On the equivalence of Gaussian Sobolev norms and some weighted Wiener-Chaos norms

Sobre la equivalencia de normas de Sobolev gaussianas y normas de Wiener-caos con pesos

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Abstract. When working with Gaussian measures, in some applications, regularity of functions needs to be evaluated. In order to measure the regularity of a function in a mean square sense, Weiner-Chaos weighted norms and also Gaussian-Sobolev norms have been introduced. Some family of weights used in the Chaos norms generate norms equivalent to the Gaussian Sobolev norms. In this short paper we review this fact and a recent proof of it presented in [5, 6] that sharpened the equivalence to and equality of one of the norms in terms of the other. We note that we review the case of spaces of functions of infinity many variables.

Keywords: White noise analysis, Wiener-Chaos expansions, finite elements, stochastic partial differential equations, stochastic elliptic equations.

Resumen. Cuando trabajamos con medidas gausianas, en algunas aplicaciones, se necesita evaluar la regularidad de la solución. Para medir la regularidad de la solución en el sentido medio cuadrático, normas del tiempo de Weiner-caos con peso y normas de Sobolev gaussianas han sido introducidas. Algunas familias de pesos usados en las normas de caos generan normas equivalentes a las normas de Sobolev gaussianas. En este artículo corto revisamos esta equivalencia y una demostración reciente que obtiene igualdad entre estas normas. Notamos que se revisa el caso de espacios de funciones en infinitas variables.

Palabras claves: Normas de Sobolev gaussianas, Normas de Weiner caos, cálculo de dimensión infinita.

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1. Introduction

In this short paper we review the known fact that Gaussian-Sobolev norms are equivalent to a particular weighted Weiner-chaos norm. The weighted chaos norms depend on the choice of a sequence of weights. The corresponding norm measures the decay of the coefficients in the chaos expansion of a random function. We recall that the chaos expansion of a random function is its expansion in terms of Fourier-Hermite orthogonal polynomials. A main issue is that the computation of the weighted chaos norms turns out to be difficult when the chaos expansion are not available explicitly. On the other hand, *Gaussian Sobolev spaces* have been also used in the literature, [4, 10, 12]. The Gaussian Sobolev norms involve (L^2) norms of derivatives of random functions. We prove that a particular weighted chaos can be characterized using Sobolev type norms for Gaussian measure as in [4, 12, 10]. In particular we prove that using partial derivatives, we can compute this particular norm $\|\cdot\|_{\frac{k}{2}}$ defined by using a decay of the Chaos expansion of z with particular sequence of weights with power $\frac{k}{2}$. More precisely we establish that for every $k \in \mathbb{N}$ we have

$$||z||^2_{\frac{k}{2}} = ||z||^2_{(L^2)} + \sum_{i=1}^{k} \sum_{R\in P^{k,i}} \|D^i z\|^2_{R(\theta\mathbf{1}_k)}. \quad (1)$$

where $P^{k,i}$ is a finite subset (of indexes) of $\mathbb{R}^i$ that will be described below. On the right there is a computation of norms of derivatives up to order k of the field z.

This equivalence of norms is useful to obtain regularity results for some stochastic partial differential equations for these norms. It might be easier to obtain bounds for partial derivatives than obtaining bounds for the coefficients of the chaos expansions of solutions.

2. White noise analysis

In order to comfortably work with infinite many variables we use the *White noise analysis*. We review the main facts related to this infinity dimensional (stochastic) calculus in this section. Let H be a real Hilbert space with inner product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H$, and let A be an operator on H such that there exists an H-orthonormal basis $\{\eta_j\}_{j=1}^{\infty}$ satisfying

1. $A\eta_j = \lambda_j \eta_j$, $j = 1, 2, \ldots$.
2. $1 < \lambda_1 \le \lambda_2 \le \cdots$, and
3. $\sum_{j=1}^{\infty} \lambda_j^{-2\theta} < \infty$ for some constant $\theta > 0$.

For $p > 0$ let

$$\mathcal{S}_p := \{\xi \in H; |\xi|_p < \infty\}$$

where

$$|\xi|_p^2 := \|A^p\xi\|_H^2 = \sum_{j=0}^{\infty} \lambda_j^{2p}(\xi, \eta_j)_H^2,$$

and for $p < 0$ let $\mathcal{S}_p$ be defined as the dual space of $\mathcal{S}_{-p}$. It is easy to see that for $p < 0$ we also have $\|\cdot\|_p = \|A^p \cdot\|_H$ and the duality pairing between $\mathcal{S}_p$ and $\mathcal{S}_{-p}$ is an extension of the H inner product. We also define

$$\mathcal{S} = \cap_{p\geq 0}\mathcal{S}_p$$

(with the projective limit topology) and let $\mathcal{S}'$ be defined as the dual space of $\mathcal{S}$, i.e., by considering the standard countably Hilbert space constructed from (H, A); see [9, 11].

Let us consider $\mathcal{S}'$ as a probability space with the sigma-field $\mathcal{B}(\mathcal{S}')$ of Borel subsets of $\mathcal{S}'$. The probability measure μ is given by the Bochner-Minlos theorem and characterized by

$$E_\mu e^{i\langle\cdot,\xi\rangle} := \int_{\mathcal{S}'} e^{i\langle\omega,\xi\rangle} d\mu(\omega) = e^{-\frac{1}{2}\|\xi\|_H^2}, \text{ for all } \xi \in \mathcal{S}.1 \tag{2}$$

Here, the pairing $\langle\omega, \xi\rangle = \omega(\xi)$ is the action of $\omega \in \mathcal{S}'$ on $\xi \in \mathcal{S}$, and E_μ denotes the expectation with respect to the measure μ. We work in the probability space $(\mathcal{S}'; \mathcal{B}(\mathcal{S}'), \mu)$. The measure μ is often called the (normalized) *Gaussian measure* on $\mathcal{S}'$.

Equation (2) says that: for any test function $\xi \in \mathcal{S}$, the random variable $\langle\cdot, \xi\rangle$ is normally distributed with zero mean and variance $\|\xi\|_H^2$. If $\xi_1, \ldots, \xi_j \in \mathcal{S}$ are orthonormal in H then the random variables $\langle\cdot, \xi_1\rangle, \ldots, \langle\cdot, \xi_j\rangle$ are independent and normally distributed with mean zero and variance equal to one; see [8, 9, 11]. We also have that for any function $\xi \in H$, the random variable $\omega \mapsto \langle\omega, \xi\rangle$ can be defined in the $L^2(\mu)$ sense and it is normally distributed with zero mean and variance $\|\xi\|_H^2$. See [1, 7, 13, 8, 9, 11] and references therein for details on the Bochner-Minlos theorem, the measure μ and (2).

3. Weighted chaos norms

We always interpret properties in the "almost surely" sense with respect to μ (even when not stated textually). We introduce the space

$$(L^2) := L^2(\mathcal{S}', d\mu(\omega)) \tag{3}$$

with norm

$$\|z\|_{(L^2)}^2 := \int_{\mathcal{S}'} |z(\omega)|^2 d\mu.$$

In this section we introduce subspaces of (L^2) consisting of smooth functions. These subspaces of smooth functions are of interest in applications involving stochastic equations or partial differential equations with random coefficients. See [5, 6].

3.1. Wiener-chaos expansion

We characterize the space (L^2) defined in (3). We need to consider multi-index of arbitrary length. To simplify the notation, we regard multi-indices as elements of the space $(\mathbb{N}_0^{\mathbb{N}})_c$ of all sequences $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots)$ with elements $\alpha_j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and with compact support, i.e., with only finitely many $\alpha_j \neq 0$. We write $\boldsymbol{\mathcal{J}} = (\mathbb{N}_0^{\mathbb{N}})_c$. Given $\boldsymbol{\alpha} \in \boldsymbol{\mathcal{J}}$ define the order and length of $\boldsymbol{\alpha}$, denoted by $d(\boldsymbol{\alpha})$ and $|\boldsymbol{\alpha}|$ respectively, by

$$d(\boldsymbol{\alpha}) = \max\{j : \alpha_j \neq 0\} \quad \text{and} \quad |\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \cdots + \alpha_{d(\boldsymbol{\alpha})}.$$

We also introduce the Hermite polynomials, h_n, $n = 0, 1, 2, \dots$. These polynomials can be defined by the generating function identity

$$e^{tx - \frac{1}{2}t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} h_n(x). \tag{4}$$

Note that $h_n'(x) = n h_{n-1}(x)$. The Hermite polynomials are an orthogonal basis for $L^2(\mathbb{R}, e^{-\frac{1}{2}x^2} dx)$. Now we define the Fourier-Hermite polynomials for $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots) \in \boldsymbol{\mathcal{J}}$ by

$$H_{\boldsymbol{\alpha}}(\omega) := \prod_{j=1}^{d(\boldsymbol{\alpha})} h_{\alpha_j}(\langle \omega, \eta_j \rangle) \quad \omega \in \mathcal{S}'.$$

We now state the Wiener-Chaos expansion theorem; see [4, 7, 13, 8, 11].

Theorem 3.1. *The Fourier-Hermite polynomials are orthogonal in* (L^2)*. Moreover,* $\|H_{\boldsymbol{\alpha}}\|^2_{(L^2)} = \boldsymbol{\alpha}!$, $\boldsymbol{\alpha} \in \boldsymbol{\mathcal{J}}$*. In addition, every polynomial in* ω *belongs to* (L^2) *and every* $z \in (L^2)$ *can be represented as a Wiener-Chaos expansion* $z = \sum_{\boldsymbol{\alpha} \in \boldsymbol{\mathcal{J}}} z_{\boldsymbol{\alpha}} H_{\boldsymbol{\alpha}}$ *with* $\|z\|^2_{(L^2)} = \sum_{\boldsymbol{\alpha} \in \boldsymbol{\mathcal{J}}} \boldsymbol{\alpha}! z^2_{\boldsymbol{\alpha}}$*.*

3.2. A family of weighted chaos norms

Now we introduce weighted chaos norms that are used to define subspaces of (L^2) consisting of smooth functions. The resulting spaces are in analogy with classical Sobolev spaces with finite independent variables.

Given a multi-index $\boldsymbol{\alpha}$ and $r \in \mathbb{R}$ we denote

$$\langle \boldsymbol{\alpha}, \boldsymbol{\lambda}^r \rangle := \sum_{j=1}^{d(\boldsymbol{\alpha})} \alpha_j \lambda_j^r.$$

Note that we have $\langle \boldsymbol{\alpha}, \boldsymbol{\lambda}^r \rangle \geq 0$. In (L^2) we can introduce the system of Hilbert norms (see [4, Chapter 10], and also [2, 13])

$$||z||_p^2 := \sum_{\boldsymbol{\alpha} \in \boldsymbol{\mathcal{J}}} (1 + \langle \boldsymbol{\alpha}, \boldsymbol{\lambda}^{2\theta} \rangle^{2p}) \boldsymbol{\alpha}! z^2_{\boldsymbol{\alpha}}, \tag{5}$$

where $z = \sum_{\boldsymbol{\alpha}\in\boldsymbol{\mathcal{J}}} z_{\boldsymbol{\alpha}} H_{\boldsymbol{\alpha}}$. Alternatively, the we can use the weight $(1 + \langle\boldsymbol{\alpha}, \boldsymbol{\lambda}^{2\theta}\rangle)^p$, instead of $1 + \langle\boldsymbol{\alpha}, \boldsymbol{\lambda}^{2\theta}\rangle^p$.

In principle, the norm $\|z\|_p$ measures the speed of the decay of the expansion of the field z in terms of Fourier-Hermite polynomials. We will see that they do measure regularity in the sense of having derivatives in (L^2). For $p > 0$ define the spaces $\mathcal{S}_p$ of smooth fields by

$$(\mathcal{S})_p = \{z \in (L^2) : \|z\|_p < \infty\}. \tag{6}$$

For $p < 0$ define $\mathcal{S}_p$ as the dual space of $\mathcal{S}_{-p}$. We define $\mathcal{S}_0 = (L^2)$ and the inclusion $\mathcal{S}_q \subset \mathcal{S}_p$ holds for all $q > p$.

Next, we recall that the weighted norms (5) can be written as square integrals using an operator acting on functions in (L^2). We can write

$$||z||_p^2 = \sum_{\boldsymbol{\alpha}\in\boldsymbol{\mathcal{J}}} (1 + \langle\boldsymbol{\alpha}, \boldsymbol{\lambda}^{2\theta}\rangle^{2p})\boldsymbol{\alpha}! z_{\boldsymbol{\alpha}}^2 = ||z||_{(L^2)}^2 + ||\Gamma_\oplus(A^{2\theta})^p z||_{(L^2)}^2 \tag{7}$$

where $\Gamma_\oplus(A^q)$ is the operator defined by

$$\Gamma_\oplus(A^q) H_{\boldsymbol{\sigma}^2,\boldsymbol{\alpha}} = \langle\boldsymbol{\alpha}, \boldsymbol{\lambda}^q\rangle H_{\boldsymbol{\alpha}}. \tag{8}$$

We point out that $\Gamma_\oplus(A^q) \neq \Gamma_\oplus(A)^q$ since $\Gamma_\oplus(A)^p H_{\boldsymbol{\alpha}} = \langle\boldsymbol{\alpha}, \boldsymbol{\lambda}\rangle^q H_{\boldsymbol{\alpha}}$ and $\Gamma_\oplus(A^q) H_{\boldsymbol{\alpha}} = \langle\boldsymbol{\alpha}, \boldsymbol{\lambda}^q\rangle H_{\boldsymbol{\alpha}}$. We observe that $||\Gamma_\oplus(A)^q \cdot ||_{(L^2)}^2$ is a norm in the space of functions in (L^2) with $u_{\mathbf{0}} = 0$ in its Fourier-Hermite expansion.

We note that a general weighted chaos norms involves arbitrary weights and it is of the form

$$||z||_{p;\rho}^2 = \sum_{\boldsymbol{\alpha}\in\boldsymbol{\mathcal{J}}} \rho(\boldsymbol{\alpha}, p)^2 \boldsymbol{\alpha}! z_{\boldsymbol{\alpha}}^2.$$

It is assumed that $\rho(\boldsymbol{\alpha}, q) \geq \rho(\boldsymbol{\alpha}, p) > 0$ for all $q > p \geq 0$ and that $\rho(\boldsymbol{\alpha}, 0)$ is a constant independent of $\boldsymbol{\alpha} \in \boldsymbol{\mathcal{J}}$. Usually, the *weights* $\rho(\boldsymbol{\alpha}, p)$ are the eigenvalues of some nonnegative operator in (L^2) with the $\sigma(s)$-Fourier-Hermite polynomials as eigenfunctions. For examples of other weights $\rho(\boldsymbol{\alpha}, p)$ we refer to [2, 4, 5, 7, 8, 9, 11, 10, 12].

The wighted chaos norm measure the decay of the coefficients in the expansion $z = \sum_{\boldsymbol{\alpha}\in\boldsymbol{\mathcal{J}}} z_{\boldsymbol{\alpha}} H_{\boldsymbol{\alpha}}$. In many applications the interest focus in weighted chaos norms that imply decay of coefficients when $|\boldsymbol{\alpha}| \to \infty$ and also when $d(\boldsymbol{\alpha}) \to \infty$. This is the reason for the choice of the weight ρ in (5) that depends on $\boldsymbol{\lambda}^{2\theta}$. We will show in Section 5 that the decay in the coefficients in the expansion implies that some partial derivatives are square integrable with respect to μ.

4. Gaussian-Sobolev norms

In this section we prove that the chaos weighted norms introduced in Section 3.2 can be computed using (L^2) norms of partial derivatives; see Section 5. In particular we review the analysis presented in [6]. We use derivative in the sense of Friedrichs as in [4, Chapter 10].

4.1. Derivatives and their norms

We follow the analysis presented in [6]. Using partial derivative (in the sense of Friedrichs as in [4, Chapter 10] we want to be able to compute a norm equivalent to the norm (5). In this section we work with differential operators acting on (L^2) and define Sobolev type norms for Gaussian measures. We refer the reader to [12, Section 2.1.5] and/or [4, Section 10.1] for details. Denote by $\partial_\ell z$ the directional derivative of z in the direction of the $\ell-$th basis function $\eta_\ell \in \mathcal{S}$. Given $z \in (L^2)_s$ we define $\partial_\ell z : \mathcal{S}' \to \mathbb{R}$

$$\partial_\ell z(\omega) := \frac{d}{dt} z(\omega + t\eta_\ell)\Big|_{t=0}. \tag{9}$$

Analogous definitions hold for higher-order partial derivatives. The definition of partial derivative in (9), and the definition of $k-$th derivative, Definition 4.1 below, are valid for functions z in a properly chosen dense subset of (L^2), e.g., the subset of exponentials of the form

$$z(\omega) = \prod_{j=1}^{N} \exp(\langle \omega, \eta_j \rangle).$$

The definition of the derivative is then extended to a subset of (L^2) using density arguments. Equation (9) is also valid for functions of the form

$$z(\omega) = F(\langle \omega, \eta_{j_1} \rangle, \ldots, \langle \omega, \eta_{j_J} \rangle)$$

where J is an integer and $F : \mathbb{R}^J \to \mathbb{R}$ is differentiable.

We compute the (L^2)-norms of some partial derivatives. For any Fourier-Hermite polynomial $H_{\boldsymbol{\alpha}}$ with $\alpha_\ell > 0$ we have that

$$\partial_\ell H_{\boldsymbol{\alpha}}(\omega) = \partial_\ell \prod_{j=1}^{d(\boldsymbol{\alpha})} h_{\alpha_j}(\langle \omega, \eta_j \rangle) = \alpha_\ell H_{\boldsymbol{\alpha}-\boldsymbol{\xi}_\ell}(\omega) \tag{10}$$

where $\boldsymbol{\xi}_\ell$ is the multi-index with one in the $\ell-$entry and zero in the other positions so that

$$\boldsymbol{\alpha} - \boldsymbol{\xi}_\ell = (\alpha_1, \ldots, \alpha_{\ell-1}, \alpha_\ell - 1, \alpha_{\ell+1}, \ldots).$$

Here we have used that $h_n' = n h_{n-1}$, see (4). For $\alpha_\ell = 0$ define $\partial_\ell H_{\boldsymbol{\alpha}}(\omega) = 0$. Then for $z = \sum_{\boldsymbol{\alpha} \in \boldsymbol{\mathcal{J}}} z_{\boldsymbol{\alpha}} H_{\boldsymbol{\alpha}}$ such that $\partial_\ell z \in (L^2)$ we have

$$\partial_\ell z(\omega) = \sum_{\boldsymbol{\alpha} \in \boldsymbol{\mathcal{J}}} \alpha_\ell z_{\boldsymbol{\alpha}} H_{\boldsymbol{\alpha}-\boldsymbol{\xi}_\ell}(\omega)$$

and therefore

$$||\partial_\ell z||^2_{(L^2)} = \sum_{\boldsymbol{\alpha} \in \boldsymbol{\mathcal{J}}} \alpha_\ell^2 z_{\boldsymbol{\alpha}}^2 (\boldsymbol{\alpha} - \boldsymbol{\xi}_\ell)! = \sum_{\boldsymbol{\alpha} \in \boldsymbol{\mathcal{J}}} \alpha_\ell z_{\boldsymbol{\alpha}}^2 \boldsymbol{\alpha}!, \tag{11}$$

where we have used that $\alpha_\ell(\boldsymbol{\alpha}-\boldsymbol{\xi}_\ell)! = \boldsymbol{\alpha}!$. Analogously, for any Fourier-Hermite polynomial the $\boldsymbol{\gamma}$ partial derivative $\partial^{\boldsymbol{\gamma}}$ can be computed as

$$\partial^{\boldsymbol{\gamma}} H_{\boldsymbol{\alpha}}(\omega) = \prod_{j=1}^{d(\alpha)} \frac{\alpha_j!}{(\alpha_j-\gamma_j)!} h_{\alpha_j-\gamma_j}(\langle\omega,\eta_j\rangle) = \frac{\boldsymbol{\alpha}!}{(\boldsymbol{\alpha}-\boldsymbol{\gamma})!} H_{\boldsymbol{\alpha}-\boldsymbol{\gamma}}$$

for every multi-index $\boldsymbol{\gamma}$ and $\boldsymbol{\alpha}$ with $\boldsymbol{\gamma} \leq \boldsymbol{\alpha}$. Then for $z = \sum_{\boldsymbol{\alpha}\in\boldsymbol{\mathcal{J}}} z_{\boldsymbol{\alpha}} H_{\boldsymbol{\alpha}}$ we have

$$\partial^{\boldsymbol{\gamma}} z(\omega) = \sum_{\boldsymbol{\alpha}\geq\boldsymbol{\gamma}} \frac{\boldsymbol{\alpha}!}{(\boldsymbol{\alpha}-\boldsymbol{\gamma})!} z_{\boldsymbol{\alpha}} H_{\boldsymbol{\alpha}-\boldsymbol{\gamma}}(\omega)$$

and the (L^2) norm of $\partial^{\boldsymbol{\gamma}} z$ is given by

$$\|\partial^{\boldsymbol{\gamma}} z\|^2_{(L^2)} = \sum_{\boldsymbol{\alpha}\geq\boldsymbol{\gamma}} \frac{\boldsymbol{\alpha}!^2}{(\boldsymbol{\alpha}-\boldsymbol{\gamma})!^2} z^2_{\boldsymbol{\alpha}} (\boldsymbol{\alpha}-\boldsymbol{\gamma})! \quad = \quad \sum_{\boldsymbol{\alpha}\geq\boldsymbol{\gamma}} \frac{\boldsymbol{\alpha}!}{(\boldsymbol{\alpha}-\boldsymbol{\gamma})!} u^2_{\boldsymbol{\alpha}} \boldsymbol{\alpha}!.$$

We define the $k-$th derivative.

Definition 4.1. Denote by $Lin(\mathcal{S})$ the set of linear functionals on $\mathcal{S}$. For $k \in \mathbb{N}$, $p \in \mathbb{R}$ and $z : \mathcal{S}' \to \mathbb{R}$ we define $D^k z : \mathcal{S}' \to Lin(\mathcal{S})^{\otimes k}$ by

$$D^k z(\omega) := \sum_{\ell_1,\ell_2,\ldots,\ell_k} \partial_{\ell_1}\ldots\partial_{\ell_k} z(\omega)\eta_{\ell_1}\otimes\cdots\otimes\eta_{\ell_k}$$

We also use the convention $D^0 z = z$.

Here and below we will use the iterated summation notation

$$\sum_{\ell_1,\ell_2,\ldots,\ell_k} := \sum_{\ell_1\in\mathbb{N}}\sum_{\ell_2\in\mathbb{N}}\cdots\sum_{\ell_k\in\mathbb{N}}.$$

As we will see in the next section, adding (L^2)-norms of derivatives yields weighted chaos norms. We are interested in weights that inherit a dependence on the eigenvalues $\{\lambda_j\}_{j=1}^\infty$. For this reason, we compute $\mathcal{S}_p-$like norms of derivatives according to the next definition.

Definition 4.2. For $k \in \mathbb{N}$ and $\boldsymbol{q} = (q_1,\ldots,q_k) \in \mathbb{R}^k$ define

$$\|D^k z\|^2_{\boldsymbol{q}} = \|A^{q_1}\otimes\cdots\otimes A^{q_k} D^k z\|^2_{L^2(\mathcal{S}',(L^2)^{\otimes k})} = \sum_{\ell_1,\ell_2,\ldots,\ell_k} \lambda^{2q_1}_{\ell_1}\ldots\lambda^{2q_k}_{\ell_k}\|\partial_{\ell_1}\ldots\partial_{\ell_k} z\|^2_{(L^2)}.$$

We also set $||D^0 z||^2 = ||z||^2_{(L^2)}$.

For instance, $Dz : \mathcal{S}' \to Lin(\mathcal{S})$. For $\omega \in \mathcal{S}'$ and $\xi \in \mathcal{S}$, the action of $Dz(\omega)$ on ξ is given by

$$\langle Dz(\omega), \xi\rangle = \sum_{\ell=1}^\infty \partial_\ell z(\omega)(\eta_\ell,\xi)_H.$$

Also, $\|Dz\|^2_{q_1} = \sum_{\ell=1}^\infty \lambda_\ell^{2q_1}\|\partial_\ell z\|^2_{(L^2)}$.

5. Equivalence of norms

We define the the action of $\Gamma_{\oplus}(A)$ on derivatives as follows.

Definition 5.1. For $k\in\mathbb{N}$ and $p\in\mathbb{R}$ define

$$\Gamma_{\oplus}(A)^{\frac{p}{2}}D^kz(\omega):=\sum_{\ell_1,\ell_2,\ldots,\ell_k}\Gamma_{\oplus}(A)^{\frac{p}{2}}\partial_{\ell_1}\ldots\partial_{\ell_k}z(\omega)\eta_{\ell_1}\otimes\cdots\otimes\eta_{\ell_k}.$$

Now we prove some basic relations between derivatives in the ω variable and the operator $\Gamma_{\oplus}(A)$ defined in (8). See [12, Proposition 4.1] for related intertwining property of derivatives and the Ornstein-Uhlenbeck operator (instead of the $\Gamma_{\oplus}(A^q)$ operator).

Lemma 5.2. *For all $p,q\in\mathbb{R}$ we have the following relations*

$$(\Gamma_{\oplus}(A^q)+\lambda_\ell^q)^{\frac{p}{2}}\,\partial_\ell z=\partial_\ell\Gamma_{\oplus}(A^q)^{\frac{p}{2}}z, \tag{12}$$

$$\left(\Gamma_{\oplus}(A^q)+\lambda_{\ell_1}^q+\cdots+\lambda_{\ell_k}^q\right)^{\frac{p}{2}}\partial_{\ell_1}\partial_{\ell_2}\cdots\partial_{\ell_k}z=\partial_{\ell_1}\partial_{\ell_2}\cdots\partial_{\ell_k}\Gamma_{\oplus}(A^q)^{\frac{p}{2}}z, \tag{13}$$

and

$$(\Gamma_{\oplus}(A^q)+\langle\boldsymbol{\beta},\boldsymbol{\lambda}^q\rangle)^{\frac{p}{2}}\,\partial^{\boldsymbol{\beta}}z=\partial^{\boldsymbol{\beta}}\Gamma_{\oplus}(A^q)^{\frac{p}{2}}z. \tag{14}$$

Proof. Since $\partial_\ell z=\sum_{\boldsymbol{\alpha}\in\boldsymbol{\mathcal{J}}}\alpha_\ell z_{\boldsymbol{\alpha}}H_{\boldsymbol{\alpha}-\boldsymbol{\xi}_\ell}$, then

$$\begin{aligned}(\Gamma_{\oplus}(A^q)+\lambda_\ell^q)^{\frac{p}{2}}\,\partial_\ell z&=\sum_{\boldsymbol{\alpha}\in\boldsymbol{\mathcal{J}}}(\langle\boldsymbol{\alpha}-\boldsymbol{\xi}_\ell,\boldsymbol{\lambda}^q\rangle+\lambda_\ell^q)^{\frac{p}{2}}\,\alpha_\ell z_{\boldsymbol{\alpha}}H_{\boldsymbol{\alpha}-\boldsymbol{\xi}_\ell}\\&=\sum_{\boldsymbol{\alpha}\in\boldsymbol{\mathcal{J}}}(\langle\boldsymbol{\alpha},\boldsymbol{\lambda}^q\rangle-\lambda_\ell^q+\lambda_\ell^q)^{\frac{p}{2}}\,\alpha_\ell z_{\boldsymbol{\alpha}}H_{\boldsymbol{\alpha}-\boldsymbol{\xi}_\ell}\\&=\sum_{\boldsymbol{\alpha}\in\boldsymbol{\mathcal{J}}}\langle\boldsymbol{\alpha},\boldsymbol{\lambda}^q\rangle^{\frac{p}{2}}\alpha_\ell z_{\boldsymbol{\alpha}}H_{\boldsymbol{\alpha}-\boldsymbol{\xi}_\ell}=\partial_\ell\Gamma_{\oplus}(A^q)^{\frac{1}{2}}z,\end{aligned}$$

which prove (12). Note that (13) follows easily from (12) and (14) is consequence of (13) and the notation $\langle\boldsymbol{\beta},\boldsymbol{\lambda}^q\rangle=\sum_{j=1}^{d(\boldsymbol{\alpha})}\beta_j\lambda_j^q$. □

Lemma 5.3. *For $k\in\mathbb{N}$ and $\boldsymbol{q}\in\mathbb{R}^k$ we have*

$$\sum_{\ell_k}\lambda_{\ell_k}^{2q_k}\|D^{k-1}\partial_{\ell_k}z\|^2_{(q_1,\ldots,q_{k-1})}=\|D^kz\|^2_{(q_1,\ldots,q_k)}, \tag{15}$$

$$\|D\Gamma_{\oplus}(A^{2q_2})^{\frac{1}{2}}z\|^2_{q_1}=\|\Gamma_{\oplus}(A^{2q_2})^{\frac{1}{2}}Dz\|^2_{q_1}+\|Dz\|^2_{q_1+q_2} \tag{16}$$

and for $\boldsymbol{q}=(q_1,\ldots,q_k)\in\mathbb{R}^k$ and $t\in\mathbb{R}$ we have

$$\|D^k\Gamma_{\oplus}(A^{2t})^{\frac{1}{2}}z\|^2_{\boldsymbol{q}}\quad=\quad\|\Gamma_{\oplus}(A^{2t})^{\frac{1}{2}}D^kz\|^2_{\boldsymbol{q}}+\sum_{i=1}^{k}\|D^kz\|^2_{\boldsymbol{q}+t\boldsymbol{\xi}_i} \tag{17}$$

where $\boldsymbol{q}+t\boldsymbol{\xi}_i=(q_1,\ldots,q_i+t,\ldots,q_k)$.

Proof. Equation (15) follows directly from Definition 4.2. We prove (16). Using Definitions 4.1 and 4.2 together with Equation (12),

$$\begin{aligned}
\|D\Gamma_{\oplus}(A^{2q_2})^{\frac{1}{2}}z\|_{q_1}^2 &= \sum_{\ell=1} \lambda_\ell^{2q_1}\|\partial_\ell \Gamma_{\oplus}(A^{2q_2})^{\frac{1}{2}}z\|_{(L^2)}^2 \\
&= \sum_{\ell=1} \lambda_\ell^{2q_1}\| \left(\Gamma_{\oplus}(A^{2q_2}) + \lambda_\ell^{2q_2}\right)^{\frac{1}{2}} \partial_\ell z\|_{(L^2)}^2 \\
&= \sum_{\ell=1} \lambda_\ell^{2q_1} \left(\|\Gamma_{\oplus}(A^{2q_2})^{\frac{1}{2}}\partial_\ell z\|_{(L^2)}^2 + \lambda_\ell^{2q_2}\|\partial_\ell z\|_{(L^2)}^2\right) \\
&= \|\Gamma_{\oplus}(A^{2q_2})^{\frac{1}{2}}Dz\|_{q_1}^2 + \|Dz\|_{q_2+q_1}^2.
\end{aligned}$$

To prove (17) observe that using (13) we get

$$\begin{aligned}
\|D^k\Gamma_{\oplus}(A^{2t})^{\frac{1}{2}}z\|_{\boldsymbol{q}}^2 &= \sum_{\ell_1,\ldots,\ell_k} \lambda_{\ell_1}^{2q_1}\cdots\lambda_{\ell_k}^{2q_k}\|\partial_{\ell_1}\cdots\partial_{\ell_k}\Gamma_{\oplus}(A^{2t})^{\frac{1}{2}}z\|_{(L^2)}^2 \\
&= \sum_{\ell_1,\ldots,\ell_k} \lambda_{\ell_1}^{2q_1}\cdots\lambda_{\ell_k}^{2q_k}\| \left(\Gamma_{\oplus}(A^{2t}) + \lambda_{\ell_1}^{2t} + \cdots + \lambda_{\ell_k}^{2t}\right)^{\frac{1}{2}} \partial_{\ell_1}\cdots\partial_{\ell_k}z\|_{(L^2)}^2 \\
&= \sum_{\ell_1,\ldots,\ell_k} \lambda_{\ell_1}^{2q_1}\cdots\lambda_{\ell_k}^{2q_k}\|\Gamma_{\oplus}(A^{2t})^{\frac{1}{2}}\partial_{\ell_1}\cdots\partial_{\ell_k}z\|_{(L^2)}^2 \\
&+ \sum_{\ell_1,\ldots,\ell_k} \lambda_{\ell_1}^{2q_1}\cdots\lambda_{\ell_k}^{2q_k}(\lambda_{\ell_1}^{2t} + \cdots + \lambda_{\ell_k}^{2t})\|\partial_{\ell_1}\cdots\partial_{\ell_k}z\|_{(L^2)}^2 \\
&= \|\Gamma_{\oplus}(A^{2t})^{\frac{1}{2}}D^kz\|_{\boldsymbol{q}}^2 + \sum_{i=1}^{k}\|D^kz\|_{\boldsymbol{q}+t\boldsymbol{\xi}_i}^2.
\end{aligned}$$

□

The following result reveals the basic relation between norms of derivatives and the norm $||u||_p^2$ defined in (5) for the values $p = 1/2$ and $p = 1$. This result will be used as the initial induction step in the proof of the equivalence of norms for any value of p half a positive integer; see Theorem 5.6.

Theorem 5.4. *For any $k \in \mathbb{N}$ and $\boldsymbol{q} = (q_1, \ldots q_k) \in \mathbb{R}^k$ we have*

$$\|\Gamma_{\oplus}(A^{2q_1})^{\frac{1}{2}}z\|_{(L^2)}^2 = \sum_{\ell=1}^{\infty} \lambda_\ell^{2q_1}||\partial_\ell z||^2 = \|Dz\|_{q_1}^2, \tag{18}$$

$$\|\Gamma_{\oplus}(A^{2q_k})^{\frac{1}{2}}D^{k-1}z\|_{(q_1,\ldots,q_{k-1})}^2 = \|D^kz\|_{(q_1,q_2,\ldots,q_k)}^2 \tag{19}$$

and we have the identities

$$\begin{aligned}
\|\Gamma_{\oplus}(A^{2q_1})^{\frac{1}{2}}\Gamma_{\oplus}(A^{2q_2})^{\frac{1}{2}}z\|_{(L^2)}^2 &= \|D^2z\|_{(q_1,q_2)}^2 + \|\Gamma_{\oplus}(A^{2(q_1+q_2)})^{\frac{1}{2}}z\|_{(L^2)}^2 \\
&= \|D^2z\|_{(q_1,q_2)}^2 + \|Dz\|_{q_1+q_2}^2.
\end{aligned} \tag{20}$$

and

$$\|\Gamma_\oplus(A^{2q_1})^{\frac{1}{2}}\Gamma_\oplus(A^{2q_2})^{\frac{1}{2}}\Gamma_\oplus(A^{2q_3})^{\frac{1}{2}}z\|^2_{(L^2)} = \|D^3z||^2_{(q_1,q_2,q_3)}$$
$$+\|D^2z||^2_{(q_1+q_3,q_2)} + \|D^2z||^2_{(q_1,q_2+q_3)} + \|D^2z||^2_{(q_1+q_2,q_3)}$$
$$+\|Dz||^2_{(q_1+q_2+q_3)} \tag{21}$$

Proof. From Equation (11) we have that

$$\begin{aligned}\|Dz\|^2_{q_1} &= \sum_{\ell=1}^{\infty}\lambda_\ell^{2q_1}\|\partial_\ell z\|^2_{(L^2)} = \sum_{\ell=1}^{\infty}\sum_{\alpha_\ell\geq 1}\alpha_\ell\lambda_\ell^{2q_1}z^2_{\boldsymbol{\alpha}}\boldsymbol{\alpha}! \\ &= \sum_{\boldsymbol{\alpha}\in\boldsymbol{\mathcal{J}}}\left(\sum_{\ell=1}^{d(\boldsymbol{\alpha})}\alpha_\ell\lambda_\ell^{2q_1}\right)z^2_{\boldsymbol{\alpha}}\boldsymbol{\alpha}! \\ &= \sum_{\boldsymbol{\alpha}\in\boldsymbol{\mathcal{J}}}\langle\boldsymbol{\alpha},\boldsymbol{\lambda}^{2q_1}\rangle z^2_{\boldsymbol{\alpha}}\boldsymbol{\alpha}! = ||\Gamma_\oplus(A^{2q_1})^{\frac{1}{2}}z||^2_{(L^2)},\end{aligned}$$

and hence (18) holds. To prove (19) observe that from (18) and (15) we get

$$\begin{aligned}&\|\Gamma_\oplus(A^{2q_k})^{\frac{1}{2}}D^{k-1}z\|^2_{(q_1,\dots,q_{k-1})} \\ &\quad= \sum_{\ell_1,\dots,\ell_{k-1}}\lambda_{\ell_1}^{2q_1}\dots\lambda_{\ell_{k-1}}^{2q_{k-1}}\|\Gamma_\oplus(A^{2q_k})^{\frac{1}{2}}\partial_{\ell_1}\dots\partial_{\ell_{k-1}}z\|^2_{(L^2)s} \\ &\quad= \sum_{\ell_1,\dots,\ell_k}\lambda_{\ell_1}^{2q_1}\dots\lambda_{\ell_k}^{2q_k}\|\partial_{\ell_1}\dots\partial_{\ell_k}z\|^2_{(L^2)s} = \|D^kz\|^2_{(q_1,\dots,q_k)}.\end{aligned}$$

To prove (20) observe that from (18), (16) and (19) we have

$$\begin{aligned}\|\Gamma_\oplus(A^{2q_1})^{\frac{1}{2}}\Gamma_\oplus(A^{2q_2})^{\frac{1}{2}}z\|^2_{(L^2)} &= \|D\Gamma_\oplus(A^{2q_2})^{\frac{1}{2}}z||^2_{q_1} \\ &= \|\Gamma_\oplus(A^{2q_2})^{\frac{1}{2}}Dz\|^2_{q_1} + \|Dz\|^2_{q_1+q_2} \\ &= \|D^2z||^2_{(q_1,q_2)} + \|\Gamma_\oplus(A^{2(q_1+q_2)})^{\frac{1}{2}}z\|^2_{(L^2)}.\end{aligned}$$

For the proof of (21), see Theorem 5.6 where we prove the general case. □

In order to write down the general version of formula (20) we shall introduce some notation. Consider the set of indexes $\{1,2,\dots,k\}$ and its set of partitions P^k; see [3]. Recall that, given $i\in\mathbb{N}$, an $i-$partition of $\{1,2,\dots,k\}$ is a decomposition of this set into i nonempty and disjoint subsets. We denote by $P^{k,i}$ the set of all $i-$partitions of $\{1,2,\dots,k\}$. It is well known that $\#(P^{k,i}) = S(k,i)$, the Stirling number of the second kind (which is also the number of distributions of k distinguishable balls into i indistinguishable urns). Let each i-partition $R=(R_1,\dots,R_i)\in P^{k,i}$, be ordered in such a way that

$$\min R_1 < \min R_2 < \cdots < \min R_i.$$

To each i-partition and $\boldsymbol{q} = (q_1, \ldots, q_k) \in \mathbb{R}^k$ we associate a multi-index $R(\boldsymbol{q}) = (R_1(\boldsymbol{q}), \ldots, R_i(\boldsymbol{q})) \in \mathbb{R}^i$ defined by

$$R_{i'}(\boldsymbol{q}) = \sum_{i'' \in R_{i'}} q_{i''}, \qquad i' = 1, \ldots, i.$$

Example 5.5. Let $\boldsymbol{q} = (q_1, q_2, q_3)$ and consider the 2-partition $R = \{R_1 = \{1\}, R_2 = \{2,3\}\}$. Then $R(\boldsymbol{q}) = (q_1, q_2 + q_3)$.

Let $\boldsymbol{q} = (q, q, q, q)$ and consider the 3-partition $R = \{R_1 = \{1\}, R_2 = \{2,3\}, R_3 = \{4\}\}$. Then $R(\boldsymbol{q}) = (q, 2q, q)$.

The following result gives a closed formula that allows us to compute the norm $||\cdot||_p^2$ using ω-partial derivatives[1]. It shows the equivalence between the weighted chaos norms in (5), and the Gaussian Sobolev norms, defined using (L^2) norms of derivatives. Similar result for the case $k = 1$ and $k = 2$ can be found in [4]. The corresponding spaces are denoted by $W^{1,2}(H, \mu)$ and $W^{2,2}(H, \mu)$, respectively. See Theorem 10.15 in page 147 and Equations (10.54)-(10.57) in page 162.

Theorem 5.6. *Let $k \in \mathbb{N}$ and $\boldsymbol{q} = (q_1, q_2, \ldots, q_k) \in \mathbb{R}^k$. We have*

$$\|\Gamma_\oplus(A^{2q_1})^{\frac{1}{2}} \ldots \Gamma_\oplus(A^{2q_k})^{\frac{1}{2}} z\|_{(L^2)}^2 = \sum_{i=1}^{k} \sum_{R \in P^{k,i}} \|D^i z\|_{R(\boldsymbol{q})}^2. \tag{22}$$

In particular, if we take $\boldsymbol{q} = \theta \mathbf{1}_k$ where $\mathbf{1}_k := (1, \ldots, 1) \in \mathbb{N}^k$

$$\|\Gamma_\oplus(A^{2\theta})^{\frac{k}{2}} z\|_{(L^2)}^2 = \sum_{i=1}^{k} \sum_{R \in P^{k,i}} \|D^i z\|_{R(\theta \mathbf{1}_k)}^2$$

and

$$||z||_{\frac{k}{2}}^2 = ||z||_{(L^2)}^2 + ||\Gamma_\oplus(A^{2\theta})^{\frac{k}{2}} z||_{(L^2)}^2 = ||z||_{(L^2)}^2 + \sum_{i=1}^{k} \sum_{R \in P^{k,i}} \|D^i z\|_{R(\theta \mathbf{1}_k)}^2.$$

Proof. We proceed by induction on k. For $k = 1$ and $k = 2$ we already proved the result, see (18) and (20) of Theorem 5.4.

Assume that (22) is valid for the first $k \in \mathbb{N}$. Then we have

$$\|\Gamma_\oplus(A^{2q_1})^{\frac{1}{2}} \ldots \Gamma_\oplus(A^{2q_{k+1}})^{\frac{1}{2}} z\|_{(L^2)}^2 = \sum_{i=1}^{k} \sum_{R \in P^{(i)}} \|D^i \Gamma_\oplus(A^{2q_{k+1}})^{\frac{1}{2}} z||_{R(\boldsymbol{q})}^2$$
$$= \sum_{i=1}^{k} \sum_{R \in P^{(i)}} \left(\|\Gamma_\oplus(A^{2q_{k+1}})^{\frac{1}{2}} D^i z\|_{R(q_1,\ldots,q_k)}^2 + \sum_{i'=1}^{i} \|D^i z\|_{R(q_1,\ldots,q_k)+q_{k+1}\boldsymbol{\xi}_{i'}}^2 \right)$$

[1]We use derivative in the sense of Friedrichs as in [4, Chapter 10].

where we have used formula (17). Then, from (19) we get

$$\begin{aligned}
&\|\Gamma_{\oplus}(A^{2q_1})^{\frac{1}{2}} \dots \Gamma_{\oplus}(A^{2q_{k+1}})^{\frac{1}{2}} z\|^2_{(L^2)} \\
&\quad = \sum_{i=1}^{k} \sum_{R\in P^{k,i}} \left(\|D^i z\|^2_{(R(q_1,\dots,q_k),q_{k+1})} + \sum_{i'=1}^{i} \|D^i z\|^2_{R(q_1,\dots,q_k)+q_{k+1}\boldsymbol{\xi}_{i'}} \right) \\
&\quad = \sum_{i=1}^{k+1} \sum_{R\in P^{k+1,i}} \|D^i z\|^2_{R(q_1,\dots,q_{k+1})}.
\end{aligned}$$

To obtain the last equality we observe that the i-partitions $P^{k+1,i}$ of the set $\{1,\dots,k+1\}$ are of the form $\{R,\{k+1\}\}$ where $R \in P^{k,i-1}$ or $R = (R_1,\dots,R_{i'} \cup \{k+1\},\dots,R_i)$ for $1 \le i' \le i$ and $R \in P^{k,i}$. □

Note that, given $r = (r_\ell)_{\ell=1}^{i} \in \mathbb{N}^i$, (see [3])

$$\#(\{R \in P^{k,i} : \theta r = R(\theta \mathbf{1}_k)\} = \prod_{j=1}^{i-1} \binom{\sum_{\ell=j}^{i} r_\ell - 1}{r_j - 1}.$$

6. On Kondratiev like norms

In this section we study another classical weighted norm. See [8, 9, 11] for more details on these norms. We also show that this norm can be computed by adding norm of derivatives, but this time we need all partial derivatives of all orders to be finite. Given a multi-index $\boldsymbol{\alpha}$ we denote

$$\boldsymbol{\lambda}^{\boldsymbol{\alpha}} := \prod_{j=1}^{d(\boldsymbol{\alpha})} \lambda_j^{\alpha_j}.$$

Take $\nu \in [0,1)$ and

$$\rho(\boldsymbol{\alpha},p) = (\boldsymbol{\alpha}!)^{\nu} \widetilde{\boldsymbol{\lambda}}^{p\alpha} = (\boldsymbol{\alpha}!)^{\nu} (\boldsymbol{\lambda}^{2\theta})^{p\alpha} = (\boldsymbol{\alpha}!)^{\nu} \boldsymbol{\lambda}^{2p\theta\boldsymbol{\alpha}}, \quad \boldsymbol{\alpha} \in \boldsymbol{\mathcal{J}} \tag{23}$$

in (5). Here and below we denote $\widetilde{\boldsymbol{\lambda}} = \boldsymbol{\lambda}^{2\theta}$ ($\widetilde{\lambda}_j = \lambda_j^{2\theta}$). The weight

$$\rho(\boldsymbol{\alpha},p)^2 = (\boldsymbol{\alpha}!)^{\nu} \boldsymbol{\lambda}^{2p\boldsymbol{\alpha}}$$

can be used as well. Let us denote by $|||u|||_p^2$ the resulting weighted norm. Note that we can write

$$|||u|||_p^2 = ||\Gamma_{\otimes,\nu}(A^{2\theta})^p u||^2_{(L^2)}$$

where $\Gamma_{\otimes,\nu}(A^{2\theta})$ is the operator defined by

$$\Gamma_{\otimes,\nu}(A) H_{\boldsymbol{\alpha}} = (\alpha!)^{\nu} \widetilde{\boldsymbol{\lambda}}^{\boldsymbol{\alpha}} H_{\boldsymbol{\alpha}}.$$

Now we show how to compute the norm $|||\cdot|||_p$ defined above for the case $\nu = 0$. We use the notation

$$(\widetilde{\boldsymbol{\lambda}}^p - \mathbf{1})^{\boldsymbol{\gamma}} = \prod_{j=1}^{d(\boldsymbol{\gamma})} (\widetilde{\lambda}_j^p - 1)^{\gamma_j}.$$

Recall that $1 < \lambda_1 \leq \lambda_2 \leq \dots$. We have

$$\begin{aligned}\widetilde{\boldsymbol{\lambda}}^{2p\boldsymbol{\alpha}} &= \prod_{j=1}^{d(\boldsymbol{\alpha})} (\widetilde{\lambda}_j^{2p} - 1 + 1)^{\alpha_j} = \prod_{j=1}^{d(\boldsymbol{\alpha})} \left(\sum_{\gamma_j \leq \alpha_j} \binom{\alpha_j}{\gamma_j} (\widetilde{\lambda}_j^{2p} - 1)^{\gamma_j} \right) \\ &= \sum_{\boldsymbol{\gamma} \leq \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\gamma}} (\widetilde{\boldsymbol{\lambda}}^{2p} - \mathbf{1})^{\boldsymbol{\gamma}}.\end{aligned}$$

Then, from (12), we have

$$\begin{aligned}\sum_{\boldsymbol{\gamma}\in\boldsymbol{\mathcal{J}}} \frac{(\widetilde{\boldsymbol{\lambda}}^{2p} - \mathbf{1})^{\boldsymbol{\gamma}}}{\boldsymbol{\gamma}!} \|\partial^{\boldsymbol{\gamma}} u\|^2_{(L^2)} &= \sum_{\boldsymbol{\gamma}\in\boldsymbol{\mathcal{J}}} \frac{(\widetilde{\boldsymbol{\lambda}}^{2p} - \mathbf{1})^{\boldsymbol{\gamma}}}{\boldsymbol{\gamma}!} \sum_{\boldsymbol{\alpha}\geq\boldsymbol{\gamma}} \frac{\boldsymbol{\alpha}!}{(\boldsymbol{\alpha}-\boldsymbol{\gamma})!} u^2_{\boldsymbol{\alpha}} \boldsymbol{\alpha}! \\ &= \sum_{\boldsymbol{\alpha}\in\boldsymbol{\mathcal{J}}} \left(\sum_{\boldsymbol{\gamma}\leq\boldsymbol{\alpha}} \frac{\boldsymbol{\alpha}!}{\boldsymbol{\gamma}!(\boldsymbol{\alpha}-\boldsymbol{\gamma})!} (\widetilde{\boldsymbol{\lambda}}^{2p} - \mathbf{1})^{\boldsymbol{\gamma}} \right) u^2_{\boldsymbol{\alpha}} \boldsymbol{\alpha}! \\ &= \sum_{\boldsymbol{\alpha}\in\boldsymbol{\mathcal{J}}} \widetilde{\boldsymbol{\lambda}}^{2p\boldsymbol{\alpha}} u^2_{\boldsymbol{\alpha}} \boldsymbol{\alpha}! = |||u|||^2_p.\end{aligned}$$

Summarizing we have

$$|||u|||^2_p = \sum_{\boldsymbol{\gamma}\in\boldsymbol{\mathcal{J}}} \frac{(\widetilde{\boldsymbol{\lambda}}^{2p} - \mathbf{1})^{\boldsymbol{\gamma}}}{\boldsymbol{\gamma}!} \|\partial^{\boldsymbol{\gamma}} u\|^2_{(L^2)}.$$

We conclude that in order to have $|||u|||_p < \infty$ we need all partial derivative of all orders to be (L^2) functions with the series above being finite. Moreover, we need a bound for the weighted sum $\sum_{\boldsymbol{\gamma}\in\boldsymbol{\mathcal{J}}} \frac{(\widetilde{\boldsymbol{\lambda}}^{2p} - \mathbf{1})^{\boldsymbol{\gamma}}}{\boldsymbol{\gamma}!} \|\partial^{\boldsymbol{\gamma}} u\|^2_{(L^2)}$. In general is then, easier to consider the norm $||u||^2_p$ (introduced in Section 3.2) to analyze and measure the stochastic regularity of a field.

7. Conclusions

We review the fact that Gaussian-Sobolev norms are equivalent to a particular weighted Weiner-chaos norms. We have shown that for particular set of weights in the Chaos norms the computation of the norm can be done by combining norms of partial derivatives up to certain oder. This allows the computation of Chaos norms when the chaos expansion are not available explicitly. This equivalence of norms is useful to obtain regularity results for some stochastic partial differential equations for these norms.

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