NUMERICAL QUENCHING SOLUTIONS OF LOCALIZED SEMILINEAR PARABOLIC EQUATION

DIABATE NABONGO (*)
THÉODORE K. BONI (**)
Resumen. En este artículo se estudia la aproximación numérica para el siguiente problema de valor de frontera inicial:
\[
\begin{align*}
u_t(x,t) &= u_{xx}(x,t) + \varepsilon(1 - u(0,t))^{-p}, \quad (x,t) \in (-l,l) \times (0,T), \\
u(-l,t) &= 0, \quad u(l,t) = 0, \quad t \in (0,T), \\
u(x,0) &= u_0(x) \geq 0, \quad x \in (-l,l),
\end{align*}
\]
donde \( p > 1, \ l = \frac{1}{2} \) y \( \varepsilon > 0 \). Bajo algunas hipótesis, probamos que la solución de una forma semidiscreta del problema anterior se satisface en un tiempo finito y estimamos su tiempo semidiscreto de enfriamiento. También mostramos que el tiempo de enfriamiento converge en ciertos casos a un real cuando la malla tiende a cero. Finalmente, presentamos algunos experimentos numéricos para ilustrar nuestro análisis.

Palabras claves. Semidiscretizaciones, ecuación parabólica semilineal localizada, tiempo semidiscreto de enfriamiento, convergencia.

1. INTRODUCTION

We consider the following initial-boundary value problem:
\[
\begin{align*}
u_t(x,t) &= u_{xx}(x,t) + \varepsilon(1 - u(0,t))^{-p}, \quad (x,t) \in (-l,l) \times (0,T), \\
u(-l,t) &= 0, \quad u(l,t) = 0, \quad t \in (0,T), \\
u(x,0) &= u_0(x) \geq 0, \quad x \in (-l,l),
\end{align*}
\]
where \( p > 1, \ l = \frac{1}{2} \), \( \varepsilon > 0 \), \( u_0(x) \) is a function which is bounded and symmetric. In addition, \( u_0(x) \) is nondecreasing on the interval \( (-l,0) \) and \( u''_0(x) + \varepsilon(1 - u(0,t))^{-p} \geq 0 \) on \( (-l,l) \). Here \( (0,T) \) is the maximal time interval on which \( \|u(x,t)\|_{\infty} < 1 \), where \( \|u(x,t)\|_{\infty} = \max_{0 \leq x \leq 1} |u(x,t)| \). The time \( T \) may be finite or infinite. When \( T \) is infinite, we say that the solution \( u \) exists globally. When \( T \) is finite, then we have
\[
\lim_{t \to T} \|u(x,t)\|_{\infty} = 1.
\]
In this case, we say that the solution \( u \) quenches in a finite time and the time \( T \) is called the quenching time of the solution \( u \).

The above problem is related to a popular model arising in the study of polarization phenomenon. It also represents a model which is related via transformation to a certain class of physical problem of ignition where the reaction is driven by the temperature at a single site. This kind of phenomena is observed in biological systems and in chemical reaction diffusion processes in which the reaction takes place only at some local sites. For more physical motivation see for instance [4] and [7].
In this paper, we are interested in the numerical study of the above problem. Let $I$ be a positive integer, and consider the grid $x_i = ih$, $0 \leq i \leq I$, where $h = 2I/I$. We approximate the solution $u$ of (1)–(3) by the solution $U_h(t) = (U_0(t), U_1(t), \ldots, U_I(t))^T$ of the following semidiscrete equations

$$
\frac{d}{dt} U_i(t) = \delta^2 U_i(t) + \varepsilon (1 - U_k(t))^{-p}, \quad 1 \leq i \leq I - 1, \quad t \in (0, T_h^q),
$$

(4)$$
U_0(t) = 0, \quad U_I(t) = 0, \quad t \in (0, T_h^q),
$$

(5)$$
U_i(0) = \varphi_i \geq 0, \quad 0 \leq i \leq I,
$$

(6)$$
\delta^2 U_i(t) = \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{h^2}, \quad 1 \leq i \leq I - 1,
$$

where $k$ is the integer part of the number $I/2$,

$$
\varphi_0 = 0, \quad \varphi_I = 0, \quad \varphi_i = \varphi_{I-i}, \quad 0 \leq i \leq I, \quad \delta^+ \varphi_i > 0, \quad 0 \leq i \leq k - 1,
$$

$$
\delta^+ \varphi_i = \frac{\varphi_{i+1} - \varphi_i}{h}.
$$

Here $(0, T_h^q)$ is the maximal time interval on which $\|U_h(t)\|_\infty < 1$ with $\|U_h(t)\|_\infty = \max_{0 \leq i \leq I} |U_i(t)|$. When the time $T_h^q$ is finite, we say that $U_h(t)$ quenches in a finite time and the time $T_h^q$ is called the quenching time of the solution $U_h(t)$.

The theoretical study of quenching solutions for semilinear parabolic equations has been the subject of investigations of many authors (see [3], [6], [7], [8] and the references cited therein). In particular in [6] and [7], the author has proved that under some assumptions, the solution of (1)–(3) quenches in a finite time and the quenching time is estimated. Here we are interested in the numerical study using the semidiscrete form defined in (4)–(6). We give some assumptions under which the solution of (4)–(6) quenches in a finite time and estimate its semidiscrete quenching time. We also show that the semidiscrete quenching time converges to the theoretical one when the mesh size goes to zero. A similar study has been undertaken in [1] and [6] for the phenomenon of blow-up where the authors have considered the problem (1)–(3) in the case where the reaction term $\varepsilon (1 - u(x, t))^{-p}$ is replaced by $(u(x, t))^q$ with $q > 1$ (we say that a solution blows up in a finite time if it attains the value infinity in a finite time). In the same way in [2] the numerical extinction has been studied using some discrete and semidiscrete schemes (we say that a solution $u$ extincts in a finite time if it reaches the value zero in a finite time).

Our paper is written in the following manner. In the next section, we prove some results about the discrete maximum principle. In the third section, under some hypotheses, we show that the solution of the semidiscrete problem quenches in a finite time and estimate its semidiscrete quenching time. In the fourth section, we give a result about the convergence of semidiscrete quenching...
times in some cases where the quenching occurs. Finally, in the last section, we give some numerical results to illustrate our analysis.

2. Properties of the semidiscrete scheme

In this section, we give some lemmas which will be used later. The following lemma is a semidiscrete version of the maximum principle.

**Lemma 2.1.** Let $a_h(t) \in C^0([0, T], \mathbb{R}^{I+1})$ and let $V_h(t) \in C^1([0, T], \mathbb{R}^{I+1})$ such that

\[
\frac{d}{dt} V_i(t) - \delta^2 V_i(t) + a_k(t)V_k(t) \geq 0, \quad 1 \leq i \leq I - 1, \quad t \in (0, T),
\]

\[
V_0(t) \geq 0, \quad V_I(t) \geq 0, \quad t \in (0, T),
\]

\[
V_i(0) \geq 0, \quad 0 \leq i \leq I.
\]

Then we have $V_i(t) \geq 0, \quad 0 \leq i \leq I, \quad t \in (0, T)$.

**Proof.** Let $T_0 < T$ and let $m = \min_{0 \leq i \leq I, 0 \leq t \leq T_0} V_i(t)$. Since for $i \in \{0, ..., I\}$, $V_i(t)$ is a continuous function, there exists $t_0 \in [0, T_0]$ such that $m = V_{i_0}(t_0)$ for a certain $i_0 \in \{0, ..., I\}$. Assume that $m < 0$. If $i_0 = 0$ or $i_0 = I$, we have a contradiction because of (8). For $i_0 \in \{1, ..., I - 1\}$, it is not hard to see that

\[
\frac{dV_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{V_{i_0}(t_0) - V_{i_0}(t_0 - k)}{k} \leq 0,
\]

\[
\delta^2 V_{i_0}(t_0) = \frac{V_{i_0+1}(t_0) - 2V_{i_0}(t_0) + V_{i_0-1}(t_0)}{h^2} \geq 0.
\]

Define the vector $Z_h(t) = e^{\lambda t} V_h(t)$ where $\lambda$ is large enough that $a_k(t_0)V_k(t_0) - \lambda m < 0$. Use (10) and (11) to obtain $\frac{dZ_{i_0}(t_0)}{dt} \leq 0$ and $\delta^2 Z_{i_0}(t_0) \geq 0$, which implies that

\[
\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + e^{\lambda t_0}(a_k(t_0)V_k(t_0) - \lambda m) < 0.
\]

On the other hand, from (7), we derive the following inequality

\[
\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + e^{\lambda t_0}(a_k(t_0)V_k(t_0) - \lambda m) \geq 0.
\]

Therefore, we have a contradiction because of (12).

Another form of the maximum principle for semidiscrete equations is the comparison lemma below.
Lemma 2.2. Let \( V_h(t), U_h(t) \in C^1([0, T], \mathbb{R}^{I+1}) \) and \( f \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R}) \) such that for \( t \in (0, T) \)
\[
\frac{dV_i(t)}{dt} + \delta^2 V_i(t) + f(V_i(t), t) \geq \frac{dU_i(t)}{dt} - \delta^2 U_i(t) + f(U_i(t), t), \quad 1 \leq i \leq I - 1,
\]
(14) \( V_0(t) \geq U_0(t), \quad V_I(t) \geq U_I(t), \)
(15) \( V_i(0) \geq U_i(0), \quad 0 \leq i \leq I. \)
Then we have \( V_i(t) \geq U_i(t), \quad 0 \leq i \leq I, \quad t \in (0, T). \)

Proof. Introduce the vector \( Z_h(t) = V_h(t) - U_h(t). \) A direct calculation yields
\[
\frac{dZ_i(t)}{dt} - \delta^2 Z_i(t) + f_\varphi(\theta_k(t), t) Z_k(t) \geq 0,
\]
\[
Z_0(t) \geq 0, \quad Z_I(t) \geq 0,
\]
\[
Z_i(0) \geq 0,
\]
where \( \theta_k \) is an intermediate value between \( U_k \) and \( V_k \) and \( f_\varphi \) is the partial derivative of \( f \) with respect to the second variable. Since \( f \in C^1 \) then \( f_\varphi(\theta_k(t), t) \) is bounded on \((0, T)\). Use Lemma 2.1 to complete the rest of the proof.

The lemma below shows that when \( i \) is between 1 and \( I - 1 \), then \( U_i(t) \) is positive where \( U_h(t) \) is the solution of the semidiscrete problem.

Lemma 2.3. Let \( U_h \) be the solution of (4)–(6). Then we have
\[
U_i(t) > 0, \quad 1 \leq i \leq I - 1.
\]

Proof. Let \( \alpha = \min_{1 \leq i \leq I - 1} \varphi_i \) and introduce the vector \( V_h \) defined by \( V_i = \alpha e^{-\lambda_h t} \sin(i\pi h), \) \( 0 \leq i \leq I, \) where \( \lambda_h = \frac{2-\cos(k\pi)}{h^2}. \) It is not hard to see that
\[
\frac{dU_i(t)}{dt} - \delta^2 U_i(t) \geq \frac{dV_i(t)}{dt} - \delta^2 V_i(t) = 0,
\]
\[
U_0(t) = V_0(t), \quad U_I(t) = V_I(t) = 0,
\]
\[
U_i(0) \geq V_i(0), \quad 1 \leq i \leq I - 1.
\]
We deduce from Lemma 2.2 that \( U_i(t) \geq \alpha e^{-\lambda_h t} \sin(i\pi h), \) \( 0 \leq i \leq I. \) This implies that \( U_i(t) > 0, \) \( 1 \leq i \leq I - 1, \) and the proof is complete.

The following lemma reveals that the solution \( U_h(t) \) of the semidiscrete problem is symmetric and \( \delta^+ U_i(t) \) is positive when \( i \) is between 1 and \( k - 1. \)

Lemma 2.4. Let \( U_h \) be the solution of (4)–(6). Then we have for \( t \in (0, T^h_q) \)
\[
U_{I-1}(t) = U_i(t), \quad 0 \leq i \leq I, \quad \delta^+ U_i(t) > 0, \quad 0 \leq i \leq k - 1.
\]
Proof. Introduce the vector $V_i$ defined as follows $V_i(t) = U_{i-1}(t)$ for $0 \leq i \leq I$. It is not hard to see that $V_i(t)$ is a solution of (4)–(6). It follows from Lemma 2.1 that $V_i(t) = U_i(t)$. Now, define the vector $Z_i(t)$ such that

$$Z_i(t) = U_{i+1}(t) - U_i(t), \quad 0 \leq i \leq k - 1,$$

and let $t_0$ be the first $t > 0$ such that $Z_i(t) > 0$ for $t \in [0,t_0)$ but $Z_i(t_0) = 0$. Without loss of the generality, we assume that $i_0$ is the smallest integer such that $Z_{i_0}(t_0) = 0$. If $i_0 = 0$ then we have $U_1(t_0) = U_0(t_0) = 0$, which is a contradiction because from Lemma 2.3. $U_1(t_0) > 0$. It is easy to see that

$$dZ_{i_0}(t_0) = 0, \quad \text{if} \quad 1 \leq i_0 \leq k - 1. \quad (17)$$

On the other hand, we observe that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \geq 0, \quad 1 \leq i_0 \leq k - 2,$$

and we know that if $i_0 = k - 1$,

$$\delta^2 Z_{k-1}(t_0) = \frac{U_{k+1}(t_0) - 2U_k(t_0) + U_{k-1}(t_0) - U_k(t_0) + 2U_{k-1}(t_0) - U_{k-2}(t_0)}{h^2}.$$

Since $k$ is the integer part of the number $I/2$, using the fact that the discrete solution is symmetric, we have either $U_{k+1}(t) = U_{k-1}(t)$ or $U_{k+1}(t) = U_k(t)$. In both cases, we find that

$$\delta^2 Z_{k}(t_0) = \frac{Z_{k-2}(t_0)}{h^2} > 0.$$

The above inequalities imply that $\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) < 0$, which is a contradiction because of (17) and the proof is complete.

A discrete version of the Green’s formula is the following

Lemma 2.5. Let $U_h(t)$ and $V_h(t)$ two vectors such that $U_0(t) = 0$, $U_I(t) = 0$, $V_0(t) = 0$, $V_I(t) = 0$. Then we have

$$\sum_{i=1}^{I-1} hU_i \delta^2 V_i + \sum_{i=1}^{I-1} hV_i \delta^2 U_i. \quad (18)$$

Proof. A routine calculation yields

$$\sum_{i=1}^{I-1} hU_i \delta^2 V_i = \sum_{i=1}^{I-1} hV_i \delta^2 U_i + \frac{V_I U_{I-1} - U_I V_{I-1} + V_0 U_1 - U_0 V_I}{h}$$

and the result follows using the assumptions of the lemma.

Now, let us state a result on the operator $\delta^2$. 

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Lemma 2.6. Let $U_h \in C^0([0,T], R^{I+1})$ such that $U_h \geq 0$. Then we have
\[
\delta^2(1 - U_i)^p \geq p(1 - U_i)^{p-1}\delta^2U_i(t) \quad \text{for} \quad 1 \leq i \leq I - 1.
\]

Proof. Using Taylor’s expansion, we get
\[
\delta^2(1 - U_i)^p = p(1 - U_i)^{p-1}\delta^2U_i(t) + (U_{i+1} - U_i)^2 \frac{p(p + 1)}{2h^2} \eta_i^{p-2}
\]
\[
+ (U_{i-1} - U_i)^2 \frac{p(p + 1)}{2h^2} \eta_i^{p-2} \quad \text{if} \quad 1 \leq i \leq I - 1,
\]
where $\eta_i$ is an intermediate value between $U_i$ and $U_{i+1}$ and $\eta_i$ the one between $U_i$ and $U_{i-1}$. The result follows taking into account the fact that $U_h$ is non-negative.

To end this section, let us give another property of the operator $\delta^2$.

Lemma 2.7. Let $V_h$ and $U_h \in C^1([0,T], R^{I+1})$. If
\[
\delta^+(U_i)\delta^+(V_i) \geq 0, \quad \text{and} \quad \delta^-(U_i)\delta^-(V_i) \geq 0,
\]
then
\[
\delta^2(U_i, V_i) \geq \frac{U_{i+1} - U_i}{h} \delta^2(V_i) + V_i \delta^2(U_i),
\]
where $\delta^+(U_i) = \frac{U_{i+1} - U_i}{h}$ and $\delta^-(U_i) = \frac{U_{i-1} - U_i}{h}$.

Proof. A straightforward computation yields
\[
h^2\delta^2(U_i, V_i) = U_{i+1}V_{i+1} - 2U_iV_i + U_{i-1}V_{i-1}
\]
\[
= (U_{i+1} - U_i)(V_{i+1} - V_i) + V_i(U_{i+1} - U_i) + U_i(V_{i+1} - V_i) + U_iV_i - 2U_iV_i
\]
\[
+ (U_{i-1} - U_i)(V_{i-1} - V_i) + (U_{i-1} - U_i)V_i + U_i(V_{i-1} - V_i) + U_iV_i,
\]
which implies that
\[
\delta^2(U_i, V_i) = \delta^+(U_i)\delta^+(V_i) + \delta^-(U_i)\delta^-(V_i) + U_i\delta^2(V_i) + V_i\delta^2(U_i).
\]
Using (19), we obtain the desired result.

3. Quenching solutions

In this section, under some assumptions, we show that the solution of the semidiscrete problem quenches in a finite time and estimate its semidiscrete quenching time.

The statement of our first result on quenching is the following

Theorem 3.1. Suppose that there exists a constant $A > 0$ and such that the initial data at (4) satisfies
\[
\delta^2\varphi_i + \varepsilon(1 - \varphi_i)^{-p} \geq A \sin(i\pi)(1 - \varphi_i)^{-p}, \quad 0 \leq i \leq I,
\]
compute reveals that
\begin{equation}
\delta^n \leq \frac{2\pi^2}{A(p+1)}(1-\|\varphi_h\|_\infty)^{p+1} > 0.
\end{equation}

Then the solution $U_h(t)$ of (4)–(6) quenches in a finite time $T^h_q$ which estimated as follows
\begin{equation*}
T^h_q \leq -\frac{1}{\pi^2} \ln(1-\frac{2\pi^2}{A(p+1)}(1-\|\varphi_h\|_\infty)^{p+1}).
\end{equation*}

**Proof.** Since $(0,T^h_q)$ is the maximal time interval on which $\|U_h(t)\|_\infty < 1$, our aim is to show that $T^h_q$ is finite and satisfies the above inequality. Introduce the function $J_h(t)$ defined as follows
\begin{equation*}
J_i(t) = \frac{d}{dt}U_i(t) - C_i(t)(1-U_i(t))^{-p}, \quad 0 \leq i \leq I,
\end{equation*}
where $C_i(t) = Ae^{-\lambda t^i}\sin(i\pi t_i)$, with $\lambda_h = \frac{2-2\cos(i\pi t_i)}{\pi^2}$. A straightforward computation reveals that
\begin{equation*}
\frac{dJ_i}{dt} - \delta^2 J_i = \frac{d}{dt}(\frac{dU_i}{dt} - \delta^2 U_i) - (1-U_i)^{-p}\frac{dC_i}{dt} - pC_i(1-U_i)^{-p-1}\frac{dU_i}{dt} + \delta^2(C_i(1-U_i)^{-p}).
\end{equation*}

From Lemma 2.6 and 2.7, the last term on the right hand side of the equality $\delta^2(C_i(1-U_i)^{-p})$ is bounded from below by $(1-U_i)^{-p}\delta^2 C_i + p(1-U_i)^{-p-1}C_i\delta^2 U_i$ due to the fact $\delta^+(C_i)\delta^+(1-U_i)^{-p}$ and $\delta^-(C_i)\delta^-(1-U_i)^{-p}$ are nonnegative because the results of Lemma 2.4 hold for $U_h(t)$ and $C_i(t)$. We deduce that
\begin{equation*}
\frac{dJ_i(t)}{dt} - \delta^2 J_i(t) \geq \frac{d}{dt}(\frac{dU_i(t)}{dt} - \delta^2 U_i(t)) - (1-U_i)^{-p}(\frac{dC_i(t)}{dt} - \delta^2 C_i(t))
- pC_i(1-U_i)^{-p-1}(\frac{dU_i(t)}{dt} - \delta^2 U_i(t)).
\end{equation*}

Using (4) and the fact that $\frac{d}{dt}C_i(t) - \delta^2 C_i(t) = 0$, we find that
\begin{equation*}
\frac{dJ_i}{dt} - \delta^2 J_i \geq \varepsilon p(1-U_k)^{-p-1}\frac{dU_k}{dt} - \varepsilon p(1-U_k)^{-p-1}C_i(1-U_k)^{-p-1}
\geq \varepsilon p(1-U_k)^{-p-1}(J_k + C_k(1-U_k)^{-p}) - \varepsilon p(1-U_k)^{-p-1}C_i(1-U_k)^{-p-1}
\geq \varepsilon p(1-U_k)^{-p-1}J_k + \varepsilon p(1-U_k)^{-p}(C_k(1-U_k)^{-p-1} - C_i(1-U_k)^{-p-1}).
\end{equation*}

From Lemma 2.4, $U_k \geq U_i$. We also observe that $C_k \geq C_i$. We deduce that
\begin{equation*}
\frac{dJ_i}{dt} - \delta^2 J_i \geq \varepsilon p(1-U_k)^{-p-1}J_k, \quad 1 \leq i \leq I - 1.
\end{equation*}

It is not hard to see that $J_0(t) = 0$, $J_1(t) = 0$ and the relation (20) implies that $J_h(0) \geq 0$. It follows from Lemma 2.1 that $J_h(t)$ is nonnegative, which implies
that
\[ \frac{dU_i}{dt} \geq C_i (1 - U_i)^{-p}, \quad 0 \leq i \leq I. \]

Using Taylor’s expansion, we find that \( \cos(h\pi) \geq 1 - \pi^2 h^2 \), which implies that \( \lambda_h \leq \pi^2 \). Obviously \( \sin(kh\pi) \geq \frac{1}{2} \). We deduce that
\[ \frac{dU_k}{dt} \geq A \frac{e^{-\pi^2 t}(1 - U_k)^{-p}}{2}, \quad 0 \leq i \leq I. \]

This inequality may be rewritten as follows

(22) \[ (1 - U_k)^p dU_k \geq A \frac{2 e^{-\pi^2 t}}{(p + 1)}, \quad 0 \leq i \leq I. \]

Integrating this inequality over \( (0, T_q) \), we arrive at
\[ A(1 - e^{-\pi^2 T_q}) \leq \frac{(1 - U_k(0))^{p+1}}{p + 1}, \]
which implies that
\[ e^{-\pi^2 T_q} \geq 1 - \frac{2\pi^2}{A(p + 1)} (1 - U_k(0))^{p+1}. \]

Since \( \|U_h(0)\|_\infty = U_k(0) \), the restriction on the initial data in (21) implies that the term on the right hand side of the above inequality is positive. Therefore we find that
\[ T_q \leq -\frac{1}{\pi^2} \ln(1 - \frac{2\pi^2}{A(p + 1)} (1 - \|\varphi_h\|_\infty)^{p+1}). \]

**Remark 3.1.** Integrating the inequality in (22) over \( (t_0, T_q) \), we get
\[ T_q - t_0 \leq -\frac{1}{\pi^2} \ln(1 - \frac{2\pi^2}{A(p + 1)} e^{\pi^2 t_0} (1 - \|U_h(t_0)\|_\infty)^{p+1}). \]

The proof of the above theorem allows us to establish the estimation in Remark 3.1 which is crucial to prove the convergence of the semidiscrete quenching time. When the initial data is null, the hypotheses of Theorem 3.1 are satisfied if the parameter \( \varepsilon \) is large enough. The theorem below also shows that \( \varepsilon \) is large enough, then the semidiscrete solution quenches in a finite time. In addition, in this case the restriction on \( \varepsilon \) is better than the one of Theorem 3.1.

**Theorem 3.2.** Let \( v(0) = \sum_{i=1}^{I} \tan(\frac{\pi}{2} h) \sin(i \pi h) \varphi_i \) and \( \lambda_h = \frac{2 - 2 \cos(\pi h)}{\pi^2} \). Assume that \( \varepsilon > \lambda_h \frac{p^p}{(p + 1)^p} \). Then the solution \( U_h(t) \) of (4)–(6) quenches in a finite time \( T_q \) which is estimated as follows
\[ T_q \leq \frac{(p + 1)^{p+1}(1 - v(0))^{p+1}}{\varepsilon(p + 1) p^p - \lambda_h p^p}. \]
Proof. Since \((0, T_q^h)\) is the maximal time interval on which \(\|U_h(t)\|_\infty < 1\), our aim is to show that \(T_q^h\) is finite and satisfies the above inequality. Introduce the function \(v(t)\) defined as follows

\[ v(t) = \sum_{i=1}^{l-1} \tan\left(\frac{\pi}{2} h\right) \sin(i\pi h) U_i(t). \]

Take the derivative of \(v\) with respect to \(t\) and use (4) to obtain

\[ v'(t) = \sum_{i=1}^{l-1} \tan\left(\frac{\pi}{2} h\right) \sin(i\pi h) (\delta^2 U_i(t) + \varepsilon(1 - U_k(t))^{-p}). \]

We observe that \(\delta^2 \sin(i\pi h) = -\lambda h \sin(i\pi h)\). From the above equality and Lemma 2.5, we arrive at

\[ v'(t) = -\lambda_h v(t) + \varepsilon (1 - v(t))^{-p} \sum_{i=1}^{l-1} \tan\left(\frac{\pi}{2} h\right) \sin(i\pi h). \]

By a routine calculation, we find that \(\sum_{i=1}^{l-1} \tan\left(\frac{\pi}{2} h\right) \sin(i\pi h)\) equals one. Due to the fact that \(U_k\) is bigger than \(v(t)\), we get

\[ v'(t) \geq \varepsilon (1 - v(t))^{-p} \left(1 - \frac{\lambda_h v(t)}{\varepsilon} (1 - v(t))^{-p}\right). \]

It is not hard to see that \(v(t)(1 - v(t))^{-p}\) is bounded from above by \(\sup_{0 \leq s \leq 1} s(1 - s)^p = \frac{p^p}{(p+1)^{p+1}}\). We deduce that

\[ v'(t) \geq \varepsilon (1 - \frac{\lambda_h p^p}{(p+1)^{p+1}})(1 - v(t))^{-p}, \]

which implies that

\[ (1 - v(t))^{-p} dv \geq \varepsilon (1 - \frac{\lambda_h p^p}{(p+1)^{p+1}}) dt. \]

Integrating this inequality over \((0, T_q^h)\), we find \(T_q^h \leq \frac{(p+1)^{p+1}(1 - v(0))^{p+1}}{\varepsilon(p+1)^{p+1} - \lambda_h p^p}\). We conclude that \(T_q^h\) is finite and the proof is complete.

Remark 3.2. Since \(U_k(t) = \|U_h(t)\|_\infty\), it is easy to see that

\[ \delta^2 U_k(t) = \frac{U_{k+1}(t) - 2U_k(t) + U_{k-1}(t)}{h^2} \leq 0. \]

Therefore, using (4), we get \(\frac{dU_k}{dt} \geq \varepsilon (1 - U_k)^{-p}\) which implies that \((1 - U_k)^p dU_k \geq \varepsilon dt\). Integrating this inequality over \((0, T_q^h)\), we arrive at \(T_q^h \geq \frac{(1 - \|U_h(0)\|_\infty)^{p+1}}{(p+1)^{p+1}}\). Thus we have a lower bound of the semidiscrete quenching time.
Remark 3.3. Consider the following semidiscrete scheme
\[ \frac{d}{dt} V_i(t) = \delta^2 V_i(t) + \varepsilon(1 - V_i(t))^{-p}, \quad 1 \leq i \leq I - 1, \quad t \in (0, T_h), \]
\[ V_0(t) = 0, \quad V_I(t) = 0, \quad t \in (0, T_h), \]
\[ V_i(0) = \varphi_i, \quad 0 \leq i \leq I, \]
where \((0, T_h)\) is the maximal time interval on which \(\| V_h(t) \|_\infty < 1\). We observe that the above scheme is a semidiscretization of the continuous problem below
\[ u_t(x,t) = u_{xx}(x,t) + \varepsilon(1 - u(x,t))^{-p}, \quad (x,t) \in (-l,l) \times (0,T) \]
\[ u(-l,t) = 0, \quad u(l,t) = 0, \quad t \in (0,T), \]
\[ u(x,0) = u_0(x) \geq 0, \quad x \in (-l,l). \]
Let \(U_h(t)\) be the solution of (4)–(6), we know from Lemma 2.4 that \(U_k \geq U_i\) for \(1 \leq i \leq I - 1\). We deduce that
\[ \frac{d}{dt} U_i(t) \geq \delta^2 U_i(t) + \varepsilon(1 - U_i(t))^{-p}, \quad 1 \leq i \leq I - 1, \quad t \in (0, T_h^q), \]
\[ U_0(t) = 0, \quad U_I(t) = 0, \quad t \in (0, T_h^q), \]
\[ U_i(0) = \varphi_i, \quad 0 \leq i \leq I. \]
Setting \(Z_h(t) = U_h(t) - V_h(t)\), it is not hard to see that
\[ \frac{d}{dt} Z_i - \delta^2 Z_i - \varepsilon p(1 - \xi_i)^{-p-1} Z_i \geq 0, \quad 1 \leq i \leq I - 1, \quad t \in (0, T_h^*), \]
\[ Z_0(t) = 0, \quad Z_I(t) = 0, \quad t \in (0, T_h^*), \]
\[ Z_i(0) = 0, \quad 0 \leq i \leq I, \]
where \(T_h^* = \min\{T_h, T_h^q\}\). \(\xi_i\) is an intermediate value between \(U_i(t)\) and \(V_i(t)\). Modifying slightly the proof of Lemma 2.1, we find that \(Z_h(t) \geq 0\) for \(t \in (0, T_h^*)\). In other words, we have \(U_h(t) \geq V_h(t)\) for \(t \in (0, T_h^*)\) and we conclude that \(T_h^q \leq T_h\).
4. Convergence of semidiscrete quenching times

In this section, under some assumptions, we show that the semidiscrete quenching time for the solution of the semidiscrete problem converges to the real one when the mesh size goes to zero. In order to prove this result, firstly, we prove the convergence of the semidiscrete scheme by the following theorem on the convergence of the semidiscrete scheme which is crucial for the proof on the convergence of the semidiscrete quenching time.

We denote by \( u_h(t) = (u(x_0, t), ..., u(x_I, t))^T \).

**Theorem 4.1.** Assume that (1)-(3) has a solution \( u \in C^4([0, 1] \times [0, T]) \) such that \( \|u(x, t)\|_{\inf} = \alpha > 0 \). Suppose that the initial data at (6) satisfies
\[
\|\varphi_h - u_h(0)\|_{\infty} = o(1) \quad \text{as} \quad h \to 0.
\]

Then, for \( h \) sufficiently small, the problem (4)-(6) has a unique solution \( U_h \in C([0, T], \mathbb{R}^I) \) such that
\[
\max_{0 \leq t \leq T} \|U_h(t) - u_h(t)\|_{\infty} = O(\|\varphi_h - u_h(0)\|_{\infty} + h^2) \quad \text{as} \quad h \to 0.
\]

**Proof.** Since \( u \in C^4 \), there exist two positive constants \( K \) and \( M \) such that
\[
\frac{\|u_{xxxx}\|_{\infty}}{12} \leq K, \quad \|u\|_{\infty} \leq K, \quad \varepsilon p(1 - \frac{\alpha}{2})^{-p-1} \leq M.
\]

The problem (4)-(6) has for each \( h \), a unique solution \( U_h \in C([0, T_h^q], \mathbb{R}^I) \).

Let \( t(h) \) the greatest value of \( t > 0 \) such that
\[
\|U_h(t) - u_h(t)\|_{\infty} < \frac{\alpha}{2} \quad \text{for} \quad t \in (0, t(h)).
\]

Since the value of the term on the left hand side of the inequality is null when \( t \) is equal zero, we deduce that \( t(h) > 0 \) for \( h \) sufficiently small. Let \( t^*(h) = \min\{t(h), T\} \). By the triangle inequality, we obtain
\[
\|U_h(t)\|_{\infty} \geq \|u(x, t)\|_{\infty} - \|U_h(t) - u_h(t)\|_{\infty} \quad \text{for} \quad t \in (0, t^*(h)),
\]
which implies that
\[
\|U_h(t)\|_{\infty} \geq \alpha - \frac{\alpha}{2} = \frac{\alpha}{2} \quad \text{for} \quad t \in (0, t^*(h)).
\]

Let \( e_h(t) = U_h(t) - u_h(x, t) \) be the error of discretization. Using Taylor's expansion, we have for \( t \in (0, t^*(h)) \),
\[
\frac{d}{dt} e_i(t) - \delta^2 e_i(t) = \frac{\varepsilon}{12} e_{xxxx}(\tilde{x}, t) + \varepsilon p(1 - \xi_i)^{-p-1} e_i(t),
\]
where \( \xi_i \) is an intermediate value between \( U_i(t) \) and \( u(x_i, t) \). Using (24) and (26), we arrive at
\[
\frac{d}{dt} e_i(t) - \delta^2 e_i(t) \leq M|e_i(t)| + Kh^2, \quad 1 \leq i \leq I - 1.
\]
Let $z_h$ the vector defined by

$$z_i = e^{(M+1)t}||\varphi_h - u_h(0)||_{\infty} + Kh^2, \quad 0 \leq i \leq I.$$ 

A direct calculation yields

$$\frac{d}{dt}z_i - \delta^2 z_i > M|z_i(t)| + Kh^2, \quad 1 \leq i \leq I - 1, \quad t \in (0, t^*(h)),$$

$$z_0 > e_0, \quad z_I > e_I,$$

$$z_i(0) > e_i(0), \quad 0 \leq i \leq I.$$ 

It follows from Lemma 2.2 that $z_i > e_i(t)$ for $t \in (0, t^*(h))$, $0 \leq i \leq I$. By the same way, we also prove that $z_i > -e_i(t)$ for $t \in (0, t^*(h))$, $0 \leq i \leq I$, which implies that

$$\|U_h(t) - u_h(t)\|_{\infty} \leq e^{(M+1)t}\|\varphi_h - u_h(0)\|_{\infty} + Kh^2, \quad t \in (0, t^*(h)).$$

Let us show that $t^*(h) = T$. Suppose that $T > t(h)$. From (25), we obtain

$$\frac{\alpha}{2} = \|U_h(t(h)) - u_h(t(h))\|_{\infty} \leq e^{(M+1)t}(\|\varphi_h - u_h(0)\|_{\infty} + Kh^2).$$

Since the term in the right hand side of (28) goes to zero as $h$ tends to zero, we deduce that $\frac{\alpha}{2} \leq 0$, which is impossible. Consequently $t^*(h) = T$, and we obtain the desired result.

Now, we are in a position to prove our main theorem of this section.

**Theorem 4.2.** Suppose that the problem (1)–(3) has a solution $u$ which quenches in a finite time $T_q$ such that $u \in C^{4,1}([0,1] \times [0, T_q])$. Under the assumption of Theorem 4.1, the problem (4)–(6) has a solution $U_h(t)$ which quenches in a finite time $T^h_q$ and $\lim_{h \to 0} T^h_q = T_q$.

**Proof.** Letting $\varepsilon > 0$, there exists a positive constant $\rho$ such that

$$\frac{\varepsilon}{2} \leq e^{2\pi^2 T_q}(1 - x)^{p+1} \leq \frac{\varepsilon}{2} \quad \text{for} \quad x \in [1 - \rho, 1).$$

Since $\lim_{t \to T_q} \|u(x, t)\|_{\infty} = 1$, there exist $T_1 < T_q$ and $|T_q - T_1| < \frac{\varepsilon}{2}$ such that $1 > \|u(x, t)\|_{\infty} \geq 1 - \frac{\varepsilon}{2}$ for $t \in [T_1, T_q]$. From Theorem 4.1, the problem (4)–(6) has a solution $U_h(t)$ such that $\|U_h(t) - u_h(t)\|_{\infty} < \frac{\varepsilon}{2}$ for $t \in [0, T_2]$ where $T_2 = \frac{T_1 + T_q}{2}$. Using triangular inequality, we get $\|U_h(t)\|_{\infty} \geq \|u_h(t)\|_{\infty} - \|U_h(t) - u_h(t)\|_{\infty} \geq 1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} \geq 1 - \rho$, for $t \in (0, T_2)$. From Theorem 3.1, $U_h(t)$ quenches at time $T^h_q$. Using (29), we arrive at $|T^h_q - T_q| \leq |T^h_q - T_2| + |T_2 - T_q| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, which leads us to the desired result.
5. Numerical results

In this section, we consider the following explicit scheme.

\[
\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} + \varepsilon(1 - U_k^{(n)})^{-p}, \quad 1 \leq i \leq I - 1,
\]

\[
U_0^{(n)} = 0, \quad U_I^{(n)} = 0,
\]

\[
U_0^0 = 0, \quad 0 \leq i \leq I,
\]

and the following implicit scheme

\[
\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{h^2} + \varepsilon(1 - U_k^{(n)})^{-p}, \quad 1 \leq i \leq I - 1,
\]

\[
U_0^{(n+1)} = 0, \quad U_I^{(n+1)} = 0,
\]

\[
U_0^0 = 0, \quad 0 \leq i \leq I,
\]

where \(n \geq 0, k = \frac{I}{2}, \Delta t_n = h^2(1 - \|U_h^{(n)}\|_\infty)^{p+1}, \Delta t_n^e = \min\{h^2, \Delta t_n\}, T^n = \sum_{j=0}^{n-1} \Delta t_j.\)

In the following tables, in rows, we present the numerical quenching times, the numbers of iterations, CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128, 256. The numerical quenching time \(T^n = \sum_{j=0}^{n-1} \Delta t_j\) is computed at the first time when \(|T^{n+1} - T^n| \leq 10^{-16}\). The order(s) of the method is computed from

\[
s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.
\]

**First case: \(\varepsilon = 0.9\).**

**Table 1:** Numerical quenching times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method

<table>
<thead>
<tr>
<th>I</th>
<th>(T^n)</th>
<th>(n)</th>
<th>CPU time</th>
<th>(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0.059321</td>
<td>448</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>32</td>
<td>0.058758</td>
<td>1736</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>64</td>
<td>0.058617</td>
<td>6649</td>
<td>-</td>
<td>2.00</td>
</tr>
<tr>
<td>128</td>
<td>0.058582</td>
<td>25353</td>
<td>1</td>
<td>2.02</td>
</tr>
<tr>
<td>256</td>
<td>0.058573</td>
<td>96383</td>
<td>4</td>
<td>1.97</td>
</tr>
</tbody>
</table>

**Table 2:** Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method
Second case: \( \varepsilon = 10 \).

**Table 3:** Numerical quenching times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method.

<table>
<thead>
<tr>
<th>I</th>
<th>( T^n )</th>
<th>n</th>
<th>CPU time</th>
<th>s</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0.060274</td>
<td>444</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>32</td>
<td>0.058988</td>
<td>1719</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>64</td>
<td>0.058674</td>
<td>6579</td>
<td>-</td>
<td>2.04</td>
</tr>
<tr>
<td>128</td>
<td>0.058596</td>
<td>25072</td>
<td>2</td>
<td>2.02</td>
</tr>
<tr>
<td>256</td>
<td>0.058577</td>
<td>95259</td>
<td>12</td>
<td>2.04</td>
</tr>
</tbody>
</table>

**Table 4:** Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method.

<table>
<thead>
<tr>
<th>I</th>
<th>( T^n )</th>
<th>n</th>
<th>CPU time</th>
<th>s</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0.052837</td>
<td>402</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>32</td>
<td>0.052267</td>
<td>1557</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>64</td>
<td>0.052125</td>
<td>5966</td>
<td>-</td>
<td>2.01</td>
</tr>
<tr>
<td>128</td>
<td>0.052089</td>
<td>22746</td>
<td>1</td>
<td>1.99</td>
</tr>
<tr>
<td>256</td>
<td>0.052080</td>
<td>86459</td>
<td>4</td>
<td>2.01</td>
</tr>
</tbody>
</table>

Third case: \( \varepsilon = 11 \).

**Table 5:** Numerical quenching times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method.

<table>
<thead>
<tr>
<th>I</th>
<th>( T^n )</th>
<th>n</th>
<th>CPU time</th>
<th>s</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0.047688</td>
<td>364</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>32</td>
<td>0.047112</td>
<td>1412</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>64</td>
<td>0.046969</td>
<td>5411</td>
<td>-</td>
<td>2.02</td>
</tr>
<tr>
<td>128</td>
<td>0.046933</td>
<td>20629</td>
<td>-</td>
<td>2.00</td>
</tr>
<tr>
<td>256</td>
<td>0.046924</td>
<td>78404</td>
<td>4</td>
<td>2.01</td>
</tr>
</tbody>
</table>

**Table 6:** Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit
Euler method

<table>
<thead>
<tr>
<th>I</th>
<th>$T^n$</th>
<th>n</th>
<th>CPU time</th>
<th>s</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0.048435</td>
<td>360</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>32</td>
<td>0.047291</td>
<td>1398</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>64</td>
<td>0.047013</td>
<td>5353</td>
<td>-</td>
<td>2.05</td>
</tr>
<tr>
<td>128</td>
<td>0.046944</td>
<td>20400</td>
<td>2</td>
<td>2.02</td>
</tr>
<tr>
<td>256</td>
<td>0.046927</td>
<td>77487</td>
<td>10</td>
<td>2.03</td>
</tr>
</tbody>
</table>

From Remark 3.3, we have seen that the quenching time of the solution of our problem is smaller than the one of the problem where the reaction term is not local. In order to verify this assertion, we do the same experiments when the reaction term is not local and is replaced by $(1 - U_i^{(n)})^{-p}$.

**First case:** $\varepsilon = 9$.

**Table 7:** Numerical quenching times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method

<table>
<thead>
<tr>
<th>I</th>
<th>$T^n$</th>
<th>n</th>
<th>CPU time</th>
<th>s</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0.061242</td>
<td>462</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>32</td>
<td>0.060855</td>
<td>1802</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>64</td>
<td>0.060760</td>
<td>6952</td>
<td>-</td>
<td>2.03</td>
</tr>
<tr>
<td>128</td>
<td>0.060737</td>
<td>26709</td>
<td>-</td>
<td>2.05</td>
</tr>
<tr>
<td>256</td>
<td>0.060731</td>
<td>102312</td>
<td>4</td>
<td>1.95</td>
</tr>
</tbody>
</table>

**Table 8:** Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

<table>
<thead>
<tr>
<th>I</th>
<th>$T^n$</th>
<th>n</th>
<th>CPU time</th>
<th>s</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0.062410</td>
<td>458</td>
<td>-</td>
<td>-</td>
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<td>32</td>
<td>0.061141</td>
<td>1785</td>
<td>1</td>
<td>-</td>
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<td>64</td>
<td>0.060832</td>
<td>6883</td>
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<td>128</td>
<td>0.060755</td>
<td>26428</td>
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<td>256</td>
<td>0.060736</td>
<td>101187</td>
<td>13</td>
<td>2.03</td>
</tr>
</tbody>
</table>

**Second case:** $\varepsilon = 10$.

**Table 9:** Numerical quenching times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method
Table 10: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

<table>
<thead>
<tr>
<th>I</th>
<th>( T^n )</th>
<th>n</th>
<th>CPU time</th>
<th>s</th>
</tr>
</thead>
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<tr>
<td>16</td>
<td>0.054197</td>
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<td>-</td>
</tr>
<tr>
<td>32</td>
<td>0.053779</td>
<td>1608</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>64</td>
<td>0.053677</td>
<td>6199</td>
<td>-</td>
<td>2.04</td>
</tr>
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<td>128</td>
<td>0.053651</td>
<td>23796</td>
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<td>1.98</td>
</tr>
<tr>
<td>256</td>
<td>0.053645</td>
<td>91070</td>
<td>4</td>
<td>2.12</td>
</tr>
</tbody>
</table>

Third case: \( \varepsilon = 11 \).

Table 11: Numerical quenching times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method

<table>
<thead>
<tr>
<th>I</th>
<th>( T^n )</th>
<th>n</th>
<th>CPU time</th>
<th>s</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0.055242</td>
<td>408</td>
<td>-</td>
<td>-</td>
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<tr>
<td>32</td>
<td>0.054034</td>
<td>1592</td>
<td>-</td>
<td>-</td>
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<tr>
<td>64</td>
<td>0.053740</td>
<td>6137</td>
<td>-</td>
<td>2.05</td>
</tr>
<tr>
<td>128</td>
<td>0.053667</td>
<td>23544</td>
<td>2</td>
<td>2.02</td>
</tr>
<tr>
<td>256</td>
<td>0.053649</td>
<td>90060</td>
<td>11</td>
<td>2.03</td>
</tr>
</tbody>
</table>

Table 12: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

<table>
<thead>
<tr>
<th>I</th>
<th>( T^n )</th>
<th>n</th>
<th>CPU time</th>
<th>s</th>
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<tr>
<td>16</td>
<td>0.048673</td>
<td>372</td>
<td>-</td>
<td>-</td>
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<tr>
<td>32</td>
<td>0.048228</td>
<td>1451</td>
<td>-</td>
<td>-</td>
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<td>64</td>
<td>0.048119</td>
<td>5594</td>
<td>-</td>
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<tr>
<td>128</td>
<td>0.048092</td>
<td>21456</td>
<td>1</td>
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<td>256</td>
<td>0.048086</td>
<td>82050</td>
<td>3</td>
<td>2.17</td>
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References


