OBSERVER DESIGN FOR THE DUFFING EQUATION USING GERSGORIN'S THEOREM

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RESUMEN: Se diseñó un observador para la ecuación de Duffing, con el fin de estimar la trayectoria solución y su derivada temporal. La convergencia del observador queda garantizada al localizar los autovalores de la ecuación del error en el semiplano izquierdo del plano complejo usando el Teorema de Gersgorin.

PALABRAS CLAVES: Sistemas caóticos, Ecuación de Duffing, Teorema de Gersgorin, Observadores no lineales.

ABSTRACT: An observer for the Duffing equation is designed to estimate the solution trajectory and its time derivative. The convergence of the observer is guaranteed by locating the eigenvalues of the error equation on the left half of the complex plane using Gersgorin's theorem.

KEYWORDS: Chaotic systems, Duffing equation, Gersgorin's theorem, Nonlinear observers.

1 INTRODUCTION

Observers are an important tool in control theory because for most practical systems it is not always possible to fully measure the state vector, in such cases the observer allows to estimate the state vector, if the system is observable, using the measured outputs. The observer theory is well known for linear systems (*Friedland,* 1986) and it is a research topic for nonlinear systems (*Isidori,* 1989).

On the other hand, chaotic systems are a class of nonlinear systems with many interesting properties; the most remarkable is the difficulty to predict the future state for a given initial condition. The interested reader may find a basic introduction to chaotic systems in (*Parke and Chuar,* 1987) and a more formal development in (*Holmes,* 1983).

In the last years there has been a growing interest in the control and cooperation of nonlinear chaotic systems (*Chen and Dong,* 1993; *Ogorzalek,* 1993). Chaos is an undesired operating condition that may be found in real physical systems, so it is important to develop techniques to observe and control this dynamic behavior.

Many control methodologies need to know the state of the system to be controlled, in practical situations this is not always possible and a state estimator or observer is required as part of the control design. The observer may be viewed as a chaotic system synchronized with the original one, this cooperation can also be exploited in signal processing and communications (*Ogorzalek,* 1993).

2 OBSERVER DESIGN

Consider the second order nonlinear differential equation,

$$\ddot{y}+\alpha_1.\dot{y}+\alpha_0.y+\Im(y,\dot{y})=u \quad (1)$$

where, u is the external forcing term, y is the solution trajectory, $\Im$ is a smooth nonlinear function, and α_0, α_1 are constant parameters.

Transforming the differential equation (1) into a state representation yields,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\alpha_0 & -\alpha_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 0 \\ \Im(x_1, x_2) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2}$$

with, $y = x_1$, and $dy/dt = x_2$, the pair (x_1, x_2) is called the state of (2).

An observer with gains $k_1(\xi_1, \xi_2)$ and $k_2(\xi_1, \xi_2)$ for system (2) is given by,

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\alpha_0 & -\alpha_1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} - \begin{bmatrix} 0 \\ \Im(\xi_1, \xi_2) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} k_1(\xi_1, \xi_2) \\ k_2(\xi_1, \xi_2) \end{bmatrix} (y - y_o)$$

$$y_o = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \tag{3}$$

The error between the actual state (x_1, x_2) and the observed state (ξ_1, ξ_2) is described by,

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} -k_1(\xi_1, \xi_2) & 1 \\ -k_2(\xi_1, \xi_2) - \alpha_0 & -\alpha_1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} - \begin{bmatrix} 0 \\ \Im(x_1, x_2) - \Im(\xi_1, \xi_2) \end{bmatrix} \tag{4}$$

where, $e_1 = x_1 - \xi_1$, $e_2 = x_2 - \xi_2$.

The additional term in (4) can be approximated using the mean value theorem (*Khalil*, 1976),

$$\begin{bmatrix} 0 \\ \Im(x_1, x_2) - \Im(\xi_1, \xi_2) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \Im_{\xi_1}(\xi_1, \xi_2) & \Im_{\xi_2}(\xi_1, \xi_2) \end{bmatrix} \cdot \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \tag{5}$$

the terms in the Jacobian are,

$$\Im_{\xi_1}(\xi_1, \xi_2) = \frac{\partial \Im(\xi_1, \xi_2)}{\partial \xi_1}$$

$$\Im_{\xi_2}(\xi_1, \xi_2) = \frac{\partial \Im(\xi_1, \xi_2)}{\partial \xi_2}$$

Replacing (5) into (4) yields the error equation,

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} -k_1(\xi_1, \xi_2) & 1 \\ -k_2(\xi_1, \xi_2) - \alpha_0 - \Im_{\xi_1}(\xi_1, \xi_2) & -\alpha_1 - \Im_{\xi_2}(\xi_1, \xi_2) \end{bmatrix} * \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \tag{6}$$

The estimation errors (e_1, e_2) will decrease asymptotically to zero if the matrix in (6) has both eigenvalues with negative real part. Using Gersgorin's theorem (*Pullman*, 1976) the eigenvalues are on the left half of the complex plane if the gains are chosen such that,

$$\begin{aligned} k_1(\xi_1, \xi_2) &> 1 \\ k_2(\xi_1, \xi_2) &= -\alpha_0 - \Im_{\xi_1}(\xi_1, \xi_2) \end{aligned} \tag{7}$$

and the inequality,

$$-\alpha_1 - \Im_{\xi_2}(\xi_1, \xi_2) < 0 \tag{8}$$

is satisfied.

3 DUFFING EQUATION

A popular model to study nonlinear oscillations, bifurcations and chaos is the Duffing differential equation,

$$\ddot{y} + \alpha_1 . \dot{y} + \alpha_0 . y + y^3 = q . \cos(\omega . t) \tag{9}$$

Equation (9) belongs to the family (1) and it describes a nonlinear mechanical oscillator with a cubic stiffness term, i.e., $\Im(dy/dt,\ y)\ =\ y^3$; a typical set of parameters for the Duffing equation are:

$$\alpha_0 = -1.$$
$$\alpha_1 = 0.4$$
$$\omega = 1.8$$

When the magnitude q of the external periodic forcing term is varied, the solution trajectories display changes in the dynamic behavior, for example $q\ =\ 0.620$ produce a periodic solution, and $q =\ 1.800$ produce a chaotic solution.

The following simulations compare the actual state of (9) with the estimated state, using the observer (3). The initial conditions are:

$$x_1(0) = -\ 0.5$$
$$x_2(0) = -\ 0.5$$
$$\xi_1(0) = 0.0$$
$$\xi_2(0) = 0.0$$

and the observer gains are chosen to satisfy (7),

$$k_1 = 5$$

$$k_2(\xi_1, \xi_2) = 1.1 - 3 * \xi_1^2$$

Notice that the Duffing equation with the typical parameters satisfies (8). Figures 1(a) and 1(b) show the actual and estimated states for the Duffing equation when $q\ =\ 0.620$. Figures 2(a) and 2(b) show the actual and estimated states for the Duffing equation when $q = 1.800$.

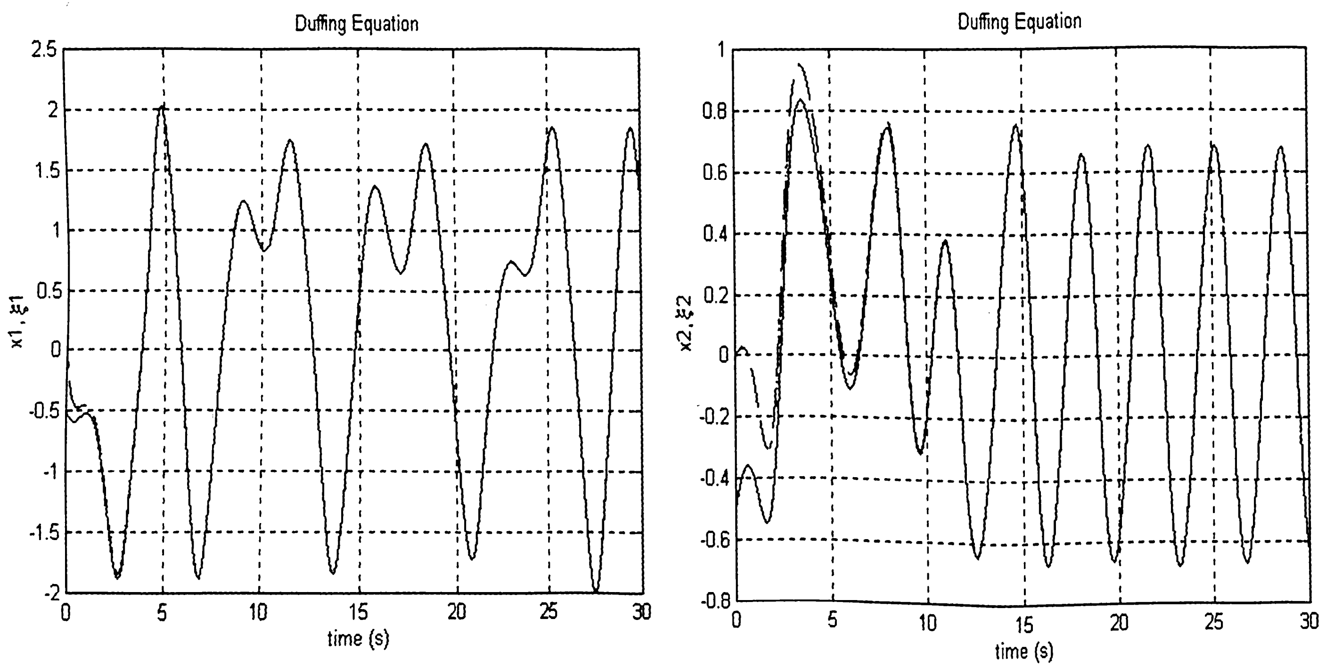
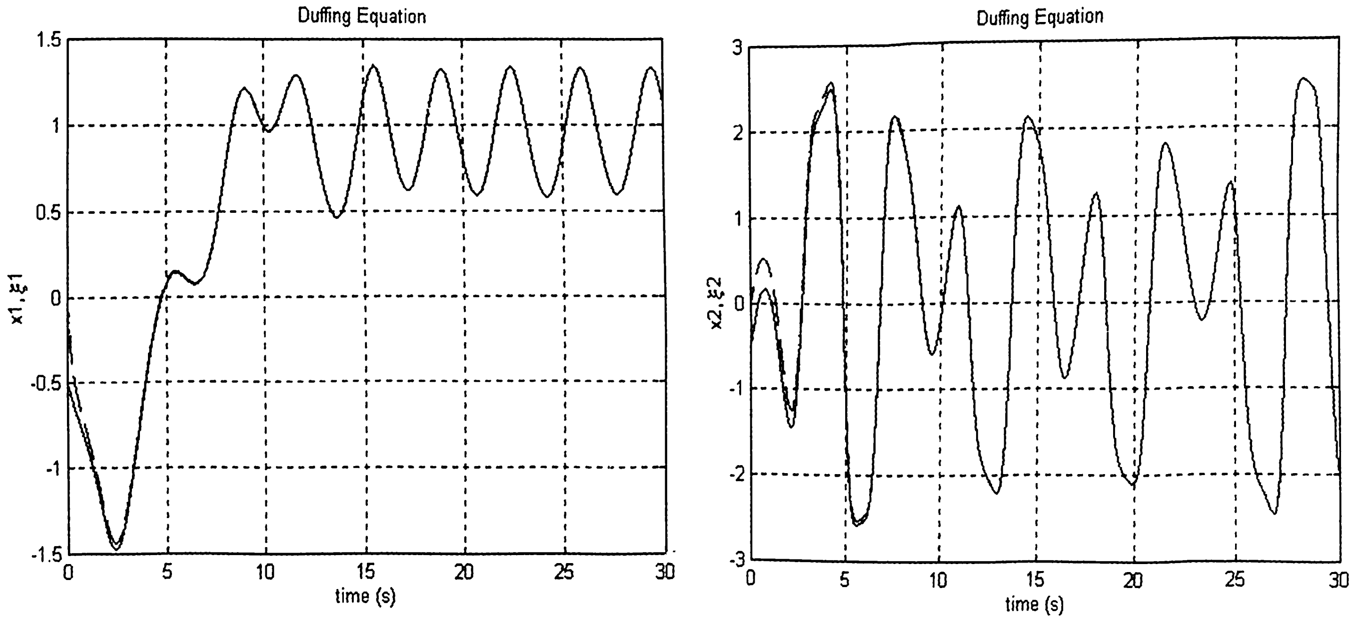


Figure 1. Actual and observed (dashed) states of the Duffing equation during a periodic behavior, $q = 0.620$. (a) $x_1(0) = -\ 0.5$, $\xi_1(0) = 0.0$; (b) $x_2(0) = -\ 0.5$, $\xi_2(0) = 0.0$.



(a) (b)

Figure 2. Actual and observed (dashed) states of the Duffing equation during a chaotic behavior, $q = 1.800$. (a) $x_1(0) = -0.5$, $\xi_1(0) = 0.0$; (b) $x_2(0) = -0.5$, $\xi_2(0) = 0.0$.

4 CONCLUSIONS

An observer has been designed for second order nonlinear differential equations of class (1), the theoretical results have been illustrated with the Duffing equation. In the simulations, after a brief transient, the observed states (ξ_1, ξ_2) match the actual states ($x_1 = y$, $x_2 = dy/dt$). The observer (3) can be used to provide an estimate of the actual states for a control law, or it can be viewed as a system synchronized with the one described by differential equation (1).

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