Openness of the induced map $C_n(f)$

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Given a map between compact metric spaces $f : X \to Y$, it is possible to induce a map between the $n$-fold hyperspaces $C_n(f) : C_n(X) \to C_n(Y)$ for each positive integer $n$. Let $\mathcal{A}$ and $\mathcal{B}$ be classes of maps. A general problem is to find the interrelations between the following two statements: 1. $f \in \mathcal{A}$; 2. $C_n(f) \in \mathcal{B}$. It is known that 1 and 2 are equivalent conditions if both $\mathcal{A}$ and $\mathcal{B}$ are the class of homeomorphisms. If $\mathcal{A}$ and $\mathcal{B}$ are the class of open maps, then the openness of $C_n(f)$ implies the openness of $f$. Furthermore, there exists an open map $f$ such that $C_n(f)$ is not open. Moreover, if $C_n(f)$ is open and $n \geq 3$, then $f$ is both open and monotone. Our main result is Theorem 3.2, where we prove that if the induced map $C_n(f)$ is an open map, for $n \geq 2$, then $f$ is a homeomorphism.

Keywords: continua, hyperspaces of continua, induced maps, open maps.

MSC: 54B20, 54E40.
1 Introduction

A continuum is a nonempty, compact, connected and metric space. For a continuum $X$ and for a positive integer $n$ we denote by $C_n(X)$ the hyperspace of all nonempty closed subsets of $X$ with at most $n$ components. Given a map $f : X \to Y$ between continua $X$ and $Y$, we define the induced map $C_n(f) : C_n(X) \to C_n(Y)$ by $C_n(f)(A) = f(A)$ [4, p. 783].

A map $f : X \to Y$ is called open if $f$ maps every open set in $X$ onto an open set in $Y$. It is known that if $C_n(f)$ is open, for some positive integer $n$, then $f$ is also an open map [4, Theorem 8, p. 786].

There are some works in which the openness of $C_n(f)$ has been studied ([1], [2], [3], [5] and [6] for $n = 1$, and [4] for $n \geq 1$). In [3, Question 4, p. 68] the following question is asked:

**Question 1.1.** What (locally connected) continua $X$ have the property that if $f$ is a map of $X$ onto a continuum $Y$ such that the induced map $C_1(f) : C_1(X) \to C_1(Y)$ is open, then $f$ is a homeomorphism?

There are some partial answers to Question 1.1: In [5, Theorem 1, p. 3729] it is proved that if $C_1(f)$ is open, where $f$ is defined between locally connected continua, then $f$ is monotone. Thus, if $f$ is defined between hereditarily locally connected continua such that $C_1(f)$ is open, then $f$ is a homeomorphism [5, Corollary 2, p. 3730]. It is known that if $X$ is a fan, then we have a positive answer to Question 1.1 [3, Theorem 9, p. 70]. Recently, we expanded [3, Theorem 9, p. 70] proving that if $f$ is defined between dendroids and $C_1(f)$ is open, then $f$ is a homeomorphism [2, Theorem 3.4, p. 233]. It is important to emphasize that there are maps $f$ defined between continua such that $C_1(f)$ is open and $f$ is not a homeomorphism [5, Example 3, p. 3730], even when $f$ is defined between locally connected continua [3, Corollary 19, p. 73].

If $n \geq 1$, then we know that [4, Theorem 10, p. 786] generalizes [5, Theorem 1, p. 3729] proving that $f$ is monotone, if $C_n(f)$ is open and $f$ is defined between locally connected continua, for any $n \in \mathbb{N}$.

It is very natural to ask, what does it happen if we change 1 by any positive integer $n$ in Question 1.1?

Our goal is to prove that if $C_n(f)$ is an open map and $n \geq 2$, then $f$ is a homeomorphism (see Theorem 3.2).
2 Definitions

If \((X, d)\) is a metric space, then given \(A \subset X\), the interior, the closure and the boundary of \(A\) are denoted by \(\text{Int}_X(A)\), \(\text{Cl}_X(A)\) and \(\text{Bd}_X(A)\), respectively. The cardinality of \(A\) is denoted by \(|A|\). The symbol \(\mathbb{N}\) denotes the set of positive integers. A map is assumed to be a continuous function.

Remark 2.1. Here every map will be assumed surjective and defined between nondegenerate continua.

Let \(X\) be a continuum and let \(A\) and \(B\) be closed subsets of \(X\). We say that \(C \subset X\) is irreducible between \(A\) and \(B\) provided that \(C \setminus A \neq \emptyset\), \(C \setminus B \neq \emptyset\), and no proper subcontinuum of \(C\) intersects both \(A\) and \(B\).

Definition 2.2. Let \(\{A_i\}_{i=1}^{\infty}\) be a sequence of subsets of \(X\). We define the inferior limit of \(\{A_i\}_{i=1}^{\infty}\), denoted by \(\liminf_{i \to \infty} A_i\), and the superior limit of \(\{A_i\}_{i=1}^{\infty}\), denoted by \(\limsup_{i \to \infty} A_i\), as follows:

1. \(\liminf_{i \to \infty} A_i = \{x \in X : \text{for any open } U \text{ in } X \text{ such that } x \in U, U \cap A_i \neq \emptyset \text{ for all but finitely many } i\}\);

2. \(\limsup_{i \to \infty} A_i = \{x \in X : \text{for any open } U \text{ in } X \text{ such that } x \in U, U \cap A_i \neq \emptyset \text{ for infinitely many } i\}\);

We say that \(\{A_i\}_{i=1}^{\infty}\) is convergent to \(A\) in \(X\), which we denote by \(\lim_{i \to \infty} A_i = A = \limsup_{i \to \infty} A_i\).

A proof of the following result may be found in [11, Theorem 4.32, p. 130].

Theorem 2.3 (Eilenberg). Let \(f : X \to Y\) be a map between continua. Then \(f\) is an open map if and only if \(\lim_{n \to \infty} f^{-1}(y_n) = f^{-1}(y)\) for each sequence \(\{y_n\}_{n=1}^{\infty}\) such that \(\lim_{n \to \infty} y_n = y\).

Eilenberg’s theorem is a characterization of open maps and may be written in the following way:

Theorem 2.4. Let \(f : X \to Y\) be a map between continua. Then \(f\) is open if and only if for each sequence \(\{y_n\}_{n \in \mathbb{N}}\) in \(Y\) such that \(\lim_{n \to \infty} y_n = y\), for some point \(y \in Y\), and for any \(x \in f^{-1}(y)\) there exists a sequence \(\{x_n\}_{n \in \mathbb{N}}\) in \(X\) such that \(\lim_{n \to \infty} x_n = x\) and \(x_n \in f^{-1}(y_n)\), for each \(n \in \mathbb{N}\).
Proof. Suppose that \( f : X \to Y \) is an open map between continua. Let \( \{y_n\}_{n \in \mathbb{N}} \) be a sequence in \( Y \) such that \( \lim_{n \to \infty} y_n = y \) for some \( y \in Y \). Let \( x \in f^{-1}(y) \). We know that \( \lim_{n \to \infty} f^{-1}(y_n) = f^{-1}(y) \), by Theorem 2.3.

Let \( \{U_n\}_{n \in \mathbb{N}} \) be a sequence of open subsets of \( X \) such that \( x \in U_n \), \( U_{n+1} \subset U_n \), for each \( n \in \mathbb{N} \), and \( \bigcap_{n=1}^{\infty} U_n = \{x\} \). Since \( x \in \lim \inf_{n \to \infty} f^{-1}(y_n) \), for each \( m \in \mathbb{N} \), there exists \( k_m \in \mathbb{N} \) such that \( U_m \cap f^{-1}(y_n) \neq \emptyset \) for each \( n \geq k_m \). Without loss of generality, we may assume that \( k_j < k_{j+1} \), if \( j < \ell \). We define the sequence \( \{x_n\}_{n=1}^\infty \) in \( X \) as follows:

1. \( x_n \in f^{-1}(y_n) \) if \( n < k_1 \);
2. \( x_n \in f^{-1}(y_n) \cap U_m \) if \( k_m \leq n < k_{m+1} \).

Clearly, \( \lim_{n \to \infty} x_n = x \) and \( f(x_n) = y_n \) for all \( n \in \mathbb{N} \). The converse implication follows from Theorem 2.3.

Given a continuum \( X \), we consider the following hyperspaces of \( X \):

1. \( 2^X = \{A \subset X : A \text{ is closed and nonempty}\} \);
2. \( C_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ components} \}, n \in \mathbb{N} \).

\( 2^X \) is topologized with the Vietoris topology [7, p. 3], which is generated by the collection of sets \( \langle U_1, U_2, \ldots, U_\ell \rangle \), where \( U_1, U_2, \ldots, U_\ell \) are open sets in \( X \) and

\[
\langle U_1, U_2, \ldots, U_\ell \rangle = \{A \in 2^X : A \subset \bigcup_{i=1}^\ell U_i \text{ and } A \cap U_i \neq \emptyset \text{ for each } i \}.
\]

Clearly, \( C_n(X) \) is a subspace of \( 2^X \). The space \( C_n(X) \) is called the \( n \)-fold hyperspace of \( X \). It is well known that if \( X \) is a continuum, then both \( 2^X \) and \( C_n(X) \) are continua. The reader may consult [7] and [8] for general information about hyperspaces.

**Notation 2.5.** We denote \( \langle U_1, U_2, \cdots, U_\ell \rangle \cap C_n(X) \) by \( \langle U_1, U_2, \cdots, U_\ell \rangle_n \) and write \( C(X) \) instead of \( C_1(X) \).

Let \( X \) be a continuum. A **Whitney map** for \( C(X) \) is a map \( \mu : C(X) \to [0, 1] \) that satisfies the following two conditions:
1. for each \( A, B \in C(X) \) such that \( A \subseteq B \) and \( A \neq B \), \( \mu(A) < \mu(B) \);

2. \( \mu(A) = 0 \) if and only if \( |A| = 1 \).

In [7, Theorem 13.4, p. 107] the following important theorem is proved.

**Theorem 2.6.** If \( X \) is a continuum, then there exists a Whitney map for the hyperspace \( C(X) \).

Let \( \mu \) be a Whitney map for \( C(X) \). Let \( A_0, A_1 \in C(X) \). A map \( \sigma : [0, 1] \to C(X) \) is said to be a segment in \( C(X) \) with respect to \( \mu \) from \( A_0 \) to \( A_1 \) provided that \( \sigma \) has the following three properties:

1. \( \sigma(0) = A_0 \) and \( \sigma(1) = A_1 \);
2. \( \mu(\sigma(t)) = (1-t)\mu(\sigma(0)) + t\mu(\sigma(1)) \) for each \( t \in [0, 1] \);
3. \( \sigma(t) \subseteq \sigma(s) \) whenever \( 0 \leq t \leq s \leq 1 \).

The following result may be found in [7, Theorem 16.9, p. 131].

**Theorem 2.7.** Let \( X \) be a continuum and let \( \mu \) be a Whitney map for \( C(X) \). Let \( A_0, A_1 \in C(X) \). Then there is a segment with respect to \( \mu \) from \( A_0 \) to \( A_1 \) if and only if \( A_0 \subseteq A_1 \).

Let \( f : X \to Y \) be a map between continua. Then the function \( 2^f : 2^X \to 2^Y \) given by \( 2^f(A) = f(A) \) for each \( A \in 2^X \), is called the induced map between \( 2^X \) and \( 2^Y \). The function \( 2^f|_{C_n(X)} \) is denoted by \( C_n(f) \) and it is called the induced map between the hyperspaces \( C_n(X) \) and \( C_n(Y) \). In [7, Lemma 13.3, p. 106] it is shown that \( 2^f \) is a map. Since \( 2^f(C_n(X)) \subseteq C_n(Y) \), \( C_n(f) \) is a map between \( C_n(X) \) and \( C_n(Y) \), for each \( n \in \mathbb{N} \).

3 Openness of \( C_n(f) \) for \( n \geq 2 \)

We begin this section with a theorem that will be used in the proof of our main result.

**Theorem 3.1.** Let \( f : X \to Y \) be a map between continua and let \( n \in \mathbb{N} \), such that \( C_n(f) : C_n(X) \to C_n(Y) \) is open. If there is a point \( y \in Y \) such that \( |f^{-1}(y)| > 1 \), then there exist two subcontinua \( D \) and \( E \) of \( X \) such that \( D \cap E = \emptyset \) and \( f(D) = f(E) = L \), where \( L \) is a proper and irreducible subcontinuum between \( y \) and some point \( y_0 \neq y \) of \( Y \).
Proof. Let \( f : X \to Y \) be a map between continua and let \( n \in \mathbb{N} \) such that \( C_n(f) : C_n(X) \to C_n(Y) \) is open. Let \( b_1, b_2, \ldots, b_{n-1} \) and \( y \) be different points in \( Y \) such that \( |f^{-1}(y)| > 1 \). Let \( a_1, a_2, \ldots, a_{n-1}, x_1 \) and \( x_2 \) be points in \( X \) such that \( f(a_i) = b_i \), for each \( i \in \{1, 2, \ldots, n-1\} \), and \( x_1, x_2 \in f^{-1}(y) \), where \( x_1 \neq x_2 \). We denote by \( B_m \) the open ball in \( Y \) about \( y \) of radius \( \frac{1}{m} \). Let \( C_m \) be the component of \( \text{Cl}(B_m) \) such that \( y \in C_m \). Notice that \( C_m \cap \text{Bdy}(B_m) \neq \emptyset \), by [10, Theorem 5.4, p. 73]. Let \( D_m \) be an irreducible subcontinuum of \( C_m \) between \( y \) and some point of \( \text{Bdy}(B_m) \) [10, Proposition 11.30, p. 212]. Observe that \( \lim_{m \to \infty} D_m = \{y\} \). Hence, we have that:

\[
\lim_{m \to \infty} (D_m \cup \{b_1, b_2, \ldots, b_{n-1}\}) = \{y, b_1, b_2, \ldots, b_{n-1}\}.
\]

Since \( \{x_1, a_1, \ldots, a_{n-1}\} \) and \( \{x_2, a_1, \ldots, a_{n-1}\} \) both belong to \( C_n(f)^{-1}\{y, b_1, \ldots, b_{n-1}\} \), by Theorem 2.4, there are two sequences \( \{E_m\}_{m \in \mathbb{N}} \) and \( \{F_m\}_{m \in \mathbb{N}} \) of \( C_n(X) \) such that \( \lim_{m \to \infty} E_m = \{x_1, a_1, \ldots, a_{n-1}\} \), \( \lim_{m \to \infty} F_m = \{x_2, a_1, \ldots, a_{n-1}\} \) and:

\[
E_m, F_m \in C_n(f)^{-1}(D_m \cup \{b_1, b_2, \ldots, b_{n-1}\}), \text{ for each } m \in \mathbb{N}.
\]

Let \( U_1, U_2, V_1, \ldots, V_{n-2} \) and \( V_{n-1} \) be open and pairwise disjoint subsets of \( X \) such that \( x_1 \in U_1, x_2 \in U_2 \) and \( a_i \in V_i \) for each \( i \in \{1, 2, \ldots, n-1\} \). Thus, there is \( \ell \in \mathbb{N} \) such that \( E_k \subset U_1 \cup V_1 \cup \cdots \cup V_{n-1} \) and \( F_k \subset U_2 \cup V_1 \cup \cdots \cup V_{n-1} \), for each \( k \geq \ell \) [11, Theorem 7.2, p. 12]. Hence, both \( E_k \) and \( F_k \) have exactly \( n \) components, for each \( k \geq \ell \). Let \( k_0 \) be a sufficiently large number such that if \( E \) and \( F \) are the components of \( E_{k_0} \) and \( F_{k_0} \), respectively, such that \( E \subset U_1 \) and \( F \subset U_2 \), then \( f(E) = f(F) = D_{k_0} \). Note that \( E \cap F = \emptyset, D_{k_0} \) is irreducible between \( y \) and some point \( y_0 \neq y \) and we may suppose that \( D_{k_0} \neq Y \). The proof is complete.

\[\square\]

In [6, Theorem 4.3, p. 243] it is proved that \( 2^\ell \) is open if and only if \( f \) is open. Furthermore, there is an open map \( f \) such that \( C_n(f) \) is not open, for any \( n \in \mathbb{N} \) [4, Remark 9, p. 786].

**Theorem 3.2.** Let \( f : X \to Y \) be a map between continua and let \( n \geq 2 \). If \( C_n(f) : C_n(X) \to C_n(Y) \) is open, then \( f \) is a homeomorphism.
Proof. Let \( f : X \to Y \) be a map between continua and let \( n \geq 2 \), such that \( C_n(f) : C_n(X) \to C_n(Y) \) is open. Suppose that \( f \) is not a homeomorphism and take a point \( y \in Y \) such that \( |f^{-1}(y)| > 1 \), for some point \( y \in Y \). Hence, there are two subcontinua \( D \) and \( E \) of \( X \) such that \( D \cap E = \emptyset \) and \( f(D) = f(E) = L \), where \( L \) is an irreducible continuum between \( y \) and \( y_0 \), where \( y \neq y_0 \), by Theorem 3.1. Notice that \( L \) is a nondegenerate subcontinuum of \( Y \) and \( D \setminus (f^{-1}(y) \cup f^{-1}(y_0)) \neq \emptyset \).

Let \( d \in D \setminus (f^{-1}(y) \cup f^{-1}(y_0)) \). We consider two cases:

1. \( n = 2 \). Observe that \( \{d\} \) and \( f^{-1}(y) \cup f^{-1}(y_0) \) are disjoint and closed subsets of \( X \). Thus, there are two open and disjoint sets \( U \) and \( V \) of \( X \) such that \( d \in U \) and \( f^{-1}(y) \cup f^{-1}(y_0) \subset V \). We show that \( C_2(f)(\{U, V\}_2) \) is not open. Note that \( \{d\} \cup E \in \{U, V\}_2 \). Therefore, \( C_2(f)(\{d\} \cup E) = L \in C_2(f)(\{U, V\}_2) \).

Let \( \mu \) be a Whitney map in \( C(X) \) (see Theorem 2.6). Since \( \{y\} \) and \( \{y_0\} \) are subsets of \( L \), there are two segments \( \sigma_1, \sigma_2 : [0, 1] \to C(X), \) with respect to \( \mu \), from \( \{y\} \) to \( L \) and from \( \{y_0\} \) to \( L \), respectively, by Theorem 2.7. Since \( \sigma_1(1) = \sigma_2(1) = L \), it is not difficult to prove that there exists a point \( s \in [0, 1] \) such that \( \sigma_1(s) \cap \sigma_2(s) \neq \emptyset \) and \( \sigma_1(t) \cap \sigma_2(t) = \emptyset \), for each \( t < s \). Observe that \( \sigma_1(0) \cup \sigma_2(0) = \{y, y_0\} \). Hence, \( s > 0 \). Since \( L \) is irreducible between \( y \) and \( y_0 \), and \( y, y_0 \in \sigma_1(s) \cup \sigma_2(s) \), we have that \( \sigma_1(s) \cup \sigma_2(s) = L \). Let \( \{t_m\}_{m \in \mathbb{N}} \) be an increasing sequence in \( [0, 1] \) such that \( \lim_{m \to \infty} t_m = s \). Clearly, \( \sigma_1(t_m) \cup \sigma_2(t_m) \in C_2(Y) \setminus C(Y) \), for each \( m \in \mathbb{N} \). We denote \( L_m = \sigma_1(t_m) \cup \sigma_2(t_m) \). Since \( \sigma_1 \) and \( \sigma_2 \) are both continuous function, and \( \lim_{m \to \infty} t_m = s \), we have that \( \lim_{m \to \infty} L_m = L \).

We show that \( L_m \notin C_2(f)(\{U, V\}_2) \), for any \( m \in \mathbb{N} \). Suppose that there exists \( D \in \{U, V\}_2 \) such that \( C_2(f)(D) = L_k \), for some \( k \in \mathbb{N} \). Since \( U \cap V = \emptyset \), \( D \cap U \neq \emptyset \) and \( D \cap V \neq \emptyset \), \( D \) has two components. Let \( E \) and \( F \) be the components of \( D \) such that \( E \subset U \) and \( F \subset V \). Note that \( y \) and \( y_0 \) belong to \( L_k \). Thus, \( D \cap f^{-1}(y) \neq \emptyset \) and \( D \cap f^{-1}(y_0) \neq \emptyset \). Since \( U \cap (f^{-1}(y) \cup f^{-1}(y_0)) = \emptyset \) and \( E \subset U \), we have that \( F \cap f^{-1}(y) \neq \emptyset \) and \( F \cap f^{-1}(y_0) \neq \emptyset \). Hence, \( f(F) \) is connected such that \( y, y_0 \in f(F) \subset L_k \), but this contradicts the fact that \( y \) and \( y_0 \) belong to different components of \( L_k \). Thus, \( L_m \notin C_2(f)(\{U, V\}_2) \), for any \( m \in \mathbb{N} \). Since \( \lim_{m \to \infty} L_m = L \) and \( L \in C_2(f)(\{U, V\}_2) \), we have that \( L \) is not an interior point of \( C_2(f)(\{U, V\}_2) \).

2. \( n > 2 \). Observe that \( L \neq Y \) by Theorem 3.1. Hence, \( X \setminus f^{-1}(L) \neq \emptyset \).
Let $x_1, x_2, \ldots, x_{n-1}$ and $x_{n-2}$ be different points in $X \setminus f^{-1}(L)$. Let $U, V, W_1, \ldots, W_{n-1}$ and $W_{n-2}$ be open and disjoint subsets of $X$ such that $d \in U, f^{-1}(y) \cup E \cup f^{-1}(y_0) \subset V$ and $x_i \in W_i$, for each $i \in \{1, 2, \ldots, n-2\}$. Let us remind that each point in $(U, V, W_1, \ldots, W_{n-2})_n$ has $n$ components. Therefore, using an argument similar to that in the case $n = 2$ in 1, we may conclude that $C_n(f)(U, V, W_1, \ldots, W_{n-2})_n$ is not an open set.

Thus $C_n(f)$ is not open, for any $n \geq 2$, by 1 and 2. Hence, $|f^{-1}(y)| = 1$ for each $y \in Y$. Since $f$ is defined between continua, $f$ is closed. Therefore, $f$ is a homeomorphism.

\[ \square \]

**Corollary 3.3.** Let $f : X \to Y$ be a map between continua and let $n \geq 2$. The following conditions are equivalent:

1. $C_n(f)$ is open;
2. $f$ is a homeomorphism;
3. $C_n(f)$ is a homeomorphism.

**Proof.** That 1 implies that 2 follows from Theorem 3.2. Using [4, Theorem 46, p. 801], we have that 2 implies that 3. Finally, since every homeomorphism is open, we have that 3 implies that 1.

\[ \square \]

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**References**


