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ON STRONGLY LAWSON AND I-LAWSON MONADS

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ABSTRACT. We introduce classes of strongly Lawson monads and I-Lawson monads and show that these monads have functional representation. We investigate the connection of these classes of monads whith the class of Lawson monads introduced in [1].

Key words and phrases. Strongly Lawson monad, I-Lawson monad.

0. Introduction

The algebraic aspect of the theory of functors in categories of topological spaces and continuous maps was investigated in the 60's. It is based, mainly, on the existence of a monad (or triple) structure in the sense of S.Eilenberg and J.Moore [2].

Many classical constructions lead to monads: hyperspaces, spaces of probability measures, superextensions etc. There were many investigations of monads in categories of topological spaces and continuous maps(see for example the survey [3]). But it seems that the main difficulty to obtain general results in the theory of monads is the different nature of specific functors.

Some functional representations of the hyperspace functor were found in [4] and [5]. There was introduced a class of Lawson monads in [1] which contains sufficiently wide class of monads. Lawson monads have a functional representation, i.e., their functorial part FX can be naturally imbedded in $\mathbb{R}^{CX}$. In this

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paper we investigate two other approaches to the problem of functional representation of monads. We introduce the classes of strongly Lawson monads and I-Lawson monads and compare them with the class of Lawson monads.

The paper is arranged in the following manner. In 1 we construct the monads $\mathbb{V}_s$ and $\mathbb{V}_I$ which will play in our paper the role of universal monads. In 2 we introduce the notions of strongly Lawson monad and I-Lawson monad and show that each strongly Lawson (I-Lawson) monad is isomorphic to some submonad of $\mathbb{V}_s$ ($\mathbb{V}_I$) and in 3 we compare introduced classes with the class of Lawson monads.

1. Universal monads

By *Comp* we denote the category of compact Haussdorff spaces (compacta) and continuous maps.

We denote by I the segment $[0,1]$. Let $X \in Comp$. We denote by CX the Banach space of all continuous functions $\varphi : X \to \mathbb{R}$ with the usual sup-norm: $\|\varphi\| = \sup\{|\varphi(x)| \mid x \in X\}$ and by $C(X,I)$ we denote the subspace of $C(X)$ consisting of all functions with codomain I. In what follows, all spaces and maps are assumed to be in *Comp* except for $\mathbb{R}$ and maps in sets CX with X compact Hausdorff.

We need some definitions concerning monads and algebras. A *monad* $\mathbf{T} = (T, \eta, \mu)$ in a category $\mathcal{E}$ consists of an endofunctor $T : \mathcal{E} \to \mathcal{E}$ and natural transformations $\eta : \mathrm{Id}_{\mathcal{E}} \to T$ (unity), $\mu : T^2 \to T$ (multiplication) satisfying the relations $\mu \circ T\eta = \mu \circ \eta T = \mathbf{1}_T$ and $\mu \circ \mu T = \mu \circ T\mu$.

A natural transformation $\psi : T \to T'$ is called a *morphism* from a monad $\mathbf{T} = (T, \eta, \mu)$ into a monad $\mathbf{T}' = (T', \eta', \mu')$ if $\psi \circ \eta = \eta'$ and $\psi \circ \mu = \mu' \circ \eta T' \circ T\psi$. If all the components of ψ are monomorphisms then the monad $\mathbf{T}$ is called a *submonad* of $\mathbf{T}'$ and ψ is called a monad embedding.

Let $\mathbf{T} = (T, \eta, \mu)$ be a monad in a category $\mathcal{E}$. The pair (X, ξ) is called a $\mathbf{T}$-*algebra* if $\xi \circ \eta X = id_X$ and $\xi \circ \mu X = \xi \circ T\xi$. Let (X,ξ), (Y,ξ') be two $\mathbf{T}$-algebras. A map $f : X \to Y$ is called a $\mathbf{T}$-algebras morphism if $\xi' \circ Tf = f \circ \xi$.

The following fact is well-known [6].

Lemma 1. Let $\mathbb{F} = (F, \eta, \mu)$ be a monad in a category $\mathcal{S}$ and X is an object of $\mathcal{S}$. Let $f, g : (FX, \mu) \to (Y, \xi)$ be $\mathbb{F}$-algebras morphism with $f \circ \eta X = g \circ \eta X = h$. Then $f = g = \xi \circ Fh$.

By $V_I X$ we denote the power $I^{C(X,I)}$. For a map $\varphi \in C(X,I)$ we denote by π_φ or $\pi(\varphi)$ the corresponding projection $\pi_\varphi : V_I X \to I$. Then any map $f : Z \to V_I X$ in *Comp* is uniquely determined by its projections $f_\varphi = \pi_\varphi \circ f$ in $C(Z,I)$ for every $\varphi \in (X,I)$. For each map $f : X \to Y$ we define the map $V_I f : V_I X \to V_I Y$ by the formula $\pi_\varphi \circ V_I f = \pi_{\varphi \circ f}$ for $\varphi \in C(Y,I)$. One can check that V_I is a covariant functor in the category *Comp*.

For $\varphi \in CX$, by $\max\varphi$ ($\min\varphi$) we denote $\max_{x\in X}\varphi(x)$ ($\min_{x\in X}\varphi(x)$). By V_sX we denote the product $\prod_{\varphi\in CX}[\min\varphi, \max\varphi]$, i.e. the set of all maps (not necessarily continuous) $\nu : CX \to \mathbb{R}$ which satisfy the condition $\min\varphi \leq \nu(\varphi) \leq \max\varphi$ for each $\varphi \in CX$.

For a map $\varphi \in CX$ we denote by π_φ or $\pi(\varphi)$ the corresponding projection $\pi_\varphi : VX \to \mathbb{R}$. Then any map $f : Z \to VX$ in *Comp* is uniquely determined by its projections $f_\varphi = \pi_\varphi \circ f$ in CZ for every $\varphi \in CX$ provided only that $[\min f_\varphi, \max f_\varphi] \subset [\min\varphi, \max\varphi]$ for all φ.

Now, for each map $f : X \to Y$ define a map $V_sf : V_sX \to V_sY$ by the formula $\pi_\varphi \circ V_sf = \pi_{\varphi\circ f}$ for $\varphi \in CY$. Since $[\min\pi(\varphi\circ f), \max\pi(\varphi\circ f)] = [\min\varphi\circ f, \max\varphi\circ f] \subset [\min\varphi, \max\varphi]$, the map V_sf is well defined. One can check that V_s is a covariant functor on the category *Comp*.

Now we shall build the natural transformations $h_I : Id_{Comp} \to V_I$, $h_s : Id_{Comp} \to V_s$ and $m_I : V_IV_I \to V_I$, $m_s : V_sV_s \to V_s$ of units and multiplications which complete the functors V_I and V_s to the monads $\mathbb{V}_I = (V_I, h_I, m_I)$ and $\mathbb{V}_s = (V_s, h_s, m_s)$ correspondingly.

For a compactum X we define components h_IX and m_IX (h_sX and m_sX) by $\pi_\varphi \circ h_IX = \varphi$ and $\pi_\varphi \circ m_IX = \pi(\pi_\varphi)$ for all $\varphi \in C(X, I)$ ($\pi_\varphi \circ h_sX = \varphi$ and $\pi_\varphi \circ m_sX = \pi(\pi_\varphi)$ for all $\varphi \in CX$). The map m_sX is well defined because $[\min\pi(\varphi), \max\pi(\varphi)] = [\min\varphi, \max\varphi]$.

Proposition 1. The triples $\mathbb{V}_I = (V_I, h_I, m_I)$ and $\mathbb{V}_s = (V_s, h_s, m_s)$ form monads in the category *Comp*.

Proof. We will prove the proposition only for $\mathbb{V}_I$. For $\mathbb{V}_s$ the proof is analogous.

Let us check the naturality of h_I and m_I. Let $f : X \to Y$ be a map. Then we have $\pi_\varphi \circ h_IY \circ f = \varphi\circ f = \pi_{\varphi\circ f} \circ h_IX = \pi_\varphi \circ V_I(f) \circ h_IX$ and $\pi_\varphi \circ m_IY \circ V_IV_If = \pi(\pi_\varphi) \circ V_IV_If = \pi(\pi_\varphi \circ V_If) = \pi(\pi_{\varphi\circ f}) = \pi_{\varphi\circ f} \circ m_IX = \pi_\varphi \circ V_If \circ m_IX$ for each $\varphi \in C(X, I)$. Hence h_I and m_I are natural transformation.

The equality $m_IX \circ h_IV_IX = m_IX \circ V_Ih_IX = \mathrm{id}_{V_IX}$ follows from the next two equalities: $\pi_\varphi \circ m_IX \circ h_IV_IX = \pi(\pi_\varphi) \circ h_IV_IX = \pi_\varphi = \pi_\varphi \circ \mathrm{id}_{V_IX}$ and $\pi_\varphi \circ m_IX \circ V_Ih_IX = \pi(\pi_\varphi) \circ V_Ih_IX = \pi(\pi_\varphi \circ h_IX) = \pi_\varphi = \pi_\varphi \circ \mathrm{id}_{V_IX}$.

The equality $m_IX \circ V_Im_IX = m_IX \circ m_IV_IX$ follows from the equality $\pi_\varphi \circ m_IX \circ V_Im_IX = \pi(\pi_\varphi) \circ V_Im_IX = \pi(\pi_\varphi \circ m_IX) = \pi(\pi(\pi_\varphi)) = \pi(\pi_\varphi) \circ m_IV_IX = \pi_\varphi \circ m_IX \circ m_IV_IX$ for each $\varphi \in C(X, I)$. The proposition is proved. □

2. Classes of monads with functional representations

We introduce classes of I-Lawson and strongly Lawson monads in this section and prove that the monads $\mathbb{V}_I$ and $\mathbb{V}_s$ are universal in corresponding classes.

Definition 1. A monad $\mathbb{F} = (F, \eta, \mu)$ is an *I-Lawson* monad if there exists a map $\xi : FI \to I$ such that the pair (I, ξ) is an $\mathbb{F}$-algebra and for each

$X \in Comp$ there exists a point-separating family of F-algebras morphisms $\{f_\alpha : (FX, \mu X) \to (I, \xi) \mid \alpha \in A\}$. (Let us recall that a family of maps $\{f_\alpha : X \to I \mid \alpha \in A\}$ is called *point-separating* if for each pair of distinct points x_1, $x_2 \in X$ there exists $\alpha \in A$ such that $f_\alpha(x_1) \neq f_\alpha(x_2)$.)

Definition 2. A monad $\mathbb{F} = (F, \eta, \mu)$ is a *strongly Lawson* monad if for each $t_1, t_2 \in \mathbb{R}$ with $t_1 \leq t_2$ there exists a map $\xi_{[t_1,t_2]} : F[t_1, t_2] \to [t_1, t_2]$ such that the pair $([t_1, t_2], \xi_{[t_1,t_2]})$ is an $\mathbb{F}$-algebra, for each t_1, t_2, t_3, $t_4 \in \mathbb{R}$ with $t_1 \leq t_2 \leq t_3 \leq t_4$ the natural embedding $j : [t_2, t_3] \to [t_1, t_4]$ is an F-algebras morphism and for each $X \in Comp$ there exists a point-separating family of F-algebras morphisms $\{f_\alpha : (FX, \mu X) \to ([t_1(\alpha), t_2(\alpha)], \xi_{[t_1(\alpha),t_2(\alpha)]}) \mid \alpha \in A\}$.

Theorem 1. Let $\mathbb{F} = (F, \eta, \mu)$ be a monad. Then there exists a monad embedding $l : \mathbb{F} \to \mathbb{V}_I$ iff $\mathbb{F}$ is I-Lawson.

Proof. Sufficiency. Fix a map $\xi : F(I) \to I$ from the definition of I-Lawson monad. For $X \in Comp$ define a map $lX : FX \to V_I X$ by the formula $\pi_\varphi \circ lX = \xi \circ F\varphi$, for $\varphi \in C(X, I)$.

Let us show that $l = \{lX\} : F \to V_I$ is a natural transformation. Let $f : X \to Y$ be a map. Then for each $\psi \in C(Y, I)$ we have $\pi_\psi \circ V_I f \circ lX = \pi_{\psi \circ f} \circ lX = \xi \circ F(\psi \circ f) = \xi \circ F\psi \circ Ff = \pi_\psi \circ lY \circ Ff$. Hence $V_I f \circ lX = lY \circ Ff$ and l is a natural transformation.

Now we have to show that l is a monad morphism. The equality $l \circ \eta = h_I$ follows from the equalities $\pi_\psi \circ lX \circ \eta X = \xi \circ F\psi \circ \eta X = \xi \circ \eta I \circ \psi = \psi = \pi_\psi \circ h_I X$ for every $X \in Comp$ and $\psi \in C(X, I)$.

For every $X \in Comp$ and $\varphi \in C(X, I)$ we also have $\pi_\varphi \circ m_I X \circ lVX \circ FlX = \pi(\pi_\varphi) \circ lVX \circ FlX = \xi \circ F(\pi(\varphi) \circ FlX = \xi_c ircF(\pi(\varphi) \circ lX) = \xi \circ F(\xi \circ F(\varphi)) = \xi \circ F\xi \circ FF\varphi = \xi \circ \mu I \circ FF\varphi = \xi \circ F\varphi \circ \mu X = \pi_\varphi \circ lX \circ \mu X$. Hence $m_I \circ lV_I \circ Fl = l \circ \mu$ and l is a monad morphism.

Finally we have to show that the map lX is injective. Let α_1, $\alpha_2 \in FX$ and $\alpha_1 \neq \alpha_2$. Since $\mathbb{F}$ is an I-Lawson monad, there exists an $\mathbb{F}$-algebras morphism $f : (FX, \mu X) \to (I, \xi)$ for some $t > 0$ with $f(\alpha_1) \neq f(\alpha_2)$. Since f is an $\mathbb{F}$-algebras morphism, we have $f =($ by Lemma 1 $)= \xi \circ F(f \circ \eta X) = \pi(f \circ \eta X) \circ lX$. Hence $\pi(f \circ \eta X) \circ lX(\alpha_1) = f(\alpha_1) \neq f(\alpha_2) = \pi(f \circ \eta X) \circ lX(\alpha_2)$.The sufficiency is proved.

Necessity. Let $l : \mathbb{F} \to \mathbb{V}_I$ be a monad embedding. Define a map $\xi : FI \to I$ by the formula $\xi = \pi(\mathrm{id}_I) \circ lI$. Then we have $\xi \circ \eta I = \pi(\mathrm{id}_I) \circ lI \circ \eta I = \pi(\mathrm{id}_I) \circ hI = \mathrm{id}_I$ and $\xi \circ \mu I = \pi(\mathrm{id}_I) \circ lI \circ \mu I = \pi(\mathrm{id}_I) \circ m_I I \circ lV_I I \circ FlI = \pi(\pi(\mathrm{id}_I)) \circ lV_I I \circ FlI = \pi(\mathrm{id}_I \circ \pi(\mathrm{id}_I)) \circ lV_I I \circ FlI = \pi(\mathrm{id}_I) \circ V_I(\pi(\mathrm{id}_I)) \circ lV_I I \circ FlI = \pi(\mathrm{id}_I) \circ lI \circ F(\pi(\mathrm{id}_I) \circ lI) = \xi \circ F\xi$. Hence (I, ξ) is an $\mathbb{F}$-algebra.

Finally one can check that for each $X \in Comp$ the family $\{(\pi_\varphi \circ lX) \mid \varphi \in CX\}$ is a point separating family of $\mathbb{F}$-algebras morphisms. The theorem is proved. □

The proof of the following theorem is analogous to the previous one.

Theorem 2. Let $\mathbb{F} = (F, \eta, \mu)$ be a monad. Then there exists a monad embedding $l : \mathbb{F} \to \mathbb{V}_s$ iff $\mathbb{F}$ is strongly Lawson.

3. Connections between the introduced classes of monads

In this section we discuse some connections between the classes of strongly Lawson, I-Lawson and Lawson monads. We need the definition of Lawson monad and the construction of monad $\mathbb{V}$ which is universal for the class of Lawson monads. For any real $t \geq 0$,we denote by I_t the segment $[-t, t]$. By VX we denote the product $\prod_{\varphi \in CX} I_{\|\varphi\|}$, i.e. the set of all mappings (not necessarily continuous) $\nu : CX \to \mathbb{R}$ which satisfy the condition $-\|\varphi\| \leq \nu(\varphi) \leq \|\varphi\|$ for each $\varphi \in CX$.

Now, for each map $f : X \to Y$ define a map $Vf : VX \to VY$ by the formula $\pi_\varphi \circ Vf = \pi_{\varphi \circ f}$ for $\varphi \in CY$.

For a compactum X we define components hX and mX of natural transformations $h : I_{Comp} to V$ and $m : VV \to V$ by $\pi_\varphi \circ hX = \varphi$ and $\pi_\varphi \circ mX = \pi(\pi_\varphi)$ for all $\varphi \in C(X)$. It is proved in [1] that the triple $\mathbb{V} = (V, h, m)$ forms a monad.

If t_1, t_2 are real numbers with $0 \leq t_1 \leq t_2$, by $j_{t_1}^{t_2}$ we denote the natural embedding $j_{t_1}^{t_2} : I_{t_1} \to I_{t_2}$.

A monad $\mathbb{F} = (F, \eta, \mu)$ is called Lawson if for each $t \geq 0$ there exists a map $\xi_t : FI_t \to I_t$ such that the pair (I_t, ξ_t) is an $\mathbb{F}$-algebra, for each t_1, $t_2 \in \mathbb{R}$ with $0 \leq t_1 \leq t_2$ the embedding $j_{t_1}^{t_2}$ is an $\mathbb{F}$-algebras morphism and for each $X \in Comp$ there exists a point-separating family of $\mathbb{F}$-algebras morphisms $\{f_\alpha : (FX, \mu X) \to (I_{t(\alpha)}, \xi_{t(\alpha)}) \mid \alpha \in A\}$[1].

It is proved in [1] that there exists a monad embedding $l : \mathbb{F} \to \mathbb{V}$ iff $\mathbb{F}$ is Lawson.

It is evidently that each strongly Lawson monad is a Lawson monad. Since the functor V does not preserve one-point spaces, $\mathbb{V}$ can not be represented as a submonad of $\mathbb{V}_s$. Hence $\mathbb{V}$ is not a strongly Lawson monad.

Now we are going to prove that $\mathbb{V}$ is not an I-Lawson monad. For $t \geq 0$ define the map $\xi_t : VI_t \to I_t$ by the formula $\xi_t = \pi(\mathrm{id}_{I_t})$.

Lemma 2. The pair (I_t, ξ_t) is a $\mathbb{V}$-algebra such that $\pi_\psi = \xi_t \circ V(\psi)$ for each $X \in Comp$ and $\psi \in C(X, I_t)$.

Proof. Evidently that $\xi_t \circ \eta I_t = \mathrm{id}_{I_t}$. We also have $\xi_t \circ V(\xi_t) = \pi(\mathrm{id}_{I_t}) \circ V(\pi(\mathrm{id}_{I_t})) = \pi(\pi(\mathrm{id}_{I_t})) = \pi(\mathrm{id}_{I_t}) \circ \mu I_t = \xi_t \circ \mu I_t$. Hence the pair (I_t, ξ_t) is a $\mathbb{V}$-algebra.

Let $X \in Comp$ and $\psi \in C(X, I_t)$. It follows from Lemma 1 that it is enough to prove that $\pi_\psi : (VX, \mu X) \to (I_t, \xi_t)$ is an $\mathbb{V}$-algebras morphism. The next equalities finish the proof of the lemma: $\xi_t \circ V(\pi_\psi) = \pi(\mathrm{id}_{I_t}) \circ V(\pi_\psi) = \pi(\pi_\psi) = \pi_\psi \circ \mu X$. □

Lemma 3. Each $\mathbb{V}$-algebra (I, ξ) is isomorphic to (I_t, ξ_t) for some $t > 0$.

Proof. Let us consider the set $M = \{f : I \to I_t \mid t > 0$ and $f : (I, \xi) \to (I_t, \xi_t)$ is a $\mathbb{V}$-algebras morphism $\}$. Assume that $f(0) = f(1)$ for all $f \in M$. Let us consider two subsets $A_1, A_2 \subset C(VI)$ defined as follows: $A_1 = \{\pi_\psi \mid \psi \in CI\}$ and $A_2 = \{\varphi \circ \xi \mid \varphi \in CI\}$.

Consider any $\Phi \in A_1 \cap A_2$. Then we have that $\pi_\psi = \Phi = \varphi \circ \xi$ for some ψ, $\varphi \in CI$. Using the previous equality for the points from $hI(I)$, we obtain that $\varphi = \psi$. We have $\pi_\varphi = \varphi \circ \xi$ or, by Lemma 2, $\xi_t \circ V(\varphi) = \varphi \circ \xi$. Hence $\varphi \in M$ and we obtain that $A_1 \cap A_2 = \{\pi_\varphi \mid \varphi \in M\} = \{\varphi \circ \xi \mid \varphi \in M\}$. Then we have by our assumption that $\Phi(hI(0)) = \Phi(hI(1))$ for each $\Phi \in A_1 \cap A_2$ and we can define $W \in VVI$ such that $\pi(\pi_\psi)(W) = \pi_\psi(hI(0)) = \psi(0)$ and $\pi(\psi \circ \xi)(W) = \psi \circ \xi(hI(1)) = \psi(1)$ for each $\psi \in CI$. We obtain $\xi(hI(0)) = \xi \circ mI(W) = \xi \circ V\xi(W) = \xi(hI(1))$. This is a contradiction with the properties of the map ξ and our assumption is false. Hence there exists a $\mathbb{V}$-algebras morphism $g : (I, \xi) \to (I_t, \xi_t)$ with $g(0) \neq g(1)$. We can suppose that $t = \|g\|$. Let us show that g is a homeomorphism.

Let $a \in I_t$. Consider $\nu \in VI$ such that $\pi_g(\nu) = a$. Then we have $a = \pi_g(\nu) = \pi(\mathrm{id}_{I_t} \circ g)(\nu) = \pi(\mathrm{id}_{I_t}) \circ Vg(\nu) = \xi_t \circ Vg(\nu) = g \circ \xi(\nu)$. Hence g is a surjective map.

Consider b, $c \in I$ with $b \neq c$. For each $t \in I$ consider $\nu_t \in VI$ defined by $\nu_t = (1-t)hI(0) + thI(1)$. We have that $\xi(\nu_0) = 0$ and $\xi(\nu_1) = 1$. There exist l_1, $l_2 \in I$ with $\xi(\nu_{l_1}) = b$ and $\xi(\nu_{l_2}) = c$. Then we have $g(b) = g \circ \xi(\nu_{l_1}) = \xi_t \circ Vg(\nu_{l_1}) = \xi_t((1-l_1)hI_t(g(0)) + l_1 hI_t(g(1))) = (1-l_1)g(0) + l_1 g(1) \neq$ (since $l_1 \neq l_2$ and $g(0) \neq g(1)$) $\neq (1-l_2)g(0) + l_2 g(1) = \xi_t((1-l_2)hI_t(g(0)) + l_2 hI_t(g(1))) = \xi_t \circ Vg(\nu_{l_2}) = g \circ \xi(\nu_{l_2}) = g(b)$. Hence g is a homeomorphism and the lemma is proved. □

Theorem 3. The monad $\mathbb{V}$ is not I-Lawson.

Proof. Assume the contrary. Let $\xi : VI \to I$ be a map from the definition of I-Lawson monad. By Lemma 3 there exists a $\mathbb{V}$-algebra isomorphism $g : (I, \xi) \to (I_t, \xi_t)$ for some $t > 0$.

Consider any α_1, $\alpha_2 \in VI$ such that $\alpha_1 \neq \alpha_2$ but $\pi_\psi(\alpha_1) = \pi_\psi(\alpha_2)$ for each $\psi \in CI$ with $\|\psi\| \leq t$. There exists a $\mathbb{V}$-algebras morphism $f : (VI, \mu I) \to (I, \xi)$ such that $f(\alpha_1) \neq f(\alpha_2)$. Then we have that $g \circ f : (VI, \mu I) \to (I_t, \xi_t)$ is a $\mathbb{V}$-algebras morphism with $g \circ f(\alpha_1) \neq g \circ f(\alpha_2)$. Put $\psi = g \circ f \circ hI$. By Lemma 1 we have $g \circ f = \xi_t \circ V\psi$ and $\xi_t \circ V\psi(\alpha_1) \neq \xi_t \circ V\psi(\alpha_2)$. On the other hand it follows from Lemma 3 that $\xi_t \circ V\psi(\alpha_1) = \pi_\psi(\alpha_1) =$ (since $\|\psi\| \leq t$) $= \pi_\psi(\alpha_2) = \xi_t \circ V\psi(\alpha_2)$. Hence we obtain a contradiction and the theorem is proved. □

Now we are going to prove that $\mathbb{V}_I$ is not a Lawson monad, thus the classes of Lawson and I-Lawson monads are incomparable.

Define the map $\xi_0 : V_I I \to I$ by the formula $\xi_0 = \pi(\mathrm{id}_I)$. The proofs of two following lemmas are similar to the proofs of Lemmas 2 and 3 correspondly.

Lemma 4. The pair (I, ξ_0) is a $\mathbb{V}_I$-algebra such that $\pi_\psi = \xi_0 \circ V_I(\psi)$ for each $X \in Comp$ and $\psi \in C(X, I)$.

Lemma 5. Each $\mathbb{V}_I$-algebra (I, ξ) is isomorphic to (I, ξ_0).

Lemma 6. Let (X, ξ) be a $\mathbb{V}_I$-algebra and $f : (X, \xi) \to (I, \xi_0)$ be a morphism. Then f is a surjective map.

Proof. Consider any $t \in I$. Choose $\nu \in V_I X$ such that $\pi_f(\nu) = t$. Then we have $f \circ \xi(\nu) = \xi_0 \circ Vf(\nu) = \pi(\mathrm{id}_I) \circ Vf(\nu) = \pi_f(\nu) = t$. The lemma is proved. □

Theorem 4. The monad $\mathbb{V}_I$ is not Lawson.

Proof. Assume the contrary. Consider a family (I_t, ξ_t) of $\mathbb{V}_I$-algebras satisfying the definition of a Lawson monad. Choose any $t_1, t_2 \geq 0$ with $t_1 < t_2$. The algebra (I_{t_2}, ξ_{t_2}) is isomorphic to (I, ξ_0) by Lemma 6. Since the inclusion $j_{t_1}^{t_2} : I_{t_1} \to I_{t_2}$ is a V_I-algebras morphism, there exists a morphism from (I_{t_1}, ξ_{t_1}) to (I, ξ_0) which is not onto. We obtain a contradiction with Lemma 6 and the theorem is proved. □

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