The asymptotics of the kernel functions associated to orthogonal polynomials in several variables on the unit ball

Comportamiento asintótico de polinomios ortogonales en varias variables sobre la bola unidad

Herbert Dueñas¹, Wilmer M. Gómez²

Abstract. We consider a family of orthogonal polynomials in several variables with respect to a Sobolev-type inner product, obtained from adding a gradient operator of order \( j \), evaluated in a fixed point to a standard inner product. We study explicit relations between the Sobolev-type polynomials and the standard polynomials, among the kernel functions associated to the Sobolev-type polynomials and the kernel functions associated to the standard polynomials. In addition, an example for a particular choice of a classical measure \( \sigma \in \mathbb{R}^d \) is analyzed. Finally, we obtain the asymptotics of the some derivatives of the kernel functions evaluated in some points of the unit ball in \( d \) variables.

Keywords: Orthogonal polynomials in several variables, Asymptotics behavior, Sobolev inner products.

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¹Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá, Colombia
²Escuela de Matemáticas y Estadística, Universidad Pedagógica y Tecnológica de Colombia, Tunja, Colombia
¹haduenaasr@unal.edu.co
²wilmer.gomez@uptc.edu.co
1. Introduction

The study of Sobolev orthogonal polynomials on the real line has been evolving in the last fifty years, over all in the last twenty years, when an extensive interest by the researchers has increased because of the uses in some subjects, in particular, in the approximation theory. The study of orthogonal polynomials of several variables has been increasing as a consequence of the results on the real line, as the properties of the zeros, differential equations, and asymptotic behavior. However, one of the motivations of study of asymptotic behavior of orthogonal polynomials on several variables is that this theory has become a tool for numerical solutions of partial differential equations. In [6], we can find an interesting survey of Sobolev orthogonal polynomials.

A special attention has been paid to the families of non-standard orthogonal polynomials, within the Sobolev-type orthogonal polynomials, see [6, 7, 8] and the references therein, which are associated with an inner product defined on the linear space of polynomials with real coefficients. These inner products

$$\langle p, q \rangle = \langle p, q \rangle_\sigma + \sum_{i=0}^j \lambda_i p^{(i)}(\xi)q^{(i)}(\xi) = \int_E p(x)q(x)d\sigma(x) + \sum_{i=0}^j \lambda_i p^{(i)}(\xi)q^{(i)}(\xi),$$

(1)

where $E \subseteq \mathbb{R}$, $j \in \mathbb{N}$, $\xi \in \mathbb{R}$, $\lambda_i \in \mathbb{R}^+$ and $\sigma$ is a classical measure.

The non-standard features of this kind of inner products is the presence of derivatives and that the operator associated with the multiplication by $x$ is not symmetric; that is, for any pair of polynomials $p$ and $q$,

$$\langle xp, q \rangle \neq \langle p, xq \rangle.$$

The family of orthogonal polynomials with respect to (1) are called Sobolev-type orthogonal polynomials and the family of orthogonal polynomials with respect to $\langle p, q \rangle_\sigma$ are called standard polynomials.

We will study a particular case of the inner product (1), for orthogonal polynomials in several variables. As in [4], the derivative is replaced by a gradient operator of order $j$. We analyze the relation among the Sobolev-type polynomials and the standard polynomials, such the relation between the kernel functions associated to the Sobolev-type polynomials and the kernel functions associated to the standard polynomials.

In [4], the authors analyze the asymptotic behavior of the kernel functions associated with the Sobolev-type orthogonal polynomials on the unit ball in $d$ variables on the point $(0,0)$, where $0 = (0,0,\ldots,0)$. They obtain expressions for the kernel functions and the fourth order derivatives (after [3]). We continue with this particular choice of measure, but we analyze the asymptotic behavior not only in $(0,0)$, but also we extend the analysis in any point on the unit sphere.
2. Preliminaries: Orthogonal polynomials in several variables

Let $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_d) \in \mathbb{N}^d$ be a multi-index and $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$. A monomial in $d$ variables is given by:

$$x^\kappa = x_1^{\kappa_1} x_2^{\kappa_2} \cdots x_d^{\kappa_d}.$$ 

The integer number $|\kappa| = \kappa_1 + \kappa_2 + \cdots + \kappa_d$ is called the total degree of $x^\kappa$.

A polynomial $P(x)$ in $d$ variables is a linear combination of monomials,

$$P(x) = \sum_\kappa c_\kappa x^\kappa,$$

where the coefficients $c_\kappa$ are in a field $\mathbb{C}$, usually the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. The degree of a polynomial is defined as the highest total degree of its monomials.

For a polynomial $P(x)$ in several variables the partial derivative with respect to the $i$-th component is denoted by

$$\partial_i P(x) = \frac{\partial P(x)}{\partial x_i}.$$ 

We will denote,

- $\Pi^d$, the set of the polynomials in $d$ variables with real coefficients,

$$\Pi^d = \left\{ \sum_\kappa c_\kappa x^\kappa : c_\kappa \in \mathbb{R}, \ x \in \mathbb{R}^d \right\}.$$ 

- $\Pi^d_n$, the subspace of $\Pi^d$ consisting on the polynomials of total degree at most $n$,

$$\Pi^d_n = \left\{ \sum_{|\kappa| \leq n} c_\kappa x^\kappa : c_\kappa \in \mathbb{R}, \ x \in \mathbb{R}^d \right\}.$$ 

A polynomial is called homogeneous if its monomials have the same total degree. $\mathcal{H}^d_n$ denote the space of homogeneous polynomials of degree $n$ in $d$ variables, i.e.,

$$\mathcal{H}^d_n = \left\{ \sum_{|\kappa| = n} c_\kappa x^\kappa : c_\kappa \in \mathbb{R}, \ x \in \mathbb{R}^d \right\}.$$ 

Every polynomial in $\Pi^d_n$ can be written as a linear combination of homogeneous polynomials. Therefore, for every $P(x) \in \Pi^d_n$, we have,

$$P(x) = \sum_{i=0}^n \sum_{|\kappa| = i} c_\kappa x^\kappa.$$
A basis of $H_n^d$ is $\{x^\kappa : |\kappa| = n\}$ and the dimension of $H_n^d$ denoted by $r_n^d$ is given by

$$\dim H_n^d = r_n^d = \left(\frac{n + d - 1}{n}\right),$$

see [5]. An essential difference between polynomials in one variable and in several variables is the fact that in the last ones we do not have an obvious natural order. The natural order among monomials of one variable is the increasing degree, which is: $1, x, x^2, x^3, \ldots$. For polynomials in several variables, there are many choices of well-defined total orders. We present two possibilities.

1. **The lexicographic order.** Let $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_d)$, $\eta = (\eta_1, \eta_2, \ldots, \eta_d)$, be two multi-indexes. $\kappa \succ L \eta$ if the first non-zero entry in the difference $\kappa - \eta = (\kappa_1 - \eta_1, \kappa_2 - \eta_2, \ldots, \kappa_d - \eta_d)$ is positive. Is clear that the lexicographic order does not respect the total degree of the polynomials.

2. **The graded lexicographic order.** Let $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_d)$, $\eta = (\eta_1, \eta_2, \ldots, \eta_d)$, be two multi-indexes. $\kappa \succ GL \eta$ if $|\kappa| > |\eta|$, or in the case that $|\kappa| = |\eta|$ then we have the first non-zero entry in the difference $\kappa - \eta = (\kappa_1 - \eta_1, \kappa_2 - \eta_2, \ldots, \kappa_d - \eta_d)$ is positive. The graded lexicographic order respect the total degree of the polynomials.

We consider the orthogonality only in terms of polynomials of different degrees; that is, equal degree polynomials are orthogonal polynomials of lower degree; but equal degree polynomials are not orthogonal between each other.

Let $d\sigma(x)$ be a positive measure defined on $E \subset \mathbb{R}^d$ a domain with a nonempty interior. Let $\langle \cdot, \cdot \rangle$ denote the inner product defined on $\Pi_n^d$ by

$$\langle p(x), q(x) \rangle_{\sigma} = \int_E p(x)q(x)d\sigma(x),$$

see [7]. We will say that the polynomial $p(x) \in \Pi_n^d$ is orthogonal with respect to (2) if $\langle p(x), q(x) \rangle = 0$, $\forall q(x) \in \Pi_{n-1}^d$.

Let $V_n^d$ be linear space of orthogonal polynomials of total degree $n$ with respect to (2). Then

$$\dim V_n^d = \dim H_n^d.$$ 

If $\{ P_n^{\kappa_i}(x) : |\kappa_i| = n, 1 \leq i \leq r_n^d \}_{n \geq 0}$ is a basis of $V_n^d$, we will write the (column) polynomial vector as:

$$P_n(x) = \begin{pmatrix} P_n^{\kappa_1}(x) \\ P_n^{\kappa_2}(x) \\ \vdots \\ P_n^{\kappa_{r_n^d}}(x) \end{pmatrix} \in \mathbb{R}_n^{r_n^d \times 1},$$

where $\kappa_1, \kappa_2, \ldots, \kappa_{r_n^d}$ are arranged according to the reverse lexicographic order.
We denote the partial derivative with respect to the $i$-th component of a polynomial vector
\[ \partial_i \mathbf{P}_n(x) = \begin{pmatrix} \partial_i P_{n_1}(x) \\
\partial_i P_{n_2}(x) \\
\vdots \\
\partial_i P_{n_d}(x) \end{pmatrix} \in \mathbb{R}^{d \times 1}. \]

We will say that \( \{\mathbf{P}_n(x)\}_{n \geq 0} \) is an orthogonal polynomial system, if,
\[ \langle \mathbf{P}_n(x), \mathbf{P}_m^T(x) \rangle = \int_S \mathbf{P}_n(x) \mathbf{P}_m^T(x) d\sigma(x) = \begin{cases} 0 & \text{if } m \neq n \\ H_n & \text{if } m = n, \end{cases} \tag{3} \]
where \( H_n \) is a symmetric and positive \( d \times d \) matrix. If \( H_n \) is the identity matrix \( \forall n \geq 0 \) then \( \{\mathbf{P}_n(x)\}_{n \geq 0} \) is called an orthonormal polynomial system (OPS). A step-by-step method of constructing a corresponding orthonormal polynomial sequence is known as the Gram-Schmidt process. It produces real orthonormal polynomials, see [2], generalizing this method it is always possible from an orthogonal polynomials system to obtain a system of orthonormal polynomials.

We define the kernel function of \( V_j^d \) by
\[ P_j(x, y) = \mathbf{P}_j^T(x) H_j^{-1} \mathbf{P}_j(y) = P_j(y, x), \quad j \geq 0, \tag{4} \]
and the kernel function of \( \Pi_n^d \) by
\[ K_n(x, y) = \sum_{j=0}^n P_j(x, y) = \sum_{j=0}^n \mathbf{P}_j^T(x) H_j^{-1} \mathbf{P}_j(y) = K_n(y, x), \quad n \geq 0. \tag{5} \]

The definition of \( K_n(x, y) \) does not depend on a particular basis, but it is often more convenient to work with an orthonormal basis, see [5]. Thus, the kernel function adopts a simpler expression
\[ K_n(x, y) = \sum_{j=0}^n \mathbf{P}_j^T(x) \mathbf{P}_j(y). \]

Let \( f \) be a real valued function on \( d \) variables. We will use the gradient operator \( \nabla \) defined as usual
\[ \nabla f(x) = (\partial_1 f(x), \partial_2 f(x), \ldots, \partial_d f(x)) \in \mathbb{R}^{1 \times d}. \]

The gradient operator can be extended for vector polynomials. If \( \{\mathbf{P}_n(x)\}_{n \geq 0} \)
is an OPS, for \( n \geq 0 \), we introduce, see [7]

\[
\nabla P_n(x) = (\partial_1 P_n(x) | \partial_2 P_n(x) | \cdots | \partial_d P_n(x))
\]

\[
= \begin{bmatrix}
\partial_1 \frac{\partial}{\partial x_1} P_n(x) & \partial_2 \frac{\partial}{\partial x_2} P_n(x) & \cdots & \partial_d \frac{\partial}{\partial x_d} P_n(x) \\
\partial_1 \frac{\partial}{\partial x_1} P_n(x) & \partial_2 \frac{\partial}{\partial x_2} P_n(x) & \cdots & \partial_d \frac{\partial}{\partial x_d} P_n(x) \\
\vdots & \vdots & \ddots & \vdots \\
\partial_1 \frac{\partial}{\partial x_1} P_n(x) & \partial_2 \frac{\partial}{\partial x_2} P_n(x) & \cdots & \partial_d \frac{\partial}{\partial x_d} P_n(x)
\end{bmatrix} \in M_{d	imes d}(\Pi^d),
\]

and, for a higher order gradient, we define, see [4]

\[
\nabla^j P_n = \sum_{i=0}^{\infty} \left( \nabla^j P_i(x) \right)^T H^{-1}_i \nabla^j P_i(y) \in M_{d\times 1}(\Pi^d),
\]

where, \( \frac{\partial}{\partial x^j} = \frac{\partial^j}{\partial x_1 \cdots \partial x_d} \) and \( \beta \) runs through all \( d^j \) combinations of \( j \) total derivatives with respect to \( d \) different variables (i.e., all different combinations of \( \gamma_1, \gamma_2, \ldots, \gamma_d \in \mathbb{N} \) such that \( \gamma_1 + \gamma_2 + \cdots + \gamma_d = j \), arranged according to the lexicographic order).

Moreover, we define the vectors

\[
K_n^{(j,0)}(x,y) = \sum_{i=0}^{\infty} \left( \nabla^j P_i(x) \right)^T H^{-1}_i \nabla^j P_i(y) \in M_{d\times 1}(\Pi^d),
\]

\[
K_n^{(0,j)}(x,y) = \sum_{i=0}^{\infty} \nabla^j P_i(x) H^{-1}_i \nabla^j P_i(y) \in M_{1\times d}(\Pi^d),
\]

which satisfy \( K_n^{(j,0)}(x,y) = (K_n^{(0,j)}(y,x))^T \), and the matrix

\[
K_n^{(j,j)}(x,y) = \left( \partial_{\beta_l} \partial_{\eta_k} K_n(x,y) \right)_{l,k=1}^{d^j},
\]

where, as above, \( \beta \) (resp. \( \eta \)) runs through all \( d^j \) combinations of \( j \) total derivatives with respect to \( d \) different variables in \( x \) (resp. \( y \)).

In [4], the authors show the following lemma.

**Lemma 2.1.** Let \( \lambda \in \mathbb{R}^+ \) and \( \xi \in \Pi^d \) a fixed point. For \( n \geq 0 \), \( I_d + \lambda K_n^{(j,j)}(\xi,\xi) \) is a symmetric and non singular matrix.

### 3. A higher order Sobolev-type inner product

We consider the following Sobolev-type inner product, see [4]

\[
\langle p(x), q(x) \rangle_\mu = \langle p(x), q(x) \rangle_\sigma + \lambda \nabla^{(j)} p(\xi) \nabla^{(j)} q(\xi)^T
\]

where \( \xi \in \Pi^d \), \( \lambda \in \mathbb{R}^+ \) and \( j \in \mathbb{N} \).

We will denote the OPS with respect to (10) by \( \{\mathbb{Q}_n(x)\}_{n \geq 0} \) and the OPS with respect to (2) by \( \{\mathbb{P}_n(x)\}_{n \geq 0} \).

We define the kernel function of \( V_j^d \) associated with \( \{\mathbb{Q}_n(x)\}_{n \geq 0} \) by

\[
Q_j(x,y) = \mathbb{Q}_j^T(x) G^{-1}_j \mathbb{Q}_j(y) = Q_j(y,x), \quad j \geq 0,
\]

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see [4], and the kernel function of $\Pi_n^d$ associated with $\{Q_n(x)\}_{n \geq 0}$ by

$$\tilde{K}_n(x, y) = \sum_{i=0}^{n} Q_i(x, y) = \sum_{i=0}^{n} Q_i^G(x) G^{-1}_i Q_i(y), \quad n \geq 0. \quad (12)$$

The following two theorems whose proof appears in [4] establish an explicit relation between $\{Q_n(x)\}_{n \geq 0}$ and $\{P_n(x)\}_{n \geq 0}$ and their respective kernel functions.

**Theorem 3.1.** Let $\{P_n(x)\}_{n \geq 0}$ be the OPS associated with the inner product (2). Define a Sobolev-type inner product as in (10). Then if we denote by $\{Q_n(x)\}_{n \geq 0}$ its corresponding OPS, normalized such that $Q_0(x) = P_0(x)$, and for $n > 0$,

$$Q_n(x) = P_n(x) - \lambda \nabla^{(j)} \nabla^{(j)} (I_1 + \lambda K^{(j,j)}_{n-1}(\xi, \xi))^{-1} K^{(j,j)}_{n-1}(\xi, x). \quad (13)$$

Conversely, if we define $\{Q_n(x)\}_{n \geq 0}$ as in (13), then they are an OPS with respect to (10).

**Theorem 3.2.** For $i \geq 1$, we have:

$$Q_i(x, y) = P_i(x, y) - \lambda (K^{(j,0)}_i(\xi, x))^T (I_1 + \lambda K^{(j,j)}_{n-1}(\xi, \xi))^{-1} K^{(j,j)}_{n-1}(\xi, x)$$

$$+ \lambda (K^{(j,0)}_j(\xi, x))^T (I_1 + \lambda K^{(j,j)}_{n-1}(\xi, \xi))^{-1} K^{(j,j)}_{n-1}(\xi, x), \quad (14)$$

where we assume $K^{(j,0)}_0(x, y) = 0$. Furthermore,

$$\tilde{K}_n(x, y) = K_n(x, y) - \lambda (K^{(j,0)}_n(\xi, x))^T (I_1 + \lambda K^{(j,j)}_{n-1}(\xi, \xi))^{-1} K^{(j,j)}_{n-1}(\xi, x). \quad (15)$$

### 4. An example on the unit ball in $\mathbb{R}^d$

Let $B^d = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$ the unit ball in $\mathbb{R}^d$, where $\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}$ is the Euclidean norm. We denote $(x, y) = x_1 y_1 + x_2 y_2 + \cdots + x_d y_d$ the standard inner product on $\mathbb{R}^d$.

As in [4], we analyze a particular case of the Sobolev-type inner product defined in (10). We consider the weight function

$$W_\mu(x) = (1 - \|x\|^2)^{\mu - \frac{1}{2}}, \quad \mu \geq -\frac{1}{2}, \quad x \in B^d. \quad (16)$$

Associated with $W_\mu(x)$, we define the inner product on the unit ball

$$\langle f, g \rangle_\sigma = N_\mu \int_{B^d} f(x)g(x)W_\mu(x)dx_1\cdots dx_d, \quad (17)$$

where $N_\mu$ is the normalizing constant in order to have $\langle 1, 1 \rangle_\sigma = 1$ and it is given by

$$N_\mu = \left( \int_{B^d} W_\mu(x)dx \right)^{-1} = \frac{\Gamma\left(\mu + \frac{d+1}{2}\right)}{\pi^{\frac{d}{2}}\Gamma\left(\mu + \frac{1}{2}\right)}. \quad (18)$$

The family of orthogonal polynomials with respect to (17) is called the classical orthogonal polynomials on the unit ball, see [9].

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As a consequence, when \( n \to \infty \),

\[
P_n^{(\alpha, \beta)}(1) = \frac{1}{\Gamma(\alpha + 1)} n^{\alpha} \left( 1 + \mathcal{O}(n^{-1}) \right),
\]

(21)

\[
P_n^{(\alpha, \beta)}(-1) = \frac{1}{\Gamma(\beta + 1)} (-1)^n n^{\beta} \left( 1 + \mathcal{O}(n^{-1}) \right),
\]

(22)

\[
C_{n, \alpha, \beta} = n \left( 1 + \mathcal{O}(n^{-1}) \right).
\]

(23)

In [4] the authors studied a family of orthogonal polynomials associated to the Sobolev-type inner product defined in (10), for a particular case where the measure is \((1 - |x|^2)^{\mu - \frac{1}{2}}\). They analyzed the asymptotic behavior of the corresponding kernel function \(K_n(x, y)\) and the matrix (9), for \( j = 2 \) and \( x = y = 0 \in \mathbb{R}^d \), i.e., \(K_n^{(2,2)}(0, 0)\).

As a continuation, we will first perform an analysis of the asymptotic behavior of the kernel, not only evaluating in the points \( x = y = 0 \in \mathbb{R}^d \), but evaluating in the points \( x = 0 \in \mathbb{R}^d \) and \( y \in \mathbb{R}^d \) with \( ||y|| = 1 \).

Then we evaluate in the points \( x, y \in \mathbb{R}^d \) such that \( ||x|| = 1 = ||y|| \), i.e., we analyze the asymptotic behavior of the kernel function \(K_n^{(2,2)}(0, y|y| = 1)\) and \(K_n^{(2,2)}(x|x| = 1, y|y| = 1)\) respectively, where \( y|y| = 1 \) means a point \( y \in \mathbb{R}^d \) such that \( ||y|| = 1 \) (similar to \( x||x|| = 1 \)).

4.1. The kernel function \(K_n^{(2,2)}(0, y|y| = 1)\)

We consider the orthogonal polynomials with respect to the Sobolev-type inner product (10), where \( \langle f, g \rangle_p \) is defined in (17). We are interested in analyzing the asymptotic behavior of the corresponding kernel functions for \( j = 2 \), \( \xi_1 = 0 \) and \( ||\xi_2|| = 1 \). At the first time we find an expression for both, \(K_n^{(0,2)}(0, y)\) and \(K_n^{(2,2)}(0, y|y| = 1)\).

Taking into account that

\[
K_n(x, y) = b_n A_n \int_{-1}^{1} P_n^{(\alpha, \beta - 1)}(w)(1 - t^2)^{\mu - 1} dt,
\]

(24)
Moreover, using,

\[ b_\mu = \left[ \int_{-1}^{1} (1 - t^2)^{\mu - 1} dt \right]^{-1} = \frac{\Gamma(\mu + \frac{1}{2})}{\sqrt{\pi} \Gamma(\mu)}, \quad (25) \]

\[ \alpha(\mu, d) = \alpha = \mu + \frac{d}{2}, \]

\[ A_n = \frac{2\Gamma(\alpha + 1)\Gamma(n + 2\alpha)}{\Gamma(2\alpha + 1)\Gamma(n + \alpha)}, \quad (26) \]

\[ w(x, y, t) = w = \langle x, y \rangle + \sqrt{1 - ||x||^2} \sqrt{1 - ||y||^2} t. \]

Taking partial derivatives with respect to the variables \( x_r \) and \( x_s \), integrating by parts and evaluating at \( x = 0 \),

\[ \frac{\partial^2}{\partial x_r \partial x_s} K_n(0, y) = y_r y_s b_\mu A_n C_{n,\alpha,\alpha-1} C_{n-1,\alpha,\alpha} - \int_{-1}^{1} P_{n-2}^{(\alpha+2,\alpha+1)} \left( \sqrt{1 - ||y|^2} t \right) (1 - t^2)^{\mu - 1} dt \]

\[ - \frac{1 - ||y||^2}{2\mu} \delta_{r,s} b_\mu A_n C_{n,\alpha,\alpha-1} C_{n-1,\alpha,\alpha}, \]

\[ \int_{-1}^{1} P_{n-2}^{(\alpha+2,\alpha+1)} \left( \sqrt{1 - ||y||^2} t \right) (1 - t^2)^{\mu} dt. \quad (27) \]

Moreover, using,

\[ \int_{-1}^{1} P_{n-2}^{(\mu+\frac{1}{2},\mu+\frac{1}{2})} \left( \sqrt{1 - ||y||^2} t \right) (1 - t^2)^{\mu - 1} dt = h_{n,\mu,d} P_{\frac{\mu+1}{2}}^{(\mu+\frac{1}{2})} (1 - 2||y||^2), \quad (28) \]

see [7], with

\[ h_{n,\mu,d} = \frac{4\Gamma(\mu + \frac{1}{2})\Gamma\left(\frac{n+1}{2}\right) + \mu + \frac{d+1}{2}}{\Gamma(\mu + \frac{d+1}{2})\Gamma\left(\frac{n+1}{2}\right) + \mu + \frac{1}{2}} \Gamma(n + 2\mu + d) \]

\[ \Gamma(n + 2\mu + d + 1)b_\mu A_n, \quad (29) \]

from (27), (28) and as \( \alpha = \mu + \frac{d}{2} \), it follows:

\[ \frac{\partial^2}{\partial x_r \partial x_s} K_n(0, y) = y_r y_s B_1 P_{\frac{\mu+1}{2}}^{(\mu+\frac{1}{2})} (1 - 2||y||^2) \]

\[ + \frac{\partial^2}{\partial x_r \partial x_s} K_n(0, y) = y_r y_s B_2 \]

\[ P_{\frac{\mu+1}{2}}^{(\mu+\frac{1}{2})} (1 - 2||y||^2), \quad (30) \]

with

\[ B_1 = b_\mu A_n C_{n,\mu+\frac{1}{2},\mu+\frac{1}{2}} C_{n-1,\mu+\frac{1}{2},\mu+\frac{1}{2}} h_{n-1,\mu,d+2}, \]

\[ B_2 = (2\mu)^{-1} b_\mu A_n C_{n,\mu+\frac{1}{2},\mu+\frac{1}{2}} C_{n-1,\mu+\frac{1}{2},\mu+\frac{1}{2}} h_{n-1,\mu+1,d}. \]

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Now, taking partial derivatives of $\frac{\partial^4}{\partial y_i \partial y_j \partial x_s \partial x_r} K_n(0, y)$ with respect to the variables $y_i$ and $y_j$ and evaluating at $y_{\|y\| = 1}$

$$\frac{\partial^4}{\partial y_i \partial y_j \partial x_s \partial x_r} K_n(0, y_{\|y\| = 1}) = -4 \left[ y_i y_j \delta_{s,j} + y_i y_s \delta_{r,j} + y_j y_s \delta_{r,i} + y_j y_r \delta_{s,i} \right] + y_i y_j \delta_{r,i}.$$

$$B_1 C_{\frac{n-2}{2}, \frac{d+2}{2}, \frac{d-1}{2}} \left\langle \frac{\partial^4}{\partial y_i \partial y_j \partial x_s \partial x_r} \right\rangle K_n(0, y_{\|y\| = 1}) = -4 \left[ y_i y_j \delta_{s,j} + y_i y_s \delta_{r,j} + y_j y_s \delta_{r,i} + y_j y_r \delta_{s,i} \right] B_1 P_{\frac{d+2, \mu - \frac{1}{2}}{2}} \left( -1 \right)$$

$$+ 16 y_i y_j y_s C_{\frac{n-2}{2}, \frac{d+2}{2}, \frac{d-1}{2}} C_{\frac{n-2}{2}, \frac{d+3}{2}, \frac{d+1}{2}} P_{\frac{d+4, \mu + \frac{1}{2}}{2}} \left( -1 \right)$$

$$+ 2 \delta_{r,i} \delta_{s,j} B_2 P_{\frac{d+1+1}{2}, \frac{d+1+1}{2}} \left( -1 \right) - 16 y_i y_j y_s B_2 C_{\frac{n-2}{2}, \frac{d+2}{2}, \frac{d+1+1}{2}} P_{\frac{d+2, \mu + \frac{1}{2}}{2}} \left( -1 \right).$$

(31)

We denote,

$$z_1 = -4 B_1 C_{\frac{n-2}{2}, \frac{d+2}{2}, \frac{d-1}{2}} P_{\frac{d+3, \mu + \frac{1}{2}}{2}} \left( -1 \right),$$

$$z_2 = B_1 P_{\frac{d+2, \mu - \frac{1}{2}}{2}} \left( -1 \right),$$

$$z_3 = 16 B_1 C_{\frac{n-2}{2}, \frac{d+3}{2}, \frac{d+1}{2}} P_{\frac{d+4, \mu + \frac{1}{2}}{2}} \left( -1 \right),$$

$$z_4 = 2 B_2 P_{\frac{d+1+1}{2}, \frac{d+1+1}{2}} \left( -1 \right),$$

$$z_5 = -16 B_2 C_{\frac{n-2}{2}, \frac{d+2}{2}, \frac{d+1+1}{2}} P_{\frac{d+2, \mu + \frac{1}{2}}{2}} \left( -1 \right).$$

(32)

(33)

(34)

(35)

(36)

As a consequence, the $d^2 \times d^2$ matrix $K^{(2,2)}_n(0, y_{\|y\| = 1})$, whose entries are arranged according to the orders of their partial derivatives, has elements

$$z_1 (y_i y_j \delta_{s,j} + y_i y_s \delta_{r,j} + y_j y_s \delta_{r,i} + y_j y_r \delta_{s,i})$$

$$+ z_2 (\delta_{r,i} \delta_{s,j} + \delta_{s,i} \delta_{r,j}) + z_3 (y_j y_i y_r y_s + z_4 \delta_{r,i} \delta_{s,j} + z_5 y_i y_j y_r y_s,$$

where $1 \leq r, s, i, j \leq d$.

From Kronecker’s deltas we can observe,

1. Let $i, j, r, s$, $z_1$ survives if at least two are equal.
2. $z_2$ survives if $(i = s \land r = j) \lor (j = s \land r = i)$.
3. $z_3$ survives in all entries of the matrix.
4. If $i, j, r, s$ are different two by two only $z_3$ survive.
5. $z_4$ survives if $r = s \land i = j$.
6. $z_5$ survives if $r = s$.

Below we give a first conjecture, regarding the distribution of the $d^4$ entries of the matrix $K^{(2,2)}_n(0, y_{\|y\| = 1})$.

**Conjecture.** The $d^4$ entries of the matrix $K^{(2,2)}_n(0, y_{\|y\| = 1})$ are distributed as follows

1. $d^2$ entries appear only once.
2. $\frac{d(d-1)(d+2)}{2}$ entries appear 2 times each one.

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3. \( \frac{d(d-1)}{2} \) entries appear 4 times each one.

4. \( \frac{d(d-1)(d-2)}{2} \) entries appear 10 times each one.

5. \( \frac{d(d-1)(d-2)(d-3)}{24} \) entries appear 24 times each one.

**Lemma 4.1.** Let \( b_n, A_n \) and \( h_{n,\mu,d} \) defined in (25), (26) and (29) respectively. When \( n \to \infty \),

\[
b_n A_n = \frac{2\Gamma(\mu + \frac{1}{2})\Gamma(\mu + \frac{d}{2} + 1)}{\sqrt{\pi} \Gamma(2\mu + d + 1)} n^{\mu+\frac{d}{2}} (1 + O(n^{-1})),
\]

\[
h_{n,\mu,d} = \frac{2\sqrt{\pi}\Gamma(\mu)\Gamma(2\mu + d + 1)}{\Gamma(\mu + \frac{d+1}{2})\Gamma(\mu + \frac{d}{2} + 1)} n^{-\mu} (1 + O(n^{-1})).
\]

**Proof.** It is obtained using the Stirling formula in the definitions of \( b_n, A_n \) and \( h_{n,\mu,d} \).

**Theorem 4.2.** Let \( z_1, z_2, z_3, z_4 \) and \( z_5 \) defined from (32) to (36). When \( n \to \infty \),

\[
z_1 = k_1 \cdot n^{\frac{d}{2} + \mu + \frac{7}{2}} (1 + O(n^{-1})), \quad (39)
\]

\[
z_2 = k_2 \cdot n^{\frac{d}{2} + \mu + \frac{5}{2}} (1 + O(n^{-1})), \quad (40)
\]

\[
z_3 = k_3 \cdot n^{\frac{d}{2} + \mu + \frac{1}{2}} (1 + O(n^{-1})), \quad (41)
\]

\[
z_4 = k_4 \cdot n^{\frac{d}{2} + \mu + \frac{1}{2}} (1 + O(n^{-1})), \quad (42)
\]

\[
z_5 = k_5 \cdot n^{\frac{d}{2} + \mu + \frac{1}{2}} (1 + O(n^{-1})), \quad (43)
\]

where \( k_1, k_2, k_3, k_4 \) and \( k_5 \) are constants depending of \( \mu \) and \( d \).

**Proof.** It is obtained directly by applying (16), (17), (37) and (38) to the definitions given.

On the other hand, we have the following result.

**Theorem 4.3.** Let \( K_n(x,y) \) be the kernel function associated to \( \{P_n(x)\}_{n \geq 0} \), the \textit{OPS} with respect to (17), then for \( n \to \infty \)

(i) \( K_n^{(2,0)}(0,0) = n^{\frac{d+2}{2}} (1 + O(n^{-1})) \) if \( y = 0 \).

(ii) \( K_n^{(2,0)}(0,0) \) uniformly in compacts inside \( B^d \) if \( ||y|| = 1 \).

(iii) \( K_n^{(2,0)}(0,0) = n^{\mu + \frac{d+1}{2}} (1 + O(n^{-1})) \) if \( ||y|| = 1 \).

**Proof.** (i) From (30), evaluating in \( ||y|| = 0 \) the first term vanishes and considering (21), (23) together with the Lemma 4.1. we get the desired expression.

(ii) We get the result using the fact that \( |F_n^{(a,b)}(t)| \leq Cn^{-1/2} \) uniformly compact subsets of \((-1,1)\), see [8] and (23) together with the Lemma 4.1.

(iii) From (30), evaluating in \( ||y|| = 1 \) the second term vanishes and considering (22), (23) together with the Lemma 4.1. we get the desired expression.
4.2. The kernel function $K_n^{(2,2)}(x \| x \|=1, y \| y \|=1)$

We consider the orthogonal polynomials with respect to the Sobolev-type inner product (10), where $\langle f, g \rangle_\alpha$ is defined in (17). We are interested in analyzing the asymptotic behavior of the corresponding kernel functions for $j = 2$, $\| x \|=1$ and $\| y \|=1$.

First, we find an expression for both $K_n^{(0,2)}(x \| x \|=1, y)$ and $K_n^{(2,2)}(x \| x \|=1, y \| y \|=1)$.

Taking partial derivatives with respect to the variables $x_r$ and $x_s$ in (24), integrating by parts, evaluating at $\| x \|=1$, using (25) and the fact that $2\mu b_\mu + 1 = (2\mu + 1)b_\mu$, we get,

$$\frac{\partial^2}{\partial x_s \partial x_r} K_n(x \| x \|=1, y) = \left( y_r y_s - \frac{1 - \| y \|^2}{2\mu + 1} \delta_{r,s} \right) A_n C_{n,\alpha,\alpha-1} C_{n-1,\alpha+1,\alpha} \left( (x \| x \|=1, y) \right)
- \frac{1 - \| y \|^2}{2\mu + 1} (x_r y_r + x_s y_s) A_n C_{n,\alpha,\alpha-1} C_{n-1,\alpha+1,\alpha} C_{n-2,\alpha+2,\alpha+1} P_{n-1}^{(\alpha+2,\alpha+1)}
+ \left( (x \| x \|=1, y) \right)
+ \frac{(1 - \| y \|^2)^2}{(2\mu + 1)(2\mu + 3)} x_r x_s A_n C_{n,\alpha,\alpha-1} C_{n-1,\alpha+1,\alpha} C_{n-2,\alpha+2,\alpha+1} C_{n-3,\alpha+3,\alpha+2} P_{n-4}^{(\alpha+4,\alpha+3)} \left( (x \| x \|=1, y) \right).$$

Now, taking partial derivatives of $\frac{\partial^2}{\partial x_s \partial x_r} K_n(x \| x \|=1, y)$ with respect to the variables $y_i$ and $y_j$ and evaluating at $\| y \|=1$, we have

$$\frac{\partial^4}{\partial y_i \partial y_j \partial x_s \partial x_r} K_n \left( x \| x \|=1, y \| y \|=1 \right) = K_n^{(2,2)}(x \| x \|=1, y \| y \|=1)
+ \left[ \delta_{r,s} \left( x_r y_r + \frac{2}{2\mu + 1} x_r y_j \right) \right] g_1
+ \left[ \delta_{r,s} \left( x_s y_s + \frac{2}{2\mu + 1} x_s y_j \right) \right] g_2
+ \left[ \delta_{r,s} \left( x_s y_s + \frac{2}{2\mu + 1} x_i y_i \right) \right] g_2
+ \left[ \left( x_r x_r y_r y_r + \frac{4}{2\mu + 1} x_r x_s y_j y_i + \frac{2}{2\mu + 1} (x_r x_j y_j y_r + x_i x_j y_j y_i) \right) \right] g_3,$$

where $1 \leq r, s, i, j \leq d$ and

$$g_1 = A_n C_{n,\alpha,\alpha-1} C_{n-1,\alpha+1,\alpha} P_{n-2}^{(\alpha+2,\alpha+1)} \left( (x \| x \|=1, y \| y \|=1) \right).$$

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\[ g_2 = A_n C_{n, \alpha, \alpha-1} C_{n-1, \alpha+1, \alpha} C_{n-2, \alpha+2, \alpha+1} P_{n-3}^{(\alpha+3, \alpha+2)} \left( \langle x \| x \rangle = 1, y \| y \rangle = 1 \right), \quad (45) \]

\[ g_3 = A_n C_{n, \alpha, \alpha-1} C_{n-1, \alpha+1, \alpha} C_{n-2, \alpha+2, \alpha+1} C_{n-3, \alpha+3, \alpha+2} P_{n-4}^{(\alpha+4, \alpha+3)} \left( \langle x \| x \rangle = 1, y \| y \rangle = 1 \right). \quad (46) \]

From Kronecker’s deltas we can observe,

1. \( j, i, r, s, g_1 \) survive only in two cases: when all of them are equal and when two of them are equal to each other and the other two are equal to each other.
2. If \( j, i, r, s \) are different two by two, \( g_2 \) does not survive.
3. \( g_3 \) survives in all entries of the matrix.

Below we give a second conjecture, regarding the distribution of the \( d^4 \) entries of the matrix \( K^{(2,2)}_{\mu} (x \| x \rangle = 1, y \| y \rangle = 1) \).

**Conjecture.** The \( d^4 \) entries of the matrix \( K^{(2,2)}_{\mu} (x \| x \rangle = 1, y \| y \rangle = 1) \) are distributed as follows

1. \( d^2 \) entries appear only once.
2. \( d^2 (d-1) \) entries appear 2 times each one.
3. \( \left[ \frac{d(d-1)}{2} \right]^2 \) entries appear 4 times each one.

**Lemma 4.4.** Let \( A_n \) defined as in (26), then for \( n \to \infty \),

\[ A_n = \frac{2\Gamma(\mu + \frac{d}{2} + 1)}{\Gamma(2\mu + d + 1)} n^{\mu + \frac{d}{2}} (1 + O(n^{-1})), \quad (47) \]

**Proof.** It is obtained using the Stirling formula in the definitions in (26).

The behavior of \( g_1, g_2 \) and \( g_3 \) is analyzed in two situations: when \( x = y \) and when \( x = -y \).

**Theorem 4.5.** Let \( g_1, g_2 \) and \( g_3 \) defined in (44), (45) and (46), when \( n \to \infty \).

(i) If \( x = y \)

\[ g_1 = k_6 \cdot n^{2\mu+d+4} \left( 1 + O(n^{-1}) \right), \quad (48) \]
\[ g_2 = k_7 \cdot n^{2\mu+d+6} \left( 1 + O(n^{-1}) \right), \quad (49) \]
\[ g_3 = k_8 \cdot n^{2\mu+d+8} \left( 1 + O(n^{-1}) \right), \quad (50) \]

where \( k_6, k_7 \) and \( k_8 \) are constants depending of \( \mu \) and \( d \).

(ii) \( x = -y \)

\[ g_1 = k_9 \cdot n^{2\mu+d+3} \left( 1 + O(n^{-1}) \right), \quad (51) \]
\[ g_2 = k_{10} \cdot n^{2\mu+d+5} \left( 1 + O(n^{-1}) \right), \quad (52) \]
\[ g_3 = k_{11} \cdot n^{2\mu+d+7} \left( 1 + O(n^{-1}) \right), \quad (53) \]

where \( k_9, k_{10} \) and \( k_{11} \) are constants depending on \( \mu \) and \( d \).
Proof. (i) When \( x = y \), \( g_1 \), \( g_2 \) and \( g_3 \) are given by

\[
g_1 = A_n C_n, \alpha - 1 C_{n-1}, \alpha + 1, \alpha P_n^{(\alpha + 2, \alpha + 1)}(1),
\]

\[
g_2 = A_n C_n, \alpha - 1 C_{n-1}, \alpha + 1, \alpha C_n - 2, \alpha + 2, \alpha + 1 P_n^{(\alpha + 3, \alpha + 2)}(1),
\]

\[
g_3 = A_n C_n, \alpha - 1 C_{n-1}, \alpha + 1, \alpha C_n - 2, \alpha + 2, \alpha + 1 C_n - 3, \alpha + 3, \alpha + 2 P_n^{(\alpha + 4, \alpha + 3)}(1).
\]

using (21), (23) and the Lemma 4.4., when \( n \to \infty \),

\[
g_1 = k_6 \cdot n^{2\mu + d + 4} (1 + O(n^{-1})),
\]

\[
g_2 = k_7 \cdot n^{2\mu + d + 6} (1 + O(n^{-1})),
\]

\[
g_3 = k_8 \cdot n^{2\mu + d + 8} (1 + O(n^{-1})).
\]

(ii) When \( x = -y \), \( g_1 \), \( g_2 \) and \( g_3 \) are given by

\[
g_1 = A_n C_n, \alpha - 1 C_{n-1}, \alpha + 1, \alpha P_n^{(\alpha + 2, \alpha + 1)}(-1),
\]

\[
g_2 = A_n C_n, \alpha - 1 C_{n-1}, \alpha + 1, \alpha C_n - 2, \alpha + 2, \alpha + 1 P_n^{(\alpha + 3, \alpha + 2)}(-1),
\]

\[
g_3 = A_n C_n, \alpha - 1 C_{n-1}, \alpha + 1, \alpha C_n - 2, \alpha + 2, \alpha + 1 C_n - 3, \alpha + 3, \alpha + 2 P_n^{(\alpha + 4, \alpha + 3)}(-1),
\]

using (22), (23) and the lemma 4.4., when \( n \to \infty \),

\[
g_1 = k_9 \cdot n^{2\mu + d + 3} (1 + O(n^{-1})),
\]

\[
g_2 = k_{10} \cdot n^{2\mu + d + 5} (1 + O(n^{-1})),
\]

\[
g_3 = k_{11} \cdot n^{2\mu + d + 7} (1 + O(n^{-1})).
\]

\[\square\]

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References


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