On partitions into four cubes

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Usamos conjuntos parcialmente ordenados (posets) y grafos para obtener una fórmula para el número de particiones de un entero positivo n en cuatro cubos con dos de ellos iguales.

Palabras Clave: composiciones, números cúbicos, grafos, particiones, trayectorias, posets.

We use partially ordered sets (posets) and graphs in order to obtain a formula for the number of partitions of a positive integer n into four cubes with two of them equal.

Keywords: compositions, cubic numbers, graphs, partitions, paths, posets.

MSC: 05A17, 11D45, 11D85, 11E25, 11P83, 16G20.

1 Introducción

We consider that part of Waring's problem regarding cubes. We must recall that Waring, in his book *Meditationes Algebraicae*, published in 1770, stated without proof that every nonnegative integer is the sum of four squares, nine cubes, 19 fourth powers and so on [19].

Waring's problem for cubes is to prove that every nonnegative integer is the sum of a finite number of nonnegative cubes. The minimum such number is denoted $g(3)$. Wieferevich and Kempner proved that $g(3) = 9$ [14]. This is clearly best possible, since there are integers, such as 23 and 239, that cannot be written as sum of eight cubes.

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Immediately after Wieferevich published his theorem, Landau observed that, in fact, only finitely many positive integers actually require nine cubes, that is, every sufficiently large integer is the sum of eight cubes, with the only exceptions being 23 and 239.

Linnik proved that every sufficiently large integer is a sum of 7 cubes; Watson simplified the proof and McCurley gave an effective and explicit proof of this result [17, 18, 21].

Demjanenko [7] proved that every number $n \not\equiv \pm 4 \bmod 9$ can be expressed as the sum of four positive or negative cubes: $x^3 + y^3 + z^3 + w^3$. It is possible that every sufficiently large integer is the sum of four nonnegative cubes. Lukes [12, 16] has found representations of all $n \leq 10^7$ as sums of four positive or negative cubes. Let $E(x)$ denote the number of positive integers up to x that cannot be written as the sum of four positive cubes. Brüdern [2] proved that $E_{4,3}(x) \ll x^{37/42+\varepsilon}$ (*i.e.*, there exists a constant $c > 0$ such that $|E_{4,3}(x)| \leq cx^{37/42+\varepsilon}$, for all x in the domain of $E_{4,3}$) and so almost all positive integers can be represented as the sum of four positive cubes. Actually, in [8] Deshouillers et al, conjectured that 7373170279850 is the largest integer which cannot be expressed as the sum of four nonnegative integral cubes and Brüdern and Wooley have proved that almost all natural numbers n are the sum of four cubes of positive integers, one of which is no larger than $n^{5/36}$ [4].

Concerning the number $r(n)$ of representations of n as the sum of four positive cubes, the work of Hardy and Litlewood [13] on Waring's problem led to the more precise formulation that $r(n)$ should satisfy an asymptotic formula of the form

$$r(n) = \Gamma\left(\frac{4}{3}\right)^3 \mathcal{G}(n)\, n^{1/3} + O\left(n^{1/3}\,(\log n)^{-\varrho}\right), \tag{1}$$

where ϱ is some positive constant and $\mathcal{G}(n)$ is the familiar singular series usually associated with four cubes. One has

$$(\log\log n)^{-c} < \mathcal{G}(n) < (\log\log n)^{c},$$

for some $c > 0$ and all natural numbers $n \geq 4$, so that (1), if true, would confirm that all sufficiently large natural number can be expressed as the sum of four cubes of natural numbers.

Brüdern and Watt [3] proved the following result:

Theorem. *Let* $M = N^{\Theta}$ *where* $5/6 < \Theta < 1$. *Then, for all but*

$O(M(\log N)^{-1/4})$ *integers* n *in the range* $N < n \leq N+M$ *the asymptotic formula*

$$r(n) = \Gamma\left(\frac{4}{3}\right)^3 \mathcal{G}(n)\, n^{1/3} + O\left(n^{1/3}(\log n)^{-1/5}\right),$$

holds.

More recently Brüdern and Wooley [4] have proved the following results:

1. *When* $\Theta \geq \frac{5}{36}$ *one has* $r_\Theta(n) \geq 1$ *for almost all* n.

2. *Suppose that* $\frac{1}{4} < \Theta \leq \frac{1}{3}$. *Then, for almost all* n, *one has*

$$r_\Theta(n) = \Gamma\left(\frac{4}{3}\right)^3 \mathcal{G}(n)\, n^\Theta + O\left(n^\Theta\,(\log n)^{-1}\right).$$

In this case, for $0 < \Theta \leq \frac{1}{3}$ and n a natural number, $r_\Theta(n)$ is the number of representations of n in the form $n = x_1^3 + x_2^3 + x_3^3 + x_4^3$ with x_1, x_2, x_3, x_4 natural numbers satisfying $x_4 \leq n^\Theta$.

In [12] Guy mentioned the following open problems: Does each of the following Diophantine equations have integer solutions:

$$x^3 + y^3 + z^3 = n\,, \tag{2}$$
$$x^3 + y^3 + 2\,z^3 = n\,, \tag{3}$$

where n is a fixed positive integer and x, y, z can be any integers with minus signs allowed?

Koyama in [16] proposed an efficient search algorithm to solve the equation (3). His algorithm obtains $|x|$ and $|y|$ (if they exist) by solving a quadratic equation derived from divisors of $2|z|^3 \pm n$.

Remark 1. *There are other interesting problems concerning cubes, for example the Pollock octahedral numbers conjecture which claims that every number is the sum of at most seven octahedral numbers, where the* n*–th octahedral number* $\mathcal{O}_n$ *is given by the formula* $\mathcal{O}_n = \frac{n(2n^2+1)}{3}$. *The corresponding conjecture for tetrahedral numbers claims that every number is the sum of at most five tetrahedral numbers, where the* n*–th tetrahedral number* σ_n *is given by the formula* $\sigma_n = \frac{n(n+1)(n+2)}{6}$. *In fact,* *on tetrahedral numbers, Chou and Deng [5] believe that all numbers > 343867, are expressible as the sum of four such numbers.*

In order to obtain a contribution to the resolution of the problem (3) we develop a formula for $\mathcal{Q}_4(n)$, the number of partitions of a natural number n into four cubes with two of them equal, by using representations of posets over $\mathbb{N}$ and some graphs associated with this kind of representations.

The paper is organized as follows. Preliminary notations and definitions are included in section 2. In section 3 we define representations of posets over the set of natural numbers and its associated graphs. In section 4 we give a characterization for natural numbers which are the sum of four cubes with two of them equal and a formula for $\mathcal{Q}_4(n)$ is given as well.

Remark 2. *We use the customary symbols* $\mathbb{N}$, $\mathbb{Z}$ *and* $\mathbb{R}$ *for the set of natural, integer and real numbers, respectively.*

2 Preliminaries

This section introduces some basic definitions and notation to be used throughout the paper.

2.1 Partitions

A *partition* λ of a positive integer n is a finite nonincreasing sequence of positive integers $(\lambda_1, \lambda_2, \cdots, \lambda_r)$ such that $\sum_{i=1}^{r} \lambda_i = n$. The λ_i are the *parts* of the partition [1]. A *composition* is a partition in which the order of the summands is considered.

Sometimes it is useful to use a notation that makes explicit the number of times that a particular integer occurs as a part. Thus if $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r)$ is a partition of n, we sometimes write $\lambda = (1^{f_1} 2^{f_2} 3^{f_3} \cdots)$ where f_i of the λ_j are equal to i. For example, there are five partitions of $4 : 4 = (4)$, $3+1 = (13)$, $2+2 = (2^2)$, $2+1+1 = (1^2 2)$, $1+1+1+1 = (1^4)$; there are eight compositions of 4 : (4), (13), (31), (22), (112), (121), (211), (1111).

The *partition function* $p(n)$ is the number of partitions of n. Clearly $p(n) = 0$ when n is negative and $p(0) = 1$, where the empty sequence forms the only partition of zero.

The partition function increases quite rapidly with n. For example $p(10) = 42$, $p(20) = 627$, $p(50) = 204226$, $p(100) = 190569292$ and $p(200) = 3972999029388$.

Associated with each partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r)$ we have its *graphical representation* $\mathcal{G}_\lambda$ (or Ferrers graph) which formally is the set of points with integral coordinates (i, j) in the plane such that if $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r)$, then $(i, j) \in \mathcal{G}_\lambda$ if and only if $0 \geq i \geq -r + 1$, $0 \leq j \leq \lambda_{|i|+1} - 1$. For example

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\end{array}$$

is the graphical representation of the partition $5+4+4+4+4+3+1+1$.

Ferrers graphs are perhaps the most useful graphical device for studying partitions. For example with the help of them it is possible to prove that, the number of partitions of n with at most m parts equals the number of partitions of n in which no part exceeds m.

A *multiset* is a set with possibly repeated elements. To be precise, we define a multiset as an ordered pair (M, f) where M is a set and f is a function from M to the nonnegative integers; for each $m \in M$, $f(m)$ is the *multiplicity* of m. When M is a finite set, say $\{m_1, m_2, \cdots, m_r\}$, we write

$$(M, f) = \{m_1^{f(m_1)} m_2^{f(m_2)} \cdots m_r^{f(m_r)}\}\,.$$

Let us begin by considering permutations of multisets (a permutation of (M, f) is a word in which each letter belongs to M and for each $m \in M$ the total number of appearances of m in the word is $f(m)$). Thus 3212232112 is a permutation of the multiset $\{1^3 2^5 3^2\}$.

Let $\mathrm{inv}(m_1, m_2, \cdots, m_r; n)$ be the number of permutations $\xi_1 \xi_2 \cdots \xi_{m_1+m_2+\cdots+m_r}$ of $\{1^{m_1} 2^{m_2} \cdots r^{m_r}\}$ in which there are n pairs (ξ_i, ξ_j) such that $i < j$ and $\xi_i > \xi_j$.

Theorem 3 shows how a graphical representation can be used directly to obtain a relationship between $\mathrm{inv}(m_1, m_2, \cdots, m_r; n)$ and some restricted partitions [1]:

Theorem 3. $\mathrm{inv}(m_1, m_2; n) = p(m_1, m_2, n)$, *where* $p(m_1, m_2, n)$ *is the number of partitions of* n *with at most* m_2 *parts, each* $\leq m_1$.

Proof. Let us define a bijection between the permutations enumerated by the number $\mathrm{inv}(m_1, m_2; n)$ and the partitions enumerated by $p(m_1, m_2, n)$.

Let us use the box graphical representation for the Ferrers graph of a partition with each part ≤ 11 and at most seven parts (here 8+6+6+1+1, see Figure 1).

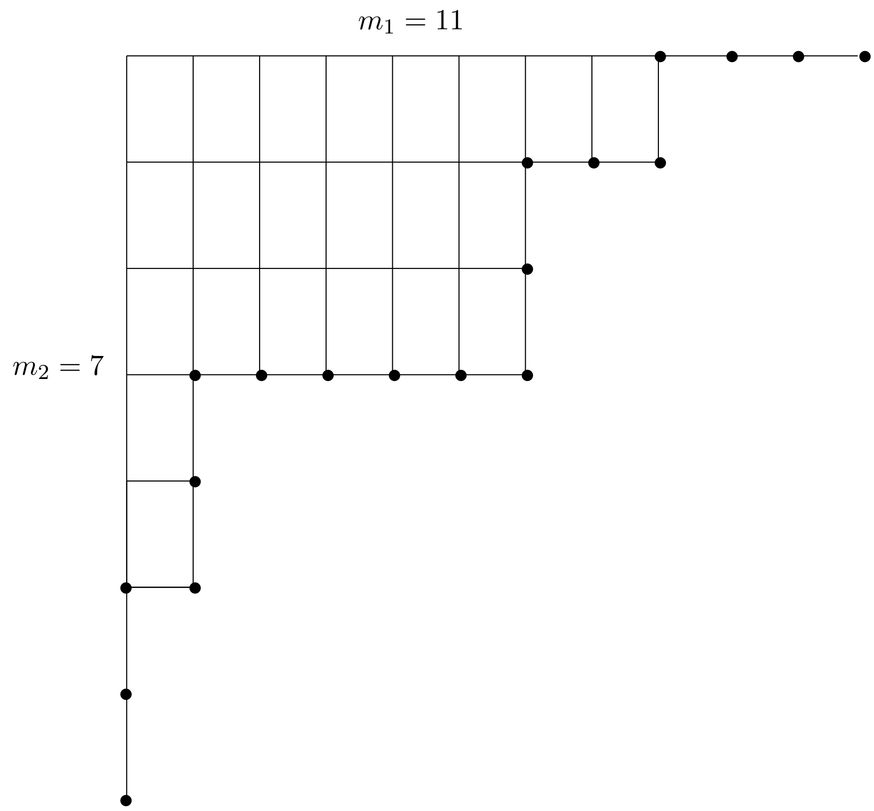

Figure 1.

We follow the path indicated by the dots in Figure 1, starting with the upper right node and moving to the left and downward: if the path moves vertically, we write a 2 and if horizontally we write 1. Hence the sequence corresponding to this graph is 111211221111122122. Notice that the number of 1's to the right of the first 2 tell us the largest part of our partition; the number of 1's to the right of our second 2 tell us the second part of our partition and in general the number of 1's to the right of the

ith 2 tell us the ith part of our partition. Clearly the above relationship between partitions and permutations establishes a bijection between the permutations of $\{1^{m_1}2^{m_2}\}$ with n inversions and the partitions of n with at most m_2 parts, each $\leq m_1$. Hence, $\text{inv}(m_1, m_2; n) = p(m_1, m_2, n)$. $\square$

In this paper we use paths associated with posets in order to obtain a formula for $\mathcal{Q}_4(n)$.

2.2 Posets

An *ordered set* (or *partially ordered set* or *poset*) is an ordered pair of the form $(\mathcal{P}, \leq)$ of a set $\mathcal{P}$ and a binary relation $\leq$ contained in $\mathcal{P} \times \mathcal{P}$, caalled the *order* (or the *partial order*) on $\mathcal{P}$, such that $\leq$ is reflexive, antisymmetric and transitive [6]. The elements of $\mathcal{P}$ are called the *points* of the ordered set. We write $x < y$ when $x \leq y$ and $x \neq y$, in this case we say that x *strictly less than* y. An ordered set is called *finite (infinite)* if the underlying set is finite (infinite). Usually we will be a little slovenly and simply say that $\mathcal{P}$ *is an ordered set*. When it is necessary to specify the order relation explicitly we write $(\mathcal{P}, \leq)$.

Let $\mathcal{P}$ be an ordered set and let $x, y \in \mathcal{P}$; we say that x is covered by y if $x < y$ and $x \leq z < y$ implies $z = x$.

Let $\mathcal{P}$ be a finite ordered set. We can represent $\mathcal{P}$ by a configuration of circles (representing the elements of $\mathcal{P}$) and interconnecting lines (indicating the covering relation). The construction goes as follows.

1. With each point $x \in \mathcal{P}$ we associate a point $p(x)$ of the Euclidean plane $\mathbb{R}^2$ depicted as a small circle.

2. For each covering pair $x < y$ in $\mathcal{P}$, take a line segment $l(x, y)$ joining the circle at $p(x)$ to the circle at $p(y)$.

3. Carry out **1** and **2** such that

   a. if $x < y$, then $p(x)$ is below than $p(y)$,

   b. the circle at $p(z)$ does not intersect the line segment $l(x, y)$ if $z \neq x$ and $z \neq y$.

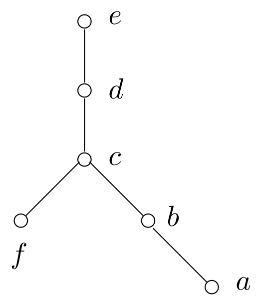

Figure 2.

A configuration satisfying **1**–**3** is a *Hasse diagram* or *diagram* of $\mathcal{P}$. In the other direction, a diagram may be used to define a finite ordered set; an example is given below, for the ordered set $\mathcal{P} = \{a, b, c, d, e, f\}$, in which $a < b < c < d < e$ and $f < c$.

We have only defined diagrams for finite ordered sets. It is not possible to represent the whole of an infinite ordered set by a diagram, but if its structure is sufficiently regular it can often be suggested diagrammatically. Of course, the same ordered set may have different diagrams. Diagram–drawing is as much an art as a science and, as we will see, good diagrams can be a real asset to understanding and to theorem–proving.

An ordered set C is a *chain* (or a *totally ordered set* or a *linearly ordered set*) if for all $p, q \in C$ we have $p \leq q$ or $q \leq p$ (*i.e.*, p and q are comparable).

Let $\prec \subseteq \mathcal{P} \times \mathcal{P}$ be a binary relation on a set $\mathcal{P}$. Then the *transitive closure* $\prec^t$ of $\prec$ is defined by $a \prec^t b$ if

$$(\exists n \in \mathbb{N})(\exists z_0, z_1, \cdots, z_n \in \mathcal{P})(a = z_0 \prec z_1 \prec z_2 \prec \cdots \prec z_{n-1} \prec z_n = b)\,.$$

Note that, the transitive closure of a binary relation $\prec$ is transitive [6, 20].

Suppose that $\mathcal{P}_1$ and $\mathcal{P}_2$ are (disjoint) ordered sets. The *disjoint union* $\mathcal{P}_1 + \mathcal{P}_2$ of $\mathcal{P}_1$ and $\mathcal{P}_2$ is the ordered set formed by defining $x \leq y$ in $\mathcal{P}_1 + \mathcal{P}_2$ if $x, y \in \mathcal{P}_1$ and $x \leq y$ in $\mathcal{P}_1$ or $x, y \in \mathcal{P}_2$ and $x \leq y$ in $\mathcal{P}_2$. A diagram for $\mathcal{P}_1 + \mathcal{P}_2$ is formed by placing side by side diagrams for $\mathcal{P}_1$ and $\mathcal{P}_2$.

2.3 Graphs

A *graph* is a pair $G = (V, E)$ of sets satisfying $E \subseteq V^2$; thus, the elements of E are 2–elements subsets of V such that $V \cap E = \emptyset$. The elements of V are the vertices of the graph G, the elements of E are its edges or arrows. If $G' = (V', E')$, $V' \subseteq V$, $E' \subseteq E$, then G' is a *subgraph* of G [9, 19].

A graph with vertex set V is said to be a graph on V. The vertex set of a graph is referred to as $V(G)$, its edge set as $E(G)$. We write $v \in G$ for a vertex $v \in V(G)$ and $e \in G$ for an edge $e \in E(G)$; an edge $\{x, y\}$ is usually written as xy or yx.

A *path* is a non–empty graph $P = (V, E)$ of the form

$$\begin{aligned} V &= \{x_0, x_1, \cdots, x_k\}, \\ E &= \{x_0x_1, x_1x_2, \cdots, x_{k-1}x_k\}, \end{aligned}$$

where the x_i are all different. The vertices x_0 and x_k are *linked* by P and are its *ends*, the vertices $x_1, \cdots, x_{k-1}$ are the inner vertices of P. The number of edges of a path is its *length* and a path of length k is denoted by P^k. We often refer to a path by the natural sequence of its vertices writing $P = x_0x_1 \cdots x_k$ and calling P a path from x_0 to x_k. If it is not necessary to make explicit the inner vertices of a path $P = x_0x_1 \cdots x_k$ then we write simply $x_0||x_k$. Note that if P is a path from x_0 to x_k without inner vertices then $x_0||x_k$ is the edge $\{x_0, x_k\}$. If $P = x_0x_1 \cdots x_k$, $Q = x_kx_{k+1} \cdots x_n$ are paths then the *concatenation* PQ of P and Q is a path such that $PQ = x_0x_1 \cdots x_kx_{k+1} \cdots x_n$.

A non–empty graph $G = (V, E)$ is *connected* if any two of its vertices are linked by a path in G. A maximal connected subgraph of G is a *component* of G.

The vertices of a connected graph can always be enumerated, let us say as $v_1, \cdots, v_n$. Furthermore, a component, being connected, is always non–empty.

The *degree* $d_G(v) = d(v)$ of a vertex v is the number $|E(v)|$ of edges at v.

A *directed graph* is a pair (V, E) of disjoint sets (of vertices and edges) together with two maps, init : $E \to V$, ter : $E \to V$, assigning to every edge an *initial vertex*, init(e) and a *terminal vertex*, ter(e). The edge e is said to be directed from init(e) to ter(e). Note that a directed graph may have several edges between the same two vertices x, y. Such edges are called *multiples edges* ; if they have the same direction, they are *parallel*.

If $\mathrm{init}(e) = \mathrm{ter}(e)$, then the edge e is called a *loop*.

A directed graph D is an *orientation* of an (undirected) graph G if $V(D) = V(G)$ and $E(D) = E(G)$ and if $\{\mathrm{init}(e), \mathrm{ter}(e)\} = \{x, y\}$ for every edge $e = xy$. Put differently, *oriented graphs* are directed graphs without loops or multiple edges.

A *directed path* is a directed graph $P \neq \emptyset$ with different vertices $x_0, \cdots, x_k$ and edges $e_0, \cdots, e_{k-1}$ such that e_i is a directed edge from x_i to x_{i+1}, for all $i < k$. We denote the last vertex x_k of P by $\mathrm{ter}(P)$. Henceforth, *path* always means directed path.

3 Partitions and representations of posets over $\mathbb{N}$

The following definition of a partition of a natural number n is given in [15].

Definition. A partition λ of *weight* n is a nonincreasing sequence of nonnegative integers $(\lambda_1, \lambda_2, \cdots, \lambda_k, \cdots)$ whose sum is n. The weight is denoted by $|\lambda|$. The number of nonzero elements in the sequence is the *length* of the partition denoted by $l(\lambda)$. There is one partition of weight 0, the partition of length 0.

There are several partial orders on the set of all partitions; for example, the *dominance order* is defined in such a way that the partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k, \cdots)$ is dominated by the partition $\mu = (\mu_1, \mu_2, \cdots, \mu_k, \cdots)$, denoted by $\lambda \ltimes \mu$, if $\sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i$, for all $k > 0$ [15].

Remark 3. *A partition λ is called graphical if it is the degree sequence of an undirected simple graph. In [15] a characterization of graphical partitions is given by using the dominance order of partitions.*

Let H be a set of nonnegative integers. Let $\underline{H}$ (respectively, $\overline{H}$) denote the set of all partitions (respectively, compositions) whose parts lie in H.

3.1 Representations of posets

Let $(\mathbb{N}, \leq)$ be the set of natural numbers equipped with its natural ordering and $(\mathcal{P}, \trianglelefteq)$ a non–empty poset. A *representation* of $\mathcal{P}$ over $\mathbb{N}$ is a system of the form

$$\Delta = (\Delta_0;\, (n_x, \lambda_x) \mid x \in \mathcal{P})\,, \tag{4}$$

where $\Delta_0 \subseteq \mathbb{N}$, $\Delta_0 \neq \emptyset$, $n_x \in \mathbb{N}$, for each $x \in \mathcal{P}$, λ_x is a partition of n_x, $\lambda_x \in \underline{\Delta_0}$ (*i.e.*, λ_x is a partition of n_x with parts in Δ_0). In particular, if $n_x = 0$, then we consider $\lambda_x = 0$. Furthermore,

$$x \trianglelefteq y \,\text{in}\, \mathcal{P} \quad \text{implies} \quad n_x \leq n_y \quad \text{and} \quad \lambda_x \ltimes \lambda_y\,. \tag{5}$$

We assume $\Delta_0 = \mathbb{N}$ if it is not explicitly mentioned in a representation Δ of a poset $\mathcal{P}$.

For example if $\Delta_0 = \{\rho_k^5 = \frac{k(3k-1)}{2} \mid k > 0\}$ is the set of pentagonal numbers of positive rank and $\mathcal{P}$ is the poset given in figure 2 then a representation of $\mathcal{P}$ over $\mathbb{N}$ is given by the following formulae:

$$\begin{aligned}
(n_a, \lambda_a) &= (5, (1^5))\,,\\
(n_b, \lambda_b) &= (25, (5^5))\,,\\
(n_c, \lambda_c) &= (60, (12^5))\,,\\
(n_d, \lambda_d) &= (110, (22^5))\,,\\
(n_e, \lambda_e) &= (175, (35^5))\,,\\
(n_f, \lambda_f) &= (17, (5^1 12^1))\,.
\end{aligned} \tag{6}$$

We say that two representations over $\mathbb{N}$, $\Delta_1 = \{\Delta_0;\, (n_x, \lambda_x) \mid x \in \mathcal{P}\}$ and $\Delta_2 = \{\Delta_0';\, (n_x', \lambda_x') \mid x \in \mathcal{P}\}$ of a given poset $(\mathcal{P}, \trianglelefteq)$ are equivalent if $\Delta_0 = \Delta_0'$ and $n_x = n_x'$ for each $x \in \mathcal{P}$. This definition allows us to formulate the following question:

Fundamental Problem. To fully characterize the equivalence classes of representations over $\mathbb{N}$ of a given poset $(\mathcal{P}, \trianglelefteq)$, giving for each $x \in \mathcal{P}$, the number of partitions of n_x with parts in a fixed $\Delta_0 \subseteq \mathbb{N}$.

3.2 The associated graph

Given a set $H \subseteq \mathbb{N}$, a subset $\mathcal{H} \subseteq \overline{H}$ and a representation

$$\Delta = (\Delta_0;\, (n_x, \lambda_x) \mid x \in \mathcal{P})\,,$$

for a poset $(\mathcal{P}, \leq)$, sometimes it is possible to associate with it a suitable oriented graph $\mathcal{I}$ which contains information about compositions of all numbers n_x with parts in the set H. In this case, $V(\mathcal{I}) = \mathcal{P}$ and there exist an oriented edge $e = \{x, y\} \in \mathcal{I}$ if and only if x is covered by y in $\mathcal{P}$ and there are compositions $\gamma_x, \gamma_y \in \mathcal{H}$ for the numbers $n_x, n_y \in \mathbb{N}$, respectively, which represent points x, y in the representation Δ. If such a graph $\mathcal{I}$ exists then we say that $\mathcal{H}$ associates the graph $\mathcal{I}$ with the representation Δ. In other words, $\mathcal{I}$ is an *associated graph* associated with the representation Δ by $\mathcal{H}$. Henceforth, if there exist an associated graph $\mathcal{I}$ associated with a given representation Δ of a poset $\mathcal{P}$ then we say that the points of $\mathcal{P}$ are vertices of $\mathcal{I}$, conversely that vertices of $\mathcal{I}$ are points of $\mathcal{P}$.

4 Partitions into four cubes

Theorem 4 below shows that the graph Γ suggested in figure 3 is an associated graph associated with a suitable representation of the infinite poset, $(\mathcal{R}, \trianglelefteq^t)$, where $\mathcal{R} = \{v_{ij} \mid i, j \in \mathbb{N}\}$ and $\trianglelefteq^t$ is the transitive closure of the binary relation $\trianglelefteq$ defined on $\mathcal{R}$ by the following rules (see, Remark 1), with $\epsilon_t = 2(\sigma_t - t)$, $t \geq 1$:

a. $v_{ij} \trianglelefteq v_{ij}$ for all $i, j \geq 0$,

b. $v_{(\sigma_h-h)(h)} \trianglelefteq v_{(\sigma_{h+1}-(h+1))(h+1)}$, $v_{(\sigma_h-h)(h+2)} \trianglelefteq v_{(\sigma_{h+1}-(h+1))(h+3)}$ for all $h \geq 0$. Furthermore

$$v_{(\sigma_h-h)(h)} \trianglelefteq v_{(\sigma_h-h)(h+1)} \trianglelefteq v_{(\sigma_h-h)(h+2)} \trianglelefteq v_{(\sigma_h-h)(h+3)}\,;$$

c. $v_{(\sigma_h+\sigma_i-(h+i))(h+i+2)} \trianglelefteq v_{(\sigma_h+\sigma_j-(h+j))(h+j+2)}$, for all $0 \leq i \leq j \leq h$,

d. for fixed $i \geq 0$,

$$v_{(\sigma_i-i)(i)} \trianglelefteq v_{(\sigma_i+\sigma_j-(i+j))(i+j)} \trianglelefteq v_{(\sigma_i+\sigma_k-(i+k))(i+k)}\,,$$

for all j, k such that $0 \leq j \leq k$,

e. $v_{(\epsilon_h)(2h+1)} \trianglelefteq v_{(\epsilon_i)(2i+1)}$, for all $1 \leq h \leq i$,

f. For fixed $h \geq 2$, $k \geq 1$,

$$\begin{aligned} v_{(\epsilon_h+\sigma_k-k)(2h+k)} &\trianglelefteq v_{(\epsilon_h+\sigma_{k+1}-(k+1))(2h+k+1)}\,, \\ v_{(\epsilon_h+\sigma_k+\sigma_i-(k+i))(2h+k+i)} &\trianglelefteq v_{(\epsilon_h+\sigma_k+\sigma_j-(k+j))(2h+k+j)}\,, \end{aligned}$$

if $0 \leq i \leq j \leq k$.

There are no other relations of type $\trianglelefteq$ between points of $\mathcal{R}$. Actually, $(\mathcal{R}, \trianglelefteq)$ induces chains given by the following formulae in $(\mathcal{R}, \trianglelefteq^t)$:

$$\begin{array}{rcll}
v_{(\sigma_h-h)(h)} & \trianglelefteq & v_{(\sigma_i-i)(i)}\,, & \\
v_{(\sigma_h-h)(h+2)} & \trianglelefteq & v_{(\sigma_i-i)(i+2)}\,, & 0 \leq h \leq i\,, \\
v_{(\sigma_i-i)(i)} & \trianglelefteq & v_{(\sigma_i-i)(i+1)} \trianglelefteq v_{(\sigma_i-i)(i+2)} & \\
 & \trianglelefteq & v_{(\sigma_i-i)(i+3)}\,, & i \geq 0\,, \\
v_{(\sigma_h-h)(h)} & \trianglelefteq & v_{(\sigma_i-i)(i)} & \\
 & \trianglelefteq & v_{(\sigma_i+\sigma_j-(i+j))(i+j)}, & 0 \leq h \leq i,\ j \geq 0\,, \\
v_{(\sigma_i+\sigma_j-(i+j))(i+j)} & \trianglelefteq & v_{(\sigma_i+\sigma_k-(i+k))(i+k)}, & i \geq 0,\ 0 \leq j \leq k\,, \\
v_{(\sigma_h+\sigma_i-(h+i))(h+i+2)} & \trianglelefteq & v_{(\sigma_h+\sigma_j-(h+j))(h+j+2)}, & 0 \leq i \leq j \leq h\,, \\
v_{(\epsilon_h)(2h+1)} & \trianglelefteq & v_{(\epsilon_i)(2i+1)}\,, & 1 \leq h \leq i\,.
\end{array} \tag{7}$$

Furthermore, for fixed $h \geq 2$, $i \geq 1$:

$$\begin{array}{rcll}
v_{(\epsilon_h+\sigma_i+\sigma_j-(i+j))(2h+i+j)} & \trianglelefteq & v_{(\epsilon_h+\sigma_i+\sigma_k-(i+k))(2h+i+k)}\,, & \\
 & & \qquad 0 \leq j \leq k \leq i\,, & \\
v_{(\epsilon_h+\sigma_i-i)(2h+i)} & \trianglelefteq & v_{(\epsilon_h+\sigma_{i+1}-(i+1))(2h+i+1)}\,, & \\
v_{(\epsilon_h+\sigma_i-i)(2h+i)} & \trianglelefteq & v_{(\epsilon_h+\sigma_i+\sigma_j-(i+j))(2h+i+j)}, & j \geq 1\,.
\end{array} \tag{8}$$

For example:

$$\begin{array}{rcl}
v_{(0,0)} & \trianglelefteq & v_{(0,1)} \trianglelefteq v_{(0,2)} \trianglelefteq v_{(0,3)} \trianglelefteq v_{(0,4)}\,, \\
v_{(0,0)} & \trianglelefteq & v_{(0,1)} \trianglelefteq v_{(2,2)} \trianglelefteq v_{(7,3)} \cdots\,, \\
v_{(0,0)} & \trianglelefteq & v_{(0,1)} \trianglelefteq v_{(0,2)} \trianglelefteq v_{(2,3)} \trianglelefteq v_{(7,4)} \cdots\,, \\
v_{(0,3)} & \trianglelefteq & v_{(4,5)} \trianglelefteq v_{(14,7)} \trianglelefteq v_{(32,9)} \trianglelefteq \cdots\,.
\end{array} \tag{9}$$

A representation over $\mathbb{N}$ is defined for $(\mathcal{R}, \trianglelefteq^t)$ by assigning to each $v_{ij} \in \mathcal{R}$ the pair $(n_{ij}, \lambda_{ij}) = (4+6i+7j, ((4+6i+7j)^1))$; we denote this representation by $\mathcal{R}_\mathcal{Q}$ and write $v_{ij} \in \mathcal{R}_\mathcal{Q}$ if in this representation the number $n_{ij} = 4+6i+7j$ is assigned to the point $v_{ij} \in \mathcal{R}$.

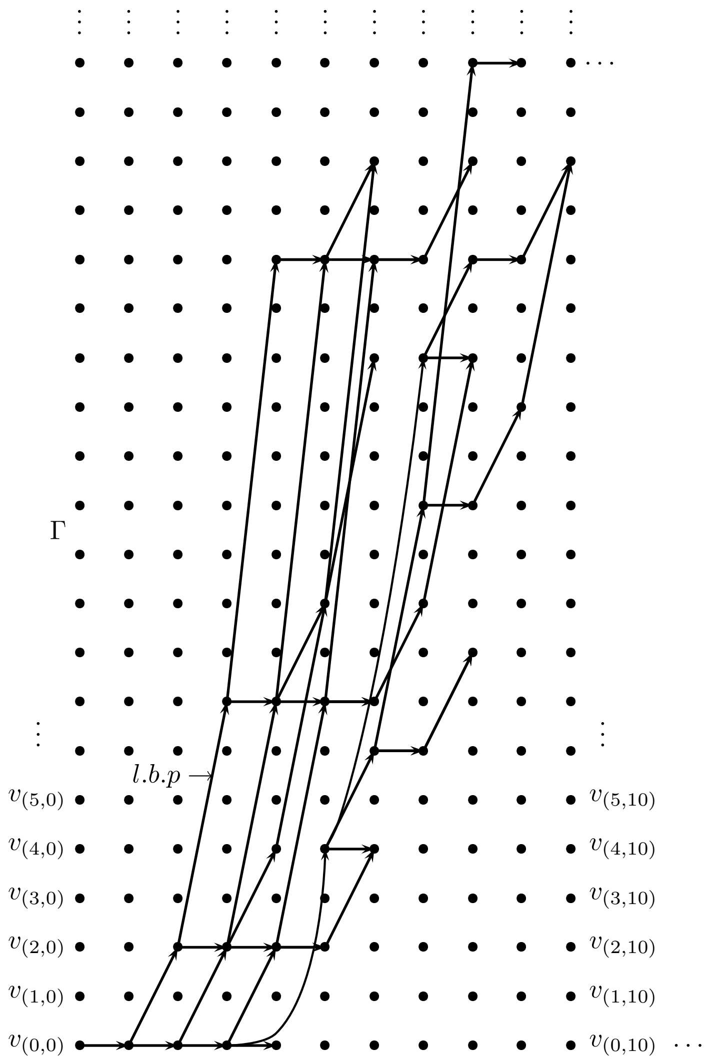

Figure 3.

We now define an oriented graph Γ such that $V(\Gamma) = \mathcal{R}$ and there is an oriented edge $e = \{v_{ij}, v_{kl}\}$ linking the vertices $v_{ij}, v_{kl} \in V(\Gamma)$ if and only if v_{ij} is covered by v_{kl} in the poset $(\mathcal{R}, \trianglelefteq^t)$.

Remark 4. *Note that, there is an oriented path $P = v_{ij}||v_{kl}$ linking vertices $v_{ij}, v_{kl} \in V(\Gamma)$, $v_{ij} \neq v_{kl}$ if and only if $v_{ij} \triangleleft^t v_{kl}$.*

Let *l.b.p.* denote the *left boundary path* of Γ such that $V(l.b.p.) = \{v_{(\sigma_i - i)(i)} \mid i \geq 0\}$ and $e \in E(l.b.p.)$ if and only if e has the form $\{v_{(\sigma_h - h)(h)}, v_{(\sigma_{h+1} - (h+1))(h+1)}\}$, $h \geq 0$.

The following theorem proves that the set $\mathcal{H}_{\mathcal{Q}}^{+} \subset \overline{\mathbb{N}}$ consisting of compositions of four cubes with two of them equal associates the graph Γ with the representation $\mathcal{R}_{\mathcal{Q}}$.

Theorem 4. *A number $m \in \mathbb{N}\backslash\{0\}$ is the sum of four positive cubes with two of them equal if and only if there exists a vertex v_{ij} in a non–trivial component of Γ for which $m = n_{ij}$ in the representation $\mathcal{R}_{\mathcal{Q}}$.*

Proof. If m represents a vertex v_{rs} in a non–trivial component of Γ, $v_{rs} \in \mathcal{R}_{\mathcal{Q}}$, $n_{rs} = m$ then we have the following four cases (see equations (7) and (8)):

i. $v_{rs} = v_{(\sigma_i - i)(i)} \in l.b.p.$ for some $i \geq 0$; in this case $n_{rs} = 4 + 6(\sigma_i - i) + 7i = 4 + 6\sigma_i + i = 2 + (i+1)^3 + 1$. Therefore, m can be written as a sum of four positive cubes with two of them equal.

ii. $v_{rs} = v_{(\sigma_i + \sigma_j - (i+j))(i+j)}$, for some $i, j > 0$; in this case $n_{rs} = 4 + 6(\sigma_i + \sigma_j) + i + j = (i+1)^3 + (j+1)^3 + 2$. Thus, m is a sum of four cubes with two of them equal. Moreover, for each $i \geq 0$, vertices $v_{(\sigma_i - i)(i+2)}$ and $v_{(\sigma_i - i)(i+3)}$, in a non–trivial component of Γ can be represented by numbers of the form $n_{(\sigma_i - i)(i+2)} = 16 + (i+1)^3 + 1$ and $n_{(\sigma_i - i)(i+3)} = 16 + (i+1)^3 + 8$, respectively.

iii. $v_{rs} = v_{(\sigma_i + \sigma_j - (i+j))(i+j+2)}$, for some $i > 0$, $0 \leq j \leq i$; in this case $n_{rs} = 18 + 6(\sigma_i + \sigma_j) + i + j = 16 + (j+1)^3 + (i+1)^3$. Therefore, m is a sum of four cubes with two of them equal.

iv. $v_{rs} = v_{(\epsilon_h + \sigma_i + \sigma_j - (i+j))(2h+i+j)}$, for some $h \geq 2$, $i \geq 1$ and $0 \leq j \leq i$; this case $n_{rs} = 4 + 6(\epsilon_h + \sigma_i + \sigma_j) + i + j + 14h = 2(h+1)^3 + (i+1)^3 + (j+1)^3$. Thus m is a sum of four cubes with two of them equal.

Conversely, if $m \in \mathbb{N}$ can be written as a sum of four positive cubes with two of them equal then there are seven cases (see the identities given above):

i. $m = 2 + (i+1)^3 + 1$, for some $i \geq 0$; in this case m represents the vertex $v_{(\sigma_i - i)(i)} \in l.b.p.$, (Note that, if $i \neq 0$ then $v_{(\sigma_i - i)(i)}$ belongs to the oriented path $v_{(0,0)} || v_{(\sigma_i - i)(i)}$, with oriented edges of the form $\{v_{(\sigma_h - h)(h)}, v_{(\sigma_{h+1} - (h+1))(h+1)}\}$, $0 \leq h \leq i - 1$).

ii. $m = 2 + (i+1)^3 + (j+1)^3$, for some $i, j > 0$; in this case m represents the vertex $v_{(\sigma_i + \sigma_j - (i+j))(i+j)}$ in the oriented path $v_{(\sigma_i - i)(i)} || v_{(\sigma_i + \sigma_j - (i+j))(i+j)}$ with oriented edges of the form

$$\{v_{(\sigma_i+\sigma_h-(i+h))(i+h)}, v_{(\sigma_i+\sigma_{h+1}-(i+h+1))(i+h+1)}\},$$

$0 \leq h \leq j-1$.

iii. $m = 16 + (i+1)^3 + 1$, for some $i > 0$; in this case m represents the vertex $v_{(\sigma_i - i)(i+2)}$ which belongs to the oriented path $v_{(0,2)}||v_{(\sigma_i-i)(i+2)}$ with oriented edges of the form

$$\{v_{(\sigma_h-h)(h+2)}, v_{(\sigma_{h+1}-(h+1))(h+3)}\},$$

where $0 \leq h \leq i-1$.

iv. $m = 16 + (i+1)^3 + (j+1)^3$, $i, j > 0$; in this case m represents the vertex $v_{(\sigma_i+\sigma_j-(i+j))(i+j+2)}$ in the oriented path

$$v_{(\sigma_i-i)(i+2)}||v_{(\sigma_i+\sigma_j-(i+j))(i+j+2)},$$

with oriented edges of the form

$$\{v_{(\sigma_i+\sigma_h-(i+h))(i+h+2)}, v_{(\sigma_i+\sigma_{h+1}-(h+i+1))(i+h+3)}\},$$

where $0 \leq h \leq j-1$.

v. $m = 2(i+1)^3 + 8 + 1 = 11 + 6\epsilon_i + 14i$, for some $i > 1$; in this case m represents the vertex $v_{(\epsilon_i)(2i+1)}$ in the path $v_{(0,3)}||v_{(\epsilon_i)(2i+1)}$, with oriented edges of the form $\{v_{(0,3)}, v_{(4,5)}\}$ if $i = 2$ and of the form

$$\{v_{(0,3)}, v_{(4,5)}\}, \qquad \{v_{(\epsilon_h)(2h+1)}, v_{(\epsilon_{h+1})(2h+3)}\},$$

where $2 \leq h \leq i-1$ if $i > 2$.

vi. $m = 2(i+1)^3 + (j+1)^3 + 1$, for some $i > 1$, $j > 1$; in this case m represents the vertex $v_{(\epsilon_i+\sigma_j-j)(2i+j)}$, in the oriented path $v_{(\epsilon_i)(2i+1)}||v_{(\epsilon_i+\sigma_j-j)(2i+j)}$, with oriented edges of the form

$$\{v_{(\epsilon_i+\sigma_h-h)(2i+h)}, v_{(\epsilon_i+\sigma_{h+1}-(h+1))(2i+h+1)}\},$$

$1 \leq h \leq j-1$.

vii. $m = 2(i+1)^3 + (j+1)^3 + (k+1)^3$, for some $i > 1$, $j > 0$, $0 < k \leq j$, therefore m represents the vertex $v_{(\epsilon_i+\sigma_j+\sigma_k-(j+k))(2i+j+k)}$, in the oriented path $v_{(\epsilon_i+\sigma_j-j)(2i+j)}||v_{(\epsilon_i+\sigma_j+\sigma_k-(j+k))(2i+j+k)}$, with oriented edges of the form

$$\{v_{(\epsilon_i+\sigma_j+\sigma_h-(h+j))(2i+h+j)}, v_{(\epsilon_i+\sigma_j+\sigma_{h+1}-(h+j+1))(2i+h+j+1)}\},$$

with $0 \leq h \leq k-1$.

This finishes the proof.

Remark 7. *Note that the representation $\mathcal{R}_{\mathcal{Q}}$ described above induces an equivalence relation $\sim$ on $\mathcal{R}$ such that $v_{ij} \sim v_{kl}$ if $n_{ij} = n_{kl}$. Let $\overline{v_{ij}} = [v_{ij}]$ denote the equivalence class of the point $v_{ij} \in \mathcal{R}_{\mathcal{Q}}$.*

We say that a directed path $P \in \Gamma$ is *admissible* either:

1. P is of type I. That is, $P = P_{i0} = v_{00}||v_{(\sigma_i-i)(i)}$, $i > 0$, with inner vertices of the form $v_{(\sigma_h-h)(h)}$, $0 < h < i$ and directed edges of the form $\{v_{(\sigma_m-m)(m)}, v_{(\sigma_{(m+1)}-(m+1))(m+1)}\}$, $0 \leq m \leq i-1$,

2. P is of type II. That is, P a concatenation of the form $P = P_{i0}P_{ij}$, where $P_{ij} = v_{(\sigma_i-i)(i)}||v_{(\sigma_i+\sigma_j-(i+j))(i+j)}$, $0 < j \leq i$ with inner vertices of the form $v_{(\sigma_i+\sigma_l-(i+l))(i+l)}$, $0 < l < j$ and directed edges of the form $\{v_{(\sigma_i+\sigma_m-(i+m))(i+m)}, v_{(\sigma_i+\sigma_{m+1}-(i+m+1))(i+m+1)}\}$, $0 \leq m \leq j-1$,

3. P is of type III. That is, P is a concatenation of the form P_{2i} or $P_{2i}P_{2ij}$, where P_{2i} is a translation of the left boundary path of the form $v_{(0,2)}||v_{(\sigma_i-i)(i+2)}$ for $i \geq 1$ and for fixed $i \geq 1$, $0 \leq j \leq i$, P_{2ij} is an oriented path such that, $V(P_{2ij}) = \{v_{(\sigma_i+\sigma_n-(i+n))(i+n+2)} \mid 0 \leq n \leq j\}$ (we assume, $P_{2i}P_{2ij} = P_{2i}$ if $j = 0$) and $e \in E(P_{2ij})$ if and only if e is an directed edge of the form

$$\{v_{(\sigma_i+\sigma_m-(i+m))(i+m+2)}, v_{(\sigma_i+\sigma_{m+1}-(i+m+1))(i+m+3)}\},$$

$0 \leq m \leq i-1$,

4. P is of type IV. That is, P is a concatenation of the form

$$P_{\epsilon_h}, P_{\epsilon_h} P_{\epsilon_h k}\,,$$

or $P_{\epsilon_h} P_{\epsilon_h k} P_{\epsilon_h kj}$, where P_{ϵ_h} is an oriented path of the form

$$v_{(0,3)} || v_{(\epsilon_h)(2h+1)}\,,$$

$h \geq 2$, with inner vertices of the form $v_{(\epsilon_j)(2j+1)}$, $2 \leq j \leq h-1$ and oriented edges of the form $\{v_{(0,3)}, v_{(4,5)}\}$, $\{v_{(\epsilon_r)(2r+1)}, v_{(\epsilon_{r+1})(2r+3)}\}$, if $2 \leq r \leq h-1$ and $h > 2$.

For fixed $k \geq 1$, $h \geq 2$, $P_{\epsilon_h k}$ is an oriented path such that (in the particular case $k=1$, we assume $P_{\epsilon_h k} = v_{(\epsilon_h)(2h+1)}$ and $P_{\epsilon_h} P_{\epsilon_h k} = P_{\epsilon_h}$), $V(P_{\epsilon_h k}) = \{v_{(\epsilon_h + \sigma_m - m)(2h+m)} \mid 1 \leq m \leq k\}$, with oriented edges of the form, $\{v_{(\epsilon_h+\sigma_n-n)(2h+n)}, v_{(\epsilon_h+\sigma_{n+1}-(n+1))(2h+n+1)}\}$, $1 \leq n \leq k-1$.

Finally, $P_{\epsilon_h kj}$ is an oriented path such that

$$V(P_{\epsilon_h kj}) = \{v_{(\epsilon_h+\sigma_k+\sigma_m-(k+m))(2h+k+m)} \mid 0 \leq m \leq j\}\,,$$

$0 \leq j \leq k$, (we assume, $P_{\epsilon_h P_{\epsilon_h k} P_{\epsilon_h kj}} = P_{\epsilon_h} P_{\epsilon_h k}$, if $j=0$), $e \in E(P_{\epsilon_h kj})$ if and only if e has the form

$$\{v_{(\epsilon_h+\sigma_a+\sigma_k-(k+a))(2h+k+a)}, v_{(\epsilon_h+\sigma_k+\sigma_{a+1}-(k+a+1))(2h+k+a+1)}\}\,,$$

$0 \leq a \leq j-1$.

Note that if $v_{ij} \in \Gamma$ belongs to a non–trivial component and n_{ij} represents v_{ij}, $v_{ij} \in \mathcal{R}_{\mathcal{Q}}$ then there exists a set $\mathcal{A}$ of admissible paths of the form $P = v_{(a_0 b_0)} || v_{rs}$, with $v_{rs} \in [v_{ij}]$ and $v_{(a_0 b_0)} \in \{v_{(0,0)}, v_{(0,2)}, v_{(0,3)}\}$. In this case we can associate with each $P \in \mathcal{A}$ a composition of n_{ij} into four positive cubes with two of them equal in the following way:

a. A composition of the form $1+1+(i+1)^3+1$, is associated with an admissible path of type I with last vertex $v_{(\sigma_i - i)(i)}$, $i \geq 0$. We assume that the composition $1+1+1+1$ is associated with the vertex $v_{(0,0)}$.

b. A composition of the form $1+1+(i+1)^3+(j+1)^3$ is associated with an admissible path P of type II with last vertex $v_{(\sigma_i+\sigma_j-(i+j))(i+j)}$, $0<j\leq i$.

c. A composition of the form $8+8+(i+1)^3+(j+1)^3$ is associated with an admissible path of type III with last vertex $v_{(\sigma_i+\sigma_j-(i+j))(i+j+2)}$, for $i>0$ and $0\leq j\leq i$.

d. A composition of the form $(h+1)^3+(h+1)^3+(i+1)^3+(j+1)^3$ is associated with an admissible path $P_{\epsilon_h}P_{\epsilon_h i}P_{\epsilon_h ij}$ of type IV with last vertex $v_{(\epsilon_h+\sigma_i+\sigma_j-(i+j))(2h+i+j)}$, $h\geq 2$, $0<j\leq i$. In particular, a composition of the form $(h+1)^3+(h+1)^3+8+1$ is associated with an admissible path P_{ϵ_h} of type IV with last vertex $v_{(\epsilon_h)(2h+1)}$. Finally, a composition of the shape $(h+1)^3+(h+1)^3+(j+1)^3+1$ is associated with an admissible path $P_{\epsilon_h}P_{\epsilon_h j}$ with last vertex $v_{(\epsilon_h+\sigma_j-j)(2h+j)}$, $h\geq 2$, $j>1$.

We say that compositions given by **a**–**d** are *compositions of type* $\Gamma_{\mathcal{Q}}$. Let $c(n,\Gamma_{\mathcal{Q}})$ denote the number of compositions of type $\Gamma_{\mathcal{Q}}$ for the natural number n. Note that $c(4,\Gamma_{\mathcal{Q}})=c(25,\Gamma_{\mathcal{Q}})=1$.

Figure 4 below shows examples of the four types of admissible paths ($v_{(0,0)}||v_{(7,3)}$, $v_{(0,0)}||v_{(9,5)}$, $v_{(0,2)}||v_{(4,6)}$, $v_{(0,3)}||v_{(8,8)}$) with associated compositions of the form $1+1+64+1$, $1+1+64+27$, $8+8+27+27$, $27+27+27+27$, respectively.

If $\sim$ is an equivalence relation defined on a set A then we say that a set $\{g_i\mid i\in I\}\subset A$ is a *minimal set of representatives* if contains one element of each equivalence class.

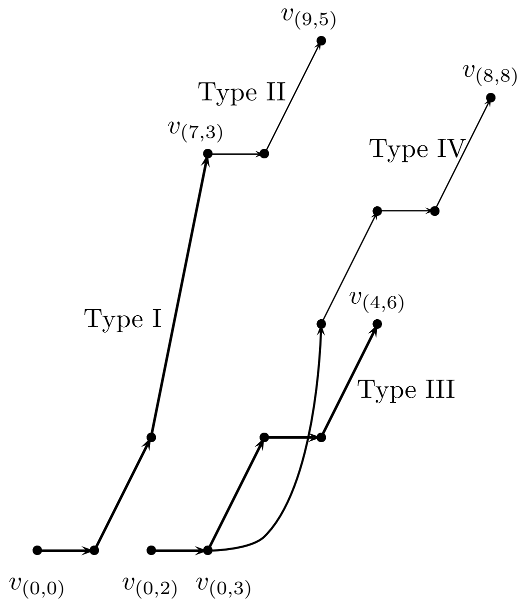

Figure 4.

Given the representation $\mathcal{R}_{\mathcal{Q}}$, let $\mathcal{G}^{\mathcal{Q}} = \{g_{ij} \mid g_{ij} \in [v_{ij}]\}$ denote a fixed minimal set of representatives of classes $[v_{ij}]$ of points $v_{ij} \in \mathcal{R}_{\mathcal{Q}}$ and for each $g_{ij} \in \mathcal{G}^{\mathcal{Q}}$, $g_{ij} \neq v_{(0,0)}$ equivalent to a vertex in a non–trivial component of Γ, let $\mathcal{A}^{\mathcal{Q}}(g_{ij})$ denote the set of admissible paths with final vertex in $[g_{ij}]$. We assume $\mathcal{A}^{\mathcal{Q}}(g_{(0,0)}) = \{v_{(0,0)}\}$ and $\mathcal{A}^{\mathcal{Q}}(g_{ij}) = \emptyset$ if $[g_{ij}]$ contains no vertex in a non–trivial component of Γ. Since by definition for each $g_{ij} \in \mathcal{G}^{\mathcal{Q}}$ equivalent to a vertex in a non–trivial component of Γ there exists a bijection between the set of compositions of type $\Gamma_{\mathcal{Q}}$ for $n = n_{ij}$ representing the vertex g_{ij} and the set of admissible paths with final vertices in $[g_{ij}]$, $v_{(0,0)} \notin [g_{ij}]$. $|\mathcal{A}^{\mathcal{Q}}(g_{ij})| = 0$, if $[g_{ij}]$ contains no vertex in a non–trivial component of Γ and $c(4, \Gamma_{\mathcal{Q}}) = c(18, \Gamma_{\mathcal{Q}}) = c(25, \Gamma_{\mathcal{Q}}) = 1$, we can conclude the following consequence of Theorem 4.

Corollary 8. *For each $n \in \mathbb{N}$;*

$$c(n, \Gamma_{\mathcal{Q}}) = \begin{cases} |\mathcal{A}^{\mathcal{Q}}(g_{ij})|, & \textit{if for some } i, j \geq 0,\ n = n_{ij} \\ & \textit{represents } g_{ij} \in \mathcal{G}^{\mathcal{Q}}, \\ 0, & \textit{otherwise.} \end{cases}$$

For example, if $v_{(11,0)} = g_{(11,0)} \sim v_{(4,6)}$ then $\mathcal{A}^{\mathcal{Q}}(g_{(11,0)}) = \{P, Q\}$, where P, Q are admissible paths such that

$$
\begin{aligned}
P &= \{v_{(0,2)}, v_{(0,3)}, v_{(2,4)}, v_{(2,5)}, v_{(4,6)}\}\,, \\
Q &= \{v_{(0,3)}, v_{(4,5)}, v_{(4,6)}\}\,.
\end{aligned}
$$

Therefore $c(70, \Gamma_{\mathcal{Q}}) = 2$. Note that $8+8+27+27 = 27+27+8+8$ are the compositions of type $\Gamma_{\mathcal{Q}}$ for 70.

If $\mathcal{A}^{\mathcal{Q}}(g_{ij}) \neq \emptyset$, $(g_{ij} \neq v_{(0,0)})$ is, as before, a fixed set of admissible paths then we say that two admissible paths $P, Q \in \mathcal{A}^{\mathcal{Q}}(g_{ij})$ are *equivalent* if they have compositions of type $\Gamma_{\mathcal{Q}}$ associated with the same four parts of the form (h^3, h^3, i^3, j^3). If $P = v_{(a_0 b_0)}||v_{ij}$ is an admissible path then let $[P]$ denote the class of P.

Now, let $\mathcal{L}^{\mathcal{Q}}(g_{ij}) \subseteq \mathcal{A}^{\mathcal{Q}}(g_{ij})$, $T^{\mathcal{Q}}(g_{ij})$ denote a fixed minimal set of representatives of admissible paths and the set of final vertices of such representatives respectively. We assume $\mathcal{L}^{\mathcal{Q}}(g_{(0,0)}) = \mathcal{A}^{\mathcal{Q}}(g_{(0,0)}) = \{v_{(0,0)}\}$ and $T^{\mathcal{Q}}(g_{(0,0)}) = \emptyset$. If $g_{ij} \in \mathcal{G}^{\mathcal{Q}}$ and $[g_{ij}]$ does not contain a vertex in a non–trivial component of Γ then $\mathcal{L}^{\mathcal{Q}}(g_{ij}) = T^{\mathcal{Q}}(g_{ij}) = \emptyset$.

For each $g_{ij} \in \mathcal{G}^{\mathcal{Q}}$, $g_{ij} \neq g_{(0,0)}$, let $\delta^{\mathcal{Q}}_{\Gamma(g_{ij})} : V(\Gamma) \to \mathbb{N}$ be a map such that

$$
\delta^{\mathcal{Q}}_{\Gamma(g_{ij})}(v_{rs}) = \begin{cases} N & \text{if } g_{ij} \text{ is equivalent to a vertex in a non–trivial} \\ & \quad \text{component of } \Gamma \text{ and } v_{rs} \in T^{\mathcal{Q}}(g_{ij}), \\ 0, & \textit{otherwise}, \end{cases}
$$

where N is the number of representatives of admissible paths with last vertex v_{rs}.

We define $\delta^{\mathcal{Q}}_{\Gamma(g_{(0,0)})}$ by $\delta^{\mathcal{Q}}_{\Gamma(g_{(0,0)})}(g_{(0,0)}) = 1$ and $\delta^{\mathcal{Q}}_{\Gamma(g_{(0,0)})}(v_{rs}) = 0$ for the remaining vertices $v_{rs} \in V(\Gamma)$.

The following result is a consequence of Theorem 4, the definition of compositions of type $\Gamma_{\mathcal{Q}}$ and functions $\delta^{\mathcal{Q}}_{\Gamma_{(g_{ij})}}$.

Corollary 9. *Let $\mathcal{Q}_4(n)$ denote the number of partitions of $n \in \mathbb{N}$ into four positive cubes with two of them equal; then*

$$
\mathcal{Q}_4(n) = \begin{cases} \sum_{v_{rs} \in [g_{ij}]} \delta^{\mathcal{Q}}_{\Gamma(g_{ij})}(v_{rs}), & \textit{if for some } i, j \geq 0,\ n = n_{ij} \\ & \textit{represents } g_{ij} \in \mathcal{G}^{\mathcal{Q}}, \\ 0, & \textit{otherwise}. \end{cases}
$$

Let us observe, for example, that the admissible paths

$$
\begin{aligned}
P &= \{v_{(0,2)}, v_{(0,3)}, v_{(2,4)}, v_{(2,5)}, v_{(4,6)}\}, \\
Q &= \{v_{(0,3)}, v_{(4,5)}, v_{(4,6)}\},
\end{aligned}
$$

are equivalent because they have associated the compositions $8+8+27+27$ and $27+27+8+8$, respectively. Thus, we can consider $\mathcal{L}^{\mathcal{Q}}(g_{(11,0)}) = \{P\}$ and $T^{\mathcal{Q}}(g_{(11,0)}) = \{v_{(4,6)}\}$. Therefore $\mathcal{Q}_4(70) = \delta^{\mathcal{Q}}_{\Gamma_{g_{(11,0)}}}(v_{(4,6)}) = 1$.

The following list shows the number of partitions into four positive cubes with two of them equal and compositions of type $\Gamma_{\mathcal{Q}}$ of some positive integers. Actually, it is easy to see that $\mathcal{Q}_4(n) \leq 1$ if $n \leq 150$.

n	$\mathcal{Q}_4(n)$	$c(n, \Gamma_{\mathcal{Q}})$	n	$\mathcal{Q}_4(n)$	$c(n, \Gamma_{calQ})$
4	1	1	56	1	1
11	1	1	70	1	2
15	0	0	93	1	1
20	0	0	107	1	1
30	1	1	140	0	0
40	0	0	144	1	2
44	1	1	148	0	0
54	0	0	156	1	1

References

[1] G. Andrews, *The Theory of Partitions* (Cambridge University Press, Cambridge, 1998).

[2] J. Brüdern, *On Waring's problem for cubes*, Math. Proc. Cambridge Phil. Soc. **109**, 229–256 (1991).

[3] J. Brüdern and N. Watt, *On Waring's problem for four cubes*, J. Duke Math **77**, 583–606 (1995).

[4] J. Brüdern and T.D Wooley, *On Waring's problem: Three cubes and a minicube*, preprint (2009).

[5] Ch. Chou and Y. Deng, *Decomposing 40 billion integers by four tetrahedral numbers*, Math. Comp. **66**, 893–901 (1997).

[6] B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order* (Cambridge University Press, 2002).

[7] V. Demjanenko, *On sums of four cubes*, Izv. vischik outchetsnik Zavedenii Math. **54**, 64–69 (1966).

[8] J. M. Deshouillers, F. Hennecart, B. Landreau and I. G. Putu, *7373170279850*, Math. Comput. **69**, 421–439 (2000).

[9] R. Diestel, *Graph Theory* (Springer, New York, 2005).

[10] E. Grosswald, *Representations of Integers as Sums of Squares* (Springer, New York, 1985).

[11] R. Guy, *Every number is expressible as the sum of how many polygonal numbers?*, Am. Math. Monthly **101**, 169–172 (1994).

[12] R. Guy, *Unsolved Problems in Number Theory* (Springer, New York, 2004).

[13] G. H. Hardy and J.E. Litlewood, *Some problems of Partitio Numerorum IV. The singular series in Waring's problem and the value of the number $G(k)$*, Math. Z. **12**, 161–188 (1922).

[14] A. Kempner, *Bemerkungen zum Waringschen Problem*, Math. Ann. **72**, 387–399 (1912).

[15] A. Kohnert, *Dominance order and graphical partitions*, Elect. J. Comb. **11**, 1–17 (2004).

[16] K. Koyama, *On searching for solutions of the diophantine equation $x^3 + y^3 + 2z^3 = n$*, Math. Comput. **69**, 1735–1742 (2000).

[17] Y.V. Linnik, *On the representations of large numbers as sums of seven cubes*, Mat. Sb. **12**, 218–224 (1943).

[18] K.S. McCurley, *An effective seven cube theorem*, J. Number Theory **19**, 176–183 (1984).

[19] M. Nathanson, *Additive Number Theory. The Classical Bases* (Springer, 1996).

[20] B. S. W. Schröder, *Ordered Sets. An Introduction* (Birkhäuser, Boston, 2003).

[21] G. L. Watson, *A proof of the seven cube theorem*, J. London Math. Soc **2**, 153–156 (1951).