}$. Furthermore, if they are, then their geodesic distance it is zero. The Lorentzian distance does not seem to give a direct answer because is zero on one side of the light cone, and singular at the light cone, where it vanishes.

In a similar fashion, one can study the scattering relation, but this time with timelike vectors. This is the purpose of [17]. To explore the results, first we will define a new scattering relation, which fixes the "mass" and the "momentum".

Definition 3.9. The *scattering relation of momentum* ρ *and mass* m $\mathcal{S}_{\rho,m}\colon \mathcal{U} \to \mathcal{V}$ given by $\mathcal{S}_{\rho,m}(p, v') = (q, w')$ is defined as follows. Let v be the timelike vector of mass m at p with orthogonal projection v' on T_pU, and pointing to the interior; then $q \in V$ is the point where the geodesic $\gamma_{p,v}$ with $(\partial_t, \dot\gamma_{p,v})_g = \rho$, issued from (p, v) meets V, and w' is the orthogonal projection on T_qV of its direction there.

We obtained the following result.

Theorem 3.10 ([17])**.** *Consider two metrics* $g = g_{h,\omega,\lambda}$ *and* $g' = g_{h',\omega',\lambda'}$ *on* $M = \mathbb{R} \times N$*, so that* $h|_{\partial M} = h'|_{\partial M}$*,* $i^*\omega = i^*\omega'$*,* $\lambda|_{\partial M} = \lambda'|_{\partial M}$*, where* i *is the embedding* $i\colon \partial M \to M$*. Assume that the* $\mathcal{MP}$*-systems* $(N, h, d\rho\omega, \rho^2/(-2\lambda))$ *and* $(N, h', \rho d\omega', \rho^2/(-2\lambda'))$ *are simple, and one of the following conditions holds:*

1. h' *is conformal to* h*;*
2. *The* $\mathcal{MP}$*-systems are real-analytic;*
3. $\dim N = 2$.

If $\mathcal{S}_{\rho,m} = \mathcal{S}'_{\rho,m}$*, then there exist a diffeomorphism* $f\colon N \to N$ *fixing* ∂N*, a positive function* $\mu \in C^\infty(N)$ *with* $\mu|_{\partial M} = 1$*, and a function vanishing on the boundary* $\varphi \in C^\infty(N)$ *so that*

$$h' = \frac{1}{\mu} f^*h, \quad \omega' = f^*\omega + d\left(\frac{\varphi}{\rho}\right), \quad \frac{1}{\lambda'} = \mu\left(\frac{1}{f^*\lambda} - \frac{m^2}{\rho^2}\right) + \frac{m^2}{\rho^2}. \qquad (3)$$

Furthermore, if we assume that the previous conditions hold for two different values of ρ, then $\mu = 1$ in (3)*.*

Remark 3.11*.* As in the case of magnetic systems (see Remark 3.8), simplicity for $\mathcal{MP}$-systems means that the boundary is $\mathcal{MP}$-strictly convex and that the exponential map defined by the $\mathcal{MP}$-flow (at the energy level of interest) is a diffemorphism at every point. We refer to [1] and [15] for further details about simplicity of $\mathcal{MP}$-systems, and to [17] to the relation between symplicity of $\mathcal{MP}$-systems its relation with standard stationary geometries.

The proof relies on the Hamiltonian reduction and the use of previous results on $\mathcal{MP}$-systems (compact Riemannian manifolds endowed with a 2-closed form (the magnetic part) and an a smooth function (the potential)). We shown that all the geometry of the geodesic flow with fixed mass and momentum on the manifold (M, g) can be reduced to the $\mathcal{MP}$-flow on an associated $\mathcal{MP}$-system on the "base". Then, one shows that the scattering relation $\mathcal{S}_{m,\rho}$ determines the scattering relation of the related $\mathcal{MP}$-system and vice versa. The result now follows from theorems obtained by the author in [15], and by Y. M. Assylbekov and H. Zhou [1]. As in the $\mathcal{MP}$-case [16], we also have a generic local rigidity result. Note that, unlike Theorem 3.7, using timelike vectors gives information of λ. We would like to point out similar reductions argument have been used recently by L. Oksanen, G. P. Paternain and M. Sarkkinen to relate the light ray transform with the magnetic X-ray transform, see [18].

We mention a recent result by P. Stefanov [23]:

Theorem 3.12 ([23])**.** *Let g and $\hat{g}$ be two Lorentzian metrics defined near some $x_0 \in \partial M$ so that $\hat{g} = \mu_0 g$ on $T\partial M \times T\partial M$ locally with some $0 < \mu_0 \in C^\infty(\partial M)$. Let $(x_0, v_0) \in T\partial M$ be lightlike for g. Assume that ∂M is strictly convex with respect to g in the direction of (x_0, v_0). If $\hat{\mathcal{S}} = \mathcal{S}$ in a neighborhood of(x_0, v_0), and μ_0 is constant, then there exists $\mu(x) > 0$ with $\mu = \mu_0$ on ∂M, and a local diffeomorphism ψ near ∂M preserving it pointwise, so that the jets of g and $\mu\psi^*\hat{g}$ coincide on ∂M near x_0.*

In the analytic setting one can prove that this gives $g = \mu\psi^*\hat{g}$ near x_0. To prove the result, one first show that the metrics coincide on a timelike vector field ∂_{x^0} tangent to ∂M, and at the same time to be both in boundary normal coordinates, both up to $O((x^n)^\infty)$ (as $x^n \to 0^+$). This gives a quasilinear PDE, which can be solved up to $O((x^n)^\infty)$. Finally, one applies a Taylor series argument using the maximizing property of timelike geodesics.

Finally, we mention that one can also study the linearized problem, i.e., the ray transform on time-spaces. This gives rise to the light-ray transform. This operator also appears naturally when one studies the inverse scattering problem for a wave equation with a time dependent potential. In the stationary case, we refer to the works by M. Lassas, L. Oksanen, P. Stefanov and G. Uhlmann [10], by A. Feizmohammadi, J. Ilmavirta, and L. Oksanen [5], the recent work by P. Stefanov [24], and references therein. We also mention the work by M. Lassas, L. Oksanen, P. Stefanov, and G. Uhlmann [9], where the authors develop microlocal methods for tomographic problems and how they can be used to detect singularities of the Lorentzian metric of the Universe using measurements of the cosmic microwave background radiation. We also mention a recent result result for the Lorentzian Calderón problem by L. Oksanen, Rakesh, and M. Salo, [19], where the authors proved that if a globally hyperbolic metric agrees with the Minkowski metric outside a compact set and has the same hyperbolic Dirichlet-to-Neumann map as the Minkowski metric, then it must be the Minkowski metric up to diffeomorphism. Regarding works on inverse problems related to Maldacena's AdS/CFT correspondence, we refer to the work by R. Graham, C. Guillarmou, P. Stefanov and G. Uhlmann on asymptotically hyperbolic manifolds [6], and the work by C. Cârstea, T. Liimatainen, and L. Tzou on minimal surfaces [2].

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