



SLOWNESS SURFACE CALCULATION FOR DIFFERENT MEDIA USING THE SYMBOLIC MATHEMATICS LANGUAGE MAPLE®

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ABSTRACT

Starting from the equation in different media, we obtain the different type of waves that can exist in such media. The evaluation of the eigenvalues and eigenvectors let us construct the slowness surfaces. In general complex calculations case have to be made. In this work, routines were implemented in the symbolic language MAPLE® and the slowness surfaces were plotted. This work is relevant for the modelling of equivalent media that simulates natural fractured reservoirs, like those common in the Colombian foothills. It is important to understand the seismic response of this reservoirs for exploration of this areas.

Key Words: Christoffel matrices, slowness vector, cubic symmetry, hexagonal symmetry, isotropic, symmetry, slowness surfaces, eigenvalues, eigenvectors

RESUMEN

Partiendo de la ecuación en diferentes medios, obtenemos los diferentes tipos de ondas que pueden existir en tales medios. La evaluación de los valores propios y los vectores propios permite construir las superficies de lentitud. En general, es necesario hacer complejos casos de cálculos. En este trabajo se implementaron algunas rutinas en el lenguaje simbólico MAPLE® y se señalaron las superficies de lentitud. Este trabajo es pertinente para el modelado de medios equivalentes que simulan los depósitos fracturados naturales, como los que aparecen en las zonas de piedemonte colombianos. Es importante entender la reacción sísmica de estos depósitos para la exploración de estas áreas.

Palabras clave: isotropía, matriz de Christoffel, simetría cúbica, simetría hexagonal, superficies de lentitud, valores propios, vectores propios, vector de lentitud.

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METHODOLOGY

This work was developed entirely in the field of the linear theory of elasticity. For homogeneous media the solution of the general elastodynamic equation reduces to the solution of the Christoffel Matrix eigenvalues problem. The Christoffel equation is solved numerically using the symbolic mathematics programming language Maple®. The phase velocity is obtained as a function of the anisotropic parameters and the propagation direction. The phase velocity $1/v$ is used to obtain the slowness, which is plotted using the standard plotting Maple function in polar coordinates. The resulting graphic shows the relation between the slowness vector and the propagation angle.

FORMULATION OF THE PROBLEM AND RESULTS

Theory Research – Fundamentals

The deformation laws for a linear elastic body are determined by the linear momentum equation (1), the constitutive equation or the Hooke's Law (2) and the small strain – displacement linear relation (3) (Sen, 2002), (Krishnaswamy, 2004), (Pšenčík, 1994):

$$\nabla \cdot \underline{\underline{\tau}} = \rho \underline{\underline{\ddot{u}}} \leftrightarrow \frac{\partial \tau_{ij}}{\partial x_j} = \rho \frac{\partial u_i}{\partial t^2} \quad (1),$$

$$\underline{\underline{\tau}} = \underline{\underline{C}} : \underline{\underline{\varepsilon}} \leftrightarrow \tau_{ij} = C_{ijkl} \varepsilon_{kl} \quad (2),$$

$$\underline{\underline{\varepsilon}} = \frac{1}{2} (\nabla \underline{\underline{u}} + \underline{\underline{u}} \nabla) \leftrightarrow \varepsilon_{kl} = \frac{1}{2} \left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right) \quad (3).$$

Where:

$\underline{\underline{\tau}}$ is second order stress tensor,

ρ is density

$\underline{\underline{u}}$ is displacement vector

$\underline{\underline{C}}$ is fourth order elasticity tensor or stiffness tensor

$\underline{\underline{\varepsilon}}$ is second order linearized finite strain tensor or tensor of small strain

x_i, x_j, x_k – Cartesian coordinates.

Strain and stress tensor are symmetric. This assumption leads us to a contracted form of the

above equations, the Voigt notation. Equation (3) defines 6 different equations for only three components of the displacement vector u_i , therefore some other conditions (4) need to be imposed over the small strain components in order to guarantee the existence of the displacement field (Mase, 1970).

$$\nabla \times \underline{\underline{\varepsilon}} \times \nabla = 0 \leftrightarrow \frac{\partial^2 \varepsilon_{ij}}{\partial x_k \partial x_m} + \frac{\partial^2 \varepsilon_{km}}{\partial x_i \partial x_j} - \frac{\partial^2 \varepsilon_{ik}}{\partial x_j \partial x} - \frac{\partial^2 \varepsilon_{jm}}{\partial x_i \partial x_k} = 0 \quad (4)$$

Thus, combining (1), (2), (3) and (4) the most general form of the elastodynamic equation is obtained:

$$\frac{\partial}{\partial x_j} \left(C_{ij} \frac{\partial u_k}{\partial x_l} \right) = \rho \frac{\partial^2 u_i}{\partial t^2} \quad (5)$$

Note that the stiffness tensor C_{ijkl} reduces to a 6x6 symmetric matrix C_{ij} due to the symmetry of the stress and strain tensors. The condition (5) gives a simpler strain – displacement relation.

The stiffness matrix can be obtained by combination of coordinate transformations corresponding to the material lattice symmetry properties. In the most general case the stiffness matrix has got 21 independent elastic constants. This is the monoclinic material. Hexagonal symmetry, for example, has got 5 different elastic constants. This system has got one of the axes of symmetry such that a rotation around this axis does not change the stiffness matrix.

For homogeneous materials in which elastic constants and density are constant, (5) reduces to (6):

$$C_{ij} \frac{\partial^2 u_k}{\partial x_j \partial x_l} = \rho \frac{\partial^2 u_i}{\partial t^2} \quad (6)$$

First consider harmonic plane waves moving through homogeneous anisotropic medium. Then the solution must be sought in the next form:

$$\underline{\underline{u}}(\underline{\underline{x}}, t) = \underline{\underline{U}} e^{-ik(vt - \underline{\underline{l}} \cdot \underline{\underline{x}})} \leftrightarrow u_i(\underline{\underline{x}}, t) = U_i e^{-ik(vt - l_i x_i)} \quad (7)$$

Where:

$\underline{\underline{U}}$ is the displacement amplitude vector,

integration constant,

i is an imaginary unit,

$k = \frac{2\pi}{\lambda}$ is the wave number,

λ is wavelength,

v is phase velocity,

t is time,

\underline{l} is unit vector along the propagation direction,

\underline{x} is position vector.

Replacing the displacement vector (7) in the elastodynamic equation (6) and applying the Fourier transform, we get:

$$L_{ij} C_{jk} L_{kl} U_i - \frac{\rho \omega^2}{k^2} U_l = 0 \quad (8)$$

Where:

L_{ij} is Director cosines matrix of the propagation vector,

$$\omega = kv = \frac{2\pi v}{\lambda} - \text{angular frequency.}$$

The equation (8) can be rewritten as:

$$\left(\Gamma_{ik} - \delta_{ik} \rho v^2 \right) U_k = 0 \quad (9)$$

Where:

$$\Gamma_{il} = L_{ij} \cdot C_{jk} \cdot L_{kl} - \text{Christoffel matrix} \quad (10)$$

The slowness $p = \frac{1}{v}$ is obtained by solving the next equation for the three values of v^2 .

$$\det\left(\Gamma_{ik} - \delta_{ik} \rho v^2\right) = 0 \quad (11)$$

Therefore, three types of plane harmonic waves along the propagation direction \underline{l} are obtained, with three different phase velocities $v^{(i)}$. The corresponding displacement vectors $U_j^{(i)}$ are mutually orthogonal; they are also called polarization vectors. The vectors \underline{l} and $U_j^{(i)}$ can be parallel, in that case the corresponding wave to that displacement direction is called pure longitudinal wave. Perpendicular directions of displacement correspond to pure shear waves.

Body of the programming routine

This routine plots the wave surfaces for all cases of anisotropy

```
> restart;
with(LinearAlgebra):
```

```
unprotect(gamma,GAMMA):
```

Stiffness matrices

Stiffness matrix in the Voigt form

```
> C:=Matrix(6,6,shape=symmetric):
```

Now the user must choose the anisotropy case:

Isotropic case

(2 elastic constants)

```
> C[1,1]:=lambda+2*mu;
C[2,2]:=lambda+2*mu;
C[3,3]:=lambda+2*mu;
C[4,4]:=mu;
C[5,5]:=mu;
C[6,6]:=mu;
C[1,2]:=lambda;
C[1,3]:=lambda;
C[2,3]:=lambda;
evalm(C);
```

Cubic anisotropy

(3 elastic constants)

```
> C[1,1]:=C11;
C[2,2]:=C11;
C[3,3]:=C11;
C[4,4]:=C44;
C[5,5]:=C44;
C[6,6]:=C44;
C[1,2]:=C12;
C[1,3]:=C12;
C[2,3]:=C12;
evalm(C);
```

Hexagonal anisotropy, VTI

(5 elastic constants)

```
> C[1,1]:=C11;
C[2,2]:=C11;
C[3,3]:=C33;
C[4,4]:=C44;
C[5,5]:=C44;
C[6,6]:=C66;
C[1,2]:=C11-2*C66;
C[1,3]:=C13;
C[2,3]:=C13;
eval(C);
```

Orthorhombic anisotropy

(9 elastic constants)

```
> C[1,1]:=C11;
C[2,2]:=C22;
C[3,3]:=C33;
C[4,4]:=C44;
C[5,5]:=C55;
C[6,6]:=C66;
C[1,2]:=C12;
C[1,3]:=C13;
C[2,3]:=C23;
evalm(C);
```

Monoclinic anisotropy

(13 elastic constants)

```

> C[1,1]:=C11:
C[1,2]:=C12:
C[1,3]:=C13:
C[1,4]:=C14:
C[1,5]:=C15:
C[1,6]:=C16:
C[2,2]:=C22:
C[2,3]:=C23:
C[2,4]:=C24:
C[2,5]:=C25:
C[2,6]:=C26:
C[3,3]:=C33:
C[3,4]:=C34:
C[3,5]:=C35:
C[3,6]:=C36:
C[4,4]:=C44:
C[4,5]:=C45:
C[4,6]:=C46:
C[5,5]:=C55:
C[5,6]:=C56:
C[6,6]:=C66:
evalm(C);

Ray director cosines matrix
> L:=Matrix(6,3):
L[1,1]:=l[1]:
L[2,2]:=l[2]:
L[3,3]:=l[3]:
L[4,2]:=l[3]:
L[4,3]:=l[2]:
L[5,1]:=l[3]:
L[5,3]:=l[1]:
L[6,1]:=l[2]:
L[6,2]:=l[1]:
evalm(L);

Christoffel Matrix
> GAMMA:=Transpose(L).C.L;
Eigenvalues and Eigenvectors
The elastic constants should be declared according
to the chosen anisotropy case:
consts:=C11=16.57*10^10,C12=6.39*10^10,C44
=7.956*10^10,rho=2332:
(Cubic anisotropy)
>
consts:=C11=3.46E10,C13=1.07E10,C33
=2.84E10,C44=8.36E9,C66=1.26E10,rho=2500:
(VTI elastic constants)
>dir_1:=l[1]=sin(theta),l[2]=0,l[3]=cos(theta):
>Ch:=CharacteristicMatrix(subs(dir_1,GAMMA),
k);
> d:=Determinant(Ch);
> factor(d);
> K:=[solve(d,k)];
> map(simplify,%,trig);

> A:=subs(k=K[3],Ch);
> A:=map(simplify,subs(k=K[3],Ch),trig);
> solve({A[1,1]*u1+A[1,2]*u2=0},{u1});
> simplify(%,trig);
> solve({A[2,1]*u1+A[2,2]*u2=0},{u1});
> simplify(%,trig);
> Rank(A);
> eg:=map(simplify,ColumnSpace(A));
>
plot([subs(consts,dir_1,sqrt(rho/K[1])),subs(consts
,dir_1,sqrt(rho/K[2])),subs(consts,dir_1,sqrt(rho/K[
3]))],theta=0..2*Pi,color=[red,blue,green],coords=
polar);
>
plot([subs(consts,dir_1,sqrt(K[1]/rho)),subs(consts
,dir_1,sqrt(K[2]/rho)),subs(consts,dir_1,sqrt(K[3]/r
ho))],theta=0..2*Pi,color=[red,blue,green],coords=
polar);
>
phi_1:=theta+arctan(diff(sqrt(K[1]/rho),theta)/sqrt
(K[1]/rho));
>
v_1:=sqrt(sqrt(K[1]/rho)^2+diff(sqrt(K[1]/rho),the
ta)^2);
>
phi_2:=theta+arctan(diff(sqrt(K[2]/rho),theta)/sqrt
(K[2]/rho));
>
v_2:=sqrt(sqrt(K[2]/rho)^2+diff(sqrt(K[2]/rho),the
ta)^2);
>
phi_3:=theta+arctan(diff(sqrt(K[3]/rho),theta)/sqrt
(K[3]/rho));
>
v_3:=sqrt(sqrt(K[3]/rho)^2+diff(sqrt(K[3]/rho),the
ta)^2);
>
plot({[subs(consts,dir_1,v_1),subs(consts,dir_1,phi
_1),theta=0..2*Pi],[subs(consts,dir_1,v_2),subs(co
nsts,dir_1,phi_2),theta=0..2*Pi],[subs(consts,dir_1,
v_3),subs(consts,dir_1,phi_3),theta=0..2*Pi]},colo
r=[red,blue,green],coords=polar);
>
plot({subs(consts,dir_1,sqrt(K[3]/rho)),[subs(const
s,dir_1,v_3),subs(consts,dir_1,phi_3),theta=0..Pi/2]
},theta=0..Pi/2,color=[red,blue],coords=polar);

In the anterior routine the term VTI means
Vertical and Transversally Isotropic media.

Routine work examples

This program plots the three wave surfaces for
any anisotropic material: slowness surface, phase
velocity surface and group velocity surface. The

```

program was tested using the next stiffness matrix for the VTI case:

$$C_{ij} = \begin{pmatrix} 3.46 & 0.94 & 1.07 & 0 & 0 & 0 \\ 0.94 & 3.46 & 1.07 & 0 & 0 & 0 \\ 1.07 & 1.07 & 2.84 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.836 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.836 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.26 \end{pmatrix} \cdot 10^{10} Pa, \quad \rho = 2500 \frac{kg}{m^3}$$

Figure 1. Stiffness matrix for sandstone (Thomsen, 2000)

CONCLUSIONS

The theoretical study of anisotropic media is important for studying the effect of different type of media on wave propagation. Particularly the effect that fractures, typical of Colombian Foothills.

We expect to continue our study of this concepts and the development of computational tools that simulate wave propagation in general Transversal Inhomogeneous media.

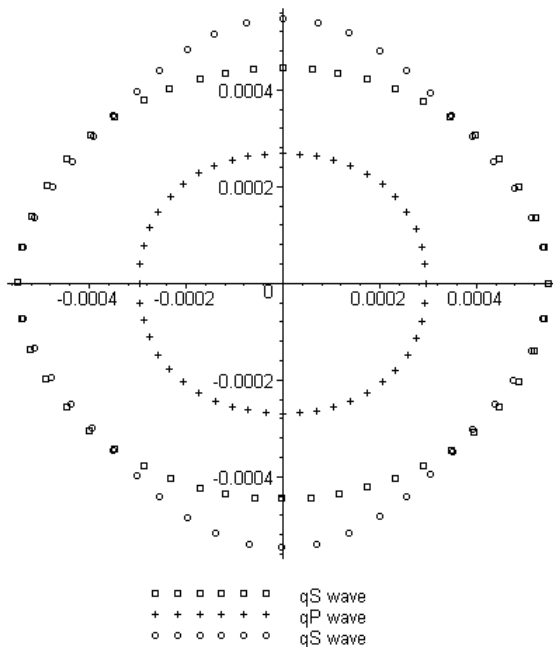


Figure 2. Slowness surfaces for sandstone. Horizontal axis x_3 . Vertical axis x_1 . Slowness given in s/m.

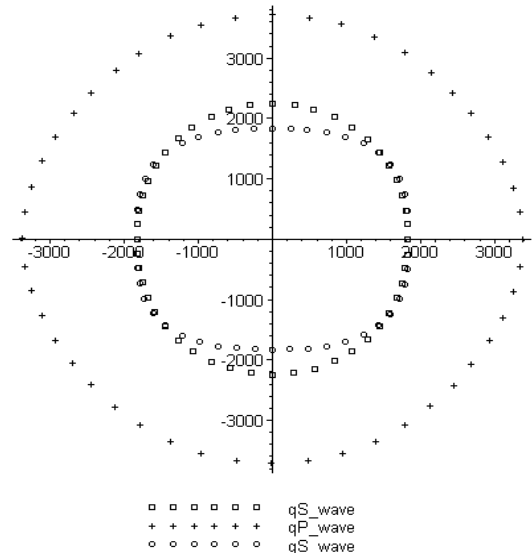


Figure 3. Slowness surfaces for sandstone. Horizontal axis x_3 . Vertical axis x_1 . Phase velocity given in m/s.

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