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A FUNCTOR THAT REFLECTS AND PRESERVES COMPACTNESS AND CONNECTEDNESS

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Abstract. We present an equivalence between the compactness of a topological space and the compactness of a quotient space obtained through a natural equivalence relation. By means of this equivalence, the study of compactness in general is reduced to that of T_0 topological spaces.

Key words and phrases. Compactness, saturated sets, T_0−spaces, connectedness, Alexandroff topology, weak topology.

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Resumen. Se presenta una equivalencia entre la compacidad de un espacio topológico y la compacidad de un espacio cociente obtenida a través de una relación de equivalencia natural. Por medio de esta relación de equivalencia, el estudio de la compacidad en general se reduce al estudio de la compacidad de espacios T_0.

Palabras claves. Compacidad, conjuntos saturados, espacios T_0, conexidad, topología de Alexandroff, topología débil.

1. Introduction

Quotient spaces, saturated sets and compactness are studied on basic courses of set theory and general topology, but in the literature results that relate these three concepts are meager. To this respect we present a very simple result, in which the saturated sets play a special role to obtain an equivalence between the compactness of an arbitrary topological space and a particular quotient space, which is always a T_0−topological space.

Using these spaces we construct a functor from the category of topological spaces to the category of T_0− topological spaces, which allows us to say that when we study compactness, we may assume, without loss of generality, that

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the topological space is T_0. Moreover, we observe that this functor preserves and reflects connectedness and concordant topologies[1] with the specialization order. To obtain more information about specialization order and concordant topologies we refer to [1], [4], [6], [8].

2. Compactness on a specific quotient space

Let (X, τ) be a topological space and S be an equivalence relation on X. Let us consider the quotient space $X/_S$ with its quotient topology $\tau/_S$.

$$\tau/_S = \{A \subseteq X/_S \mid \theta^{-1}(A) \in \tau\}$$

where $\theta : X \longrightarrow X/_S : \theta(x) = \overline{x}$ is the canonical quotient map.
We say that a subset B of X is $S-$**saturated**[2] (or simply saturated), if $\theta^{-1}(\theta(B)) = B$, that is,

$$(\forall\ y \in B)(\ y\ S\ x \Longrightarrow x \in B).$$

Theorem 2.1. *Let (X, τ) be a topological space and S be an equivalence relation on X.If the open sets of (X, τ) are saturated, then (X, τ) is compact if and only if $(X/_S, \tau/_S)$ is compact.*

Proof.

- $\Rightarrow$) Since θ is a continuous and onto function and (X, τ) is compact, then $(X/_S, \tau/_S)$ is compact.
- $\Leftarrow$) Let $\{A_i\}_{i\in I}$ be a covering of X by open sets of τ, that is, $X = \underset{i\in I}{\cup} A_i$.
  Then $X/_S = \theta(X) = \theta\left(\underset{i\in I}{\cup} A_i\right) = \underset{i\in I}{\cup}\theta(A_i)$. $\theta(A_i)$ is open in $X/_S$, for each $i \in I$, because the open sets of (X, τ) are saturated. Thus $\{\theta(A_i)\}_{i\in I}$ is an open cover of $X/_S$ and as $(X/_S, \tau/_S)$ is compact, there exists $\{i_1, ..., i_n\} \subseteq I$ such that $X/_S = \overset{n}{\underset{j=1}{\cup}}\theta(A_{i_j})$.
  Let us see now that $X = \overset{n}{\underset{j=1}{\cup}} A_{i_j}$.
  $$X = \theta^{-1}(X/_S) = \theta^{-1}\left(\overset{n}{\underset{j=1}{\cup}}\theta(A_{i_j})\right) = \overset{n}{\underset{j=1}{\cup}}\theta^{-1}(\theta(A_{i_j})) = \overset{n}{\underset{j=1}{\cup}} A_{i_j}.$$
  Therefore, (X, τ) is compact.

□

[1] Concordant topologies are also called compatible topologies, see [5].
[2] Definition taken from [2].

3. A particular case

On the space (X, τ) we define the following relation:

$$x \sim y \iff (x \in cl_\tau\{y\} \quad \wedge \quad y \in cl_\tau\{x\}),$$

where $cl_\tau Z$ denotes the closure of Z with respect to the topology τ.
$\sim$ is an equivalence relation on X, since by definition $\sim$ is symmetric and the properties of the closure make that $\sim$ is reflexive and transitive.

Proposition 3.1. *If $\tau \in Top(X)$,then the open sets of τ are $\sim$-saturated.*

Proof. Consider $A \in \tau$. If $x \in \theta^{-1}(\theta(A))$, there exists $a \in A$ such that $x \sim a$. This implies that $a \in cl_\tau\{x\}$, and hence that $x \in A$. □

Corollary 3.1. $\theta : (X, \tau) \longrightarrow (X/\sim, \tau/\sim) : \theta(x) = \overline{x}$ *is open.*

As a result of Proposition 3.1 and Theorem 2.1 we obtain the following corollary.

Corollary 3.2. *Let (X, τ) be a topological space and $\sim$ be the equivalence relation on X defined above. (X, τ) is compact if and only if $(X/\sim, \tau/\sim)$ is compact.*

Proposition 3.2. *Let (X, τ) be a topological space and $\sim$ be the equivalence relation on X,defined above. (X, τ) is connected if and only if $(X/\sim, \tau/\sim)$ is connected.*

Proof.

- $\Rightarrow$) Since θ is a continuous and onto function and (X, τ) is connected, then $(X/\sim, \tau/\sim)$ is connected.
- $\Leftarrow$) Let us suppose that (X, τ) is disconnected, that is $X = A \cup B$, where A, B are nonempty, open, disjoint subsets of X. Then

$$X/\sim = \theta(X) = \theta(A \cup B) = \theta(A) \cup \theta(B),$$

where $\theta(A)$, $\theta(B)$ are nonempty elements of $\tau/\sim$.

$$\begin{aligned} \overline{y} \in \theta(A) \cap \theta(B) &\Longrightarrow \overline{y} \in \theta(A) \wedge \overline{y} \in \theta(B) \\ &\Longrightarrow (\exists a \in A, \exists b \in B)(\overline{a} = \overline{y} = \overline{b}) \\ &\Longrightarrow (\exists a \in A, \exists b \in B)(a \in cl_\tau\{y\} \wedge b \in cl_\tau\{y\}) \\ &\Longrightarrow y \in A \wedge y \in B \\ &\Longrightarrow y \in A \cap B = \emptyset. \end{aligned}$$

Thus $\theta(A) \cap \theta(B) = \emptyset$. Therefore $\{\theta(A), \theta(B)\}$ is a disconnection of $(X/\sim, \tau/\sim)$.

□

The following result is well known. We omit its proof.

Proposition 3.3. *$(X/\sim, \tau/\sim)$ is a T_0 topological space.*

We obtain that the compactness and the connectedness of a topological space X are equivalent respectively to the compactness and the connectedness of the quotient space $X/\sim$, which always results to be T_0.

4. The functor $F : Top \longrightarrow TopT_0$

With aid of the relation $\sim$ defined in the previous section, we will construct a functor from the category of topological spaces to the category of T_0-topological spaces. In what follows, we will denote θ_X to be the canonical quotient map, where the index reminds us the space on which it acts.

Proposition 4.1. *If $f : (X, \tau) \longrightarrow (Y, \mu)$ is continuous, then the function $f_\sim : (X/\sim, \tau/\sim) \longrightarrow (Y/\sim, \mu/\sim)$, defined by $f_\sim(\overline{x}) = \overline{f(x)}$, is continuous.*

Proof. It is easy to prove that $f_\sim$ is well defined. Let $B \in \mu/\sim$, then $\theta_Y^{-1}(B) \in \mu$, and, $f^{-1}(\theta_Y^{-1}(B)) \in \tau$. We shall see that $f_\sim^{-1}(B) \in \tau/\sim$, that is, $\theta_X^{-1}(f_\sim^{-1}(B)) \in \tau$.

$$\begin{aligned}
\theta_X^{-1}(f_\sim^{-1}(B)) &= \{x \mid \overline{x} \in f_\sim^{-1}(B)\} \\
&= \{x \mid f_\sim(\overline{x}) = \overline{f(x)} \in B\} \\
&= \{x \mid f(x) \in \theta_Y^{-1}(B)\} = f^{-1}(\theta_Y^{-1}(B))\} \in \tau.
\end{aligned}$$

□

Proposition 4.2. *Let $F : Top \longrightarrow Top$ be such that $F(X, \tau) = (X/\sim, \tau/\sim)$ and for $f : (X, \tau) \longrightarrow (Y, \mu)$, $F(f) = f_\sim : (X/\sim, \tau/\sim) \longrightarrow (Y/\sim, \mu/\sim)$, as in the previous proposition. Then F is a covariant functor.*

Remark 1. *The proof of Proposition 4.1 shows that θ is a natural transformation from the identity functor in Top to the functor F:*

$$\begin{array}{ccc}
(X, \tau) & \xrightarrow{\theta_X} & (X/\sim, \tau/\sim) \\
f \downarrow & & \downarrow F(f) = f_\sim \\
(Y, \mu) & \xrightarrow{\theta_Y} & (Y/\sim, \mu/\sim)
\end{array}$$

By Proposition 3.3 we have that $F : Top \longrightarrow TopT_0 \subseteq Top$, where $TopT_0$ is the category of the T_0-topological spaces.

Proposition 4.3. *The functor $F : Top \longrightarrow TopT_0$ is left adjoint of the inclusion functor $I : TopT_0 \longrightarrow Top$.*

Proof. It is enough to observe that $\lambda = (\lambda_{X,Y})$ is a natural isomorphism, where $\lambda_{X,Y} : [F(X, \tau), (Y, \mu)] \longrightarrow [(X, \tau), (Y, \mu)] : f \longmapsto f \circ \theta_X$. □

Remark 2. *This proposition is related to the Exercise 27 a), Chapter 1, Section 8 of [3].*

Thus F is a covariant functor from Top to $TopT_0$ that preserves and reflects compactness, connectedness and is left adjoint of the inclusion functor I, so that F preserves co-limits and I preserves limits.

5. Behavior of the functor F with certain topologies

A preorder relation on a set induces several topologies on the set. We mention here two of them of special interest in the context of this article since the functor F behaves well with respect to them.

Let R be a preorder relation on a set X. We say that a subset A of X is an upper set if it satisfies

$$x \in A \quad \wedge \quad x\ R\ y \Rightarrow y \in A.$$

The Alexandroff topology on X, associated with R, is the topology whose open sets are exactly the upper sets. We will denote this topology by $\gamma(R)$.

The weakest topology on X in which the sets $\downarrow x = \{y \in X \mid y\ R\ x\}$ are closed is called the weak topology associated with R and is denoted by $\upsilon(R)$[3].

Remark 3. *Let τ be a topology on X such that $\upsilon(R) \subseteq \tau \subseteq \gamma(R)$. It is clear that $cl_\tau\{x\} = \downarrow x$. (See [1].) These topologies have been called concordant topologies with R. (See [8].)*

As a consequence of this observation, if τ and μ are two concordant topologies with R, we have:

$$x \in cl_\tau\{y\} \quad \Longleftrightarrow \quad x \in cl_\mu\{y\}.$$

Thus the relation $\sim$ previously defined is the same for the topology τ and the topology μ.

Let now τ be a concordant topology with R. We define the following relation in $X/\sim$:

$$\Re = \{(\overline{x}, \overline{y}) \mid (x, y) \in R\}$$

Note that $x \sim y$ is equivalent to $x\ R\ y$ and $y\ R\ x$; so that $(X/\sim, \Re)$ is the ordered set associated to the preorder relation R on X.

Proposition 5.1. $F(X, \gamma(R)) = (X/\sim, \gamma(\Re))$.

Proof. We shall see that $\gamma(R)/\sim\, = \gamma(\Re)$. Consider $A \in \gamma(R)/\sim$.

$$\begin{aligned} \overline{a} \in A \ \wedge\ \overline{a}\,\Re\,\overline{b} &\Rightarrow a \in \theta^{-1}(A) \in \gamma(R) \wedge a\,R\,b \\ &\Rightarrow b \in \theta^{-1}(A) \\ &\Rightarrow \overline{b} \in A. \end{aligned}$$

Then $A \in \gamma(\Re)$. Consider now $A \in \gamma(\Re)$.

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[3]For more information, see [4], [6], [7].

$$\begin{aligned} a \in \theta^{-1}(A) \wedge a\,R\,b &\Rightarrow \overline{a} \in A \wedge \overline{a}\,\Re\,\overline{b} \\ &\Rightarrow \overline{b} \in A \\ &\Rightarrow b \in \theta^{-1}(A) \\ &\Rightarrow \theta^{-1}(A) \in \gamma(R). \end{aligned}$$

Then $A \in \gamma(R)/\sim$. □

By the proposition above the functor F preserves Alexandroff spaces. Another version of this result can be seen in [7].

Remark 4. *If J is an index finite set, we have:*

$$\theta(X \setminus \bigcup_{i\in J} \downarrow x_i) = (X/\sim) \setminus \bigcup_{i\in J} \downarrow \overline{x_i}.$$

Indeed,

$$\begin{aligned} \overline{y} \notin \bigcup_{i\in J} \downarrow \overline{x_i} &\Longleftrightarrow \overline{y} \notin \downarrow \overline{x_i}, \forall i \in J \\ &\Longleftrightarrow \rceil(\overline{y}\,\Re\,\overline{x_i}), \forall i \in J \\ &\Longleftrightarrow \rceil(y\,R\,x_i), \forall i \in J \quad (\text{since } \overline{x}\,\Re\,\overline{y} \Leftrightarrow x\,R\,y) \\ &\Longleftrightarrow y \notin \downarrow x_i, \forall i \in J \\ &\Longleftrightarrow y \notin \bigcup_{i\in J} \downarrow x_i \\ &\Longleftrightarrow y \in X \setminus \bigcup_{i\in J} \downarrow x_i = \theta^{-1}\left(\theta(X \setminus \bigcup_{i\in J} \downarrow x_i)\right) \\ &\qquad (\text{since } J \text{is finite } \Rightarrow \left(X \setminus \bigcup_{i\in J} \downarrow x_i\right) \in \tau) \\ &\Longleftrightarrow \overline{y} \in \theta(X \setminus \bigcup_{i\in J} \downarrow x_i). \end{aligned}$$

Proposition 5.2. $F(X, \upsilon(R)) = (X/\sim, \upsilon(\Re))$.

Proof. We shall see that $\upsilon(R)/\sim = \upsilon(\Re)$. Let $\downarrow \overline{x}$ be closed in $\upsilon(\Re)$, that is, $(X/\sim) \setminus \downarrow \overline{x} = A \in \upsilon(\Re)$.

$$\theta^{-1}(A) = \theta^{-1}((X/\sim) \setminus \downarrow \overline{x}) = X \setminus \theta^{-1}(\downarrow \overline{x}) = X \setminus \downarrow x,$$

since $\theta^{-1}(\downarrow \overline{x}) = \downarrow x$.

As $X \setminus \downarrow x \in \upsilon(R)$, we have that $A \in \upsilon(R)/\sim$ and $\downarrow \overline{x}$ is closed in $\upsilon(R)/\sim$.

Let now $A \in \upsilon(R)/\sim$ be, that is $\theta^{-1}(A) = \bigcup_{i\in I} \bigcap_{j\in J_i} (X \setminus \downarrow x_{ij})$, where J_i is finite, then:

$$\begin{aligned} A = \theta(\theta^{-1}(A)) &= \theta\left(\underset{i\in I}{\cup}\ \underset{j\in J_i}{\cap}(X\backslash \downarrow x_{ij})\right) \\ &= \underset{i\in I}{\cup}\theta\left(\ \underset{j\in J_i}{\cap}(X\backslash \downarrow x_{ij})\right) \\ &= \underset{i\in I}{\cup}\theta\left(X \setminus \underset{j\in J_i}{\cup} \downarrow x_{ij}\right) \\ &= \underset{i\in I}{\cup}\left((X/\sim) \setminus \underset{j\in J_i}{\cup} \downarrow \overline{x_{ij}}\right) \quad \text{(by the previous remark.)} \\ &= \underset{i\in I}{\cup}\left(\underset{j\in J_i}{\cap}((X/\sim) \setminus \downarrow \overline{x_{ij}})\right) \in \upsilon(\Re). \end{aligned}$$

□

According to this proposition, the functor F preserves spaces with the weak topology.

Proposition 5.3. *Let R be a preorder relation on X and τ, μ be two $R-$concordant topologies on X. If $\tau \subseteq \mu$, then $\tau/\sim \subseteq \mu/\sim$.*

Proof.

$$\begin{aligned} A \in \tau/\sim &\Longrightarrow \big(A \subseteq X/\sim, \wedge, \theta^{-1}(A) \in \tau \subseteq \mu\big) \\ &\Longrightarrow A \in \mu/\sim . \end{aligned}$$

□

Proposition 5.4. *Let R be a preorder relation on X. If τ is an $R-$concordant topology on X, then $\tau/\sim$ is an $\Re-$concordant topology on $X/\sim$.*

Proof.

$$\begin{aligned} \tau \text{ is } R-\text{ concordant } &\Leftrightarrow \upsilon(R) \subseteq \tau \subseteq \gamma(R) \\ &\Rightarrow \upsilon(R)/\sim \subseteq \tau/\sim \subseteq \gamma(R)/\sim \\ &\Rightarrow \upsilon(\Re) \subseteq \tau/\sim \subseteq \gamma(\Re) \\ &\Leftrightarrow \tau/\sim \text{ is } \Re-\text{concordant.} \end{aligned}$$

□

Given a preorder relation R on a set X, we will denote:

$$[\upsilon(R);\gamma(R)] = \{\tau \in Top(X) \mid \tau \text{ is } R-\text{concordant}\}.$$

Corollary 5.1. $F_X : ([\upsilon(R);\gamma(R)], \subseteq) \longrightarrow ([\upsilon(\Re);\gamma(\Re)], \subseteq) : \tau \longmapsto \tau/\sim$ *is a morphism of ordered sets.*

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