A Universal Homogeneous Simple Matroid of Rank 3

Una matroide simple homogénea universal de rango 3

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Abstract. We construct a $\wedge$-homogeneous universal simple matroid of rank 3, i.e. a countable simple rank 3 matroid M_* which $\wedge$-embeds every finite simple rank 3 matroid, and such that every isomorphism between finite $\wedge$-subgeometries of M_* extends to an automorphism of M_*. We also construct a $\wedge$-homogeneous matroid $M_*(P)$ which is universal for the class of finite simple rank 3 matroids omitting a given finite projective plane P. We then prove that these structures are not $\aleph_0$-categorical, they have the independence property, they admit a stationary independence relation, and that their automorphism group embeds the symmetric group $Sym(\omega)$. Finally, we use the free projective extension $F(M_*)$ of M_* to conclude the existence of a countable projective plane embedding all the finite simple matroids of rank 3 and whose automorphism group contains $Sym(\omega)$, in fact we show that $Aut(F(M_*)) \cong Aut(M_*)$.

Keywords: homogeneous structures, matroids, incidence structures, automorphism groups.

Resumen. Construimos una matroide $\wedge$-homogénea universal de rango 3, i.e. una matroide M^* contable simple de rango 3 en el que que se $\wedge$-sumerge toda matroide finita simple de rango 3, y tal que todo isomorfismo entre $\wedge$-subgeometrías finitas de M^* se extienden a un automorfismo de M_*. Construimos además una matroide $M_*(P)$ $\wedge$-homogénea que es universal para la clase de las matroides finitas simples de rango 3 que omiten un plano proyecto finito P dado. Entonces demostramos que estas estructuras no son $\aleph_0$-categóricas, tienen la propiedad de independencia y admiten una relación de independencia estacionaria, y que su grupo de automorfismos sumerge el grupo de simetrías $Sym(\omega)$. Finalmente, usamos la extensión productiva libre $F(M_*)$ de M_* para concluir la existencia de un plano proyecto contable que sumerge todas las matroides finitas simples de rango 3 y cuyo grupo de automorfismos contiene $Sym(\omega)$, de hecho demostramos que $Aut(F(M_*)) \cong Aut(M_*)$.

Palabras claves: estructuras homogéneas, matroides, estructuras de incidencia, grupos de automorfismos.

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1. Introduction

A countably infinite structure M is said to be *homogeneous* if every isomorphism between finitely generated substructures of M extends to an automorphism of M. The study of homogeneous combinatorial structures such as graph, digraphs and hypergraphs is a very rich field of study (see e.g. [2], [3], [6], [7], [17], [16]). On the other hand, *matroids* are objects of fundamental importance in combinatorial theory, but very little is known on homogeneous matroids. In this short note we propose a new approach to the study of homogeneous matroids, focusing on the case in which the matroid is of rank 3 and simple. In this case the matroidal structure can be defined in a very simple manner as a 3-hypergraph[1], as follows:

Definition 1.1. A *simple matroid*[2] *of rank* ≤ 3 is a 3-hypergraph (V, R) whose adjacency relation is irreflexive, symmetric and satisfies the following *exchange axiom*:

(Ax) if $R(a, b, c)$ and $R(a, b, d)$, then $\{a, b, c, d\}$ is an R-clique.

We say that the matroid has rank 3 if it contains three non-adjacent points.

As well-known (see e.g. [15, pg. 148]), simple matroids of rank ≤ 3 are in canonical correspondence (cf. Convention 1) with certain incidence structures known as *linear spaces*:

Definition 1.2. A *linear space* is a system of points and lines satisfying:

(A) every pair of distinct points determines a unique line;

(B) every pair of distinct lines intersects in at most one point;

(C) every line contains at least two points.

In [5] Devellers provides a complete classification of the countable homogeneous linear spaces. In this work it is shown that (as formulated) the theory is very poor, and in fact the only infinite homogeneous linear space is the trivial one, i.e. infinitely many points and infinitely many lines incident with exactly two points.

This situation is reflected in the context of matroid theory with the well-known observation (see e.g. [21, Example 7.2.3]) that the class of finite simple matroids of rank 3 does not have the amalgamation property, and so the construction of a homogeneous (with respect to the notion of subgeometry) simple matroid of rank 3 containing all the finite simple matroids of rank 3 as subgeometries is hopeless.

One might wonder if this is all there is to it, and no further mathematical theory is possible. In this short note we give evidence that this is not

[1] To the reader familiar with matroid theory it will be clear that in (V, R) the hyper-edges are nothing but the dependent sets of size 3 of the unique simple matroid of rank 3 coded by (V, R).

[2] For a general introduction to matroid theory see e.g. the classical references [4] and [15].

the case, and that there might be a very interesting combinatorial theory for homogeneous matroids, if the problem (viz. choice of language) is correctly formulated.

The crucial observation that underlies our approach is that (with respect to questions of homogeneity) the choice of substructure that we are considering is *too weak*, and does not take into account enough of the geometric structure encoded by these objects, i.e. their associated *geometric lattices*[3]. This inspires:

Definition 1.3. Let P be a linear space (cf. Definition 1.2). On P we define two partial functions $p_1 \vee p_2$ and $\ell_1 \wedge \ell_2$ denoting, respectively, the unique line passing through the points p_1 and p_2, and the unique point p at the intersection of the lines ℓ_1 and ℓ_2, if such a point exists, and 0 otherwise (where 0 is a new symbol). If we extend P to $\hat{P}$ adding a largest element 1 and a smallest element 0 and we extend the interpretation of $\vee$ and $\wedge$ in the obvious way, then the structure $(\hat{P}, \vee, \wedge, 0, 1)$ is a so-called geometric lattice. For details on this see [4, Chapter 2] or [15, pg. 148].

Convention 1. *When convenient, we will be sloppy in distinguishing between a simple rank 3 matroid and it associated linear space/geometric lattice (cf. Definition 1.3). This is justified by the following canonical correspondence between the two classes of structures. Given a linear space P consider the simple rank 3 matroid M_P whose dependent sets of size 3 are the triples of collinear points of P. Given a simple rank 3 matroid (V, R) consider the linear space P_M whose points p are the elements of V and whose lines ℓ are the sets of the form $\{a, b\} \cup \{c \in V : R(a, b, c)\}$, together with the incidence relation $p \in \ell$. Also, we will use freely the partial functions $\vee$ and $\wedge$ introduced in Definition 1.3 in the context of linear spaces.*

Definition 1.4. A *simple $\wedge$-matroid of rank ≤ 3* is a structure $M = (V, R, \wedge)$ such that (V, R) is a simple matroid of rank ≤ 3 (cf. Definition 1.1) and $\wedge$ is a 4-ary function defined as follows[4] (cf. Definition 1.3):

$$\wedge_M(a, b, c, d) = \begin{cases} (a \vee b) \wedge (c \vee d) & \text{if } (a \vee b) \wedge (c \vee d) \notin \{0, a, b, c, d, a \vee b, c \vee d\}, \\ a & \text{otherwise.} \end{cases}$$

In this study we will see that with respect to the new notion of substructure introduced in Definition 1.4 there is hope for a rich mathematical theory, which is potentially analogous to the situation for homogeneous graphs (see e.g. [17]). In fact, we prove:

Theorem 1.5. *There exists a homogeneous simple rank 3 $\wedge$-matroid M_* which is universal for the class of finite simple $\wedge$-matroids of rank ≤ 3.*

[3]A geometric lattice is a semi-modular point lattice without infinite chains. For more on this see e.g. [12, Section 2], [4, Chapter 2] and [15].

[4]Clearly, in the definition of $\wedge(a, b, c, d)$, the only case in which we are interested is the first case of the disjunction, i.e. when $a \vee b$ and $c \vee d$ are two distinct lines intersecting in a fifth point p, in which case the vale of $\wedge(a, b, c, d)$ is indeed p. The way the definition of $\wedge(a, b, c, d)$ is written is just a technical way to express this natural condition.

Theorem 1.6. *Let P be a finite projective plane, and M_P the corresponding simple rank 3 matroid. Then there exists a homogeneous simple $\wedge$-matroid $M_*(P)$ which is universal for the class of finite simple $\wedge$-matroids of rank ≤ 3 omitting*[5] M_P.

It might be argued that in the context of simple rank 3 matroids the homogeneous structure of Theorem 1.5 plays the role played by the random graph [22] for the class of finite graphs, while the homogeneous structure of Theorem 1.5 plays the role played by the random K_n-free[6] graph [9] for the class of finite graph omitting K_n.

We then prove several facts of interest on the automorphism groups of the homogeneous structures from Theorems 1.5 and 1.6.

Theorem 1.7. *Let M_* be as in Theorem 1.5 or Theorem 1.6. Then:*

1. *M_* is not $\aleph_0$-categorical;*
2. *M_* has the independence property;*
3. *M_* admits a stationary independence relation;*
4. *$Aut(M_*)$ embeds the symmetric group $Sym(\omega)$;*
5. *if the age of M_* has the extension property for partial automorphisms, then $Aut(M_*)$ has ample generics, and in particular it has the small index property.*

Finally, we give an application to projective geometry proving:

Corollary 1.8. *Let M_* be as in Theorem 1.5, and let $F(M_*)$ be the free projective extension of M_* (cf. [8]). Then:*

1. *$F(M_*)$ embeds all the finite simple rank 3 matroids as subgeometries;*
2. *every $f \in Aut(M_*)$ extends to an $\hat{f} \in Aut(F(M_*))$;*
3. *$f \mapsto \hat{f}$ is an isomorphism from $Aut(M_*)$ onto $Aut(F(M_*))$;*
4. *$Aut(F(M_*))$ embeds the symmetric group $Sym(\omega)$.*

We leave the following open questions:

Question 1. Let M_* be as in Theorem 1.5 or Theorem 1.6.

1. Does $Aut(M_*)$ have the small index property?
2. Does $Aut(M_*)$ have ample generics?

Question 2.

[5] By this we mean that there is no injective map $f : M_P \to N$ such that $M_P \cong f(M_P)$.
[6] K_n denotes the complete graph on n vertices.

1. Does the class of simple $\wedge$-matroids of rank 3 have the extension property for partial automorphisms?
2. Does the class of freely linearly ordered simple $\wedge$-matroids of rank 3 have the Ramsey property?

The only infinite homogeneous simple $\wedge$-matroids of rank 3 known to the author are the ones from Theorems 1.5 and 1.6, and the trivial one, i.e. infinitely many points and infinitely many lines incident with exactly two points.

Problem 1. Classify the countable homogeneous simple $\wedge$-matroids of rank 3.

Concerning $F(M_*)$, in [13] Kalhoff constructs a projective plane of Lenz-Barlotti class V embedding all the finite simple rank 3 matroids. In [1] Baldwin constructs some almost strongly minimal projective planes of Lenz-Barlotti class I.1. We leave as an open problem the determination of the Lenz-Barlotti class of $F(M_*)$.

2. Preliminaries

For background on Fraïssé theory and homogeneous structures we refer to [10, Chapter 6]. In particular, given a homogeneous structure M we refer to the closure up to isomorphisms of the collection of finitely generated substructures of M as the *age* of M and denote it by $\mathbf{K}(M)$. For background on the notions on automorphism groups occurring in Theorem 1.7 see e.g. [14]. Concerning free projective extensions see [8] and [11, Chapter XI]. Concerning the notion of stationary independence:

Definition 2.1 ([23] and [19])**.** Let M be a homogeneous structure. We say that a ternary relation $A \perp_C B$ between finitely generated substructures of M is a *stationary independence relation* if the following axioms are satisfied:

(A) (Invariance) if $A \perp_C B$ and $f \in Aut(M)$, then $f(A) \perp_{f(C)} f(B)$;

(B) (Symmetry) if $A \perp_C B$, then $B \perp_C A$;

(C) (Monotonicity) if $A \perp_C \langle BD \rangle$ and $A \perp_C B$, then $A \perp_{\langle BC \rangle} D$;

(D) (Existence) there exists $A' \equiv_B A$ such that $A' \perp_B C$;

(E) (Stationarity) if $A \equiv_C A'$, $A \perp_C B$ and $A' \perp_C B$, then $A \equiv_{\langle BC \rangle} A'$.

Definition 2.2. A *projective plane* is a linear space (cf. Definition 1.2) such that:

(A') every pair of distinct lines intersects in a unique point;

(B') there exist at least four points no three of which are collinear.

For a definition of the notion of independence property of a first-order theory see e.g. [20, Exercise 8.2.2].

3. Proofs

We will prove a series of claims from which Theorems 1.5, 1.6 and 1.7 follow.

Lemma 3.1. *The class of simple $\wedge$-matroids of rank ≤ 3 is a Fraïssé class.*

Proof. The hereditary property is clear. The joint embedding property is easy and the amalgamation property is proved in [12, Theorem 4.2]. Notice that the context of [12] is the study of geometric lattices in a language $L' = \{0, 1, \vee, \wedge\}$, but keeping in mind Definition 1.3, Convention 1, and the fact that we are considering $\wedge$-matroids it is easy to see that the two context are indeed equivalent. □

Definition 3.2 ([**?**, Definition 6])**.** Let M be a homogeneous structure and $\mathbf{K} = \mathbf{K}(M)$ its age. We say that M has *canonical amalgamation* if there exists an operation $B_1 \oplus_A B_2$ on triples from $\mathbf{K}$ satisfying the following conditions:

(a) $B_1 \oplus_A B_2$ is defined when $A \subseteq B_i$ $(i = 1, 2)$ and $B_1 \cap B_2 = A$;

(b) $B_1 \oplus_A B_2$ is an amalgam of B_1 and B_2 over A;

(c) if $B_1 \oplus_A B_2$ and $B'_1 \oplus_{A'} B'_2$ are defined, and there exist $f_i : B_i \cong B'_i$ $(i = 1, 2)$ with $f_1 \restriction A = f_2 \restriction A$, then there is:

$$f : B_1 \oplus_A B_2 \cong B'_1 \oplus_{A'} B'_2$$

such that $f \restriction B_1 = f_1$ and $f \restriction B_2 = f_2$.

Remark 3.3. Notice that the amalgamation from [12, Theorem 4.2] used to prove Lemma 3.1 is canonical in the sense of Definition 3.2. We will denote the canonical amalgam of M_1 and M_2 over M_0 from [12, Theorem 4.2] as $M_1 \oplus_{M_0} M_2$ (when we use this notation we tacitly assume that $M_0 \subseteq M_1$, $M_0 \subseteq M_2$ and $M_1 \cap M_2 = M_0$). Notice that the amalgam $M_3 := M_1 \oplus_{M_0} M_2$ can be characterized as the following $\wedge$-matroid:

1. $M_3 = M_1 \cup M_2$ (i.e. $M_1 \cup M_2$ is the domain of M_3);
2. $R^{M_3} = R^{M_1} \cup R^{M_2} \cup \{\{a, b, c\} : a \vee b = a \vee c = b \vee c = a' \vee b'$ and $\{a', b'\} \subseteq M_0\}$;
3. $\wedge_{M_3}(a, b, c, d) = a$, unless $a \vee b = a' \vee b'$, $c \vee d = c' \vee d'$ and $\wedge_{M_\ell}(a', b', c', d') \neq a'$, for some $\ell = 1, 2$ and $\{a', b', c', d'\} \subseteq M_\ell$, in which case:

$$\wedge_{M_3}(a, b, c, d) = \wedge_{M_\ell}(a', b', c', d').$$

The intuition behind (3) is that the value of the function symbol $\wedge_{M_3}(a, b, c, d)$ is trivial unless $a \vee b$ and $c \vee d$ are two intersecting lines from one of the M_ℓ $(\ell = 1, 2)$.

Lemma 3.4. *Let P be a finite projective plane, and M_P the corresponding matroid. The class of simple $\wedge$-matroids N of rank ≤ 3 omitting*[7] *M_P is a Fraïssé class.*

Proof. Also in this case, the only non-trivial part of the proof is amalgamation. Let M_0, M_1, M_2 be $\wedge$-matroids omitting M_P and such that $M_0 \subseteq M_1$, $M_0 \subseteq M_2$ and $M_1 \cap M_2 = M_0$. Let $M_3 := M_1 \oplus_{M_0} M_2$ be as in Remark 3.3. We want to show that M_3 does not embed M_P, but this is clear noticing that by Remark 3.3 we have:

(i) if $j \in \{1, 2\}$ and ℓ is a line from M_j such that there are no a_0, a_1 in M_0 with $\ell = a_0 \vee a_1$, then the number of points incident with ℓ in M_j is equal to the number of points incident with ℓ in M_3;

(ii) if ℓ is a line of M_3 which is incident with at most one point of M_1 and at most one point of M_2, respectively, then ℓ is incident with exactly two points.

□

We can now prove Theorems 1.5 and 1.6.

Proof of Theorems 1.5 and 1.6. This follows from Lemmas 3.1 and 3.4 using Fraïssé theory (see e.g. [10, Chapter 6]). □

The following lemma establishes the non $\aleph_0$-categoricity of the homogeneous structures of Theorems 1.5 and 1.6.

Lemma 3.5. *For every $n < \omega$ there exists a finite simple rank 3 $\wedge$-matroid $M(n)$ of size $6 + (n + 1)$, and 6 distinct points $p_1, .., p_6 \in M(n)$ such that $\langle p_1, ..., p_6 \rangle_{M(n)} = M(n)$, where $\langle A \rangle_B$ denotes the substructure generated by A in B. Furthermore, $M(n)$ can be taken such that it does not contain any projective plane.*

Proof. By induction on $n < \omega$, we construct a finite simple rank 3 $\wedge$-matroid $M(n)$ such that:

(a) the domain of $M(n)$ is $\{p_1^-, p_2^-, p_1^+, p_2^+, p_1^*, p_2^*, q_0, ..., q_n\}$;

(b) $|\{p_1^-, p_2^-, p_1^+, p_2^+, p_1^*, p_2^*, q_0, ..., q_n\}| = 6 + (n + 1)$;

(c) if n is even, then $p_1^- \vee q_n$ and $p_2^- \vee p_1^*$ are parallel in $M(n)$;

(d) if n is odd, then $p_1^+ \vee q_n$ and $p_2^+ \vee p_2^*$ are parallel in $M(n)$;

(e) $\langle p_1^-, p_2^-, p_1^+, p_2^+, p_1^*, p_2^* \rangle_{M(n)} = M(n)$.

[7] Recall that by this we mean that there is no injective map $f : M_P \to N$ such that $M_P \cong f(M_P)$.

$\underline{n = 0}$. Let M be the simple rank 3 $\wedge$-matroid with domain $\{p_1^-, p_2^-, p_1^+ p_2^+, p_1^*, p_2^*\}$ such that are no hyper-edges (i.e. every line is incident with exactly two points). Add to M the point q_0 which is incident only with the lines $p_1^+ \vee p_2^+$ and $p_1^* \vee p_2^*$ (which are parallel in M), and let $M(0)$ be the resulting $\wedge$-matroid.

$\underline{n = 2k + 1}$. Let $M(n-1)$ be constructed, then $M(n-1)$ contains the lines $p_1^- \vee q_{2k}$ and $p_2^- \vee p_1^*$, and, by induction hypothesis, they are parallel in $M(n-1)$. Add to $M(n-1)$ the point q_n which is incident only with the lines $p_1^- \vee q_{2k}$ and $p_2^- \vee p_1^*$ which are parallel in $M(n-1)$), and let $M(n)$ be the resulting $\wedge$-matroid.

$\underline{n = 2k > 0}$. Let $M(n-1)$ be constructed, then $M(n-1)$ contains the lines $p_1^+ \vee q_{2k-1}$ and $p_2^+ \vee p_2^*$, and, by induction hypothesis, they are parallel in $M(n-1)$. Add to $M(n-1)$ the point q_n which is incident only with the lines $p_1^+ \vee q_{2k-1}$ and $p_2^+ \vee p_2^*$, and let $M(n)$ be the resulting $\wedge$-matroid. □

Lemma 3.6. *Let M_* be as in Theorem 1.5 or Theorem 1.6. Then M_* has the independence property.*

Proof. As in [12, Theorem 4.6]. □

Lemma 3.7. *Let M_* be as in Theorem 1.5 or Theorem 1.6. For every finite substructures A, B, C of M_*, define $A \perp_C B$ if and only if $\langle A, B, C\rangle_{M_*} \cong \langle A, C\rangle_{M_*} \oplus_C \langle B, C\rangle_{M_*}$. Then $A \perp_C B$ is a stationary independence relation.*

Proof. Easy to see using Remark 3.3. □

Lemma 3.8. *Let M_* be as in Theorem 1.5 or Theorem 1.6. If $f \in Sym(M_*)$ induces an automorphism of $Aut(M_*)$ (i.e. $g \mapsto fgf^{-1} \in Aut(Aut(M_*))$), then $f \in Aut(M_*)$.*

Proof. First of all, notice that if M is a simple rank 3 $\wedge$-matroid and M^- is the reduct of M to the language $L = \{R\}$ then we have that $f \in Aut(M)$ if and only if $f \in Aut(M^-)$. Thus if $f \notin Aut(M_*)$, then $f \notin Aut(M_*^-)$, i.e. there exists a set $\{a, b, c\} \subseteq M_*$ such that either $\{a, b, c\}$ is dependent in M_* and $\{f(a), f(b), f(c)\}$ is independent in M_*, or $\{a, b, c\}$ is independent in M_* and $\{f(a), f(b), f(c)\}$ is dependent in M_*. Modulo replacing f with f^{-1}, we can assume, that $\{a, b, c\}$ is independent in M_* and $\{f(a), f(b), f(c)\}$ is dependent in M_*. Suppose now that in addition f induces an automorphism of $Aut(M_*)$. Since M_* is homogeneous, it is easy to see that f has to send dependent sets of size 3 to independent sets of size 3. Now, by Definition 1.1, if $R(a, b, c)$ and $R(a, b, d)$, then $\{a, b, c, d\}$ is an R-clique. On the other hand, trivially in M_* we can find distinct points $\{a, b, c, d\}$ such that $\{a, b, c\}$ is an independent set, $\{a, b, d\}$ is an independent set, and $\{b, c, d\}$ is *not* an independent set. Hence, we easily reach a contradiction. □

Remark 3.9. Notice that the linear space P consisting of infinitely many points and infinitely many lines incident with exactly two points satisfies $Aut(P) \cong Sym(\omega)$.

Proof of Theorem 1.7. Item (1) follows from Lemma 3.5. Item (2) is Lemma 3.6. Item (3) follows from Lemma 3.7. Item (4) follows from item (3), the main result of [19] and Remark 3.9. Item (5) follows from Remark 3.3 (JEP for partial automorphisms is easy to see), [14, Theorem 1.6], and [14, Theorem 6.2]. □

Proof of Corollary 1.8. Notice that every point and every line of M_* is contained in a copy of the Fano plane (which is a confined configuration, in the terminology of [11, pg. 220]). Thus, the result follows from [11, Theorem 11.18] or [18, Lemma 1]. □

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