Boletín de Matemáticas
Nueva Serie, Volumen XIII No. 2 (2006), pp. 100–110

IGUSA'S LOCAL ZETA FUNCTIONS OF THE DU VAL-KLEIN SINGULARITIES

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Abstract. In this note we calculate Igusa's zeta functions associated to the Du Val-Klein singularities by using the stationary phase formula due to Igusa.

Key words and phrases. Igusa's zeta functions, Du Val-Klein singularities, stationary phase formula.

Resumen. En este artículo calculamos funciones zeta de Igusa asociadas a las singularidades de Du Val-Klein utilizando la fórmula de fase estacionaria debida a Igusa.

Palabras claves Funciones zeta de Igusa, singularidades de Du Val-Klein, fórmula de fase estacionaria.

2000 Mathematics Subject Classification: 12E12

1. Introduction

Let K be a non-archimedean local field of arbitrary characteristic. Let $\mathcal{O}_K$ be the ring of integers of K and $\mathcal{P}_K$ its maximal ideal. Let π be a fixed uniformizing parameter of K, i.e. $\mathcal{P}_K = \pi\mathcal{O}_K$, and let the residue field of K, i.e. $\mathcal{O}_K/\mathcal{P}_K$, be $\mathbb{F}_q$, the finite field with $q = p^r$ elements. Let υ denote the valuation of K such that $\upsilon(\pi) = 1$. For $x \in K^\times$, let $|x|_K = q^{-\upsilon(x)}$ and $|0|_K = 0$. Let $f(x) \in \mathcal{O}_K[x]$, $x = (x_1, \ldots, x_n)$ be a nonconstant polynomial.

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This work is based on the author's undergraduate thesis, submitted at the Universidad Nacional de Colombia, Bogotá.

To these data one associates Igusa's local zeta function,

$$Z(f,s) := \int_{\mathcal{O}_K^n} |f(x)|_K^s \, |dx|, \quad s \in \mathbb{C}, \quad Re(s) > 0,$$

where $|dx|$ denotes the Haar measure on K^n, normalized such that $\mathcal{O}_K^n$ has measure 1. These local zeta functions were introduced by Weil and their basic properties for general f were first studied by Igusa [7] (see also [9]). If the $char(K) > 0$, the rationality of $Z(f,s)$ is still an open problem. Zúñiga-Galindo showed the rationality of $Z(f,s)$ in the case in which f is a non-degenerate polynomial with respect to its Newton polyhedron ([14], [13], see also [12]). Igusa has showed the rationality of the local zeta function associated with several prehomogeneous vector spaces by using the stationary phase formula, a method that works in arbitrary characteristic [9], [8, and references therein].

Igusa's local zeta functions are related to the number of solutions of congruences modulo $\pi^m \mathcal{O}_K$ and to exponential sums modulo $\pi^m \mathcal{O}_K$ (see e.g. [6], [9]), more precisely, if

$$N_m = Card \left\{ x \in \left(\mathcal{O}_K / \pi^m \mathcal{O}_K \right)^n \mid f(x) \equiv 0 \mod \pi^m \mathcal{O}_K \right\},$$

and $P(t)$ is the Poincaré series $P(t) = \sum_{m=0}^{\infty} N_m \left(q^{-n} t \right)^m$, where $N_0 = 1$, then

$$P(t) = \frac{1 - tZ(f,s)}{1 - t},$$

where $t = q^{-s}$ [9, Theorem 8.2.2]. We might mention that the Poincaré series of $f(x)$ was introduced and its rationality was conjectured by Borevich and Shafarevich [4].

In this paper, we compute explicitly the Igusa's zeta functions corresponding to the Du Val-Klein singularities (see Theorems 4, 5, 6 and 7)

Type	*Equation*
E_6	$x^4 + y^3 + z^2 = 0$
E_7	$x^3 y + y^3 + z^2 = 0$
E_8	$x^5 + y^3 + z^2 = 0$
A_r	$x^{r+1} - yz = 0, \quad r \in \mathbb{N}$
D_r	$x^{r-1} - xy^2 + z^2 = 0, \quad r \in \mathbb{N} \smallsetminus \{0\}$,

(see e.g. [3]).

This is a quasi-homogeneous singularities, in [13] Zúñiga-Galindo studied the local zeta functions of this type of polynomials, particulary, they gave explicitly the denominator of such a zeta function. The denominator of $Z(f,s)$ coincides with the one given by Zúñiga-Galindo for these cases.

2. The stationary phase formula, a short account

In [10], Ono introduced the stationary phase formula (SPF). Igusa has used this formula systematically in the calculation of the local zeta functions associated with prehomogeneous vector spaces [8]. The formula takes its name from the classical method of stationary phase that is used in the study of the asymptotic behavior of oscillatory integrals of the form

$$\int ue^{i\omega f}\,dx,$$

where f and u are smooth, $Im\, f \geq 0$ and $\omega \to +\infty$, see for example the surveying article by Acosta [1].

The following is the exact form of the stationary phase formula SPF.

Theorem 1. (Igusa's Stationary Phase Formula [9, pag 168])
Let $\overline{E}$ be a subset of $\mathbb{F}_q^n$ and let $\overline{S}$ its subset consisting of all $\overline{a}$ in $\overline{E}$ such that $\overline{f}(\overline{a}) = \nabla\overline{f}(\overline{a}) = 0$. Let E and S denote the preimages of $\overline{E}$, $\overline{S}$ under the canonical homomorphism $\mathcal{O}_K \to \mathcal{O}_K/\pi\mathcal{O}_K$ and let N be the number of zeros of $\overline{f}(x)$ in $\overline{E}$. Then we have

$$\begin{aligned}\int_E |f(x)|_K^s\,|dx| =& q^{-n}\big(Card(\overline{E})-N\big)+q^{-n}\big(N-Card(\overline{S})\big)\frac{(1-q^{-1})q^{-s}}{1-q^{-1-s}}\\ &+\int_S |f(x)|_K^s\,|dx|\end{aligned}$$

3. The Calculations

Before starting the calculations required to prove the results mentionated in the introduction, we want to recall a theorem due to Zúñiga-Galindo [13] about the zeta function of a quasi-homogeneous polynomial. A polynomial $f(x) \in \mathcal{O}_K[x]$, $x = (x_1,\ldots,x_n)$ is called quasihomogeneous of weight d and exponents $\alpha_1,\ldots,\alpha_n$ if it satisfies

$$f(t^{\alpha_1}x_1,\ldots,t^{\alpha_n}x_n) = t^d f(x_1,\ldots,x_n), \quad \text{for every } t \in K.$$

The mentioned result is the following:

Theorem 2 (Zúñiga-Galindo [13], [14])**.** Let $f(x) \in K[x]$, $x = (x_1, \dots, x_n)$ be a quasihomogeneous polynomial of weight d and exponents $\alpha_1, \dots, \alpha_n$. If the origin of K^n is the only singular point of $f(x)$, then Igusa's local zeta function of $f(x)$ is a rational function of $t = q^{-s}$. More precisely,

$$Z(f,s) = \frac{L(t)}{(1-q^{-1}t)(1-q^{-|\alpha|}t^d)},$$

where $L(t) \in \mathbb{Q}[t]$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Although the proof of Theorem 2 given by Zúñiga-Galindo is mostly constructive, and it uses also the SPF, we use an slightly different approach. Basically we apply SPF recursively until we recover the original integral times a function of q^{-s}, then by bringing this expression to the left side we obtain an explicit expression for the desired integral.

We will use the following notation due to Igusa:

$$[a,b] = 1 - q^{-a}t^b, \qquad [a] = [a,0],$$
$$[a,b]_+ = 1 + q^{-a}t^b, \qquad [a]_+ = [a,0]_+.$$

The following result will be used in our calculations

Theorem 3. [11, pag 12] Let $q = p^r$, $p \neq 2$, then the number of solutions of $x^m + y^n = 0$ in $\mathbb{F}_q$, denoted for $|N|$, is

$$|N| = \begin{cases} 1 + (q-1)gcd(m,n,q-1), & \text{if } ord_2(q-1) > \min\{ord_2(m), ord_2(n)\}, \\ 1, & \text{otherwise.} \end{cases}$$

3.1. **Calculation for E_6.** Before the calculation, we count the number of solutions on $\mathbb{F}_q^3$ of $y^3 + z^2 = 0$, $x^4 + y^3 = 0$, $x^4 + z^2 = 0$ and $x^4 + y^3 + z^2 = 0$. By Theorem 3 for $y^3 + z^2 = 0$ and $x^4 + y^3 = 0$ this is q^2 and for $x^4 + z^2 = 0$ this is $2q^2 - q$ or q according to $q \equiv 1 \mod 4$ or $q \equiv 3 \mod 4$, now for $x^4 + y^3 + z^2 = 0$ if $y = 0$ we have $2q - 1$ or 1 solutions according to $q \equiv 1 \mod 4$ or $q \equiv 3 \mod 4$. If $y \neq 0$ and $x = uy$, $z = vy^2$, we have $y(u^4 + v^2) + 1 = 0$ and in this case we have $q^2 - 2q + 1$ or $q^2 - 1$ solutions according to $q \equiv 1 \mod 4$ or $q \equiv 3 \mod 4$, hence we always have q^2 solutions.

Now let K be a non-archimedean local field such that $char\big(\mathcal{O}_K/\pi\mathcal{O}_K\big) \neq 2, 3$, then in the first application of SPF our integral becomes

$$\int_{\mathcal{O}_K^3} |x^4+y^3+z^2|_K^s\,|dx\,dy\,dz| = [1] + \frac{q^{-1}[1][2]}{[1,1]}t + q^{-3}t^2\int_{\mathcal{O}_K^3} |\pi^2x^4+\pi y^3 + z^2|_K^s\,|dx\,dy\,dz|,$$

if we put $Z(f_{m,n,l},s) = \int_{\mathcal{O}_K^3} |f_{m,n,l}(x,y,z)|_K^s\,|dx\,dy\,dz|$, with $f_{m,n,l}(x,y,z) = \pi^m x^4 + \pi^n y^3 + \pi^l z^2$, we can write this as follows

$$Z(f_{0,0,0},s) = [1] + \frac{q^{-1}[1][2]}{[1,1]}t + q^{-3}t^2 Z(2,1,0),$$

the systematic applications of SPF produce the following

$$\begin{aligned}
Z(f_{2,1,0},s) &= [1] + q^{-1}tZ(f_{1,0,1},s),\\
Z(f_{1,0,1},s) &= [1] + q^{-1}tZ(f_{0,2,0},s),\\
Z(f_{0,2,0},s) &= N + q^{-2}t^2 Z(f_{2,0,0},s),\\
Z(f_{2,0,0},s) &= [1] + \frac{q^{-1}[1]^2}{[1,1]}t + q^{-2}t^2 Z(f_{0,1,0},s),\\
Z(f_{0,1,0},s) &= N(t) + q^{-2}tZ(f_{3,0,1},s),\\
Z(f_{3,0,1},s) &= [1] + q^{-1}tZ(f_{2,2,0},s),\\
Z(f_{2,2,0},s) &= [1] + q^{-1}t^2 Z(f_{0,0,0},s),
\end{aligned}$$

where $N(t)$ is $[1]^2 + 2\frac{q^{-1}[1]^2}{[1,1]}t$ or $[2]$ according to $q \equiv 1 \mod 4$ or $q \equiv 3 \mod 4$, we therefore obtain:

Theorem 4. Let K be a non-archimedean local field such that $char(\mathcal{O}_K/\pi\mathcal{O}_K) \neq 2, 3$ and let $Z(s) = \int_{\mathcal{O}_K^3} |x^4+y^3+z^2|_K^s\,|dx\,dy\,dz|$, then we have

i) if $q \equiv 1 \mod 4$,

$$Z(s) = \frac{[1]\Big(1 - q^{-3}t(1-t)(1+q^{-3}t^3+q^{-4}t^4+q^{-6}t^6-q^{-7}t^8) - q^{-13}t^{11}\Big)}{[1,1][13,12]}$$

ii) if $q \equiv 3 \mod 4$,

$$Z(s) = \frac{[1]\Big(1 - q^{-3}t([0,1]+q^{-2}t^3[2,2][4,3]_+) - q^{-13}t^{11}\Big) + [2]q^{-5}t^4[1,1][4,4]_+}{[1,1][13,12]}.$$

This result agrees with Theorem 2 because $x^4+y^3+z^2$ is a quasihomogeneous polynomial of weight 12 and exponents 3, 4 and 6.

3.2. **Calculation for E_7.** First count the number of solutions on $\mathbb{F}_q^3$ of $y^3 + z^2 = 0$, $x^3y + y^3 = 0$, $x^3y + z^2 = 0$ and $x^3y + y^3 + z^2 = 0$, by Theorem 3, for $y^3 + z^2 = 0$ we have q^2 zeros, for $x^3y + y^3 = 0$ we have, if $y = 0$, q^2 zeros and if $y \neq 0$ then $x^3 + y^2 = 0$ and in this case we have $q^2 - q$ zeros, for $x^3y + z^2 = 0$, if $x = 0$, q zeros and if $x \neq 0$ and $y = -v^2x$, $z = vx^2$, we have $q(q-1)$ zeros, therefore we have q^2 zeros, and for $x^3y + y^3 + z^2 = 0$ we have if $y = 0$, q zeros, and if $y \neq 0$ and $x = uy$, $z = vy^2$ we have $y(u^3 + v^2) = -1$ and then we have $q^2 - q$ zeros, we therefore obtain q^2 zeros, and if we put $Z(f_{m,n,l}, s) = \int_{\mathcal{O}_K^3} |f_{m,n,l}(x, y, z)|_K^s \, |dx\, dy\, dz|$, with $f_{m,n,l}(x, y, z) = \pi^m x^3y + \pi^n y^3 + \pi^l z^2$, then the systematic application of SPF produces

$$
\begin{aligned}
Z(f_{0,0,0}, s) &= [1] + \frac{q^{-1}[1][2]}{[1,1]}t + q^{-3}t^2 Z(f_{2,1,0}, s),\\
Z(f_{2,1,0}, s) &= [1] + q^{-1}tZ(f_{1,0,1}, s),\\
Z(f_{1,0,1}, s) &= [1] + q^{-1}tZ(f_{1,2,0}, s),\\
Z(f_{1,2,0}, s) &= [1] + q^{-1}tZ(f_{0,1,1}, s),\\
Z(f_{0,1,1}, s) &= [1]^2 + \frac{q^{-1}[1]^2}{[1,1]}t + q^{-1}tZ(f_{2,0,0}, s),\\
Z(f_{2,0,0}, s) &= [1] + \frac{q^{-1}[1]^2}{[1,1]}t + q^{-2}t^2 Z(f_{1,1,0}, s),\\
Z(f_{1,1,0}, s) &= [1] + q^{-1}tZ(f_{0,0,1}, s),\\
Z(f_{0,0,1}, s) &= [1]^2 + 2\frac{q^{-1}[1]^2}{[1,1]}t + q^{-2}tZ(f_{3,2,0}, s),\\
Z(f_{3,2,0}, s) &= [1] + q^{-1}t^2 Z(f_{1,0,0}, s),\\
Z(f_{1,0,0}, s) &= [1] + \frac{q^{-1}[1]^2}{[1,1]}t + q^{-2}t^2 Z(f_{0,1,0}, s),\\
Z(f_{0,1,0}, s) &= [1] + \frac{q^{-1}[1]^2}{[1,1]}t + q^{-2}tZ(f_{2,0,1}, s),\\
Z(f_{2,0,1}, s) &= [1] + q^{-1}tZ(f_{2,2,0}, s),\\
Z(f_{2,2,0}, s) &= [1] + q^{-1}t^2 Z(f_{0,0,0}, s),
\end{aligned}
$$

we therefore obtain

Theorem 5. Let K be a non-archimedean local field such that $char(\mathcal{O}_K/\pi\mathcal{O}_K) \neq 2, 3$ then we have

$$\int_{\mathcal{O}_K^3} |x^3y + y^3 + z^2|_K^s \, |dx\,dy\,dz| =$$
$$\frac{[1]\Big(1 - q^{-3}t(1-t)(1 + q^{-4}t^4 + q^{-6}t^6 + q^{-8}t^8 + q^{-10}t^{10} + q^{-12}t^{12}) - q^{-19}t^{17}\Big)}{[1,1][19,18]}.$$

This result agrees with Theorem 2 because $x^3y+y^3+z^2$ is a quasihomogeneous polynomial of weight 18 and exponents 4, 6 and 9.

3.3. **Calculation for E_8.** This calculations appear in Igusa's book [9, pag 172] and for the sake of completeness we present them here:

$$\int_{\mathcal{O}_K^3} |x^5 + y^3 + z^2|_K^s \, |dx\,dy\,dz| =$$
$$\frac{[1]\Big(1 - q^{-3}t(1-t)(1 + q^{-6}t^6 + q^{-10}t^{10} + q^{-12}t^{12})\Big)}{[1,1][31,30]}$$
$$+\frac{[1]\Big(-q^{-3}t(1-t)(q^{-16}t^{16} + q^{-18}t^{18} + q^{-22}t^{22}) - q^{-31}t^{29}\Big)}{[1,1][31,30]}.$$

Again $x^5+y^3+z^2$ is a quasihomogeneous polynomial of weight 30 and exponents 6, 10 and 15.

3.4. **Calculation for A_r.** Again as $x^{r+1} - yz = 0$, $r \in \mathbb{N}$ has q^2 solutions in $\mathbb{F}_q^3$, we obtain

$$Z(s) = \int_{\mathcal{O}_K^3} |x^{r+1} - yz|_K^s \, |dx\,dy\,dz| = [1] + \frac{q^{-1}[1][2]}{[1,1]}t$$
$$+ q^{-3}t^2 \int_{\mathcal{O}_K^3} |\pi^{r-1}x^{r+1} - yz|_K^s \, |dx\,dy\,dz|.$$

Let

$$I_m(s) = \int_{\mathcal{O}_K^3} |\pi^m x^{r+1} - yz|_K^s \, |dx\,dy\,dz|, \quad m \in \mathbb{N},$$

then the application of SPF produces

$$I_{2l}(s) = [1]^2\Big(1 + 2\frac{q^{-1}t}{[1,1]}\Big)\sum_{i=0}^{l-1}(q^{-1}t)^{2i} + (q^{-1}t)^{2l}Z(s),$$

$$I_{2l+1}(s) = [1]^2\Big(1 + 2\frac{q^{-1}t}{[1,1]}\Big)\sum_{i=0}^{l-1}(q^{-1}t)^{2i} + (q^{-1}t)^{2l}I_1(s),$$

and two applications of SPF produce

$$I_1 = [1]^2\Big(1 + 2\frac{q^{-1}t}{[1,1]}\Big) + q^{-2}t[1] + q^{-3}t^2 I_r(s),$$

we therefore obtain

Theorem 6. Let K be a non-archimedean local field such that $char(\mathcal{O}_K/\pi\mathcal{O}_K) \nmid r+1$, $r \in \mathbb{N}$ and let $Z(s) = \int_{\mathcal{O}_K^3} |x^{r+1} - yz|_K^s\, |dx\, dy\, dz|$, then we have

i) if $r = 2l+1$,

$$Z(s) = \frac{[1]\Big([3,1] + q^{-3}t^2[1][1,1]_+ \sum_{i=0}^{l-1}(q^{-1}t)^{2i}\Big)}{[1,1][2l+3,2l+2]};$$

ii) if $r = 2l$,

$$Z(s) = \frac{[1]\Big(q^{-1}t[2] + [1,1][2l+3,2l+1]_+ + q^{-3}t^2[1][1,1]_+[2l+1,2l]_+ \sum_{i=0}^{l-1}(q^{-1}t)^{2i}\Big)}{[1,1][4l+4,4l+2]}.$$

Again $x^{2l+2} - yz$ is a quasihomogeneous polynomial of weight $2l+2$ and exponents 1, 2 and $2l$ and $x^{2l+1} - yz$ is a quasihomogeneous polynomial of weight $4l+2$ and exponents 2, 2 and $4l$.

3.5. **Calculation for** D_r**.** Let $\nu = q^{-3}(q^3 - N)$ and $\sigma = q^{-3}(N-1)$, where N is the number of zeros of $x^{r-1} - xy^2 + z^2$, $r \in \mathbb{N} \setminus \{0\}$ in $\mathbb{F}_q^3$, then the first application of SPF gives

$$Z(s) = \int_{\mathcal{O}_K^3} |x^{r-1} - xy^2 + z^2|_K^s\, |dx\, dy\, dz| = \nu + \sigma\frac{[1]}{[1,1]}t + q^{-3}t^2\int_{\mathcal{O}_K^3} |\pi^{r-3}x^{r-1} - \pi xy^2 + z^2|_K^s\, |dx\, dy\, dz|.$$

Let

$$I_m(s) = \int_{\mathcal{O}_K^3} |\pi^m x^{r-1} - \pi xy^2 + z^2|_K^s\, |dx\, dy\, dz|, \quad m \in \mathbb{N}$$

then two applications of SPF produce the following

$$I_m(s) = [1] + q^{-1}[1]^2 t + \frac{q^{-2}[1]^2}{[1,1]} t^2 + q^{-2}t^2 I_{m-2}(s),$$

and then by an easy induction, we have

$$I_{2m}(s) = \Big([1] + q^{-1}[1]^2 t + \frac{q^{-2}[1]^2}{[1,1]} t^2\Big) \sum_{i=0}^{m-1} (q^{-1}t)^{2i} + (q^{-1}t)^{2m} I_0(s),$$

$$I_{2m+1}(s) = \Big([1] + q^{-1}[1]^2 t + \frac{q^{-2}[1]^2}{[1,1]} t^2\Big) \sum_{i=0}^{m-1} (q^{-1}t)^{2i} + (q^{-1}t)^{2m} I_1(s),$$

and then by Theorem 3, if $r = 2l+1$ we have

$$I_0(s) = \begin{cases} [1]^2 + 2\frac{q^{-1}[1]^2}{[1,1]} t + q^{-2}t^2 K_{r-3}(s) & \text{if } q \equiv 1 \mod 4, \\ [2] + q^{-2}t^2 K_{r-3}(s) & \text{if } q \equiv 3 \mod 4, \end{cases}$$

on the other hand, if $r = 2l$, three applications of SPF produce

$$I_1(s) = [1]\Big(1 + q^{-1}t + q^{-3}t^2 + 3\frac{q^{-1}[1]}{[1,1]} t\Big) + q^{-4}t^4 K_{r-4}(s),$$

where

$$K_m(s) = \int_{\mathcal{O}_K^3} |\pi^m x^{r-1} - xy^2 + z^2|_K^s \, |dx\, dy\, dz|, \quad m \in \mathbb{N}$$

then one application of SPF produces

$$K_m(s) = [1] + q^{-1}\frac{[1]^2}{[1,1]} t + q^{-2}t^2 K_{m-2}(s),$$

therefore

$$K_{2m}(s) = \Big([1] + q^{-1}\frac{[1]^2}{[1,1]} t\Big) \sum_{i=0}^{m-1} (q^{-1}t)^{2i} + (q^{-1}t)^{2m} Z(s),$$

we therefore obtain

Theorem 7. Let K be a non-archimedean local field such that $char(\mathcal{O}_K/\pi\mathcal{O}_K) \nmid r-1, 2$, $r \in \mathbb{N} \setminus \{0\}$ and let $Z(s) = \int_{\mathcal{O}_K^3} |x^{r-1} - xy^2 + z^2|_K^s \, |dx\, dy\, dz|$, then we have

i) if $r = 2l + 1$ and $q \equiv 1 \mod 4$,

$$Z(s) = \frac{\nu[1,1] + \sigma[1]t + [1]^2[1,1]_+ q^{-2l-1}t^{2l}}{[1,1][4l+1,4l]} + \frac{q^{-3}t^2[1]\left(1 - q^{-2}t + q^{-2l}t^{2l} - q^{-2l-2}t^{2l+1}\right)\sum_{i=0}^{l-2}(q^{-1}t)^{2i}}{[1,1][4l+1,4l]};$$

ii) if $r = 2l + 1$ and $q \equiv 3 \mod 4$,

$$Z(s) = \frac{\nu[1,1] + \sigma[1]t + q^{-2l-1}t^{2l}[2]}{[1,1][4l+1,4l]} + \frac{q^{-3}t^2[1]\left(1 - q^{-2}t + q^{-2l}t^{2l} - q^{-2l-2}t^{2l+1}\right)\sum_{i=0}^{l-2}(q^{-1}t)^{2i}}{[1,1][4l+1,4l]};$$

iii) if $r = 2l$,

$$Z(s) = \frac{\nu[1,1] + [1]\left(\sigma t + q^{-2l+1}t^{2l-2} + 2q^{-2l-1}t^{2l}[1] - q^{-2l-3}t^{2l+1}\right)}{[1,1][4l-1,4l-2]} + \frac{q^{-3}t^2[1]\left(1 - q^{-2}t + q^{-2l}t^{2l} - q^{-2l-2}t^{2l+1}\right)\sum_{i=0}^{l-3}(q^{-1}t)^{2i}}{[1,1][4l-1,4l-2]}.$$

Again $x^{2l} - xy^2 + z^2$ is a quasihomogeneous polynomial of weight $4l$ and exponents 2, $2l-1$ and $2l$ and $x^{2l-1} - xy^2 + z^2$ is a quasihomogeneous polynomial of weight $4l-2$ and exponents 2, $2l-2$ and $2l-1$.

Acknowledgments I would like to thank professor V. Albis for suggesting this topic for my undergraduate thesis and for his support and assistance while completing it. I would also like to thank professor W. Zúñiga for replying my e-mails on this topic.

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Recibido: Agosto de 2006. Aceptado para publicación: Noviembre de 2006