The Labour Theory of Value and the Problems of Other Theories of Value*

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Abstract

This article first shows that one of the main misunderstandings surrounding the labour theory of value arises from the non-distinction between real determination and mathematical determination: although mathematically it is possible to link value prices (values expressed in money) and production prices through a linear transformation that operates *in both directions*, from the point of view of a realistic economic theory the "real relations of value", which take place in the real world, are prior to, and determine *de facto*, *univocally*, those theoretical-mathematical relationships. Secondly, to study prices appropriately it is necessary to distinguish between *actual* prices, which are as real as labour-values, and the pair of *theoretical-hypothetical* prices that would exist if the rate of profit or the rate of surplus value, respectively, were uniform (which they are not in reality). Thirdly, after defining the equations of these prices in the correct way in which they appear in the literature, it is proven that it is possible to read the Marxian Transformation in two steps, showing that the first Marxian definition of both types of prices (defined from actual prices) gives way to a second and definitive definition in which both coincide with the correct definitions. Fourthly, the transformation is framed in a broader framework of various (linear) transformations to show that value prices and production prices are related, also exactly, on the one hand to actual prices and on the other hand to the values and quantities of direct labour represented in the commodities, so that all the vectors involved are calculable simultaneously within a single accounting system. Fifthly, it is explained why a *physical* theory of value, based on a substance of value alternative to labour, cannot prosper. And finally it is shown that the only productive factor of value is labour, and, although all factors cooperate with labour in the production of wealth (use values), this is an irrelevant question for the determination of prices.

Keywords: Value determination; Actual prices; Marxian transformation; Labor value.

JEL: B14; D46; E11; P16.

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La teoría laboral del valor y los problemas de otras teorías del valor

Resumen

Este artículo muestra, en primer lugar, que uno de los principales malentendidos en torno a la teoría laboral del valor surge de la no distinción entre determinación real y determinación matemática: aunque matemáticamente sea posible vincular precios de valor (valores expresados en dinero) y precios de producción mediante una transformación lineal que opera en ambas direcciones, desde el punto de vista de una teoría económica realista las «relaciones reales de valor», que tienen lugar en el mundo real, son anteriores a, y determinan de facto, unívocamente, esas relaciones teórico-matemáticas. En segundo lugar, para estudiar adecuadamente los precios es necesario distinguir entre los precios reales, que son tan reales como los valores-trabajo, y el par de precios teórico-hipotéticos que existirían si la tasa de ganancia o la tasa de plusvalía, respectivamente, fueran uniformes (lo que no son en la realidad). En tercer lugar, tras definir las ecuaciones de estos precios de la forma correcta en que aparecen en la literatura, se demuestra que es posible leer la Transformación Marxiana en dos pasos, mostrando que la primera definición marxiana de ambos tipos de precios (definidos a partir de los precios reales) da paso a una segunda y definitiva definición en la que ambos coinciden con las definiciones correctas. En cuarto lugar, se enmarca la transformación en un marco más amplio de varias transformaciones (lineales) para mostrar que los precios de valor y los precios de producción están relacionados, también exactamente, por un lado, con los precios reales y por otro con los valores y cantidades de trabajo directo representados en las mercancías, de modo que todos los vectores implicados son calculables simultáneamente dentro de un mismo sistema contable. En quinto lugar, se explica por qué una teoría física del valor, basada en una sustancia de valor alternativa al trabajo, no puede prosperar. Y, por último, se demuestra que el único factor productivo de valor es el trabajo y, aunque todos los factores cooperan con el trabajo en la producción de riqueza (valores de uso), ésta es una cuestión irrelevante para la determinación de los precios.

Palabras clave: determinación del valor; precios reales; transformación marxiana; valor trabajo.

A teoria do valor-trabalho e os problemas de outras teorias do valor

Resumo

Este artigo mostra, em primeiro lugar, que um dos principais mal-entendidos em torno da teoria do valor do trabalho decorre da não distinção entre determinação real e determinação matemática: embora matematicamente seja possível vincular os preços de valor (valores expressos em dinheiro) e os preços de produção por meio de uma transformação linear que opera em ambas as direções, do ponto de vista de uma teoria econômica realista, as "relações reais de valor", que ocorrem no mundo real, são anteriores e determinam de fato, univocamente, essas relações teórico-matemáticas. Em segundo lugar, para estudar os preços adequadamente, é necessário distinguir entre os preços reais, que são tão reais quanto os valores do trabalho, e o par de preços teórico-hipotéticos que existiriam se a taxa de lucro ou a taxa de mais-valia, respectivamente, fossem uniformes (o que não é o caso na realidade). Em terceiro lugar, depois de definir as equações desses preços da maneira correta em que aparecem na literatura, fica provado que é possível ler a transformação marxiana em duas etapas, mostrando que a primeira definição marxiana de ambos os tipos de preços (definidos a partir dos preços reais) dá lugar a uma segunda e definitiva definição em que ambos coincidem com as definições corretas. Em quarto lugar, a transformação é enquadrada em uma estrutura mais ampla de várias transformações (lineares) para mostrar que os preços de valor e os preços de produção estão relacionados, também de forma exata, por um lado, aos preços reais e, por outro lado, aos valores e quantidades de trabalho direto representados nas mercadorias, de modo que todos os vetores envolvidos possam ser calculados simultaneamente em um único sistema contábil. Em quinto lugar, explica-se por que uma teoria **física do valor,** baseada em uma substância de valor alternativa ao trabalho, não pode prosperar. E, por fim, é demonstrado que o único fator produtivo de valor é o trabalho e que, embora todos os fatores cooperem com o trabalho na produção de riqueza (valores de uso), essa é uma questão irrelevante para a determinação dos preços.

Palavras chave: determinação de valor; preços reais; transformação marxiana; valor-trabalho.

Introduction

The present article aims to contribute to laying the foundations for a new approach to the "problem of Transformation" based on two key elements: First, the reconsideration of the meaning of the contribution made by Karl Marx to the resolution of this problem, and second, the intention to update up to the present and develop Marx's contribution in this regard, starting from ideas that are not very common in the current debates between the various schools and best-known interpretations of the Transformation. In relation to the first point, and in our opinion, there is no inconsistency in Marx's approach, but rather two successive logical steps in his thought: in the first step, Marx starts from real prices (actual prices) to develop his model of hypothetical prices (prices that contain either a uniform rate of profit or a uniform rate of surplus value; that is, production prices and value prices respectively), and in the second step, he introduces a double assumption: he assumes that real prices are value prices in Books I and II of Capital, and are production prices in Book III. In this way, the resulting equations for Marx's prices come to coincide with those that are considered correct equations in modern times. As for the second point, it is emphasized that the relationship between value prices (and therefore values) and production prices can be analyzed as an exact linear transformation of the first price vector into the second (and of the second into the first), a transformation that also occurs between production prices and actual prices. This is how it turns out that the three types of prices are exactly connected to each other and to the vectors of direct labor coefficients and vertically integrated labor coefficients or values.

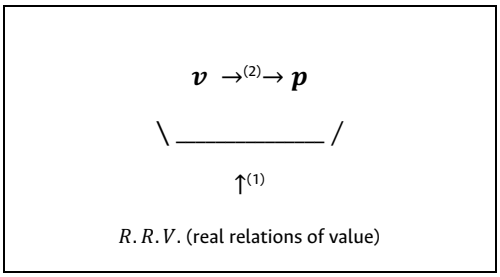
In analysing the labour theory of value (and especially the Marxian theory of value), emphasis is placed on the theory of prices contained in it (Shaikh 1981) and on the question of the substance of value, but without failing to refer to other important points about labour and value. The points that follow are among those that distinguish our approach from those advanced by other writers.

Real Determination: The Real Relations of Value

First, the distinction between real determination and the *mathematical* determination of prices is highlighted (Shaikh 1981; 1984). The question of the quantitative relationships between values and prices is of course studied, but it is emphasized from the beginning that the study should not begin with those relations but with the fact that it is the *real activity* of flesh-and-blood-people, mediated by the value relationships of the current society, which actually *produces* goods that have the form of commodities, i.e. have a value and a price. The difference between the two types of "determination" is reflected in Figure 1, where the vertical arrow $\uparrow^{(1)}$ indicates the real determination of values and prices by the "real relations of value" (RRV),[1] i.e. by the set of all really existing capitalist production (and circulation) relations.

1 Value "is a relationship between people" (Georgescu-Roegen, 1971, p. 32) and "the cause of the price" (Boudin, 1907, p. 66), to which we can add that price is the effect or result of value.

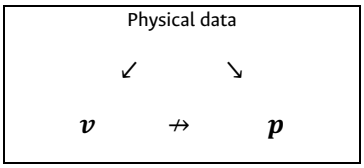
Figure 1. Real determination (1) versus mathematical determination (2)



Source: Own elaboration.

For its part, the horizontal arrow $\rightarrow^{(2)}$ indicates the mathematical determination of the prices, $\boldsymbol{p}$, by the values, $\boldsymbol{v}$,[2] a highly debated issue within the labour theory of value (and other theories of value), whose study, from our point of view, must presuppose the real determination $\uparrow^{(1)}$. To mention some of the issues discussed, there are authors who replace the arrow $\rightarrow^{(2)}$ with a truncated arrow, $\nrightarrow$, to indicate the impossibility of going from $\boldsymbol{v}$ to $\boldsymbol{p}$ (Steedman, 1977); others who would prefer to make the $\boldsymbol{v}$ and the arrow $\rightarrow^{(2)}$ disappear because they think that in the real world there are only prices but no values (although many, inconsistently, continue to analyze the values to criticize them); a third group, while defending the labour theory of value, indirectly casts doubt on it since they seem to think that, according to this theory, "$\rightarrow^{(2)}$" means that $\boldsymbol{p}$ and $\boldsymbol{v}$ are *proportional*, which is something that does not actually happen (Cogliano et al., 2018); a fourth group, which we call "*physicalists*", in fact replaces the vertical arrow going from the RRV upwards with two arrows that go downwards from the "physical data" to $\boldsymbol{v}$ and $\boldsymbol{p}$ (disconnected from each other) respectively.

Figure 2. The physicalist interpretation



Source: Own elaboration.

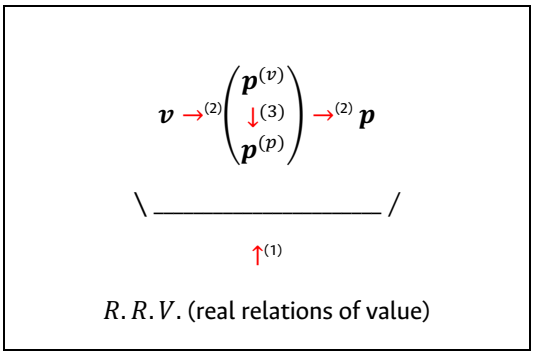
2 In this article we use from the beginning vectors and matrices, with the following notation: bold uppercase letters will be used for matrices, bold lowercase letters for vectors and non-bold lowercase letters for scalars (with some exceptions), without distinguishing between row vectors and column vectors, since the latter are very rare in the text (mainly, $\boldsymbol{x}$ = vector of sectoral gross outputs, $\boldsymbol{y}$ = vector of sectoral net outputs, and $\boldsymbol{c}$ = vector of coefficients of private or "unproductive" consumption: the components of the real wage per capita).

Finally, we could replace the arrow $\rightarrow^{(2)}$ with two, one in each direction, $\overset{(2)}{\leftarrow}\ \overset{}{\rightarrow}\ {}^{(2)}$, to indicate that from a purely mathematical point of view we can go both from values to prices as well as from prices to values: this would only be appropriate if the analysis of the double mathematical determination is framed in the context of a previous real determination that always goes upwards (from the real value relations to the values and prices). The confusion between $\uparrow^{(1)}$ and $\rightarrow^{(2)}$ is a recurring issue that has generated a lot of confusion —for example, the controversy of Samuelson (1974), Sinha (2010), etcetera, against Baumol (1973), Morishima (1973), an others, regarding the Fundamental Marxian Theorem—.

Two kinds of prices: real prices and hypothetical prices

Secondly, Figure 3 reflects our point of view on the intersecting relationships between values and the three types of prices: production prices, $\boldsymbol{p}^{(p)}$, value prices, $\boldsymbol{p}^{(v)}$, and (long term) actual prices, $\boldsymbol{p}$.[3] We mean that a third dimension must be added, represented by the parenthesis and the down arrow $\downarrow^{(3)}$. That is, from our point of view it is necessary to mentally construct a pair of *theoretical* or *ideal* prices, of a *hypothetical* nature, ontologically *abstract*, which are clearly differentiated from the other two vectors, which are *totally real*, *factual* or *concrete* entities: *actual values* (which are coefficients of total or vertically integrated labour) and *actual prices* (which are long-term prices, different from the short-term market prices that revolve around the center of gravity that are the $\boldsymbol{p}^{(p)}$ (or, at a less abstract level, the $\boldsymbol{p}$).

Figure 3. The various relationships between values and prices



Source: Own elaboration.

3 We include later a fourth type of prices, the Sraffian prices, $\boldsymbol{p}^{(S)}$, as well as the vector of quantities of labour (direct labour coefficients, $\boldsymbol{l}$). We will take the $\boldsymbol{p}^{(S)}$ as a particular version of the production prices, a version that is at present more common than others, above all when focusing the issue of the distribution of income, which is not our case.

The pair of prices within the parenthesis are hypothetical because they are *the prices that would exist* if the rate of profit, r, or the rate of surplus value, s, respectively, were uniform, something that does not occur in real actual prices (defined by the *non uniformity* of both rates) nor in the real values, which are theoretically *prior* and independent of any distribution, that is, independent of how the labour is divided between its paid and its unpaid part. It could be said that the pairs $\boldsymbol{p}^{(v)} - \boldsymbol{p}^{(p)}$ and $\boldsymbol{v} - \boldsymbol{p}$ form the two submodels of our pricing model, the first of which is a fundamental analytical tool for a better understanding of the real relationships between values and prices. We will see below that the arrow $\downarrow^{(3)}$ as well should in fact be, from a mathematical point of view, replaced by an arrow in each direction ($\downarrow^{(3)}\uparrow$).

Marx's Transformation is correct

A third contribution of our article consists of the demonstration that the vectors of value prices and production prices defined by Marx, which in a first approximation are different from the correct ones, become correct if the different assumptions that Marx makes in *Capital* (one in books I and II and other in book III) are taken into account. the correct production prices and the correct value prices (which are proportional to values and often called "monetary values" or "values expressed in monetary terms") will appear later within the set (1) to (4) presented in the section The "correct" values and prices.

$$\boldsymbol{p}^{(p)} = \boldsymbol{p}^{(p)}\boldsymbol{A}^{(+)}(1+r) \qquad (3)$$

$$\boldsymbol{p}^{(v)} = \boldsymbol{p}^{(v)}\boldsymbol{B}^{(+)}(1+s) \qquad (4),$$

whose notation and components will be explained later. The important thing is to highlight that, without having to resort to the iterative method used by Bródy, Shaikh or Morishima and starting from equations that are different from those that his critics attribute to Marx, it is shown that the first definitions of both prices (which we note with the superscript $^{(M1)}$) become the correct definitions, with superscript $^{(M2)}$, so that

$$\boldsymbol{p}^{(p)(M1)} \neq \boldsymbol{p}^{(p)(M2)} = \boldsymbol{p}^{(p)}$$

$$\boldsymbol{p}^{(v)(M1)} \neq \boldsymbol{p}^{(v)(M2)} = \boldsymbol{p}^{(v)}.$$

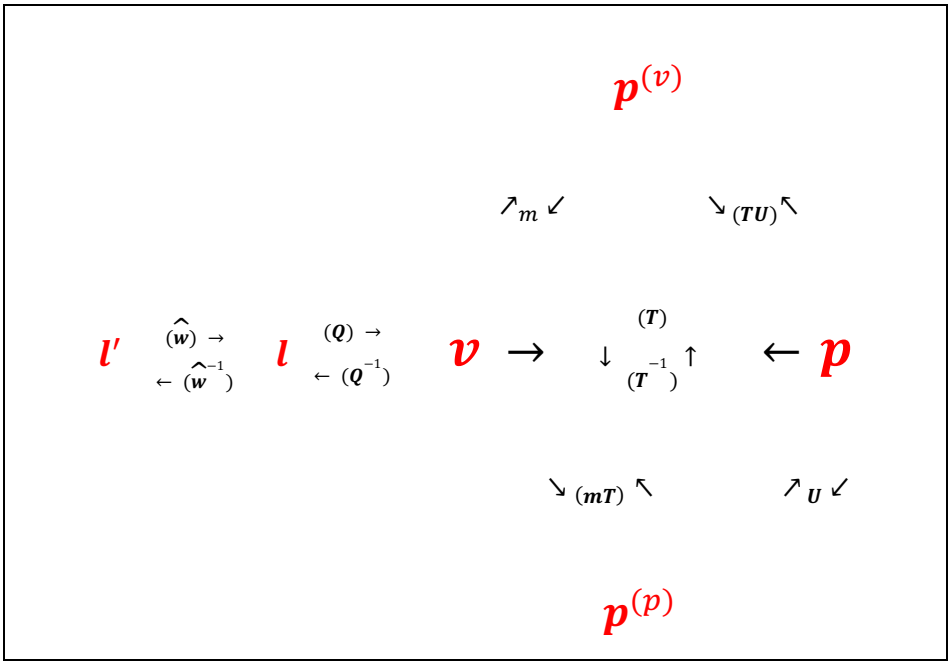
Likewise, critics are wrong when describing Marx's definition of the rate of profit, since in fact Marx's definition is different, and this allows us to write $r^{(M1)} \neq r^{(M2)} = r$.

Linear transformations in one single model of values and prices

Fourthly, the four vectors that appeared in Figure 3, plus the vectors $\boldsymbol{l}$ and $\boldsymbol{l}'$ (which should also have a double arrow going back and forth to each of the other four vectors), $\begin{matrix}(\widehat{\boldsymbol{w}}^{-1}) \rightarrow \\ \leftarrow (\widehat{\boldsymbol{w}})\end{matrix}$, are interrelated to each other, *mathematically*, through linear (direct and inverse) transformations that convert one vector into any of the others; to do this, it is necessary to find in each case the

linear operator that carries out the linear transformation of one vector into another, along with the matrices $\boldsymbol{Q}$ and $\widehat{\boldsymbol{w}}$ and the scalar m for the relationship between $\boldsymbol{v}$ and $\boldsymbol{p}^{(v)}$.

Figure 4. Relationships between direct labour, values, value prices, production prices and actual prices, by means of the scalar *m* and matrices $T, U, TU, mT, Q, \widehat{w}$



Source: Own elaboration.

In the best known case, practically the only case treated in the literature on the subject, the relationship between $\boldsymbol{p}^{(v)}$ and $\boldsymbol{p}^{(p)}$ is written as:

$$\boldsymbol{p}^{(p)} = \boldsymbol{p}^{(v)}\boldsymbol{T} \qquad (24),$$

where, as will be seen, the matrix $\boldsymbol{T} = \left(\boldsymbol{I} - (1+s)\boldsymbol{B}^{(+)}\right)\left(\boldsymbol{I} - (1+r)\boldsymbol{A}^{(+)}\right)^{-1}$ is a function of the uniform rate of profit, the uniform rate of surplus value (both connected by the formula $r = s/q$, where q = 1 + the value composition of capital) and the coefficients a_{ij} (the inputs of productive consumption) and b_{ij} (the inputs of unproductive consumption) that appear in $\boldsymbol{A}^{(+)} = \boldsymbol{A} + \boldsymbol{B}$ and $\boldsymbol{B}^{(+)} = \boldsymbol{B}(\boldsymbol{I} - \boldsymbol{A})^{-1}$.

The substance of value and the weakness of the physical *theory of value*

Fifthly, this article shows that the *substance* of value is labour, or, in other words, that although all productive factors contribute to the generation of wealth, *the only* productive factor that creates value is labour. Ricardo explained clearly the difference between use value and value, and at a macroeconomic level he distinguished between national wealth (*riches*) and the value of that wealth, exemplifying in J.-B. Say the important error which consists of not grasping that distinction, confusing value with wealth: "Many of the errors of political economy have arisen from errors on this subject, from considering an increase of riches, and an increase of value, the same thing [...]" (Ricardo, 1821/2005, p.199). Today, this error reappears in what we could call the *physical theory of value* – some of whose defenders, neoclassical or not, are anti-Ricardian at this point, included those who were once known as "*neo-Ricardian*" – a theory based on a physicalism that present two versions (soft or hard). *Soft physicalism* confuses value with use value when it implicitly denies the need for any *original* factor (labour, land, etc.), which it replaces with commodities (including labour power) understood as *intermediate* factors. This physicalism states that "labour" is a commodity like any other, and from there it makes the logical leap of wrongly equating any commodity with labour; but it forgets that labour is above all an *activity*, an activity carried out by specific men and women who act as the *subjects* of all real production processes, using for it the means of production, which are the other commodities, those which enter as *objective* elements in these processes, in their role of *objects* of *labour*. For its part, *hard physicalism*, elevating the physical to a metaphysical category (Cogliano *et al.* 2018; Ganssmann 1981), making it a *universal*, considers that *the physical*, the physical aspect, is not only separate from the labour aspect but is superior to it and must "come first" in the analysis; this contradicts Marx's position, for whom both aspects of production form a unity: the product is only the *result* of labour, and the surplus product is the result of surplus labour, i.e. it is the material form in which the surplus value created by surplus labour *is expressed*.[4]

This article shows the limitations of physicalism. First, because it treats labour as if it were not physical (when Time is a fundamental physical variable, perhaps the main one), in order to present it as the counterpoint of the physical data. Second, because it reduces the physical to the *tangible*, and at the same time the technical coefficients to the *passive* material technical coefficients (the a_{ij}), forgetting that the direct labour coefficients, the l_i, represent the *active side*, the productive *activity* within which the value of the a_{ij} is established as a set of ratios that *passively reflect* the effect of labour on the means of production and the product, so that the l_i are the *active*

4 However, some supporters of the physical theory of value are also contributors to the labour theory of value, either thanks to specific clarifications compatible with it (Bortkiewicz 1907/1952; Dmitriev 1898/1974; Steedman 1977), or because they imitate, although *on a larger scale*, Adam Smith's contradictions –keep in mind that these do not exist in Ricardo (Rubin 1929/1979; Tsoulfidis 1998), who is faithful to the labour theory of value, and not to a theory of cost price wrongly attributed to him by authors such as Stigler (1958) or Dorfman (1989)– author who, despite defending other versions at the same time, also defended the labour theory of value in tune with Ricardo an Marx. In the case of hard physicalism, it simultaneously supports theses of the physical theory of value and the labour theory of value, which reminds us that there are also defenders of Marx's theory of value who occasionally make statements that are incompatible with it. (This is the case for example of Mariolis and Tsoulfidis, who write that "it can be stated that $r = 0$ or $\boldsymbol{lA} = \lambda_{A,1}\boldsymbol{l}$ implies that the 'pure labour theory of value' holds true and $w = 0$ implies that the 'pure capital theory of value' holds true, while the 'capital-labour theory of value' [...] applies to all other cases" (Mariolis and Tsoulfidis 2016, p. 17).

(authentically primary) material technical coefficients. Thirdly, physicalism does not notice that the techniques that interests the theory of value are always the *labour* techniques linked to the *productive consumption* developed within the firms, not simple *natural* techniques (such as the one represented by the coefficients of flowers and bees in the natural production of honey) nor techniques linked to *unproductive* consumption (culinary type, for example).

Furthermore, it is false that the a_{ij} are physical ratios in the physicalist sense for at least three reasons. First, actually, they are monetary ratios of production costs and that is why matrix $\boldsymbol{A}$ does not include free inputs (sun, rain, wind...) and yet it does include "inputs" that are not physical or technical but monetary (insurances, bank interests, night watchmen...). Second, many "services" (tertiary sector) appearing in the real input-output tables, whose quantity is measured in hours of labour —since products are "things" and services are "activity" (Marx, 1863-64, p. 87)— are always "forgotten" in these authors' models. Third, and the supposed physical matrix, $\boldsymbol{A}^p$, is a fiction that is obtained from the monetary matrix, $\boldsymbol{A}^{(m)}$, by the formula $\boldsymbol{A}^{(p)} = \widehat{\boldsymbol{p}}^{-1}\boldsymbol{A}^{(m)}\widehat{\boldsymbol{p}}$ (where $\widehat{\boldsymbol{p}}$ is the diagonal matrix of actual prices) and thanks to the helpful Leontief's formula to *redefine* the physical units of commodities;[5] thanks to this procedure, all prices are made equal to 1, so $\widehat{\boldsymbol{p}} = \boldsymbol{I}$ and therefore $\boldsymbol{A}^{(p)} = \boldsymbol{A}^{(m)}$, which is nothing more than a permission to *call* a matrix "physical" which is the necessary monetary form in which value is expressed.

Labour and value in a Leontief economy

Sixth, our mathematical model represents the developed capitalist economy as a "Leontief economy" in its simplest version, with only *circulating* capital and *simple* production. It is known, however, that the same results are valid in the general case, which includes *fixed* capital and *joint* production (see Cogliano *et al*. 2018), and it is obvious that matrix algebra, *eigensystems* of equations and the theorems of Perron and Frobenius can be easily applied in both cases. Taking advantage of the fact that Leontief's input-output tables implicitly contain the labour theory of value (Cameron, 1952, p. 191; Gale, 1960, p. 302; Hicks 1960, p. 671, Lancaster, 1968, pp. 89-90), Bródy's *full matrix* $\mathbb{A} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{c} \\ \boldsymbol{l} & \boldsymbol{0} \end{pmatrix}$ can be used to synthetically show the double side of labour, since the last row, $\boldsymbol{l}$, contains the direct labour coefficients (which represent labour as *an activity*), and the last column, $\boldsymbol{c}$, contains the real per capita wage coefficients (which represent labour as the *commodity labour power*). It is clearly seen in the $\mathbb{A}_{n+1\,x\,n+1}$ matrix that there cannot be *alternative* values (the famous n *commodity-values*, or i*-values*, present in the $\boldsymbol{A}_{n\,x\,n}$ matrix) to the labour value, but rather n alternative numéraires that serve to express labour values by other means. But the *numéraire* cannot be confused with the *substance* of value

5 Leontief writes the following: "In order to obtain the corresponding physical amounts of all commodities and services, we simply define the unit of physical measurement of every particular type of product so as to make it equal to that amount of the commodity which can be purchased for one dollar at prevailing prices." (Leontief 1951, p. 72) —see also Dorfman *et al*. (1958, pp. 238-239, 207), Ochoa (1984, p.62), Chiang (1984, p. 116), Miller and Blair (1985, p. 356), and Bidard (1991/2004, p. 18) Referring to the USA input-output tables, Leontief clarifies that, "although presented in terms of millions of dollars, these figures can also be *interpreted* as representing physical quantities measured in terms of units defined as the 'amount of particular commodities or services purchasable for one million dollars at 1939 prices'." (Leontief, 1951, p. 148, emphasis added).

(given only by row n + 1), and it is obvious that if the objective of the real economy is the *social reproduction* of humans, autonomous labour activity (which is exclusive of people, the subjects[6] of production) cannot be replaced by other types of *alleged* activity, whether the "activity" of a magic wand that produces everything, including $\boldsymbol{c}$, with an $\boldsymbol{A} = \boldsymbol{0}$, or the "activity" of any of the n commodities, animate or inanimate; this point of view would only represent an *animist-fetishist* conception of social activity, with which the metaphysics of "the physical-tangible, first" reaches the *theological* level (in its version of *primitive* theism), while the real productivity of the system reaches levels close to zero.

The "Correct" Values and Prices

Labour and value

Our analysis of values and prices will focus on equations (1) to (4), that we explain below, which define the values, $\boldsymbol{v}$, the actual prices, $\boldsymbol{p}$, the production prices, $\boldsymbol{p}^{(p)}$, and the value prices, $\boldsymbol{p}^{(v)}$. (Sraffian prices, $\boldsymbol{p}^{(S)}$, will be analyzed as a variant of production prices: see equation (9) $\boldsymbol{p}^{(S)} = \boldsymbol{p}^{(S)}\boldsymbol{H}^{(+)}r^{(S)}$)

$$\boldsymbol{v} = \boldsymbol{l}\boldsymbol{Q} \quad (1)$$

$$\boldsymbol{p} = \boldsymbol{p}\boldsymbol{A}^{(+)}(\boldsymbol{I} + \hat{\boldsymbol{r}}) = \boldsymbol{p}\boldsymbol{C}^{(+)} \qquad (2)$$

$$\boldsymbol{p}^{(p)} = \boldsymbol{p}^{(p)}\boldsymbol{A}^{(+)}(1 + r) \qquad (3a)$$

$$\boldsymbol{p}^{(v)} = \boldsymbol{p}^{(v)}\boldsymbol{B}^{(+)}(1 + s) \qquad (4a).$$

6 Regarding the subject (the individual), we can ask ourselves like Sen (1978), "*who* created (Michelangelo's) David" and answer that "Michelangelo Buonarroti", without worrying about leaving aside the marble and the tools he used (chisel, hammer, awl...) since no one would think of saying that the *authors* of (Michelangelo's) David are Buonarroti *and* his chisel, his hammer, his awl, his marble..., etc. Likewise, if we ask about the work of an economist, no one would say, for example, that "Production of commodities by means of commodities" is the work of several co-authors: Sraffa, his typewriter, his library, his work table... in such or what proportion." Likewise, if a driver negligently runs over a pedestrian, no one will say that the *responsible and perpetrators* of the crime are the driver, his car, the gasoline in the car's tank..., which is why no criminal has ever defended himself in court by saying : "Your Honor, I am only 40% guilty of the crime; the remaining 60% of the penalty must correspond to my car, the gasoline..., etc." Now, if we move from the field of art, science or jurisprudence to the field of material production of society, the truth is the same. Who is the sole author and creator of the social product, and in particular of that "immense accumulation of commodities" that form "the wealth of societies in which the capitalist mode of production prevails" (Marx 1867/1996, p. 45)? The answer is: the whole of society or producers who, with their labour (and with the *help* of the means of production), *create* that *wealth* (use values) of whose production they are the *sole authors and responsible*, the subjects who allocate a certain average portion of the available labour to a certain sum of commodities, and thereby conferring to each unit a certain quantum of labour-value. That is, *all objective factors (including the subject), driven by labour, create wealth, but labour itself is the only productive factor that creates the value of that wealth*. Value is a matter of *who*, not what: that is why value corresponds exclusively to the subject and not to the objects that he or she uses.

Once we explain the notation, we will see that equations (1) and (3) are accepted as "correct" by most of the value literature. On the other hand, equations (2) and (4) are relatively unknown, since (4) may seem superfluous to those who are interested exclusively in the *proportionality* of value prices with respect to values (see equation (7) below, $\boldsymbol{p}^{(v)} = m \cdot \boldsymbol{v}$, where *m* is a scalar), and believe that it is enough to use values; and (2) is a new and strange equation for everyone who believes that it is unnecessary to talk about *long-term actual* prices (not to be confused with short-term market prices) and consequently do not see the relevance of a second transformation that converts production prices into actual prices.

The idea that the value of a commodity has two components – which come, respectively, from the direct labour that produces it, $\boldsymbol{l}$, and from the indirect labour *represented* in the means of production used directly by the workers who produce it, $\boldsymbol{vA}$ – was clearly expressed already in Ricardo and can be formalized in this generally accepted way:

$$\boldsymbol{v} = \boldsymbol{l} + \boldsymbol{vA} = \boldsymbol{l}(\boldsymbol{I} - \boldsymbol{A})^{-1} = \boldsymbol{lQ} \qquad (5),^{7}$$

where $\boldsymbol{Q}$ is another name for the Leontief inverse (Bródy 1970). The idea of *general interdependence* between all sectors of the economy is already contained in equation (5) but is easier to perceive in equation (6), of the value of the *individual* commodity i. This interdependence is the result of influences that come not only from the production process of i, but from all production processes that currently *coexist* in the *economy as a whole*. It is clearly seen how the value of i,

$$v_i = \boldsymbol{lQ}^{(i)} \qquad (6),$$

is a function of two factors: all the elements of *the vector* of *direct* labour coefficients, and the coefficients of column i of $\boldsymbol{Q}$, which actually involves the technical coefficients of all columns (sectors) of $\boldsymbol{A}$, since, as $\boldsymbol{Q}$ is the inverse of ($\boldsymbol{I}$ – $\boldsymbol{A}$), its definition includes both the reciprocal of the *determinant* of ($\boldsymbol{I}$ – $\boldsymbol{A}$), which involves all the a_{ij}, and the *adjoint* matrix of ($\boldsymbol{I}$ – $\boldsymbol{A}$), which also contains them all.[8]

Regarding direct labour, $\boldsymbol{l}$, we must begin by clearing up the doubts about what abstract labour means and how its quantification is affected by the existence of collective labour (Nagels, 1974). Abstract labour is really the same labour as concrete labour, with the difference that in it we see the labour without paying attention to the specific and concrete nature of each fragment of labour (aimed at the production of one or another particular use value), but rather taking into account what all labours, all the jobs of men and women, have in common as equal

7 It is not necessary to share the theory of value of Marx and Ricardo to admit equation (5), as seen in Tinbergen (1992), who recognizes that Seton (1992) "shows how the 'full' labour-costs of each commodity can be derived by aggregating its direct labour-cost and the indirect labour absorbed in the form of raw materials, fuel, etc." (Tinbergen 1992)

8 Opponents of any "substantialist" interpretation of value fail to see that the average value of a commodity depends on what happens in the production of *all commodities*, so that the substance of value is not something that is individually "put" in the commodity in the form of a physical injection, but rather it is *put* in it as a *representation*, in that commodity, of a certain amount or fraction of total and interdependent social labour that depends on how the latter is distributed or allocated among all sectors of the economy (See Heinrich 2004, pp. 49-50).

labour, since they form a unit in that their society[9] distributes their total labour among different destinations or uses, each individual labour representing only a fraction of the total. Now, the really existing individual labours are not those that Marx (and before him the classics) uses to illustrate simple commodity production (where artisans and small independent producers dominate),[10] but something very different, since later in *Capital* the mode of production of *large-scale industry* is studied above all,[11] where the *collective labour* is preponderant. It might seem that one hour of labour is 60 minutes of a worker's activity. However, for Marx, an hour of labour in the capitalist mode of production is not 60 minutes of an individual worker but 1 minute of labour of 60 workers (in a small firm) or 1 second of labour of 3,600 workers (in a large firm). This is because the typical labourer in this society is what Marx called the *collective labourer*,[12] a human machine that forms a unit in which each individual is nothing more than a piece in a collective body that is who really works. This is what explains why the character of each individual contribution to collective production is given by it.[13]

Regarding the quantification of labour, there are two fundamental aspects. Firstly, concrete labours cannot be added as such but only as abstract labour. Thus, one hour of Robinson's hunting + one hour of Robinson's fishing are *two* hours of Robinson's labouring day. Steedman

9 We can model society in such a way that it can accommodate two different submodels: (1) that of Robinson Crusoe, which we will call the 1 x 4 model (society of 1 person with 4 tasks for example) – a model with only one person may seem strange, but even stranger is a model with only one commodity (see Kurz & Salvadori, 1995), both being however theoretically justified – and (2) the model of a $p \; x \; t$ society (p billions people, say 4 billions working in the real world, and t thousands of tasks). Robinson's labour is the same labour although he divides it into different concrete tasks. Likewise, the labour of each of us, working individuals of the 21st century, is the same even if we divide it into different concrete manifestations of our labour day (for example, a teacher teaches classes, prepares them, gives exams, corrects them, researches, writes articles or books, etc.). The labour of society as a whole is also divided into different concrete activities that are carried out in different jobs, in each of which a specific labour is carried out that expresses in one way or another the total human labour.

10 In the first book of *Capital*, when explaining the functioning of that fictitious society called *simple commodity production* (ideally created to explain the value of commodities as such, before considering the value in the form corresponding to developed capitalist society), Marx refers especially to the figure of the master craftsman ("the" shoemaker, "the" carpenter, "the" tanner, etc.), "the" small farmer (the cattle-breeder...), etc.

11 The society corresponding to the large-scale industry is governed, from the technical point of view, by an "automatic system of machines" where all workers are wage workers (including the managers or executives who are so often seen on the street, he says, demanding employment, every time the economy goes into crisis) and where each capitalist is progressively transformed into a simple participant or shareholder in "capitalist communism", facilitated by the growing development of credit and financial capital, etc.

12 "[...] is not the individual worker but rather a *socially combined labour capacity* that is more and more the *real executor* of the labour process as a whole, and since the different labour capacities which cooperate together to form the productive machine as a whole contribute in very different ways to the direct process by which the commodity, or, more appropriate here, the product, is formed, one working more with his hands, another more with his brain, one as a MANAGER, ENGINEER, or technician, etc., another as an OVERLOOKER, the third directly as a manual worker, or even a mere assistant, more and more of the *functions of labour capacity* are included under the direct concept of *productive labour* [...]" (Marx 1863/1864, p.443-444).

13 Thus, for example, the productivity of labour in the maintenance department of a firm (made up of electricians, plumbers, etc.) is not measured in meters of cable or tubes per hour worked, but in *units* defined exclusively by the nature of the collective labourer of which he or she is a part; for example, if their firm is the "metro" (suburban) of a large city, the unit of the product has dimension "number of travelers per kilometer, per hour", and if their firm is a hotel or a private student residence, the product unit will be "number of rooms rented per day" (or "per quarter", etc.).

(1977), despite being a staunch critic of the Marxian theory of value, clearly admits that this is also the case in developed capitalism.[14] Therefore, we do not intend to "reduce concrete labour to abstract labour", but to standardize abstract labour so that the number of hours measured directly by the clock-machines of each firm —the "factory's working time" (Georgescu-Roegen 1971, p. 245)—, that is, vector $\boldsymbol{l}'$, is adjusted upwards or downwards in each sector (to get $\boldsymbol{l}$) depending on the comparative degree of skills —and intensity— in the labour carried out by the average collective worker in each sector, since differences of skill and intensity between sectors seem much greater than between the different work centers of the same firm or between the different firms in the same sector. It is customary, in the empirical studies done to test the labour theory of value, to adjust the amounts of labour by means of the sector's monetary wage compared to the economy's average wage, for which it is common to use the diagonal matrix $\widehat{\boldsymbol{w}} = (w_1, w_2, \dots, w_n)$, where $w_i = \frac{w_i'}{w'^*}$, that is, the quotient between the per capita monetary wage of the sector i and the average per capita monetary wage of the economy. In this way, we have $\boldsymbol{l} = \boldsymbol{l}'\widehat{\boldsymbol{w}}$.

The different prices

Before defining the different prices, we have to say something about the matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$, that we will use in these definitions. Matrix $\boldsymbol{A} \equiv [a_{ij}]$ is the input-output matrix, where the a_{ij} denote the technical coefficients or quantity of the commodity i required (in the process of productive consumption) to produce a unit of commodity j. Matrix $\boldsymbol{B} \equiv [b_{ij}]$ is in a certain sense *parallel* to $\boldsymbol{A}$, with the difference that the b_{ij} represent the quantity of i required per unit of j (in the process of *improductive* consumption of the workers of sector j), $\boldsymbol{A}^{(+)}$ being the *augmented* matrix, or sum of the two previous ones, $\boldsymbol{A}^{(+)} = \boldsymbol{A} + \boldsymbol{B}$. To calculate $\boldsymbol{B} = \boldsymbol{cl}$, it is usual to assume that the per capita wage (both nominal and real) is uniform, the latter being represented by the column vector $\boldsymbol{c} \equiv [c_i]$, where each $c_i = C_i/L$ is the quantity of commodity i that each worker consumes as part of his real wage, and $l_j = L_j/x_j$, so that, in the case 2 x 2, $\boldsymbol{B} = \begin{pmatrix} b_{11} = c_1 l_1 & b_{12} = c_1 l_2 \\ b_{21} = c_2 l_1 & b_{22} = c_2 l_2 \end{pmatrix}$.

Actual prices

We have already mentioned five different types of price. Compared to hypothetical prices ($\boldsymbol{p}^{(p)}$, $\boldsymbol{p}^{(v)}$ ($\boldsymbol{p}^{(S)}$)), actual prices, $\boldsymbol{p}$, are characterized by the fact that the unit profit they contain is not proportional to either the total capital or the variable capital invested in their production (or constant capital in the case of $\boldsymbol{p}^{(S)}$); that is, in them the rate of profit and the rate of surplus value are not uniform. Compared to production prices (see equation (3)), which have a uniform profit rate, actual prices are defined as

14 "All summations of labour-times are summations of quantities of abstract labour [...] The *very fact* that these different labour-times, expended in a capitalist economy, are added together means that they are treated as abstract labour-time. (One can no more add 7 hours of *concrete* coal-mining labour to 3 hours of *concrete* tea-making labour than one can add 7 apples to 3 oranges.)" (Steedman 1977, p. 19).

$$\boldsymbol{p} = \boldsymbol{pA}^{(+)}(\boldsymbol{I} + \hat{\boldsymbol{r}}) = \boldsymbol{pC}^{(+)} \qquad (2),$$

where $\boldsymbol{C}^{(+)} = \boldsymbol{A}^{(+)}(\boldsymbol{I} + \hat{\boldsymbol{r}})$ and $\boldsymbol{r} = (r_1, r_2, \dots, r_n)$ is the vector of the (generally different) sectoral profit rates.[15]

These long-term "actual" prices differ from *daily* market prices, which fluctuate in the *short term* due to very diverse causes.[16] They are simply the *average purchase prices* that the capitalist faces as something that is given by the market and he takes as a presupposition of his production process; however, they are not "disequilibrium" prices in the usual sense of the term but rather a kind of equilibrium prices *readjusted* to reflect *permanent*, or long-term, deviations from *pure* production prices.[17]

Note that (2) is an eigenequation like (3) and (4), with the singularity that the maximum eigenvalue of $\boldsymbol{C}^{(+)}$ is 1. Now, whereas (3) and (4) allow us to calculate at the same time prices (in both cases), and, depending on the case, the rate of profit in (3) or the rate of surplus value in (4), what happens in (2) is that n different sectoral rates of profit intervene, which are part of $\boldsymbol{C}^{(+)}$. Two things can be done in this regard. It can be assumed, first, that the r_i are already known data, as it is also the decomposition in each sector of the price into cost and profit ($p_i = \boldsymbol{pA}^{(+)(i)} + r_i\boldsymbol{pA}^{(+)(i)}$). But equation (2) also allows us to proceed in two steps:

[1] first we redefine *à la Leontief* all physical units to make all the p_i = 1, and therefore $\boldsymbol{p} = \boldsymbol{e} = (1, 1, \dots, 1)$, or $\boldsymbol{p} = f \cdot \boldsymbol{e}$, where f is any positive real number; in this way, the $\boldsymbol{C}^{(+)'}$ matrix (which is the matrix $\boldsymbol{C}^{(+)}$ for the *redefined* units) would be a stochastic matrix (Woods, 1978, p. 51), obtained as the product of the matrix $\boldsymbol{A}^{(+)}$ (whose columns would consist of summable numerical coefficients) by another matrix, $(\boldsymbol{I} + \hat{\boldsymbol{r}})$, where each r_i could be obtained by $r_i = \frac{(1 - \boldsymbol{eA}^{(+)(i)})}{\boldsymbol{eA}^{(+)(i)}} = (\boldsymbol{eA}^{(+)(i)})^{-1} - 1$;

[2] and, secondly, we obtain the matrix $\boldsymbol{C}^{(+)}$ for the *original* physical units, which is not stochastic despite having a maximum eigenvalue = 1 and, associated to it, a single positive left-hand

15 These prices could alternatively be defined as

$$\boldsymbol{p} = \boldsymbol{pB}^{(+)}(\boldsymbol{I} + \hat{\boldsymbol{s}}) = \boldsymbol{pD}^{(+)} \qquad (2\text{bis})$$

(where it would be $\boldsymbol{D}^{(+)} = \boldsymbol{B}^{(+)}(\boldsymbol{I} + \hat{\boldsymbol{s}})$, and $\boldsymbol{s} = (s_1, s_2, \dots, s_n)$), but we will henceforth avoid duplicating the arguments and equations that we develop for matrix $\boldsymbol{C}^{(+)}$.

16 Other authors do not distinguish between short term and long term in the case of actual prices. F. Seton refers to "actual market prices," and notes that "in the Marxian system the 'prices of production' (defined as cost plus profit at the average rate) are only the *first approximations to actual market prices.*" (Seton 1956/57:149). And H. Grossman mentions "these intermediate degrees that lead from 'value-prices' through 'production prices' to the phenomenon of 'market prices'." (Grossman, 1928/1979, p. 82).

17 These $\boldsymbol{p}$ are similar to, or a generalization of, the "actual/real prices of production" (of which Marx speaks when referring to commercial capital and differential and absolute ground rent: Marx 1894:282, 284, ...); for example, the actual price of a coffee on a terrace in the center of a city *differs* from its price in the suburbs, something that does not happen if the ground rent is not included in the production costs), but they are at the same time more *concrete*, since they include deviations that Marx does not analyze in book III of *Capital* (but whose treatment he promises for future books), such as those due to the presence of *monopolies* or the State, which do not appear in pure theory. Think of the *indirect taxes* and *subsidies* that fall unequally on commodities, causing, for example, tobacco or alcohol to be sold at prices much higher than theoretical production prices and, at the same time, some food products or public transport are sold at prices lower than these.

eigenvector that is the vector of actual prices , $\boldsymbol{p} \neq \boldsymbol{e}$, where each relative price deviates from 1 in the same proportion in which the redefined physical units deviate from the original physical units.[18]

Production prices

Turning to the production prices, we see that their eigenequation includes a uniform profit rate:[19]

$$\boldsymbol{p}^{(p)} = \boldsymbol{p}^{(p)}\boldsymbol{A}^{(+)}(1+r) \qquad (3),$$

where $\boldsymbol{A}^{(+)}$ allows us to directly solve the uniform profit rate from the maximum eigenvalue of $\boldsymbol{A}^{(+)}$, $\lambda_{\boldsymbol{A}^{(+)},1} = (1+r)^{-1}$, with $r = (\lambda_{\boldsymbol{A}^{(+)},1})^{-1} - 1$, and in addition the vector of production prices is given by the positive left-hand eigenvector associated with $(1+r)^{-1}$. Thus, for example, in Sraffa's system for surplus production (Sraffa 1960, p.7),

280 qr. wheat + 12 t. iron ⟶ 575 qr. wheat

120 qr. wheat + 8 t. iron ⟶ 20 t. iron,

matrix $\boldsymbol{A}^{(+)}$ would be $\boldsymbol{A}^{(+)} = \begin{pmatrix} 280/575 & 120/20 \\ 12/575 & 8/20 \end{pmatrix} = \begin{pmatrix} 0.487 & 6 \\ 0.021 & 0.4 \end{pmatrix}$,[20] to which correspond, as expected, a maximum eigenvalue $\lambda_{\boldsymbol{A}^{(+)},1} = 0.8, r = 25\%, \text{and } \boldsymbol{p}^{(p)} = (1, 15)$.

Value prices

As for the value prices, only a few authors consider them as a different entity from values,[21] but apart from being proportional to the latter, these prices are defined by another equation, (4), formally identical to that of the production prices, which, being $\boldsymbol{B}^{(+)} = \boldsymbol{BQ}$, can be written as

$$\boldsymbol{p}^{(v)} = \boldsymbol{p}^{(v)}\boldsymbol{A} + \boldsymbol{p}^{(v)}\boldsymbol{B}(1+s) = \boldsymbol{p}^{(v)}\boldsymbol{BQ}(1+s) = \boldsymbol{p}^{\boldsymbol{v}}\boldsymbol{B}^{(+)}(1+s) \qquad (4).$$

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18 Let's look at a 2 x 2 example: if we have $\boldsymbol{A}^{(+)} = \begin{pmatrix} 0.495 & 0.45 \\ 0.32 & 0.5 \end{pmatrix}$, then matrix $\boldsymbol{C}^{(+)\prime} = \begin{pmatrix} 0.607 & 0.474 \\ 0.393 & 0.526 \end{pmatrix}$ is stochastic if r_1 = 22.7% and r_2 = 5.26%, with a price vector $\boldsymbol{p} = f \cdot \boldsymbol{e} = (0.707, 0.707)$. This is due to the redefinition of the physical unit of sector 1 (wheat) to a "wamount" = half a quintal, and that of sector 2 (iron) to an "imount" = 1/10 of a ton; however, with the original units the relative price of iron in terms of wheat is 10/2 = 5 instead of 1, and matrix $\boldsymbol{C}^{(+)} = \begin{pmatrix} 0.607 & 2.368 \\ 0.079 & 0.526 \end{pmatrix}$ still has a maximum eigenvalue = 1 and a price vector $\boldsymbol{p} = (0.196, 0.981)$ with a relative price = 5 (which is also the quotient of the two ratios between the original and the redefined physical units in both sectors, $\frac{1/2}{1/10} = 5$). (The determinants of $\boldsymbol{C}^{(+)}$and $\boldsymbol{C}^{(+)\prime}$ coincide as well.)

19 "There is no doubt, on the other hand, that aside from unessential, incidental and mutually compensating distinctions, differences in the average rate of profit in the various branches of industry do not exist in reality, and could not exist without abolishing the entire system of capitalist production." (Marx 1894, p. 152).

20 We know that it is matrix $\boldsymbol{A}^{(+)}$, not $\boldsymbol{A}$, since the quantities in Sraffa's systems include the "sustenance for those who work." (Sraffa 1960, p.3)

21 They are generally understood only as values expressed in monetary terms, sometimes overlooking the fact that they contain a (uniform) rate of surplus value that is not part of the definition of the values, since, as said, these are logically prior and independent of the proportion in which labour is paid or not. (At most, it could be interpreted that in the values s is uniform and equal to 0.)

In this case, it is the uniform rate of surplus value, s, that appears as the reciprocal of the maximum eigenvalue of $\boldsymbol{B}^{(+)}$ minus 1, that is, $s = (\lambda_{\boldsymbol{B}^{(+)},1})^{-1} - 1$, where $\lambda_{\boldsymbol{B}^{(+)},1} = (1+s)^{-1}$; and the value price vector is the positive left-hand eigenvector associated with $(1+s)^{-1}$. Using again Sraffa's example and adding two assumptions necessary for this purpose such as $\boldsymbol{l} = (0.1 \quad 0.4)$ and $\boldsymbol{c} = \begin{pmatrix} 0.05 \\ 0.1 \end{pmatrix}$, we would have $\boldsymbol{B} = \boldsymbol{cl} = \begin{pmatrix} 0.005 & 0.02 \\ 0.01 & 0.04 \end{pmatrix}$ and $\boldsymbol{B}^{(+)} = \boldsymbol{BQ} = \begin{pmatrix} 0.013 & 0.151 \\ 0.026 & 0.302 \end{pmatrix}$, which results in $\lambda_{\boldsymbol{B}^{(+)},1} = 0.315$ and $s = 218\%$. As for value prices, they turn out to be $\boldsymbol{p}^{v} = (1, 11.8)$ which differ from $\boldsymbol{p}^{(p)} = (1, 15)$ and coincide, as we will see, with the values.

On the other hand, since $\boldsymbol{B}^{(+)} = \boldsymbol{BQ} = \boldsymbol{clQ} = \boldsymbol{cv}$, it is possible to write (4) as $\boldsymbol{p}^{(v)} = (1+s)\boldsymbol{p}^{(v)}\boldsymbol{cv}$, which shows that value prices are proportional to values by a proportionality factor given by the scalar $m = (1+s)\boldsymbol{p}^{(v)}\boldsymbol{c}$, where $\boldsymbol{p}^{(v)}\boldsymbol{c}$ is the uniform monetary wage per capita (or unit price of the labour force) and m is the value added per capita, so that

$$\boldsymbol{p}^{(v)} = m \cdot \boldsymbol{v} \qquad (7).$$

This proportionality with respect to values is verified by observing that the relative labour value is, in effect, 11.8, since $\boldsymbol{l} = (0.1 \quad 0.4)$, $\boldsymbol{Q} = \begin{pmatrix} 2.40 & 22.43 \\ 0.04 & 1.94 \end{pmatrix}$, $\boldsymbol{v} = \boldsymbol{lQ} = (0.256 \quad 3.021)$ and $\frac{v_2}{v_1} = 11.8$.

Note, furthermore, that $(1+s)^{-1}$ is what Ricardo called "relative wage"[22] (today it is called *wage share*) and Marx "value of labour power", which is the value of the means of consumption (subsistence) of the workers, the scalar $\boldsymbol{vc}$. In effect, given that $\boldsymbol{vc} + (1 - \boldsymbol{vc}) = 1$, and therefore $s = \frac{1-\boldsymbol{vc}}{\boldsymbol{vc}}$, we have that the relative wage is $\varpi = \frac{1}{1+s} = \frac{1}{1+\frac{1-\boldsymbol{vc}}{\boldsymbol{vc}}} = \frac{1}{\frac{1}{\boldsymbol{vc}}} = \boldsymbol{vc}$.

Sraffian prices

Let us say, in passing, that it is possible to define other hypothetical prices by applying different distribution rules to obtain the unit profit. The best known of those prices is the vector of Sraffian prices, $\boldsymbol{p}^{(S)}$, which are defined by applying a uniform profit factor *only* to the unit material (= non-wage) cost – since Sraffa considers that wages are not part of the capital investment but a fraction of the net income generated in each period – and adding to this the wage bill per unit of product:

$$\boldsymbol{p}^{(S)} = \boldsymbol{p}^{(S)}\boldsymbol{A}\left(1 + r^{(S)}\right) + w\boldsymbol{l} \qquad (8),$$

where the superscript $^{(S)}$ alludes to Sraffa, $r^{(S)} \neq r$ is the rate of profit that corresponds to the $\boldsymbol{p}^{(S)}$, and w is the monetary wage or price of the labour force ($w = \boldsymbol{p}^{(S)}\boldsymbol{c}$). Since $w\boldsymbol{l} = \boldsymbol{p}^{(S)}\boldsymbol{cl} = \boldsymbol{p}^{(S)}\boldsymbol{B}$, we can also write (8) in another way:

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22 Ricardo (1821, p.14) spoke of the "real value of wages", but Marx translates it by pointing out that "the concept of *relative* wages is one of Ricardo's greatest contributions" (Marx 1861/63, III:33). This wage share is the part or fraction that wages represent in the value added (or national income), which in traditional Marxian notation is $\frac{V}{V+PV} = \frac{V/V}{V/V+PV/V} = \frac{1}{1+s}$, where V is the variable capital and PV is the surplus-value.

$$\boldsymbol{p}^{(S)} = \boldsymbol{p}^{(S)}\boldsymbol{A}^{(+)} + r^{(S)}\boldsymbol{p}^{(S)}\boldsymbol{A} = r^{(S)}p^{(S)}\boldsymbol{A}\boldsymbol{Q}^{(+)} = r^{(S)}\boldsymbol{p}^{(S)}\boldsymbol{H}^{(+)} \qquad (9),$$

where, by analogy with Bródy's $\boldsymbol{Q}$, we call $\boldsymbol{Q}^{(+)}$ the augmented inverse $(\boldsymbol{I} - \boldsymbol{A}^{(+)})^{-1}$ and define $\boldsymbol{H}^{(+)} = \boldsymbol{A}\boldsymbol{Q}^{(+)}$. The eigenequation (9) of the Sraffian prices could be accepted as part of our set of theoretical and hypothetical prices, the vector of these prices being the positive left-hand eigenvector associated with the maximum eigenvalue of matrix $\boldsymbol{H}^{(+)}$, which is $(r^{(S)})^{-1}$; and the uniform (Sraffian) rate of profit would be the reciprocal of that eigenvalue, $r^{(S)}$.[23]

The Success of Marx's Transformation

Let's begin by distinguishing three different things. (A) What is known as "Marx's *Transformation*"[24] is, according to Marx's critics, a failed or logically defective transformation of our equation (1) into equation (3), where both are expressed in different units. (B) On the other hand, this transformation is for Marx, at first glance, a transformation of equation (4) into (3), both expressed in money. (C) Finally, for us (as we will develop in section IV), all the fundamental equations, from (1) to (4), along with the direct labour coefficients, are involved in our *extended transformation*. But before addressing the latter – where *the* (usual) single Transformation appears only as one among several, in a series of linear transformations operated by matrices that represent "the logical process of solving a system of equations" (Pasinetti 1981a, p. 149) we will compare with ours some of the modern interpretations of Marx's algorithm that seem most relevant from our own point of view. We do not intend here to do any history or general review of the debates about the Transformation (but see Dostaler, 1978; Desai, 1988; Foley, 2000; Mohun & Veneziani 2017), so that we will limit ourselves to contrasting four different points of view on Marx's *solution* that interest us especially: first, the one that considers it an unmitigated *error*, which is the dominant point of view; second, the one that considers Marx's solution only a *first step* that must be completed through an *iterative* procedure to reach the correct solution; thirth, the one who *denies* the need to make any transformation, insisting that Marx analyzed the issue correctly; and, compared to all, fourth, our own interpretation of Marx's equations.

First, most critics of the labour theory of value make two claims (although not in our own words). Firstly, they admit that the correct equations for production prices and value prices are

23 We can see in equation (8) what happens in two cases that are of particular interest to Sraffa. First, if $r^{(S)} = 0$ and w = 1, we obtain that these prices coincide with values, $\boldsymbol{p}^{(S)} = \boldsymbol{l}\boldsymbol{Q} = \boldsymbol{v}$; and, since the rate of surplus value is a multiple of r ($s = rq$, with q = the value composition of capital + 1), we obtain that, when $r = s = 0$, equation (4) becomes $\boldsymbol{p}^{(v)} = \boldsymbol{p}^{(v)}\boldsymbol{A} + \boldsymbol{p}^{(v)}\boldsymbol{B} = \boldsymbol{p}^{(v)}\boldsymbol{A}^{(+)}$ and (9) becomes $\boldsymbol{p}^{(S)} = \boldsymbol{p}^{(S)}\boldsymbol{A}^{(+)}$, with which we would have $\boldsymbol{p}^{(S)} = \boldsymbol{p}^{(v)}$ since $\boldsymbol{A}^{(+)}$ is the same in both cases (with both members expressed in money, not as it occurs in the "equality" $\boldsymbol{p}^{(S)} = \boldsymbol{v}$). Second, if w = 0 (and therefore $w\boldsymbol{l} = \boldsymbol{p}^{(S)}\boldsymbol{B} = 0$), then $r^{(S)} = R^{(S)}$ (Sraffa's maximum profit rate) and $r = R^{(m)}$ (maximum profit rate corresponding to *r*), so that equation (8) becomes $\boldsymbol{p}^{(S)} = \boldsymbol{p}^{(S)}\boldsymbol{A}(1 + R^{(S)}) = R^{(S)}\boldsymbol{p}^{(S)}\boldsymbol{H}$, and equation (3) becomes $\boldsymbol{p}^{(p)} = \boldsymbol{p}^{(p)}\boldsymbol{A}(1 + R^{(m)}) = R^{(m)}\boldsymbol{p}^{(p)}\boldsymbol{H}$, with $\boldsymbol{H} = \boldsymbol{A}\boldsymbol{Q}$ and $\boldsymbol{p}^{(S)} = \boldsymbol{p}^{(p)}$ if $R^{(m)} = R^{(S)}$. That is, $\boldsymbol{p}^{(S)}$ moves between $\boldsymbol{p}^{(v)}$ and $\boldsymbol{p}^{(p)}$, depending on the values we give to the distributive variables, w and r.

24 Literature calls it "transformation of values into prices," and Marx titles chapter 9 of book III of *Capital* "Transformation of the values of commodities into prices of production."

(3) and (4), although it is obvious that they have much more interest in (3) than in (4). Secondly, they think that, although equation (4) correctly reflects Marx's thought, the same does not happen with his production prices, which are $\boldsymbol{p}^{(p)(M)} \neq \boldsymbol{p}^{(p)}$, and his rate of profit, $r^{(M)} \neq r$, which, according to them, are both incorrect since Marx would be using the inconsistent equation (10) (or perhaps (10') instead of equation (3).

$$\boldsymbol{p}^{(p)(M)} = \boldsymbol{p}^{(v)}\boldsymbol{A}^{(+)} + \boldsymbol{r}^{(M)}\boldsymbol{p}^{(v)}\boldsymbol{A}^{(+)} = \boldsymbol{p}^{(v)}\boldsymbol{A}^{(+)}(1 + r^{(M)}) \qquad (10).$$

$$[\boldsymbol{p}^{(p)(M)} = \boldsymbol{p}^{(v)}\boldsymbol{A}^{(+)} + \boldsymbol{r}^{(M)}p^{(p)(M)}\boldsymbol{A}^{(+)} = \boldsymbol{p}^{(v)}\boldsymbol{A}^{(+)}(\boldsymbol{I} - r^{(M)}\boldsymbol{A}^{(+)})^{-1} \quad (10').]^{25}$$

According to these critics, equation (10) presents two problems: a) first, it "forgets" to transform, or Marx does not know how to do so, the price of inputs (and that is why he uses $\boldsymbol{p}^{(v)}$ to value the unit costs, instead of using $\boldsymbol{p}^{(p)}$); b) second, he uses or invents a profit rate, $r^{(M)} = \frac{s\boldsymbol{p}^{(v)}\boldsymbol{B}\boldsymbol{x}}{\boldsymbol{p}^{(v)}\boldsymbol{A}^{(+)}\boldsymbol{x}} = \frac{\boldsymbol{p}^{(v)}(\boldsymbol{I}-\boldsymbol{A}^{(+)})\boldsymbol{x}}{\boldsymbol{p}^{(v)}\boldsymbol{A}^{(+)}\boldsymbol{x}}$, which is not correct, since he should have used $r = \frac{\boldsymbol{p}^{(p)}(\boldsymbol{I}-\boldsymbol{A}^{(+)})\boldsymbol{x}}{\boldsymbol{p}^{(p)}\boldsymbol{A}^{(+)}\boldsymbol{x}}$. (See below equations (19) and (20))

However, it is the critics who are wrong, since (10) does not represent Marx's true thought; in fact, we will see later that Marx's *initial* equation is (17), $\boldsymbol{p}^{(p)(M)} = \boldsymbol{p}\boldsymbol{A}^{(+)} + r^{(M)}\boldsymbol{p}\boldsymbol{A}^{(+)}$, an equation that uses *actual prices in the right member*, and finally becomes the correct one, the (3), but not by the equally legitimate procedure of finding a specific iteration algorithm that transforms (10) into (3), but by the simple means of taking two successive logical steps (i.e. the usual method of *successive approximations* in only two steps).

Second, the iterative method, which was used in the 1970s by Bródy (1970), Okishio (1972), Shaikh (1973; 1977), or Morishima (1973), is summarized by Shaikh who says that equation (10) does not reflect an error in the transformation but a *provisional result* when defining production prices. In the opinion of these authors – who agree with us that Marx's approach is correct but disagree on a point that we mention below –, equation (10) can be interpreted as a useful starting point to finally obtain (3), so that Marx's solution would be only a first approximation to the correct solution, which can be completed by an iterative procedure that was anticipated at the beginning of the 20th century by G. von Charasoff 1910 (see Screpanti & Zamagni 1995, pp. 286-287). With this, Shaikh is saying two things that he shares with the other authors mentioned, one correct and the other not:

1st) he correctly states that (10) converges to (3) if the inputs initially valued at $\boldsymbol{p}^{(v)}$ (which are the starting point of the iteration, $\boldsymbol{p}^{(p)^{<0>}} = \boldsymbol{p}^{(v)}$) are replaced by inputs valued successively at $\boldsymbol{p}^{(p)^{<t_1>}}, \boldsymbol{p}^{(p)^{<t_2>}}, \ldots, \boldsymbol{p}^{(p)^{<t_{k-1}>}}, \ldots, \boldsymbol{p}^{(p)^{<\infty>}}$, which result from the revaluations necessary to apply, at each step, the profit rate resulting from the previous step..., until finally reaching $\boldsymbol{p}^{(p)^{<t_k>}} = \boldsymbol{p}^{(p)}$).

2nd) but at the same time he assumes that equation (10) is the initial equation in Marx, which in our opinion is not true.

25 These equations are not eigenequations in any case. Eq. (10) reflects better than (10') the idea, which critics attribute to Marx, that he mistakenly uses value prices, both for unit costs and unit profits, in addition to using an erroneous $r^{(M)}$ rate based on value prices instead of production prices (see below). An alternative, less likely, representation would be equation (10'), which at the end of its right member makes production prices disappear.

This iterative method has been criticized by various authors (see Hodgson, 1982, p.97; Kurz, 1979, p. 55; Lippi, 1979, pp. 67-69)[26] but the critics make a mistake: it is true that *the same iteration algorithm* can be used by starting from numbers different from those of the vector of values (including Hodgson's: those corresponding to the number of letters in the name of each commodity in Serbo-Croatian),[27] thereby reaching a certain vector of production prices, but the criticism is not correct because it is not true that *the same results* are reached regardless of the starting data, as can be verified by comparing the results obtained in the example used by Shaikh (1977) with those obtained, with the same method, starting from other numbers.[28] On the other hand, although the iterative *procedure* is the same, the specific algorithms proposed by various authors differ with respect to Marx's famous double identity (or equality). the most common has been to obtain only the equality total price = total value is obtained but not total profit = total surplus value,[29] although in Morishima and Catephores 1978:167-8 the opposite occurs.[30]

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26 Basically, it is argued that "the iterative process can be initiated from *any* price vector, so that the vector of labour values carries out the simple role of an *arbitrary* vector of exchange values." (Screpanti & Zamagni, 1995, p. 287).

27 According to Hodgson (1982), "...there are an *infinite number* of 'first approximations' which can be used, in the same iterative process, to derive de same results. We could, conceivably, start with our old friend 'embodied energy', or even the number of letters in the name of the commodity when the name is translated into Serbo-Croat. Subject to certain conditions, all these 'first approximations' will led us to the same end point. What matters in the iterative process is not the starting point but the process itself, and embodied labour plays no part in Shaikh's process. There is no evidence to show that this is anything more than a *calculation*; embodied labour values play no apparent role in the real capitalist world." (p. 97).

28 Shaikh (1979) starts from a fixed rate of surplus value of s = 2/3 and shows that Marx's initial rate of profit ($r^{(0)}$ = 29.6%) becomes the correct rate, $r^{(9)} = r$ = 25%, starting from the 9th iteration; with this, the relative prices go from (1 : 0.7 : 0.44) to (1 : 0.67 : 0.42), the mass of profit goes from $400 to $350 and all this while the total price remains constant at $1750. We have compared these results changing Shaikh's initial data, and we verified that the same iteration algorithm does not lead us to the same price vector as before, contrary to what critics say, but to other different vectors. For example, with Shaikh's data but changing the rate of surplus value to s = 1, the figures change and the rate of profit goes from an initial 44.4% to a final 37.1%, the profit from $600 to $528, and the relative prices from (1 : 0.7 : 0.44) to (1 : 0.66 : 0.41), keeping the total price constant at $1950. And the same thing happens by changing the organic compositions of capital and maintaining s = 2/3, or using the figures of the author's ID to fill in the constant and variable capital data in the three sectors.

29 For example, Bródy 1970:90 proposed an algorithm for the general case (with fixed capital) that implies, as in Shaikh, the equality total price = total value but not that of total profit = total surplus value; his algorithm is

$$\boldsymbol{p}_{j+1} = \boldsymbol{p}_j\mathbb{A} + \frac{\boldsymbol{p}_j(\boldsymbol{I}-\mathbb{A})\boldsymbol{x}}{\boldsymbol{p}_j\boldsymbol{K}\boldsymbol{x}}\boldsymbol{p}_j\mathbb{K} \quad (11),$$

where $\mathbb{A}$ is the *full matrix*, $\mathbb{K}$ is the *stock* matrix (fixed capital plus the stock of circulating capital obtained taking into account the turnover rates corresponding to each element of circulating capital) and where, postmultiplying both members of (11) by $\boldsymbol{x}$, we have $\boldsymbol{p}_{j+1}\boldsymbol{x} = \boldsymbol{p}_j\boldsymbol{x}$, and also

$$\boldsymbol{p}^{(p)}\boldsymbol{x} = \boldsymbol{p}^{(v)}\boldsymbol{x} \quad (12)$$

$$\boldsymbol{p}^{(p)}(\boldsymbol{I}-\mathbb{A})\boldsymbol{x} \neq \boldsymbol{p}^{(v)}(\boldsymbol{I}-\mathbb{A})\boldsymbol{x} \quad (13).$$

30 Finally, the "inverse iteration" method should be explored, to confirm if it is possible to operate in the opposite direction, going from the correct production prices to the $\boldsymbol{p}^{(p)(M)}$. For example, following Shaikh, who uses an algorithm based on the powers of $\boldsymbol{A}^{(+)}$ (see equation V.7 in the mathematical appendix of Shaikh 1977),

$$\boldsymbol{p}^{(v)(T)} = \boldsymbol{p}^{(v)(1)}\boldsymbol{A}^{(+)^{(T)}}\frac{\boldsymbol{p}^{(v)(1)}\cdot\boldsymbol{1}}{\boldsymbol{p}^{(v)(1)}\boldsymbol{A}^{(+)^T}\cdot\boldsymbol{1}} \quad (14)$$

(where vector $\boldsymbol{1}$ is our $\boldsymbol{e}$), if we assume that 9 iterations are needed for convergence to the predetermined level of accuracy, equation (14) would be $\boldsymbol{p}^{(v)(9)} = \boldsymbol{p}^{(v)(1)}(\boldsymbol{A}^{(+)})^9 \cdot \frac{\boldsymbol{p}^{(v)(1)}\cdot\boldsymbol{1}}{\boldsymbol{p}^{(v)(1)}(\boldsymbol{A}^{(+)})^9\cdot\boldsymbol{1}}$, so its inverse would be $\boldsymbol{p}^{(v)(1)} = \boldsymbol{p}^{(v)(9)}(\boldsymbol{A}^{(+)})^{-9} \cdot \frac{\boldsymbol{p}^{(v)(1)}(\boldsymbol{A}^{(+)})^9\cdot\boldsymbol{1}}{\boldsymbol{p}^{(v)(1)}\cdot\boldsymbol{1}}$.

Thirth, recently, Moseley (2016) has made a double statement on Marx's transformation that has had some impact. Firstly, he rejects as a principle, following Mattick (1969),[31] the *need* to use *unit* values and unit prices, preferring their expression only as *totals*, this being difficult to understand[32] since in an economy of *n* sectors – where the "unit" for Marx, says Moseley (1960, p.35), is the sectoral output, X_i,[33] with the total output of the economy (in monetary terms) equal to the scalar $X = \sum_{i=1}^{i=n} X_i = X_1 + X_2 + ... + X_n$ – nothing prevents us from also stating that each X_i is the product of the physical quantity of the individual commodity i[34] times its unit price, $X_i = p_i x_i$, and, therefore, $X = \sum_{i=1}^{i=n} p_i\, x_i = \boldsymbol{px}$. Secondly, in Moseley's opinion, the price of inputs *is not transformed* nor should it be transformed because the definition of value that Marx gives in book III of *Capital* is different from the definition that he uses in books I and II, since the new definition corresponds to developed capitalism, where the commodity is not simple but a "commodity as a product of capital";[35] as a consequence of this, Moseley thinks that inputs should be valued at production prices and not at values, generalizing something that Marx points out about the commodity labour power: that "the average price of labour, i.e., the value of labour power, is determined by the production price of the necessary means of subsistence." (Marx 1894, p.855).

Although Moseley (2016) does not use a linear model, we can *translate* his idea of Marx's monetary values (value prices) as a different equation[36] (in two versions) of our equation (3); using the superscript $^{(Mos)}$ to allude to Moseley's interpretation of Marx, the two variants are:

$$\boldsymbol{p}^{(v)(Mos)} = \boldsymbol{p}^{(p)(Mos)}\boldsymbol{A}^{(+)} + s^{(Mos)}\boldsymbol{p}^{(p)(Mos)}\boldsymbol{B} \qquad \text{(16a), or}$$

$$\boldsymbol{p}^{(v)(Mos)} = \boldsymbol{p}^{(p)(Mos)}\boldsymbol{A}^{(+)} + s^{(Mos)}\boldsymbol{p}^{(v)(Mos)}\boldsymbol{B} \qquad \text{(16b),}$$

where the only difference between the two is the type of price at which the commodities that make up the surplus value expressed in money are valued (second addend of the right member). Moseley cannot comment here on which of the two equations best reflects his thesis (in fact,

31 "The problem of individual price determination was of no real interest to Marx [...] Beyond the statement that price relations presuppose value relations [...] no need exists for a 'Marxian theory of prices' [...] The deviation of price from value [...] is not such that value is discernable in price. Aside from being a practical impossibility, it would be a superfluous undertaking" (Mattick 1969, ch. 4).

32 In fact, although it is obvious that Marx is interested in the macroeconomic questions of whether both $\boldsymbol{p}^{(v)}\boldsymbol{x} = \boldsymbol{p}^{(p)}\boldsymbol{x}$ and $\boldsymbol{p}^{(v)}(\boldsymbol{I} - \boldsymbol{A}^{(+)})\boldsymbol{x} = \boldsymbol{p}^{(p)}(\boldsymbol{I} - \boldsymbol{A}^{(+)})\boldsymbol{x}$ occur simultaneously or not, he is equally or more interested in the *microeconomic* point of view, since to study *competition* it is necessary to know the *unit prices* (their level and evolution) of all commodities. For Marx, giving unit prices up also means the inability to study competition.

33 Note that, as an exception, capital letters are used here to denote scalars.

34 Marx begin *Capital* with these words: "The wealth of those societies in which the capitalist mode of production prevails, presents itself as 'an immense accumulation of commodities', *its unit being a single commodity*." (Marx 1867, p. 45; emphasis added).

35 Indeed, "The *commodity* as it emerges from capitalist production is determined differently from the commodity as it was at the starting point, as the element, the presupposition, of capitalist production." (Marx 1863/1864, p. 362-363; emphasis added).

36 Our interpretation of Moseley, which he probably opposes, "translates" his macroeconomic equations (see chapter II of Moseley 2016) into the simplest classic multisector linear equations (systems of equations), where each sector corresponds to a single type of commodity.

it is something that he does not even consider, given that the second addend of the right member is for him simply "S", the "surplus value" without further ado), but it should not be overlooked that there is a problem with each of the two options. If, both for him as for all authors, it is decisive to specify the type of price at which inputs (costs) are valued, which in his opinion are not value prices but production prices, it should be equally important to all to specify with what type of prices the surplus value (which is the surplus labour expressed in money) is evaluated. If they are the same prices with which he evaluates wages (the paid part of labour), that is, the $\boldsymbol{p}^{(p)}$, it would result that all the components of values or value prices would be valued at production prices (equation 16a); and if they are the $\boldsymbol{p}^{(v)}$, it would be surprising that each of the two parts into which direct labour is decomposed (the paid and unpaid parts) is valued at different prices (equation 16b).[37] In any case, both problems are limited to the scope of value prices since there is no reason to believe that Moseley calls into question the standard definition of production prices, which would mean that our equation (3) of production prices is valid for him.

A Different Reading of Marx's Transformation

Finally, we will propose our own reading of Marx compared to the previous ones. When analyzing the Marxian T*ransformation*, most scholars exclusively analyze section 2 of book III of *Capital* (chapters 8 to 12) and ignore the first section. In the second section, Marx analyzes "The conversion of profit into average profit", and in chapter 9 specifically (where his famous transformation tables are found) he studies the "Formation of a general rate of profit (average rate of profit) and transformation of the values of commodities into prices of production" (here, as in many other passages of *Capital*, Marx calls "values" the monetary expression of these, the $\boldsymbol{p}^{(v)}$). But it is often forgotten that section 1 of book III deals with "The conversion of surplus value into profit and of the rate of surplus value into the rate of profit", and therefore we believe that this section must play an important role in the analysis of the Transformation and indeed plays a decisive role in our interpretation. In fact, following Marx's indications and explanations in these two sections, we are going to obtain some "provisional" or "first" equations from Marx (hence the superscript $^{(M1)}$) for production prices (17) and value prices (18), which subsequently become his "last" or "final" equations (hence the superscript $^{(M2)}$), which exactly match the correct equations (3) and (4). Since in his time it was impossible to count on theorems and mathematical resources that did not appear until the 20th century, Marx could not use equations like ours but he was aware that his "provisional" equations (those implicit in his Tables) did not exactly represent his true thought, and that is why he speaks of the "possibility of an error" since he senses that there must be a way (then out of his reach) to express his idea.

According to our interpretation of *Capital*, the transformation process as a whole is analyzed by Marx in *three successive steps*:

37 However, if (16b) is rewritten as $\boldsymbol{p}^{(v)(Mos)} = \boldsymbol{p}^{(p)(Mos)}\boldsymbol{A}^{(+)}(\boldsymbol{I} - s^{(Mos)}\boldsymbol{B})^{-1}$, the $\boldsymbol{p}^{(v)}$ disappear from the right member.

1st) firstly, Marx literally says that the *rate* of surplus value *is transformed* into the rate of profit: $s \rightarrow r$;

2nd) secondly,[38] it explicitly states that (the *mass* of) surplus value is transformed into (the mass of) profit, for which it uses, in our interpretation, actual prices: $s\boldsymbol{pB} \rightarrow r\boldsymbol{pA}^{(+)}$ (this is, for the moment, the transformation of a *part of the price*, the part that exceeds the cost price);

3º) and, finally, in chapter 9 he transforms the *full* value price into the *full* price of production: $\left[\boldsymbol{pA}^{(+)} + s\boldsymbol{pB}\right] = \boldsymbol{p}^{(v)} \quad \rightarrow \quad \left[\boldsymbol{pA}^{(+)} + r\boldsymbol{pA}^{(+)}\right] = \boldsymbol{p}^{(p)}$.

It is the taking into consideration (*for the first time*) of these three steps as a whole that leads to our thesis that, in Marx, what at the beginning (with $^{(M1)}$) *appears* to be an *erroneous* transformation of $\boldsymbol{p}^{(v)(M1)}$ into $\boldsymbol{p}^{(p)(M1)}$ (passage from equation (18) to (17). see below) finally appears as a transformation of $\boldsymbol{p}^{(v)(M2)}$ into $\boldsymbol{p}^{(p)(M2)}$, which is nothing more than the correct transformation of $\boldsymbol{p}^{(v)}$ into $\boldsymbol{p}^{(p)}$, that is, from (4) to (3) (once (17) has become (3), and (18) has become (4). see below). Let us, then, begin our modeling by writing Marx's provisional equations (or "attributing" them to him).[39]

$$\boldsymbol{p}^{(p)(M1)} = \boldsymbol{pA}^{(+)} + r^{(M1)}\boldsymbol{pA}^{(+)} = \boldsymbol{pA}^{(+)}(1 + r^{(M1)}) \qquad (17)$$

$$\boldsymbol{p}^{(v)(M1)} = \boldsymbol{pA}^{(+)} + s^{(M1)}\boldsymbol{pB} = \boldsymbol{pA} + \boldsymbol{pB}(1 + s^{(M1)}) \qquad (18),$$

where the decisive point is that the two addends of the right side of both equations *are valued at long term actual prices*, $\boldsymbol{p}$, which, as we know, contain different (non-uniform) rates of profit and rates of surplus value, which makes them different from $\boldsymbol{p}^{(p)}$ and $\boldsymbol{p}^{(v)}$. Since what Marx intends is to explain how value prices are transformed into production prices,[40] it is not surprising that he does not initially give importance to actual *purchase* prices, which are given to the capitalist *from outside* (as data from the *market*, from *circulation*), while the relevant study at this point is for him the production process and the prices that *result* from it.[41] This means that each capitalist or firm behaves here, as conventional microeconomics says, as a *price-taker*, when what Marx is interested in is the *production* process, where each of the *producers* strives to improve its technique and competitiveness, both inside and outside the sector, against rivals

38 This is the proper order for his first two steps, since Marx states that "The transformation of surplus value into profit must be deduced from the transformation of the rate of surplus value into the rate of profit, not vice versa." (Marx 1894:47)

39 This type of "attribution" (undoubtedly an anachronism) is not our invention but something perfectly generalized. For example, Morishima and Catephores, referring to the iteration algorithm that they attribute to Marx, state that "this is the formula which Marx used to transform values into prices." (Morishima & Catephores, 1978, p. 164). Better known is that Marx never wrote anything like the equation of value as vertically integrated labour, $\boldsymbol{v} = \boldsymbol{lQ}$, and everyone has accepted it, for more than half a century, as an adequate expression of his thought.

40 Classical economists and Marx's contemporaries, such as Rodbertus, etc., did not distinguish well between $\boldsymbol{p}^{(p)}$ and $\boldsymbol{p}^{(v)}$ or between *s* and *r*, which is why Marx proposes himself as a fundamental objective of book III of *Capital* to study "the unequal rates of profit derived from the same rate of surplus value" (Marx 1894:150).

41 When investigating spinning as a valorization process, says Marx, "We have no need at present to investigate the value of this cotton, for our capitalist has, we will assume, bought it at its full value, say of ten shillings. In this price the labour required for the production of the cotton is already expressed in terms of the average labour of society." (Marx 1867, p. 197)

that, like everyone else, want to become *price-setters* in order to set prices that allow them to increase their market share at the expense of the others (Guerrero, 1995; Guerrero, 2003; Shaikh 2016; Tsoulfidis & Tsaliki 2005). Each capitalist *takes actual* purchase prices (in which a certain "amount of the average labour of society" is represented) but those prices *are not yet* (are not determined as) *either value prices or production prices*; they are simply "*prices*", the purchase prices that define the capitalist's *cost price*,[42] which in equations (17) and (18) still appears as $\boldsymbol{pA}^{(+)}$. The *selling* price,[43] however, is formed by the cost price plus a surplus, and this *unit surplus* (without additional determination) can be compared either with the *total* unit cost price (unit material cost + unit wage cost), to obtain the rate of profit that we call $r^{(M1)}$ (which defines production prices), or with the wage cost only, to obtain the rate of surplus value that we call $s^{(M1)}$ (which defines value prices). As Marx says,

> "The rate of surplus value measured against the variable capital is called rate of surplus value. The rate of surplus value measured against the total capital is called rate of profit. These are two different measurements of *the same entity* [...] [but] it is altogether erroneous, as a study of the Ricardian school shows, to try to identify the laws of the rate of profit with the laws of the rate of surplus value, or vice versa." (Marx 1894:46, 49; emphasis added)[44]

It is very important to highlight that all of the above is reflected in chapter 9 of book III *before Marx defines the price of production*, and it is also important to highlight that the expression "price of production" does not appear in that chapter until after the three tables, since what Marx has

42 What interests Marx above all is "the excess of the product's *value* over its cost *price*" (Marx 1894:155; emphasis added), and in no case is it said that the advanced capital or the cost price are valued at prices of value or at production prices.

43 What initially leads Marx to distinguish the *purchase* price from the *sale* price is the fact that for each *individual* capitalist, buying and selling occur at different moments in time, but also the fact that the commodities each firm sells are different commodities from those that it buys. Subsequently, in the "final" equations, both value prices and production prices are considered from the point of view of *capital as a whole* (and not of the individual capitalist), and therefore the purchase and sale prices are equalized in the *current average social* production process, where the *average* conditions are given for all companies.

44 In our opinion, Marx's main objective in developing the Tables of Transformation is to clarify the relationship between the rate of surplus value and the rate of profit, showing that, if the former is uniform, the rate of profit cannot also be uniform, and if the rate of profit is uniform, there cannot be a uniform rate of surplus value at the same time. This does not contradict the idea that in Marx a tendency to equalize the rate of profit and a tendency to equalize the rate of surplus value operate simultaneously. Authors such as Cogliano, Foley and Shaikh have pointed out that this idea of Marx comes from Smith, who wrote:

"The whole of the advantages and disadvantages of the different employments of labour and stock must, in the same neighbourhood, be either *perfectly equal or continually tending to equality*. If in the same neighbourhood, there was any employment evidently either more or less advantageous than the rest, so many people would crowd into it in the one case, and so many would desert it in the other, that its advantages would soon return to the level of other employments." (Smith, 1776/2012, p. 104; emphasis added).

For Cogliano, Marx's interchangeable use of the expressions *rate of* exploitation and *rate of surplus value* throughout *Capital* can generate confusion, and that is why Cogliano distinguishes the "rate of exploitation" – "the ratio of surplus value to the value of labour power" (hence he speaks of the EQRE, the "Marx's equalized rate of exploitation") – from the "rate of surplus value", which is "the ratio of realized surplus value to wages" (Cogliano, 2023, p. 139). It is the latter that is not uniform in all sectors when production prices and the uniform rate of profit dominate. In Marx's third table, the variable capital of the five branches (20v, 30v, 40v, 15v, 5v) and the mass of profit that results from applying the general rate of profit (22%) to the total capital of each branch (= 100), that is, 22, lead to the resulting "surplus value" rates of 110%, 73.3%, 55%, 146.67% and 440%, respectively, whose weighted average, however, is 100% (equal to the exploitation rate).

done previously is to explain, as we have seen, that (1st) *s* is transformed into *r* (2nd) and the surplus value, $s^{(M)}\boldsymbol{pB}$, is transformed into profit, $r^{(M)}\boldsymbol{pA}^{(+)}$; that is why (3rd) it is only in a third moment, once the exposition of all his three tables is *finished*, when Marx *defines* the price of production like this:

> "The prices which obtain as the average of the various rates of profit in the different spheres of production added to the cost prices of the different spheres of production, constitute the *prices of production*." (Marx, 1894/1998, p.156).[45]

Up to that point, production prices have not yet appeared in *Capital*. It is very significant that in none of the columns of any of the three tables of chapter 9 does Marx still mention the prices of production: he only writes "cost price" and "price of commodities", but not "price of production" (Marx, 1894/1998, p.154-156). What in our opinion this means is that, in Marx's *analysis*, what truly distinguishes the production price from the value price is whether one is using *s* or *r* to define them, leaving the definition of the cost price as something of much less importance. From this point, all that is needed to understand that Marx's provisional prices finally become correct prices, without the need to use the iterative method, is to take into account the two alternative assumptions that successively (as appropriate to the general scientific method of *successive approximations*) he makes in *Capital*: in books I and II he assumes that $\boldsymbol{p} = \boldsymbol{p}^{(v)}$, while in book III he assumes $\boldsymbol{p} = \boldsymbol{p}^{(p)}$. This is how one can go from Marx's provisional pair of equations, (17) and (18), to his correct final equations, which coincide with (3) and (4), being $\boldsymbol{p}^{(v)(M2)} = \boldsymbol{p}^{(v)}$ and $\boldsymbol{p}^{(p)(M2)} = \boldsymbol{p}^{(p)}$:

$$\boldsymbol{p}^{(p)(M1)} = \boldsymbol{pA}^{(+)}\left(1 + r^{(M1)}\right) \ (17) \quad \rightarrow\rightarrow \quad \boldsymbol{p}^{(p)(M2)} = \boldsymbol{p}^{(p)} = \boldsymbol{p}^{(p)}\boldsymbol{A}^{(+)}(1 + r) \ (3)$$

$$\boldsymbol{p}^{(v)(M1)} = \boldsymbol{pA} + \boldsymbol{pB}\left(1 + s^{(M1)}\right) (18) \quad \rightarrow\rightarrow \quad \boldsymbol{p}^{(v)(M2)} = \boldsymbol{p}^{(v)} = \boldsymbol{p}^{(v)}\boldsymbol{B}^{+}(1 + s) \ (4)$$

Regarding the rate of profit, critics admit that Marx correctly defines the rate of surplus value, $s^{(M)} = s = \frac{\boldsymbol{p}^{(v)}(\boldsymbol{I}-\boldsymbol{A}^{(+)})\boldsymbol{x}}{\boldsymbol{p}^{(v)}\boldsymbol{Bx}}$, but, as we said above, they believe that he $r^{(M)} = \frac{\boldsymbol{p}^{(v)}(\boldsymbol{I}-\boldsymbol{A}^{(+)})\boldsymbol{x}}{\boldsymbol{p}^{(v)}\boldsymbol{A}^{(+)}\boldsymbol{x}} \neq r = \frac{\boldsymbol{p}^{(p)}(\boldsymbol{I}-\boldsymbol{A}^{(+)})\boldsymbol{x}}{\boldsymbol{p}^{(p)}\boldsymbol{A}^{(+)}\boldsymbol{x}}$, which is not really the case. What Marx has in mind are, *first*, a rate of profit $r^{(M1)} = \frac{\boldsymbol{p}(\boldsymbol{I}-\boldsymbol{A}^{(+)})\boldsymbol{x}}{\boldsymbol{pA}^{(+)}\boldsymbol{x}}$ and a rate of surplus value $s^{(M1)} = \frac{\boldsymbol{p}(\boldsymbol{I}-\boldsymbol{A}^{(+)})\boldsymbol{x}}{\boldsymbol{pBx}}$, both defined using actual prices; and *then* applies his two alternative assumptions to these equations, to *finally* obtain the two rates defined correctly:

$$r^{(M2)} = r = \frac{\boldsymbol{p}^{(p)}(\boldsymbol{I}-\boldsymbol{A}^{(+)})\boldsymbol{x}}{\boldsymbol{p}^{p}\boldsymbol{A}^{(+)}\boldsymbol{x}} \ (19)$$

45 Or: "The price of a commodity, which is equal to its cost price plus the share of the annual average profit on the capital *advanced* (not merely consumed) in its production that falls to it in accordance with the conditions of turnover, is called its price of production." (Marx 1894:157; emphasis added)

$$s^{(M2)} = s = \frac{\boldsymbol{p}^{(v)}(\boldsymbol{I}-\boldsymbol{A}^{(+)})\boldsymbol{x}}{\boldsymbol{p}^{(v)}\boldsymbol{B}\boldsymbol{x}} \quad (20).$$

Finally, it would be interesting to reflect on what has been the normal path —followed by Guerrero (2007) too— to "defend" the Marxian theory of value from its critics (as if this theory needed to be defended when in reality, as we are seeing, it is the other theories of value that need a rescue).[46] In particular, it seems to us a mistake to try to demonstrate that Marx's two identities – total value = total price, that is, $\boldsymbol{p}^{(v)}\boldsymbol{x} = \boldsymbol{p}^{(p)}\boldsymbol{x}$; and total surplus value = total profit, that is, $\boldsymbol{p}^{(v)}\left(\mathbf{I} - \boldsymbol{A}^{(+)}\right)\boldsymbol{x} = \boldsymbol{p}^{(p)}\left(\mathbf{I} - \boldsymbol{A}^{(+)}\right)\boldsymbol{x}$ – are fulfilled simultaneously. Actually, this double equality is not important for Marx; he simply assumes these two equalities as a (provisional) simplification – a step required by the theoretical need not to analyze everything at once – of his most complete and finished idea: that the total mass of values is *approximately* equal to that of prices, and that the total mass of surplus value is *approximately* equal to that of profit. Thus, what Marx thinks is that, simultaneously,

$$\boldsymbol{p}^{(v)}\boldsymbol{x} \approx \boldsymbol{p}^{(p)}\boldsymbol{x} \qquad (21)$$

$$\boldsymbol{p}^{(v)}(\boldsymbol{I} - \boldsymbol{A}^{(+)})\boldsymbol{x} \approx \boldsymbol{p}^{(p)}(\boldsymbol{I} - \boldsymbol{A}^{(+)})\boldsymbol{x} \qquad (22).$$

When Marx states that in the transformation "there is always the possibility of an error" (Marx 1894:164), he is not only thinking about the effects of a change in the valuation of inputs, but also that equality $\boldsymbol{p}^{(v)}\boldsymbol{x} = \boldsymbol{p}^{(p)}\boldsymbol{x}$ is not exact, but approximate: $\boldsymbol{p}^{(v)}\boldsymbol{x} \approx \boldsymbol{p}^{(p)}\boldsymbol{x}$. Therefore, the important thing is to understand that, in a context of n economic sectors with n different, non-uniform capital value compositions, Ω_i, it must occur (whatever the statistical distribution of the Ω_i) that

$$\boldsymbol{p}^{(p)} \not> \boldsymbol{p}^{(v)} \not> \boldsymbol{p}^{(p)} \qquad (23),$$

since when sector i has a higher composition than the average ($\Omega i > \Omega^*$), $p_i^{(p)} > p_i^{(v)}$ has to occur, and when the opposite happens ($\Omega i < \Omega^*$) we necessarily have $p_i^{(p)} < p_i^{(v)}$; therefore it is not possible to have $\boldsymbol{p}^{(p)} > \boldsymbol{p}^{(v)}$ nor $\boldsymbol{p}^{(v)} > \boldsymbol{p}^{(p)}$ because that would mean that all sectors of the economy have a composition greater than the average (or all less than the average), and this is logically impossible.

The Series of Linear Transformations (in a Single System)

Already in 1948, K. May pointed out that "the 'transformation problem' in the formal sense of linking value and price of production is seen to be practically trivial mathematically" (May, 1948, p. 596), a statement supported by these words from Pasinetti (1981a):[47]

46 This is so although many authors defend the "inferiority of Marx's paradigm of values" (Samuelson 1982, p.17).

47 See also Pasinetti and Garbellini (2015) and Lippi (1976, pp. 106-107).

> It may be concluded that the Marxian problem of the 'transformation of values into prices of production', which has given rise to so much controversy in the economic literature, is expressed, in analytical terms, by an algebraic operation of linear transformation. A certain vector, representing the 'values,' is transformed into another vector, representing the 'prices of production,' on being multiplied by a matrix which represents the logical process of solving a system of equations. And the inverse of this matrix effects the inverse transformation of 'prices of production' into 'values.' There is therefore a one-to-one correspondence between 'values' and 'prices of production.' This is a conclusion which should no longer be open to dispute. (p.149).

However, our agreement with Pasinetti (1981a) cannot be extended to those critics of Marx who, as Pasinetti says, draw the conclusion that:

> *on a strictly analytical ground* there is no justification whatever for giving any sort of logical priority to either 'values' over prices or to prices over 'values'. Each of them can be derived from the other; and both can be obtained from interindustry relations expressed in terms of physical quantities of commodities. (p.149).[48]

Apart from being a confession of physicalism, it must be clarified that in this paragraph "analytical" means "mathematical", so this double mutual derivation between values and prices is purely mathematical but does not reflect what actually occurs in the *reality*, where it is the *real* relations of value (see figures 1 and 3) —or the workings of real capitalist economies, including *real* competition between capitals— what defines or determines the numerical value of each vertically integrated labour coefficient and all elements of $\boldsymbol{p}^{(v)}$ and $\boldsymbol{p}^{(p)}$. *Priority therefore belongs to real values*, and production prices are only a reflection of value relations.[49]

After having analyzed in sections III and IV various interpretations of Marx's solution to the Transformation, we now proceed in this section to extend the transformation beyond $\boldsymbol{p}^{(v)}$ and $\boldsymbol{p}^{(p)}$. Our purpose is to improve the understanding of the "problem" thanks to the development of a series of linear transformations (and other matrices and scalars) that allows us to accurately quantify the relationship between the most relevant vectors of the theory of value. Given its popularity, we will start with the Marxian relationship between value prices and production prices: these vectors are of course different ($\boldsymbol{p}^{(p)} \neq \boldsymbol{p}^{(v)}$) but the values reached by the t_{ij} of matrix $\boldsymbol{T}$ (see below) make it possible to specify the exact quantitative relationship

48 A non-physicalist way to write this paragraph would be: "*On a strictly analytical ground* there is no justification whatever for giving any sort of logical priority to either value prices over production prices or to production prices over value prices. Each of them can be derived from the other; and both can be obtained from interindustry *labour* relations (or real relations of value) as physical quantities of labour that result in products that appear as ratios between quantities of commodities." As Marx says, "The process disappears in the product; the latter is a use value, Nature's material adapted by a change of form to the wants of man. Labour has incorporated itself with its subject: the former is materialised, the latter transformed. That which in the labourer appeared as movement, now appears in the product as a fixed quality without motion. The blacksmith forges and the product is a forging." (Marx, 1867, pp.190-191).

49 Without a doubt, what we have just said belongs to the field of the "discussions of greater scope" to which Pasinetti refers when he writes that "the objective of these notes has been to clarify the logical-mathematical aspects of the problem, a task which is always the essential and necessary preliminary to any discussion of greater scope." (Pasinetti, 1981a, p.150).

between the set of value prices and the set of production prices.[50] To discover this quantitative connection we need to start by rewriting equations (3) and (4) as

$$\boldsymbol{p}^{(p)}\left(\boldsymbol{I}-(1+r)\boldsymbol{A}^{(+)}\right)=\boldsymbol{0} \qquad \text{(3bis)}$$

$$\boldsymbol{p}^{(v)}\left(\boldsymbol{I}-(1+s)\boldsymbol{B}^{(+)}\right)=\boldsymbol{0} \qquad \text{(4bis);}^{[51]}$$

and then, by equating the left members of both equations, we arrive at

$$\boldsymbol{p}^{(p)}=\boldsymbol{p}^{(v)}\left(\boldsymbol{I}-(1+s)\boldsymbol{B}^{(+)}\right)\left(\boldsymbol{I}-(1+r)\boldsymbol{A}^{(+)}\right)^{-1}=\boldsymbol{p}^{(v)}\boldsymbol{T} \quad (24).$$

Therefore, we simply have $\boldsymbol{p}^{(p)}=\boldsymbol{p}^{(v)}\boldsymbol{T}$, where $\boldsymbol{T}=\left(\boldsymbol{I}-(1+s)\boldsymbol{B}^{(+)}\right)\left(\boldsymbol{I}-(1+r)\boldsymbol{A}^{(+)}\right)^{-1}$. Regarding a specific commodity, the correct formula is:

$$p_i^{(p)}=\boldsymbol{p}^{(v)}\boldsymbol{T}^{(i)} \quad (25),$$

where $\boldsymbol{T}^{(i)}$ is the i column of matrix $\boldsymbol{T}$. Equation (25) shows that each price of production *depends* on the value prices (and therefore the values) of *all* commodities and the elements of column i of $\boldsymbol{T}$, which are all interdependent; that is, it depends on rates *s* and *r* —which, as Marx says, are "two different measurements of the same entity" (Marx, 1894/1998, p.46)— and on all a_{ij} and b_{ij}. It is clear that, mathematically, this linear operator (or matrix) $\boldsymbol{T}$ can serve for both a direct and inverse transformation, since we also have

$$\boldsymbol{p}^{(v)}=\boldsymbol{p}^{(p)}\boldsymbol{T}^{-1} \quad (24\text{bis.}1),$$

so we must add to the above that each value price depends as well on all production prices in addition to s, r and all the a_{ij} and b_{ij}. This reciprocal mathematical dependence expresses nothing more than a logical co-implication,

$$\boldsymbol{p}^{(v)} \Leftrightarrow \boldsymbol{p}^{(p)},$$

which shows that $\boldsymbol{p}^{(p)}$ exists if and only if $\boldsymbol{p}^{(v)}$ exists, and vice versa.

The case of actual prices is parallel to the previous two: we need to rewrite equation (2) as

$$\boldsymbol{p}\left(\boldsymbol{I}-\boldsymbol{C}^{(+)}\right)=\boldsymbol{0} \text{ (2bis.1),}$$

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50 Actually, it is enough to start from **equations** (3) and (4) to directly obtain, once *r* and *s* are known, the relationship $\frac{p_i^{(p)}}{p_i^{(v)}}=\frac{1+r}{1+s}\cdot\frac{\boldsymbol{p}^{(p)}\boldsymbol{A}^{(+)(i)}}{\boldsymbol{p}^{(v)}\boldsymbol{B}^{(+)(i)}}$, or also $p_i^{(p)}=\left(\frac{1+r}{1+s}\cdot\frac{\boldsymbol{p}^{(p)}\boldsymbol{A}^{(+)(i)}}{\boldsymbol{p}^{(v)}\boldsymbol{B}^{(+)(i)}}\right)\cdot p_i^{(v)}=\alpha\cdot p_i^{(v)}$. However, equations (24), (26), (27) and (31)-(33) extend the relationship to the entire system.

51 To which we could add $\boldsymbol{p}^{(S)}(\boldsymbol{I}-r^{(S)}\boldsymbol{H}^{(+)})=\boldsymbol{0}$ for the Sraffian prices.

so that when comparing (2bis) with (3bis) we obtain

$$\boldsymbol{p} = \boldsymbol{p}^{(p)}\left(\boldsymbol{I} - (1+r)\boldsymbol{A}^{(+)}\right)\left(\boldsymbol{I} - \boldsymbol{C}^{(+)}\right)^{-1} = \boldsymbol{p}^{(p)}\boldsymbol{U} \quad (26)$$

and also

$$\boldsymbol{p} = \boldsymbol{p}^{(p)}\boldsymbol{U} = \boldsymbol{p}^{(v)}(\boldsymbol{TU}) \quad (27),^{52}$$

where $\boldsymbol{U} = \left(\boldsymbol{I} - (1+r)\boldsymbol{A}^{(+)}\right)\left(\boldsymbol{I} - \boldsymbol{C}^{(+)}\right)^{-1}$ and $\boldsymbol{TU} = \left(\boldsymbol{I} - (1+s)\boldsymbol{B}^{(+)}\right)\left(\boldsymbol{I} - \boldsymbol{C}^{(+)}\right)^{-1}$.

We have seen that equation (7), $\boldsymbol{p}^{(v)} = m \cdot \boldsymbol{v}$ (where $m = (1+s)\boldsymbol{p}^{(v)}\boldsymbol{c}$ is the monetary value added per capita, or per hour), shows the proportionality between value prices and values. But since the *relative* $\boldsymbol{v}$ and the relative $\boldsymbol{p}^{(v)}$ are equal, the proportionality factor can be any scalar with dimension \$/hour. What the literature knows as MELT (monetary expression of labour time) is precisely the quotient between the value added in monetary terms, mL, and the total mass of direct labour, L (that is, $\frac{mL}{L} = m$), but it could be asked if there are more appropriate ratios than the MELT.[53] If we therefore write $\boldsymbol{p}^{(v)} = m \cdot \boldsymbol{v}$, this allows us to add to (27) a new equation,

$$\boldsymbol{p} = \boldsymbol{v}(m\boldsymbol{TU}) \quad (31),$$

that specifies the relationship of actual prices with labour values through the $m\boldsymbol{TU}$ matrix. Now, remembering equation (1), $\boldsymbol{v} = \boldsymbol{lQ}$, we can see also how actual prices are linked in a quantitatively exact way with the direct labour coefficients:

$$\boldsymbol{p} = \boldsymbol{l}(m\boldsymbol{QTU}) \quad (32).$$

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52 We would also have $\boldsymbol{p}^{(v)} = \boldsymbol{p}(\boldsymbol{TU})^{-1} = \boldsymbol{pU}^{-1}\boldsymbol{T}^{-1}$.

53 According to Shaikh (2016, 241-242) it is preferable to replace the MELT (based on the "net product") with a ratio more in the spirit of Marx, based on "the whole product" (see our z below), since the MELT seems imposed by the definitional needs of the New Interpretation school. In this regard, it is interesting to observe that if we start from the equations (scalar products)

$$\boldsymbol{px} = X \quad (28)$$

$$\boldsymbol{lx} = L \quad (29)$$

$$\boldsymbol{vx} = V \quad (30),$$

where X is the total monetary output, L the total direct labour, and V the total value —what Marx calls the total "value of the product", not the "value product" or value added see (Marx, 1885/1997, p. 436)—, we can define scalars such as $y = \frac{X}{L} = \frac{\boldsymbol{px}}{\boldsymbol{lx}}$ or $z = \frac{X}{V} = \frac{\boldsymbol{px}}{\boldsymbol{vx}}$, which in both cases are expressed in \$/H. The value of y is immediately obtained empirically, for example through the national accounts, which show that in 2014 in the US X = 31 trillion US\$ and L = 0.28 billion hours, and therefore y = 110.7 US\$/H. Now, if, instead of actual prices, either $\boldsymbol{p}^{(v)}$ or $\boldsymbol{p}^{(p)}$ were used, we would have $X^{(v)}$ (gross output at value prices) or $X^{(p)}$ (at production prices), generally being $X^{(v)} \neq X$, $X^{(p)} \neq X$ and $X^{(v)} \neq X^{(p)}$. Therefore, it might seem that $z^{(v)} = \frac{X^{(v)}}{V} = \frac{\boldsymbol{p}^{(v)}\boldsymbol{x}}{\boldsymbol{vx}}$ is the most appropriate quotient to connect values with value prices, given that $X^{(v)} = z^{(v)}V$ (or $\boldsymbol{p}^{(v)}\boldsymbol{x} = z^{(v)}\boldsymbol{vx}$). Although this equality does not imply $\boldsymbol{p}^{(v)} = z^{(v)}\boldsymbol{v}$ nor $p_i^{(v)} = z^{(v)}v_i$, it can be assumed that this is the case, so $z^{(v)}$ would be a useful *average* to go from $\boldsymbol{v}$ to $\boldsymbol{p}^{(v)}$, just like the wage contained in m is also an *average*. In any case, for practical purposes it is indifferent to use $z^{(v)}$ or $m \neq z^{(v)}$ (or use $\boldsymbol{p}^{(v)} = z^{(v)}\boldsymbol{v}$ instead of $\boldsymbol{p}^{(v)} = m\boldsymbol{v}$), since any vector multiplied by a scalar keeps its proportions intact.

Finally, remembering what was said in II.1 and using matrix $\widehat{\boldsymbol{w}}$ to get

$$\boldsymbol{l} = \boldsymbol{l}'\widehat{\boldsymbol{w}} \qquad (33),$$

we arrive at

$$\boldsymbol{p} = \boldsymbol{l}'(m\widehat{\boldsymbol{w}}\boldsymbol{QTU}) \qquad (34).$$

This section can be summarized, then, as follows. Although, from a theoretical point of view, it is also possible to link values specifically to simple reproduction of capital (Bródy) or production without surplus (Sraffa), and production prices specifically to extended reproduction of capital (or production with surplus), our exposition shows how at any moment everything exists at once: each commodity has a given actual price, but it also has a certain value, a certain value price and a certain production price, all of which is the consequence of the actual expenditure of certain amounts of total labour, both direct and indirect. All of this confirms that the reciprocal mathematical relationships between all the vectors studied, which operate both directly and inversely, are not an obstacle to reaffirming that it is the reality of value that really explains and determines all these relationships (see figures 1 and 3), whose totality can and must be represented by a single accounting system (the Single System) where each price, each value and each quantity of labour simultaneously belong to the same developed capitalist relationship regulated by the law of (labour-)value.

Labour Alone Creates Value

One of the factors that has contributed most to the misunderstanding of the Marxian theory of value has been the interference, in the theoretical analysis of value, of the always fiery ideological debates on the distribution of income, which have led many authors to oppose the labour theory of value because they do not share the idea – which they believe is implicit in the labour theory of value, without being the case – that productive factors that are not labour (especially "capital") are not creators of wealth. These authors oppose this idea because they claim the right to a "fair" share (profit) in the income generated by producing and selling a commodity (they forget that neither Law nor Justice have a place in Marxian economics), arguing that without the participation of that capital in production, the latter would be impossible. However, what the labour theory of value in Ricardo and Marx affirms is that, although the other factors participate, along with labour, in the creation of wealth, only labour participates in the creation of value: labour is (1) the only productive factor of value and (2)

the only explanatory factor of value, and only for that reason the price of the commodity expresses the exclusive productivity of labour as the creator of value.[54]

This thesis should not be confused with the thesis of the exploitation of labour by capital, which comes (analytically) later, once the theory of value is completed and applied to the commodity labour force. But what concerns us here is something prior and refers to value itself: it is the explanation that (1) the price of each commodity is what it is because the amount of labour involved in its production is what it is (or it deviates from that quantity in an amount fully explained by the theory); and (2) the price of all the commodities produced is what it is because the quantity of labour involved in their production is what it is (or it deviates from it by an even smaller amount, for reasons completely explained by the theory).

Faced with this duality (one more duality) of Marx's theory of value, expressed in this way, some of Marx's critics might reply: "Okay..., but if you admit that capital contributes to the creation of wealth, then it must participate in its enjoyment, so the capitalist has the right to keep a part of the wealth created." To which it could be counter-replied that, according to this, the capitalist who 'produces' toilet paper would have at most the 'right' to a part of the wealth created, the paper rolls produced, say 50%; or the one who 'produces' car tires, entitled to 40% of the tires; or the one that 'produces' car batteries, 60% of the batteries, etc. (These figures are obviously arbitrary). The point is that this dialogue about Rights and Justice has little to do with the decisive issue of what explains the values and prices of commodities. No quantity of things (use values or wealth) can be expressed in monetary income, unless their prices are

54 Apart from economists such as the Malthusian Cazenove (1832), who came to say that the labour theory of value is false because it is "dangerous", more prestigious economists have also always protested against the "monomania" of seeing labour as the only source of value, from Cournot (1863, p.70) —who denounces "l'excès qui consiste à voir dans le travail de l'homme la source unique de la valeur"— to Tinbergen (1992, p.vii) —who finds it "arbitrary" that the "Marxian labour- values neglect non-labour-costs and utilities in their entirety"—, passing through Seton, for whom this thesis is nothing but an "act of fiat" (Seton 1956/1957, p. 160). Those who explicitly or implicitly consider the labour theory of value as socialist should explain why Ricardo also defended it. When Seton accuses Marx of not being a scientist because he "does not seek to explain the world; he seeks to change it, and nowhere is this clearer than in his theory of value" (Seton 1992:12), apart from making it clear that his own personal objective seems to be "not to change the world", he forgets that this coincidence between Ricardo and Marx (1894/1998, pp.929-930) disqualifies the idea of the socialist or exclusively Marxist character of the theory. That both Ricardo and Marx, so opposite, defend the same theory of value proves the scientific objectivity of both at the same time, and this is independent of (and more important than) the fact that Ricardo's thought served as an "arsenal" of ideas for all types of socialists and anarchists (according to H. Carey), included the "Ricardian" socialists, the communists like R. Owen (Engels, 1885, p.17) and almost all utopians —see Mariolis and Tsoulfidis (2016, pp.3-4) such as F. Tristan, all of whom supported the exclusive right of labour to the entire product (Tristan, 1843, p.i).

A second proof of Marx's objectivity is provided by the fact that his criticism of the confusion between wealth and value is applied first of all against the program of his own comrades at the Congress of the German socialist party in Gotha (Marx, 1875/2021, p.9 writes "Labour is *not the source* of all wealth. *Nature* is just as much the source of use values (and it is surely of such that material wealth consists!) as labour, which itself is only the manifestation of a force of nature, human labour power." Most authors forget that "scientific objectivity is compatible with ethical and political partiality" (Bunge, 1999, p.45) and seem unaware that "the implicit belief in the existence of a body of scientific knowledge acquired independently of all evaluations is [...] naive empiricism" (Myrdal, 1929, p.9); this is also the reason why almost everyone, like Seton, believes he or she is "impartial because they are unaware of their chains." (Steuart, 1767) [It must be remembered that, although Marx's position has been especially misinterpreted on this point —see for example Bose (1975, p.58) or Sinha (2010, p.255)—, others have understood it, both Marxists (Boudin, 1907, p.66; Grossman, 1924, p.36; Medio, 1972, p.315; Braverman, 1974, p. 51) and non-Marxists (Baumol, 1973, p. 69, Baumol & Blinder, 1988, p. 878; Kühne, 1972, Gramm, 1988, p. 230; Wolfstetter, 1973, p. 790).

known, and it is precisely these prices that any theory of value must explain. What we are asking is the *relative price* of, let us say, a battery and a tire, and we doubt that capitalist A, who owns 100,000 tires, or capitalist B, who owns 12,000 batteries, or the economists of their liking, know of any theory that, in case A and B want to exchange their commodities between them, is capable of explaining what the labour theory of value does explain: the price at which they will have to do it anyway.[55] Since they have no answers to these questions, except for chatter about justice, their criticisms of the labour theory of value become irrelevant...

But other critics will protest by arguing that, "objectively, this relative price is given, like all relative prices, by the positive left-hand dominant eigenvector associated with the maximum eigenvalue of the physical matrix $\boldsymbol{A}^{(+)}$" (remember equation (3)), to which we might respond that matrix $\boldsymbol{A}^{(+)} = \boldsymbol{A} + \boldsymbol{B}$, which critics consider 'physical', are actually monetary matrices (not $\boldsymbol{A}^{(p)}$ and $\boldsymbol{B}^{(p)}$ but $\boldsymbol{A}^{(m)}$ and $\boldsymbol{B}^{(m)}$) whose coefficients a_{ij} and b_{ij} are production costs and nothing else, so if someone wants to *interpret* both matrices as if they were physical matrices (see Leontief words in note 6) he or she must remember that *the physicality* that appears in them is the expression and real result of other more specific and relevant physical data, which are the quantities of labour actually expended in the production of those commodities.[56]

This closes the circle of our arguments in favor of the labour theory of value, which, let us not forget, starts from a double and unappealable accounting identity: if we have by definition both $\boldsymbol{x} = \boldsymbol{Qy}$ and $\boldsymbol{v} = \boldsymbol{lQ}$,[57] it is obvious that we also have

$$\boldsymbol{vy} = \boldsymbol{lQy} = \boldsymbol{lx}, = \boldsymbol{L}.$$

Therefore, the value of the net social product ($\boldsymbol{vy}$) is equal to the direct labour performed in the production of the gross product, $\boldsymbol{lx}$.

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55 "What becomes relevant for economic purposes, which means for the process of pricing, is only *the amount of human activity* which is required, whether directly or indirectly, to make a technological or a biological process work. [...] For pricing purposes, what matters is the amount of human activity which has been and has to be used. [...] *Labour* emerges from the very logic of the present analysis *as the only ultimate factor or production*. [...] Contrary to what traditional economics has maintained for a long time, it is not the 'productivity of capital', or of any commodity, that turns out to be the *raison d'être* of a rate of profit. It is the growth, and the increasing productivity, of labour!" (Pasinetti, 1981b, pp. 131-133; emphasis added).

56 Indeed, the physical coefficients a_{ij} and b_{ij} are inputs of commodity i necessary for the production of one unit of j, but they are relevant inputs here *only* because they actually enter into the *productive consumption* process of real capitalist firms that are the result of human beings who built them and continue working within them, and not because some *things* enter into a *natural* process —for example, in the natural production of honey, where it is easy to quantify the coefficients of flowers and bees per gram of honey produced— nor because they enter into a process of *unproductive* consumption (such as that which takes place when preparatory labour is carried out for private or family consumption, for example a barbecue). Consequently, inquiring more closely in the numbers of matrix $\boldsymbol{A}$ (the a_{ij}), they directly express what happens in the labour process of each sector – this is apart from what is usually said about vector $\boldsymbol{l}$ and matrix $\boldsymbol{A}$, which form the pair ($\boldsymbol{A}$, $\boldsymbol{l}$) that defines "the technique of the system" – when labour, which in the present social conditions is always subsumed in *value relations*, puts into practice its capacity as a *subject* to define and create the *labour technique* that determines, as a consequence, the average quantities and proportions of each of the *objects* and means of production available to bring production to a successful conclusion.

57 If $\boldsymbol{A}$ is productive, it is because there exists a vector like $\boldsymbol{x}$ or $\boldsymbol{v}$ such that $\boldsymbol{x} > \boldsymbol{Ax}$, and $\boldsymbol{v} > \boldsymbol{vA}$. Therefore, $\boldsymbol{x} - \boldsymbol{Ax} = \boldsymbol{y} > \boldsymbol{0}$, and $\boldsymbol{v} - \boldsymbol{vA} = \boldsymbol{l} > \boldsymbol{0}$, and so $\boldsymbol{x} = (\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{y} = \boldsymbol{Qy}$, and $\boldsymbol{v} = \boldsymbol{l}(\boldsymbol{I} - \boldsymbol{A})^{-1} = \boldsymbol{lQ}$.

Conclusions

What we have attempted to explore, or at least open up, in this article is a different path to address the question of the transformation of values into production prices. Although attention has been paid to some different interpretations —but limited to the iterative approach of Bródy-Shaikh-Morishima and to the modern interpretation of Moseley— we have not attempted to make an exhaustive study of approaches other than our own. Our approach is directed both at better understanding Marx's approach - to vindicate the logic implicit in his solution, without forgetting that it was he who raised the problem as part of the solution to the doubts and inadequacies of the contribution of the classics of the labor theory of value - and at the approach that at present seems capable of better developing Marx's approach. Having seen that Marx's final solution (the second step of his analysis) is correct, we have proposed to conceive the transformation as the result of applying a linear operator, the matrix $\boldsymbol{T}$, to the value-price vector to obtain the production-price vector exactly. Moreover, this is one of several linear transformations (others are, for example, the one linking value prices with actual prices, and production prices with actual prices) which, together with other equations referring to the quantities of direct labour expended in the production of commodities and to labour-values, allow us to connect in a mathematically exact way all the vectors we have put in connection: $\boldsymbol{l}, \boldsymbol{v}, \boldsymbol{p}^{(v)}, \boldsymbol{p}^{(p)}, \boldsymbol{p}$. It seems to us that this path can lead to the appropriate contrast between what is the solution of the system of equations represented by each of the eigenequations posed (which are the eigenvalues and eigenvectors studied, that is, the prices and the rates of surplus value and profit), and what are the *identity equations* that result from the comparison of two of these different equations (their solutions), as has been done here by means of the matrix $\boldsymbol{T}$.

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