On embedding theorems for weighted spaces of holomorphic functions in tubular domains

Sobre teoremas de inmersiones para espacios ponderados de funciones holomorfas en dominios tubulares

Romi F. Shamoyan[1,a], **Olivera R. Mihić**[2,b]

Abstract. We introduce new mixed norm analytic spaces in products of tubular domains over symmetric cones and provide new sharp embedding theorems for them extending previously known assertions in tubular domain over symmetric cone and polydisk simultaniously. Our results extend some recent theorems of Sehba, Nana and Shamoyan in tube domain and polydisk simultaneously.

Keywords: analytic functions, tubular domains over symmetric cones, Bergman type spaces.

Resumen. Introducimos nuevos espacios analíticos con norma mixta en productos de dominios tubulares sobre conos simétricos y suministra nuevos y precisos teoremas englobantes para ellos extendiendo aserciones anteriormente conocidas en dominios tubulares sobre conos simétricos y poli-discos simultáneamente. Nuestros resultados extienden algunos teoremas recientes de Sehba, Nana y Shamoyan en dominios tubulares y poli-discos simultaneamente.

Palabras claves: funciones analíticas, dominios tubulares sobre conos simétricos, espacios de tipos de Bergman.

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1. Introduction and preliminaries on geometry of tubular domains over symmetric cones

In recent paper [8] characterization of measures μ in the unit polydisc for which the differentiation operator maps anisotropic weighted space of holomorphic function with mixed norm into the Lebesgue space $L^q(\mu)$ was obtained.

[1]Department of Mathematics, Bryansk University, Bryansk, Russia
[2]Department of Mathematics, Fakultet organizacionih nauka, Jove Ilića 154, Belgrade, Serbia
[a]rshamoyan@gmail.com
[b]oliveradj@fon.rs

Various related assertions (sharp embedding theorems in analytic function spaces) can be seen in [4], [13] and various references there. The theory of analytic spaces (in particular various embeddings) on bounded pseudoconvex domains is well-developed by various authors during last decades (see [1], [4], [5], [10], [11], [12]). One of the goals of this paper, among other things, is to define for the first time in literature new mixed norm analytic spaces in products unbounded tubular domains and to establish some basic embeddings of these spaces.

In this section some basic facts and lemmas in tubular domains will be indicated which are vital for our proofs of main results of next section. Let Ω be an irreducible symmetric cone in the Euclidean space V, and $T_\Omega = V + i\Omega$ the corresponding tube domain in the complexified space $V^{\mathbb{C}}$. We shall note $\widetilde{n}$ the dimension of V and r the rank of Ω. Moreover, we shall denote by $(x|y)$ the scalar product in V, and by Δ the determinant function. For the description of such cones Ω in terms of Jordan algebras, one may use the book of Faraut and Koranyi [3]. Let $B_R(z_j) = B_j$ be the Bergman ball in tube (or we denote them below by $B_{T_\Omega}(z,r),\ z \in T_\Omega,\ r \in (0,1)$), see [1].

We write $A \asymp B$ if there are positive constants C_1, C_2 so that $C_2 B \leq A \leq C_1 B$.

For convenience, we will use the following notation $d\nu_t(z) = \delta^t(z) d\nu(z)$, $t > -1$, $\delta^t(z) = \Delta^t(Im\ z)$, $t > -1$, $z \in T_\Omega$. $K : T_\Omega \times T_\Omega \to \mathbb{C}$ will be the Bergman kernel of T_Ω. Let the K_t is a kernel of type t. Note if K is kernel of type $t,\ t \in \mathbb{N}$, then K^s is kernel of type st, $s \in \mathbb{N}, t \in \mathbb{N}$. Note also $K = K_{\widetilde{n}/r}$ (see [1], [5]) and $K_{\widetilde{n}/r}(z,w) = \Delta^{-2\widetilde{n}/r}\left(\frac{z-\bar{w}}{i}\right)$, $z, w \in T_\Omega$. Note that for every fixed a_k from our T_Ω domain $\nu_\alpha(B_{T_\Omega}(a_k,R)) = \int\limits_{B_{T_\Omega}(a_k,R)} \delta^\alpha(z) d\nu(z) \asymp (\delta^\alpha(a_k))\nu(B_{T_\Omega}(a_k,R)),\ \alpha > -1$.

The existence of r-lattice in tube domains with smooth boundary is ensured by the following: Let $T_\Omega \subset \mathbb{C}^{\widetilde{n}}$ be a tubular domain over symmetric cone. Then for every $\widetilde{r} \in (0,1)$ there exists an r-lattice in T_Ω, that is there exist $m \in \mathbb{N}$ and a sequence $\{a_k\} \subset T_\Omega$ of points such that $T_\Omega = \bigcup\limits_{k=0}^{\infty} B_{T_\Omega}(a_k, \widetilde{r})$ and no point of T_Ω belongs to more than m of the balls $B_{T_\Omega}(a_k,R)$, where $R = \frac{1}{2}(1+\widetilde{r})$. The sequence of points $\{z_j\},\ j \in \mathbb{N}$ is called a r-lattice of T_Ω. We have also (see [1]) $v_\alpha(B_j) \asymp v_\alpha(B'_j) \asymp \Delta^{\frac{2\widetilde{n}}{r}+\alpha}(Im z_j),\ \alpha > \frac{\widetilde{n}}{r} - 1,\ z_j \in T_\Omega,\ j \in \mathbb{N}$, where (B'_j) is a family of so-called enlarged Bergman balls (see [1]). Dealing with K Bergman kernel ($K = K_{\widetilde{n}/r}$) we always assume $|K(z,a_k)| \asymp |K(a_k,a_k)|$ for any $z \in B_{T_\Omega}(a_k,\widetilde{r})$, $\widetilde{r} \in (0,1)$ (see [1], [5]). Let $m = (\widetilde{n}/r)l,\ l > 0$. Then $|K_m(z,a_k)| \asymp |K_m(a_k,a_k)|$, $z \in B_{T_\Omega}(a_k,\widetilde{r}), \widetilde{r} \in (0,1)$. This fact (estimate from below of Bergman kernel) is crucial for embedding theorems in pseudoconvex domains (see also [10], [12]) and for our proofs also. We shall use a submean estimate for nonnegative plurisubharmonic function on Bergman balls. Note, in the unit ball, polydisk and unit disk this result is also well known (see, for example, [7], [8], [13]).

Lemma 1.1. *(see [1], [5], [6]) Let $T_\Omega \subset \mathbb{C}^n$ be a tubular domain over symmetric cone. Given $\widetilde{r} \in (0,1)$, set $R = \frac{1}{2}(1+\widetilde{r}) \in (0,1)$. Then there exists a $C_{\widetilde{r}} > 0$ depending on $\widetilde{r}$ such that*

$$\forall z_0 \in T_\Omega, \quad \forall z \in B_{T_\Omega}(z_0, \widetilde{r}), \quad \chi(z) \leq \frac{C_{\widetilde{r}}}{\nu(B_{T_\Omega}(z_0, \widetilde{r}))} \int\limits_{B_{T_\Omega}(z_0,R)} \chi d\nu$$

for every nonnegative subharmonic (analytic) function $\chi : T_\Omega \to \mathbb{R}^+$.

Proof. We denote by $d : T_\Omega \times T_\Omega \to \mathbb{R}_+$ a Bergman distance function defined with the help of Bergman metric H and Bergman length (see [1]). By e we denote the usual unit vector in T_Ω (see [1]). The Bergman distance is equivalent to the Euclidean distance on compact sets of $\mathbb{C}$. It is enough to show that the following estimate is valid for any analytic in T_Ω function f and for any δ, $\delta \in (0,1)$, $0 < p < \infty$, $z \in T_\Omega$, $\quad |f(z)|^p \leq \delta^{-n} \int_{\{d(z,w)<\delta\}} |f(w)|^\rho \frac{dud\nu}{\Delta^{\frac{2n}{r}}(\nu)}$.

We now recall (see [1]) the $\dfrac{dud\nu}{\Delta^{\frac{2n}{z}}(\nu)}$ measure is invariant under all automorphisms of T_Ω. Therefore it suffices simply to prove that the following is valid

$$|f(ie)|^p \leq \left(\delta^{-n}\right) \left(\int_{\{d(ie,w)<\delta\}} |f(w)|^\rho \frac{dud\nu}{\Delta^{\frac{2n}{r}}(\nu)} \right)$$

This last estimate follows directly from the mean value property and the equivalence between d distance function and the Euclidean distance in a neighborhood of ie. □

The following lemma is also vital for the proofs.

Lemma 1.2. *(Whitney decomposition of tube, see [1]) Given $R \in (0,1]$ there exist a sequence $\{z_j\}$ of points of T_Ω such that if $B_j = B_R(z_j) : B'_j = B_{R/3}(z_j)$*

1) the balls (B'_j) are pairwise disjoint;

2) the balls (B_j) form a cover of T_Ω;

3) there exists a positive integer $N = N(\Omega)$ (independent of R) such that every point of T_Ω belongs to at most N balls B_j.

The complete analogues of this lemma for various other domains are well known (see, for example, [13], [7], [11] and various references there).

2. Main results

In this section based on facts and lemmas from previous section in particular, we formulate our main sharp embedding theorems in tubular domains over symmetric cones related with the mixed norm Bergman type spaces. Theorems we provide below extend recent sharp results of Nana and Sehba, see [6], from one hand (case of one tubular domain) and extend known sharp results in the unit disk and polydisk from the other hand, see [8]. Both main theorems of this paper in context of bounded strongly pseudoconvex domains with smooth boundary are also valid and were proved recently in [11]. We consider in this paper new analytic spaces on products of unbounded tubular domains $T_\Omega^m = T_\Omega \times \cdots \times T_\Omega$. We denote by $H(T_\Omega \times \cdots \times T_\Omega) = H\left(T_\Omega^m\right), m \in \mathbb{N}$, the space of all analytic functions by each variable on T_Ω^m. Let $L^{\vec{p}}_{\vec{\alpha}}\left(T_\Omega^m\right) = \{f \in L^1_{loc}\left(T_\Omega^m\right)$:
$\left(\int_{T_\Omega} \cdots \left(\int_{T_\Omega} |f\left(z_1, \dots, z_m\right)|^{p_1} \delta^{\alpha_1}(z_1) dv(z_1)\right)^{\frac{p_2}{p_1}} \cdots \delta^{\alpha_m}(z_m) dv(z_m)\right)^{\frac{1}{p_m}} < \infty\}$,
$0 < p_i < \infty$, $\alpha_i > -1$, $i = 1, \dots, m$, $A^{\vec{p}}_{\vec{\alpha}}\left(T_\Omega^m\right) = L^{\vec{p}}_{\vec{\alpha}}\left(T_\Omega^m\right) \cap H\left(T_\Omega^m\right), m \in \mathbb{N}$.

These are Banach spaces if $\min\limits_j(p_j) \geq 1$ and complete metric spaces for other values of p. Such spaces in $\mathbb{R}^{\widetilde{n}}$ was studied in [2], in polydisk in [9]. Note for $m = 1$ these spaces studied actively (see, for example, [1], [5] and various references there).

Theorem 2.1. *Let $\vec{\mu} = (\mu_1, \dots, \mu_n)$, μ_j be Borel nonnegative measures on $T_\Omega \subset \mathbb{C}^{\widetilde{n}}$, $\widetilde{\alpha}_j > -1$, $j = 1, \dots, n$, $\widetilde{r} > 0$, $\vec{p} = (p_1, \dots, p_n)$, $\vec{q} = (q_1, \dots, q_n) \in \mathbb{R}^{\widetilde{n}}_+$ with $0 < p_j \leq q_j$, $j = 1, \dots, n$. Then the following assertions are equivalent:*

1. $\|f\|_{L^{\vec{q}}(\vec{\mu})} \leq C_{(\vec{\mu})} \|f\|_{A^{\vec{p}}(\vec{\alpha})}$;
2. $(\mu_j)\left(B_{T_\Omega}\left(a_k, \widetilde{r}\right)\right) \leq C\left(\delta(a_k)\right)^{\left(\widetilde{\alpha}_j + \frac{2\widetilde{n}}{r}\right)\frac{q_j}{p_j}}$, $j = 1, \dots, n$, $k = 0, 1, 2, \dots$ *where $\{a_k\}$ is r-lattice of T_Ω.*

In the case of measures $\widetilde{\nu}$ defined on T_Ω^n the following result holds:

Theorem 2.2. *Let $\{a_k\}$ be r-lattice of $T_\Omega \subset \mathbb{C}^{\widetilde{n}}$. Let $p_j \leq q < +\infty$, $\alpha_j > -1$, $1 \leq j \leq n$, $\widetilde{\nu}$ be the Borel nonnegative measure on T_Ω^n, $\widetilde{r} > 0$. Then the following assertions are equivalent:* (1) $\left(\int\limits_{T_\Omega^n} |f\left(z\right)|^q d\widetilde{\nu}(z)\right)^{\frac{1}{q}} \leq C_{(\widetilde{\nu})} \|f\|_{A^{\vec{p}}(\vec{\alpha})}$;
(2) $\widetilde{\nu}(B_{T_\Omega}(a_{k_1}, \widetilde{r}) \times \cdots \times B_{T_\Omega}(a_{k_n}, \widetilde{r})) \leq C \prod_{j=1}^n \delta(a_{k_j})^{\left(\alpha_j + \frac{2\widetilde{n}}{r}\right)\frac{q}{p_j}}$, $j = 1, \dots, n$, $k_j = 0, 1, 2, \dots$

3. Sketches of proofs

In this section we provide complete sketches of proofs of our theorems. We consider only biproduct domain case since arguments needed for proofs of general versions of theorems are similar.

Sketch of proof of Theorem 2.1. The basic estimates and facts for this proof and for the proof of next theorem are (for implication 2) $\Rightarrow$ 1), in particular assertion (A) can be applied to use Lemma 1.1):

(A) The Φ function is subharmonic. Here

$$\Phi(z_2) = \left(\int\limits_{\chi} |f(z_1,z_2)|^q \, (\delta(z_1))^\alpha \, d\nu(z_1) \right)^\beta ,$$

where $\beta > 0$, $\alpha > -1$, $z_2 \in T_\Omega$, $q \in (0,\infty)$, $f \in H(T_\Omega^2)$, $\chi = B_{T_\Omega}(z,r)$ or $\chi = T_\Omega$

$$(B) \quad \left(\int_{T_\Omega^2} |f(\vec{z})| \, (\delta(\vec{z}))^\alpha \, d\nu(\vec{z}) \right)^\tau \leq C_0 \int_{T_\Omega^2} |f(\vec{z})|^p \, (\delta(\vec{z}))^{\alpha\tau + \frac{2\tilde{n}}{r}\tau - \frac{2\tilde{n}}{r}} \, d\nu(\vec{z}),$$

where $0 < \tau \leq 1$, $\alpha > -1$, $f \in H(T_\Omega^2)$ and $d\nu(\vec{z}) = d\nu(z_1) \times d\nu(z_2)$, $f(\vec{z}) = f(z_1,z_2)$, $\delta(\vec{z}) = (\delta(z_1)) \times (\delta(z_1))$.

In case of simpler domains, unit disk, polydisk and unit ball these assertions are known (see [7], [13]). Proofs are similar, namely, for example, if

$$\Phi(z_m) = \left(\int_{T_\Omega} \cdots \int_{T_\Omega} \left(\int_{T_\Omega} |f(z_1,\ldots,z_m)|^q \delta(z_1)^{\alpha_1} d\nu(z_1) \right) \cdots \delta(z_{m-1})^{\alpha_{m-1}} d\nu(z_{m-1}) \right)^{\frac{1}{q_{m-1}}} ,$$

then $\Phi(z_m)$ function is subharmonic where $z_m \in T_\Omega,\ \alpha_j > -1,\ j = 1, \cdots, m - 1$.

We will also use constantly properties of r-lattices provided by us in previous section. We will use for example the following well known property also.

Let $T_\Omega \subset \mathbb{C}^{\tilde{n}}$ tubular domain over symmetric cone and $r \in$(0,1). Then $\nu(B_{T_\Omega}(\cdot,r)) \approx \delta^{2\tilde{n}/r}$, where the constant depends on r, and then there is $C > 0$ such that

$$\frac{C}{1-r}\delta(z_0) \leq \delta(z) \leq \frac{1-r}{C}\delta(z_0)$$

for all $r \in (0,1)$, $z_0 \in T_\Omega$ and $z \in B_{T_\Omega}(z_0,r)$, where C is constant which is not depend on r.

The proof of Theorem 2.1 will hold for $m = 2$, since for $m > 2$ there are similar arguments. First we note that the implication 1) $\Rightarrow$ 2) checked by standard way, using the test function $e_{\vec{z}}(\vec{\zeta}) = \prod\limits_{j=1}^{m} K_{\beta_j}(\zeta_j, z_j)$, $z = (z_1, \ldots, z_m)$, $\zeta = (\zeta_1, \ldots, \zeta_m) \in T_\Omega^m$, for sufficiently large β_j, $\beta_j > \beta_0$, $\beta_j \in \mathbb{N}$, $j = 1, \ldots, m$ (see [1], [5], [6]).

Indeed we have to choose simply an appropriate f test function to get what we need. Put $f = K_{\beta_1} \times K_{\beta_2}$ and we put this test function into estimate

$$\left(\int_{T_\Omega}\left(\int_{T_\Omega}|f(z_1,z_2)|^{q_1}d\mu_1(z_1)\right)^{\frac{q_2}{q_1}}d\mu_2(z_2)\right)^{\frac{1}{q_2}}$$
$$\leq C_1(\mu)\left(\int_{T_\Omega}\left(\int_{T_\Omega}|f(z_1,z_2)|^{p_1}d\nu_{\widetilde{\alpha}_1}(z_1)\right)^{\frac{p_2}{p_1}}d\nu_{\widetilde{\alpha}_2}(z_2)\right)^{\frac{1}{p_2}}$$

Note these type arguments were used many times in various embeddings in ball, polydisk, before. Note, using basic properties of determinant function, we have for $\beta_j > \beta_0$, $j = 1, 2$

$$\left(\int_{T_\Omega}\left(\int_{T_\Omega}|K_{\beta_1}(z_1,\widetilde{q}_1)|^{p_1}|K_{\beta_2}(z_2,\widetilde{q}_2)|^{p_1}d\nu_{\widetilde{\alpha}_1}(z_1)\right)^{\frac{p_2}{p_1}}d\nu_{\widetilde{\alpha}_2}(z_2)\right)^{\frac{1}{p_2}}$$
$$\leq C_3\left(\delta(\widetilde{q}_2)\right)^{\alpha_2}\left(\delta(\widetilde{q}_1)\right)^{\alpha_1},$$

for some α_1 and α_2.

Then using Lemma 1.2, basic properties of determinant function and remarks above on r-lattices we get the following estimates from below for $f = K_{\beta_1} \times K_{\beta_2}$

$$\widetilde{J} = \left(\int_{T_\Omega}\left(\int_{T_\Omega}|f(z_1,z_2)|^{q_1}d\mu_1(z_1)\right)^{\frac{q_2}{q_1}}d\mu_2(z_2)\right)^{\frac{1}{q_2}}$$

$$\geq C_5\left(\int_{B_{T_\Omega}(a_{k_1},\widetilde{r})}|K_{\beta_2}(z_1,\widetilde{q}_1)|^{q_2}d\mu_2(z_2)\right)^{\frac{1}{q_2}}\left(\int_{B_{T_\Omega}(a_{k_2},\widetilde{r})}|K_{\beta_1}(z_2,\widetilde{q}_2)|^{q_1}d\mu_1(z_1)\right)^{\frac{1}{q_2}}$$
$$\widetilde{J} \geq C_6\left(\prod_{j=1}^{2}\mu_j(B_{T_\Omega}(a_{k_j},\widetilde{r}))\delta^{\tau_j}(a_{k_j})\right),$$

for some τ_j, $j = 1, 2$, $\tau_j - \alpha_j = (\widetilde{\alpha}_j + 2\widetilde{n}/r)(q_j/p_j)$, $j = 1, 2$, based on vital remarks on Bergman kernel we see before Lemma 1.1. The last estimate is based on estimate from below of Bergman kernel (see previous section) on Bergman balls. Note, on diagonal (z, z), namely, we use the estimate

$$|K(z,z)|^t \geq c\delta(z)^{2\widetilde{n}/rt},$$

for any $z \in T_\Omega$. The rest is clear.

Therefore, we turn to the proof of the implication 2) $\Rightarrow$ 1). Based on Lemmas 1.2 and basic properties of determinant function, above presenting our domain as the union of dyadic domains (Bergman balls from Lemma 1.2) and using standard properties of r-lattices from previous section, we have

$$M(z_2) = \left(\int_{T_\Omega}|f(z_1,z_2)|^{q_1}\,d\mu_2(z_1)\right)^{\frac{1}{q_1}}$$

$$\leq c_1 \left(\sum_{k_1=0}^{+\infty} \max_{z_1 \in B_{T_\Omega}(a_{k_1}, \widetilde{r})} \left\{ |f(z_1,z_2)|^{q_1} (\delta(z_1))^{\frac{2\tilde{n}}{r}\frac{q_1}{p_1}} (\delta(z_1))^{\frac{q_1}{p_1}\widetilde{\alpha}_1} \right\} \right)^{\frac{1}{q_1}}.$$

Taking into account that $\dfrac{p_1}{q_1} \leq 1$, we have

$$M(z_2) \leq c_2 \left(\sum_{k_1=0}^{+\infty} \max_{z_1 \in B_{T_\Omega}(a_{k_1}, \widetilde{r})} \left\{ |f(z_1,z_2)|^{p_1} (\delta(z_1))^{\frac{2\tilde{n}}{r}} (\delta(z_1)^{\widetilde{\alpha}_1}) \right\} \right)^{\frac{1}{p_1}}.$$

Now, applying Lemma 1.1 and basic properties of determinant function to the function $f(z_1, z_2)$ for fixed $z_2 \in T_\Omega$ we obtain

$$M(z_2) \leq c_3 \left(\int_{T_\Omega} |f(z_1,z_2)|^{p_1} (\delta(z_1))^{\widetilde{\alpha}_1} d\nu(z_1) \right)^{\frac{1}{p_1}}.$$

Raising to the power q_2 both sides of the last inequality and integrating over μ_2, we obtain

$$\left(\int_{T_\Omega} [M(z_2)]^{q_2} d\mu_2(z_2) \right)^{\frac{1}{q_2}} \leq C \left(\int_{T_\Omega} \left(\int_{T_\Omega} |f(z_1,z_2)|^{p_1} (\delta(z_1))^{\widetilde{\alpha}_1} d\nu(z_1) \right)^{\frac{q_2}{p_1}} d\mu_2(z_2) \right)^{\frac{1}{q_2}}.$$

If we represent the T_Ω domain as union of dyadic domains (Bergman balls from Lemma 1.2), using standard properties of r-lattices from previous section and take into account (1), we have: $\left(\int\limits_{T_\Omega} [M(z_2)]^{q_2}\, d\mu_2(z_2) \right)^{\frac{1}{q_2}} \leq$

$$\leq C_2 \left(\sum_{k_2=0}^{+\infty} \max_{z_2 \in B_{T_\Omega}(a_{k_2}, \widetilde{r})} \left(\int_{T_\Omega} |f(z_1,z_2)|^{p_1} (\delta(z_1))^{\widetilde{\alpha}_1} d\nu(z_1) \right)^{\frac{q_2}{p_1}} (\delta(z_2))^{\widetilde{\alpha}_2 \frac{q_2}{p_2}} (\delta(z_2))^{\frac{2\tilde{n}}{r}\frac{q_2}{p_2}} \right)^{\frac{1}{q_2}}.$$

Note $\dfrac{1}{q_2} = \dfrac{p_2}{q_2}\dfrac{1}{p_2}$ and $\dfrac{p_2}{q_2} \leq 1$ and we have the estimate $\left(\int\limits_{T_\Omega} [M(z_2)]^{q_2}\, d\mu_2(z_2) \right)^{\frac{1}{q_2}} \leq$

$$\leq C_2 \left(\sum_{k_2=0}^{+\infty} \max_{z_2 \in B_{T_\Omega}(a_{k_2}, \widetilde{r})} \left(\int_{T_\Omega} |f(z_1,z_2)|^{p_1} (\delta(z_1))^{\widetilde{\alpha}_1} d\nu(z_1) \right)^{\frac{p_2}{p_1}} (\delta(z_2))^{\widetilde{\alpha}_2} (\delta(z_2))^{\frac{2\tilde{n}}{r}} \right)^{\frac{1}{p_2}}.$$

Note, now using (A) and Lemma 1.1 for inner integral we finish the proof.

Theorem 2.1 is proved. □

Sketch of proof of Theorem 2.2. As in the proof of Theorem 2.1, the implication 1) ⇒2) is verified in a standard way.

We have the following estimate for $\widetilde{f}$ test function, $\widetilde{f} = K_{\beta_1} \times K_{\beta_2}$, where $\beta_j,\ j = 1, 2$ are large enough, for $n = 2$ case:

$$I = \left(\int\limits_{T_\Omega^2} \left| \widetilde{f}(\vec{z}) \right|^q d\widetilde{\nu}(\vec{z}) \right)^{\frac{1}{q}} \leq c_1(\widetilde{\nu}) \left(\int\limits_{T_\Omega} \left(\int\limits_{T_\Omega} \left| \widetilde{f}(z_1, z_2) \right|^{p_1} d\nu(z_1) \right)^{\frac{p_2}{p_1}} d\nu(z_2) \right)^{\frac{1}{p_2}}.$$

Then we have based on remarks on Bergman kernel before Lemma 1.1

$$I \geq c_2 \left(\int\limits_{B\left(a_{k_1}, r\right)} \int\limits_{B\left(a_{k_2}, r\right)} \left| \widetilde{f}(z_1, z_2) \right|^q d\widetilde{\nu}(\vec{z}) \right)^{\frac{1}{q}} \geq$$

$$\geq c_3 \left(\widetilde{\nu} \left(B_{T_\Omega} \left(a_{k_1}, r \right) \times B_{T_\Omega} \left(a_{k_2}, r \right) \right) \right) \times \left(\left(\delta \left(a_{k_1} \right) \right)^{\tau_1} \right) \times \left(\left(\delta \left(a_{k_2} \right) \right)^{\tau_2} \right),$$

for some $\tau_1 = \tau_1(q, \beta), \tau_2 = \tau_2(q, \beta)$.

The last estimate is based on estimate from below of Bergman kernel (see previous section) on Bergman balls. Note, on diagonal (z, z) namely, we use the estimate $|K(z, z)|^t \geq c\delta(z)^{2\widetilde{n}/rt}$, for any $z \in T_\Omega$. Also, we have

$$\left\| \widetilde{f} \right\|_{A^{\vec{p}}_{\vec{\alpha}}} \leq c_4 \left(\left(\delta \left(a_{k_1} \right) \right)^{\widetilde{\beta}_1} \left(\delta \left(a_{k_2} \right) \right)^{\widetilde{\beta}_2} \right),$$

(this is follows directly from basic properties of determinant function) for some values of $\widetilde{\beta}_1, \widetilde{\beta}_2 : \widetilde{\beta}_1 = \widetilde{\beta}_1(p_2, p_1, \beta_1, \beta_2)$; $\quad \widetilde{\beta}_2 = \widetilde{\beta}_2(p_2, p_1, \beta_1, \beta_2)$.

Note that $\tau_s - \widetilde{\beta}_s = \left(- \left(\alpha_s + \dfrac{2\widetilde{n}}{r} \right) \dfrac{q}{p_s} \right)$, $s = 1, 2$. This gives one part of the theorem.

Now we proceed to the proof of 2) ⇒1). Based on lemmas above again, we prove the theorem for $n = 2$, since for $n > 2$ there are similar arguments. First suppose that

$$q \geq p_1 \geq p_2. \tag{1}$$

Using the arguments in the proof of Theorem 2.1, Lemmas 1.1, 1.2 by each variable separately and condition on measure in formulation of Theorem 2.2, we have

$$I(f) = \left(\int\limits_{T_\Omega^2} |f(z_1, z_2)|^q \, d\widetilde{\nu}(z_1, z_2) \right)^{\frac{1}{q}}$$

$$\leq C_1 \Big(\int\limits_{T_\Omega^2} |f(z_1, z_2)|^q \left(\delta(z_1)\right)^{\alpha_1 \frac{q}{p_1}} \left(\delta(z_2)\right)^{\alpha_2 \frac{q}{p_2}} \times$$

$$\times(\delta(z_1))^{\frac{2\tilde{n}}{r}\frac{q}{p_1}}(\delta(z_2))^{\frac{2\tilde{n}}{r}\frac{q}{p_2}}d\nu(z_1)d\nu(z_2)\Big)^{\frac{1}{q}}.$$

Taking into account that $p_1 \le q$, from (A) and (B) we obtain

$$I(f) \le C_2\Big(\int\limits_{T_\Omega}(\delta(z_2))^{\alpha_2\frac{p_1}{p_2}}(\delta(z_2))^{\frac{2\tilde{n}}{r}\frac{p_1}{p_2}-\frac{2\tilde{n}}{r}}\times$$

$$\times\left[\int\limits_{T_\Omega}|f(z_1,z_2)|^q\,(\delta(z_1))^{\alpha_1\frac{q}{p_1}}(\delta(z_1))^{(\frac{2\tilde{n}}{r}\frac{q}{p_1}-\frac{2\tilde{n}}{r})}d\nu(z_1)\right]^{\frac{p_1}{q}}d\nu(z_2)\Big)^{\frac{1}{p_1}}.$$

Again, using similar arguments to the inner integral, we obtain the estimate (based on (B))

$$I(f) \le C_3\Big(\int\limits_{T_\Omega}(\delta(z_2))^{\alpha_2\frac{p_1}{p_2}}\,(\delta(z_2))^{\frac{2\tilde{n}}{r}\frac{p_1}{p_2}-\frac{2\tilde{n}}{r}}\times\left[\int\limits_{T_\Omega}|f(z_1,z_2)|^{p_1}\,(\delta(z_1))^{\alpha_1}d\nu(z_1)\right]d\nu(z_2)\Big)^{\frac{1}{p_1}}.$$

Taking into account that $\dfrac{p_2}{p_1} \le 1$, as above, we obtain (based on (A) and (B))

$$I(f) \le C_4\left(\int\limits_{T_\Omega}(\delta(z_2))^{\alpha_2}\left[\int\limits_{T_\Omega}|f(z_1,z_2)|^{p_1}\,(\delta(z_1))^{\alpha_1}d\nu(z_1)\right]^{\frac{p_2}{p_1}}d\nu(z_2)\right)^{\frac{1}{p_2}}.$$

The theorem is proved under the condition (1). Now we turn to the case

$$q \ge p_2 \ge p_1. \tag{2}$$

Let $(\widetilde{z}_{k_1},\ \widetilde{z}_{k_2}) \in B_{T_\Omega}(z_{k_1},r)\times B_{T_\Omega}(z_{k_2},\widetilde{r})$ then by Lemma 1.1

$$\max|f(\widetilde{z}_{k_1},\ \widetilde{z}_{k_2})|^q \le \frac{C}{(\delta(z_{k_1}))^{\frac{2\tilde{n}}{r}\frac{q}{p_1}}}\left(\int\limits_{B^*_{T_\Omega}(z_{k_1},r)}|f(\zeta_1,\widetilde{z}_{k_2})|^{p_1}\,d\nu(\zeta_1)\right)^{\frac{q}{p_1}},$$

where $B^*_{T_\Omega}(\widetilde{z}_{k_1},\widetilde{r})$ is expansion of the dyadic domain with the same center $B_{T_\Omega}(z_{k_1},\widetilde{r})$ and $\widetilde{z}_{k_2} \in B^*_{T_\Omega}(z_{k_2},\widetilde{r})$, (see Lemma 1.2). Therefore, based on (A) and condition on measure and properties of r-lattice (see Lemmas 1.1, 1.2), as in the proof of the first part: $\quad I(f) = \left(\int\limits_{T^2_\Omega}|f(z_1,z_2)|^q\,d\widetilde{\nu}(z_1,z_2)\right)^{\frac{1}{q}} \le$

$$\le C_6\left(\sum_{k_2=0}^{+\infty}\sum_{k_1=0}^{+\infty}(\delta(z_{k_2}))^{\alpha_2\frac{q}{p_2}}(\delta(z_{k_2}))^{\frac{2\tilde{n}}{r}\frac{q}{p_2}}\left[\int\limits_{B^*_{T_\Omega}(z_{k_1},r)}|f(\zeta_1,z_{k_2})|^{p_1}\,\delta^{\alpha_1}(\zeta_1)d\nu(\zeta_1)\right]^{\frac{q}{p_1}}\right)^{\frac{1}{q}}.$$

Using the inequality (2), we obtain

$$I^{p_2}(f) \leq C_6^{p_2} \sum_{k_2=0}^{+\infty} \sum_{k_1=0}^{+\infty} (\delta(z_{k_2}))^{\alpha_2} (\delta(z_{k_2}))^{(\frac{2\tilde{n}}{r})} \left[\int\limits_{B^*_{T_\Omega}(z_{k_1},r)} |f(\zeta_1, z_{k_2})|^{p_1} \delta^{\alpha_1}(\zeta_1) d\nu(\zeta_1) \right]^{\frac{p_2}{p_1}}.$$

We have based on (A)

$$I^{p_2}(f) \leq C_6^{p_2} \int\limits_{T_\Omega} \Big(\sum_{k_1=0}^{+\infty} \Big[\int\limits_{B^*_{T_\Omega}(z_{k_1},r)} |f(\zeta_1, z_2)|^{p_1} \times (\delta(\zeta_1))^{\alpha_1} d\nu(\zeta_1) \Big]^{\frac{p_2}{p_1}} \Big) (\delta(\zeta_2))^{\alpha_2} d\nu(\zeta_2). \tag{3}$$

By the condition (2) $\dfrac{p_1}{p_2} = \alpha \geq 1$. Hence, $\sum_{k=0}^{+\infty} b_k^\alpha \leq \left(\sum_{k=0}^{+\infty} b_k\right)^\alpha$ for all $b_k \geq 0$, $k \in \mathbb{N}$. Therefore, by (3) we obtain (based on Lemma 1.2)

$$I^{p_2}(f) \leq C_7 \int\limits_{T_\Omega} \left(\int\limits_{T_\Omega} |f(z_1, z_2)|^{p_1} (\delta(z_1))^{\alpha_1} d\nu(z_1) \right)^{\frac{p_2}{p_1}} (\delta(\zeta_2))^{\alpha_2} d\nu(\zeta_2).$$

Theorem 2.2 is proved. □

References

[1] D. Bekolle, A. Bonami, G. Garrigos, C. Nana, M. Peloso, and F. Ricci, *Lecture notes on Bergman projectors in tube domain over cones, an analytic and geometric viewpoint*, Proceeding of the International Workshop on Classical Analysis, Yaounde, 2001, 75 pages.

[2] O. V. Besov, V. P. Il'in, and S. M. Nicol'skij, *Integral'nye predstavleniya funktsij i teoremy vlozheniya*, Moskva, Nauka, Fizmatlit, 2nd ed., rev. and compl. Edition, 1996.

[3] J. Faraut and A. Koranyi, *Analysis on symmetric cones*, Clarendon Press, Oxford, 1994.

[4] P. Mercer and J. Cima, *Composition operators between Bergman spaces on convex domains in $\mathbb{C}^n$*, Journal Operator Theory **33** (1995), no. 2.

[5] B. F. Sehba, *Bergman type operators in tube domains over symmetric cones*, Proc. Edinburg. Math. Society **52** (2009), no. 2, 529–544.

[6] B. F. Sehba and C. Nana, *Carleson Embeddings and two operators on Bergman spaces of tube domains over symmetric cones*, Integr. Equ. Oper. Theory **83** (2015), 151–178.

[7] F. A. Shamoyan, *Embedding theorems and a characterization of traces in the spaces $h^p(u^n)$, $0 < p < \infty$*, Mathematics of the USSR-Sbornik **35** (1979), no. 5, 709–725.

[8] ______, *Embedding theorems for weighted anisotropic spaces of holomorphic functions in the polydisk*, Journal Math. Anal. Geom. (2003), 116–125.

[9] F. A. Shamoyan and O. V. Yaroslavtseva, *Continuous projections, duality, and the diagonal mapping in weighted spaces of holomorphic functions with mixed norm*, Journal of Mathematical Sciences **101** (2000), no. 3, 3211–3215.

[10] R. Shamoyan and S. Kurilenko, *On new embedding theorems in analytic bergman spaces in bounded pseudoconvex domains*, Journal Siberian Fed. Univ. **7** (2014), no. 3, 383–388.

[11] R. Shamoyan and S. Maksakov, *Embedding theorems for weighted anisotropic spaces of holomorphic functions in strongly pseudoconvex domains*, Romai Journal **13** (2017), no. 1, 71–92.

[12] R. Shamoyan and E. Povpritz, *Multifunctional analytic spaces and new sharp embedding theorems in strongly pseudoconvex domains*, Krag. Math. Journal (2013), no. 37, 221–244.

[13] K. Zhu, *Spaces of holomorphic functions in the unit ball*, Springer-Verlag, New York, 2005.