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FEJÉR KERNEL: ITS ASSOCIATED POLYNOMIALS

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Abstract. We show that the Fejér kernel generates the fifth-kind Chebyshev polynomials.

Palabras claves. Kernels in Fourier series, Chebyshev polynomials.

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Resumen. Se demuestra que el núcleo de Fejér genera polinomios de Chebyshev de quinto orden.

Key words and phrases. Núcleos en series de Fourier, polinomios de Chebyshev.

1. Introduction

In the original approach to Fourier series, it is convenient to consider the following partial sums for the interval $[-\pi, \pi]$:

$$f_n(y) = \tfrac{1}{2}a_0 + a_1 \cos y + \cdots + a_n \cos(ny) + b_1 \sin(y) + \cdots + b_n \sin(ny), \tag{1}$$

assuming for $a_r,\ b_r$ the values:

$$a_r = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(rt)dt,\ \mathrm{b}_r = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(rt)dt, \tag{2}$$

and investigate what happens if n increases to infinity. From (1) and (2) we obtain:

$$f_n(y) = \int_{-\pi}^{\pi} f(t)\, \underset{D}{K_n}(t-y)dt, \tag{3}$$

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with the Dirichlet kernel [1-3]:

$$K_{\underset{D}{n}}(t-y) = \frac{1}{2\pi} \frac{\sin\left[\left(n+\frac{1}{2}\right)(t-y)\right]}{\sin\left(\frac{t-y}{2}\right)}. \tag{4}$$

Then we hope that with n increasing to infinity, $f_n(y)$ approaches $f(y)$ with an error which can be made arbitrarily small. This requires a very strong focusing power of $K_{\underset{D}{n}}(t-y)$, that is, we would like to have the strict property:

$$\lim_{n\to\infty} K_{\underset{D}{n}}(t-y) = \delta(t-y), \tag{5}$$

however, (4) simulates a Dirac delta only until certain approximation, then the convergence:

$$\lim_{n\to\infty} f_n(y) = f(y) \tag{6}$$

has to be restricted to a definite class of functions $f(y)$ which are conveniently smooth to counteract the insufficient focusing power of $K_{\underset{D}{n}}(t-y)$; the corresponding restrictions on $f(y)$ are the known Dirichlet conditions [1-3] for infinite convergent Fourier series.

From (4) we see that $K_{\underset{D}{n}}(\theta)$ is an even function, then here we consider it for $\theta \in [0,\pi]$:

$$K_{\underset{D}{n}}(\theta) = \frac{1}{2\pi} \frac{\sin\left(n+\frac{1}{2}\right)\theta}{\sin\left(\frac{\theta}{2}\right)}, \tag{7}$$

thus

(8)

$$K_{\underset{D}{0}}(\theta) = \tfrac{1}{2\pi},\ \ K_{\underset{D}{1}}(\theta) = \tfrac{1}{2\pi}(1+2\cos\theta),\ \ K_{\underset{D}{2}}(\theta) = \tfrac{1}{2\pi}(-1+2\cos\theta+4\cos^2\theta),$$

$$K_{\underset{D}{3}}(\theta) = \tfrac{1}{2\pi}(-1-4\cos\theta+4\cos^2\theta+8\cos^3\theta),\ \text{etc.}$$

then it is natural to introduce the polynomials:

$$W_n(x) = W_n(\cos\theta) = 2\pi\, K_{\underset{D}{n}}(\theta),\ \ x\in[-1,1], \tag{9}$$

which were named “fourth-kind Chebyshev polynomials” by Gautschi [4,5]. Therefore, see Fig. 1:

(10)

$$W_0(x) = 1,\ W_1(x) = 2x+1,\ W_2(x) = 4x^2+2x-1,$$

$$W_3(x) = 8x^3+4x^2-4x-1,\ W_4(x) = 16x^4+8x^3-12x^2-4x+1,\ \text{etc.}$$

In the next Section we exhibit a set of associated polynomials to Fejér kernel [1-3].

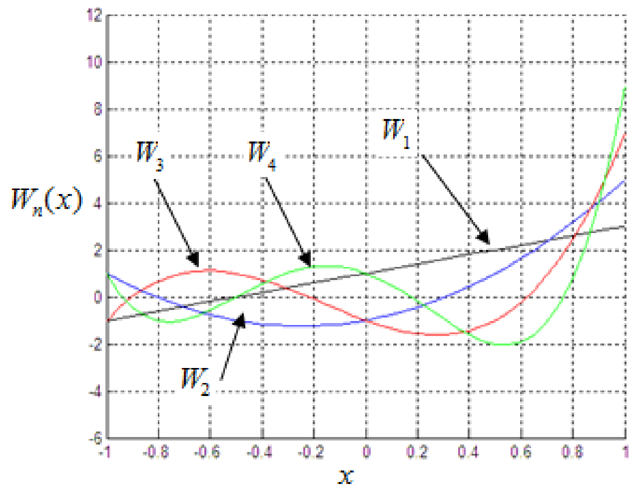

FIGURE 1. Some fourth-kind Chebyshev polynomials

2. Chebyshev-Fejér polynomials

Fejér [5] invented a new method of summing the Fourier series by which he greatly extended the validity of the series. Using the arithmetic means of the partial sums (1), instead of the $f_n(y)$ themselves, he could sum series which were divergent. The only condition the function still has to satisfy is the natural restriction that $f(y)$ shall be absolutely integrable.
Then, in the Fejér approach we construct the sequence:
(11)

$$g_1(y) = f_0(y),\ g_2(y) = \tfrac{1}{2}[(f_0(y) + f_1(y)],\ g_3(y) = \tfrac{1}{3}[(f_0(y) + f_1(y) + f_2(y)], \ldots,$$

$$g_n(y) = \tfrac{1}{n}[(f_0(y) + f_1(y) + \cdots + f_{n-1}(y)],$$

accepting the expressions (1) and (2), therefore:

$$g_n(y) = \int_{-\pi}^{\pi} f(t) \underset{F}{K_n}(t-y)dt, \tag{12}$$

thus we see that Fejér results come about by the fact that his method is related with the following kernel [1-3]:

$$\underset{F}{K_n}(t-y) = \frac{1}{2\pi n}\frac{\sin^2\left[\frac{n}{2}(t-y)\right]}{\sin^2\frac{t-y}{2}}, \tag{13}$$

which possesses a strong focusing power, that is, it satisfies (5), then a $f(y)$ absolutely integrable in $[-\pi,\pi]$ guarantees the convergence of $g_n(y)$ towards $f(y)$.

Now we consider the Fejér kernel:

$$K_{\underset{F}{n}}(\theta) = \frac{1}{2\pi n}\frac{\sin^2\left(n\frac{\theta}{2}\right)}{\sin^2\frac{\theta}{2}}, \;\theta \in [0,\pi], \tag{14}$$

that is:

$$\begin{gathered} K_{\underset{F}{0}}(\theta) = 0, \;\; K_{\underset{F}{1}}(\theta) = \tfrac{1}{2\pi}, \;\; K_{\underset{F}{2}}(\theta) = \tfrac{1}{2\pi}(1+\cos\theta), \\ K_{\underset{F}{3}}(\theta) = \tfrac{1}{6\pi}(1+4\cos\theta+4\cos^2\theta), \text{ etc.,} \end{gathered} \tag{15}$$

then it is natural the introduction of the functions:

$$\tilde{W}_n(x) = \tilde{W}_n(\cos\theta) = \frac{2\pi}{n+1}\, K_{\underset{F}{n+1}}(\theta), \; x \in [-1,1], \tag{16}$$

that we name "fifth-kind Chebyshev polynomials", which are not explicitly in the literature. Therefore:

(17)

$$\tilde{W}_0\,(x) = 1, \; \tilde{W}_1\,(x) = \tfrac{1}{2}\,(x+1)\,, \; \tilde{W}_2\,(x) = \tfrac{1}{9}\left(4x^2+4x+1\right),$$

$$\tilde{W}_3\,(x) = \tfrac{1}{2}\left(x^3+x^2\right), \; \tilde{W}_4\,(x) = \tfrac{1}{25}\left(16x^4+16x^3-4x^2-4x+1\right), \; etc.$$

thus $\tilde{W}_n(1) = 1$, see the following Figure:

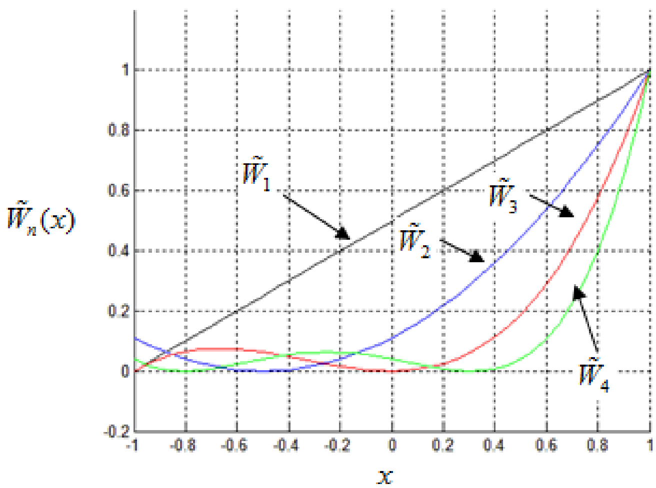

FIGURE 2. Some fifth-kind Chebyshev polynomials

which are solutions of the non-homogeneous differential equation:

$$(1-x)\left[(1-x^2)\tilde{W}_n'' - (3x+2)\tilde{W}_n' + (n+1)^2\tilde{W}_n\right] + x\tilde{W}_n = 1. \tag{18}$$

In other paper we will study topics as recurrence, Rodrigues formula, interpolation properties, orthonormality, generating function, etc., for fifth-kind Chebyshev polynomials introduced in this work.

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