

## BALKING AND RENEGING IN A VACATION QUEUEING MODEL

K. C. MADAN

Department of Mathematics  
University of Bahrain, P. O. Box 32038  
Isa Town, Bahrain (Arabian Gulf)

**ABSTRACT.** We study a single server queueing system in which arrivals follow a compound Poisson process and the service times of customers have exponential distribution. The system is subject to server vacations occurring randomly in time and in addition the impatient customers resort to balking and renegeing during the server's vacation period. The steady state probability generating functions for the system when the server is available or on vacation have been obtained explicitly as also the average number in the system. Various particular cases of interest have also been derived.

**Key words:** Poisson arrivals, exponential service, steady state, vacation period, balking, renegeing, probability generating function, average number in the system.

**AMS Primary Classification:** 60 K

### 1. INTRODUCTION.

Vacation queueing models as well as queueing systems subject to breakdowns have been studied by many authors including Shanthikumar [10], Manoharan & Krishnamoorthy [9], Madan [8] y Gaver [3], to mention a few. For a complete overview of such models the reader is referred to Doshi [2]. In the present paper we have introduced the phenomenon of balking and renegeing in a vacation queueing model. It can be commonly visualized that some customers may not enter the queue on finding the system down or the server absent from the system while others, who have been waiting in the queue may become impatient and start leaving the system. For balking and renegeing as independent topics, the reader is referred to Haight [5,6], and Barrer [1].

### 2. THE MATHEMATICAL MODEL.

(i) Customers arrive at the system in batches of variable size. Let  $\lambda c_i dt$  ( $i = 1, 2, 3, \dots$ ) denote the first order probability that a batch of  $i$  customers arrives in the short interval of time  $dt$ , where,  $0 \leq c_i \leq 1$  and  $\sum_{i=1}^{\infty} c_i = 1$  and  $\lambda > 0$  is the mean arrival rate of batches and the batches are pre-ordered for service purposes.

(ii) The system possesses a single server which provides one by service and the service times of customers have exponential distribution with mean service time  $1/\mu$ . ( $\mu > 0$ )

(iii) The server enters into vacations from time to time and a vacation may start any time, even while a service is in progress or the server is in the idle state. The server may be deemed to be analogous to a mechanical service channel which is subject to failures occurring randomly in time. The customer whose service is suspended remains at the head of the queue without affecting the total number in the system. The server's period of availability is exponentially distributed with mean length  $1/\alpha$  ( $\alpha > 0$ ). In other words  $\alpha \Delta t$  is the first order probability that the server will cease to work (i.e. the vacation period will begin) during the time interval  $(t, t + \Delta t)$ .

(iv) The length of the vacation period follows an exponential distribution with mean length  $1/\beta$  ( $\beta > 0$ ). In other words,  $\beta \Delta t$  is the first order probability that a vacation will terminate (i.e. the server will join the system) during the time interval  $(t, t + \Delta t)$ .

(v) During the server's vacation not all batches of customers who arrive at the system will join the queue. It has been assumed that an arriving batch joins the queue with probability  $\pi$  and balks (leaves as soon it arrives) with probability  $1 - \pi$ .

(vi) In addition to balking there is reneging (one by one) during the server's vacation period. It has been assumed that  $r \Delta t$  is the first order probability that a customer will renege in the time interval  $(t, t + \Delta t)$  during the server's vacation period.

(vii) Various stochastic processes involved in the system are independent of each other.

## EQUATIONS GOVERNING THE SYSTEM.

We define

$a_n(t) \equiv$  the probability that at time  $t$  there are  $n (\geq 0)$  customers in the system, including one in service, if any and the server is available (i.e. providing service if there is at least one customer or idle if there is none).

$v_n(t) \equiv$  the probability that at time  $t$  there are  $n (\geq 0)$  customers in the system and the server is on vacation.

$p_n(t) \equiv a_n(t) + v_n(t)$  is the probability that at time  $t$  there are  $n (\geq 0)$  customers in the system irrespective of whether the server is available or on vacation.

In order to study the steady state behavior of system, we drop the argument  $t$  in the above definitions of probabilities and directly have the following set of steady

state difference equations:

$$(1) \quad (\lambda + \alpha + \mu)a_n = \lambda \sum_{i=1}^n c_i a_{n-i} + \mu a_{n+1} + \beta v_n \quad n \geq 1$$

$$(2) \quad (\lambda + \alpha)a_0 = \mu a_1 + \beta v_0$$

$$(3) \quad (\lambda\pi + r + \beta)v_n = \lambda\pi \sum_{i=1}^n c_i v_{n-i} + r v_{n+1} + \alpha a_n \quad n \geq 1$$

$$(4) \quad (\lambda\pi + \beta)v_0 = r v_1 + \alpha a_0$$

We define the following probability generating functions:

$$(5) \quad a(z) = \sum_{n=0}^{\infty} a_n z^n; \quad v(z) = \sum_{n=0}^{\infty} v_n z^n$$

$$p(z) = \sum_{n=0}^{\infty} p_n z^n; \quad c(z) = \sum_{n=0}^{\infty} c_n z^n$$

#### 4. THE STEADY STATE SOLUTION.

We multiply equation (1) by  $z^{n+1}$ , sum over  $n$  from 1 to  $\infty$ , and  $z$  times equation (2), use (5) and simplify. Thus we have

$$(6) \quad \{-\lambda z c(z) + (\lambda + \mu + \alpha)z - \mu\}a(z) = \mu(z-1)a_0 + \beta z v(z)$$

Similarly, we multiply equation (3) by  $z^{n+1}$ , sum over  $n$  from 1 to  $\infty$  add  $z$  times equations (4), use (5) and simplify. Then we have

$$(7) \quad \{-\lambda\pi z c(z) + (\lambda\pi + r + \beta)z - r\}v(z) = r(z-1)v_0 + \alpha z a(z)$$

Solving equations (6) and (7) simultaneously for  $a(z)$  and  $v(z)$ , we have

$$(8) \quad a(z) = \frac{(z-1)[\mu a_0 k_2(z) + r v_0 \beta z]}{k_1(z)k_2(z) - \alpha\beta z^2}$$

$$(9) \quad v(z) = \frac{(z-1)[r v_0 k_1(z) + \mu a_0 \alpha z]}{k_1(z)k_2(z) - \alpha\beta z^2}$$

where

$$(10) \quad k_1(z) = -\lambda z c(z) + (\lambda + \alpha + \mu)z - \mu$$

$$k_2(z) = -\lambda\pi z c(z) + (\lambda\pi + r + \mu)z - r$$

Now to determine the only unknowns  $a_0$  and  $v_0$  appearing in the numerators of the right hand sides of equations (8) and (9), we proceed as follows. We note from (5) that  $c(1) = 1$  and, for that matter, (10) yields  $k_1(1) = \alpha$ ,  $k_2(1) = \beta$  and, therefore, the denominators of both  $a(z)$  and  $v(z)$  in (8) and (9) become 0 at  $z = 1$  and, therefore,  $a(z)$  and  $v(z)$  are indeterminate of  $\frac{0}{0}$  form. We will, therefore use

L'Hopital's rule in order to find out  $a(1)$  and  $v(1)$ . Noting that  $c(1) = \sum_{i=1}^{\infty} ic_i$ , we have  $k_1(1) = \alpha + \mu - \lambda \sum_{i=1}^{\infty} ic_i$  and  $k_2(1) = \beta + r - \lambda \pi \sum_{i=1}^{\infty} ic_i$ . Carrying out the L'Hopital's rule on (8) and (9), using the above values, simplifying and taking the limit as  $z \rightarrow 1$ , we have

$$(11) \quad a(1) = \frac{(\mu a_0 + r v_0) \beta}{\alpha(r - \lambda \pi \sum_{i=1}^{\infty} ic_i) + \beta(\mu - \lambda \sum_{i=1}^{\infty} ic_i)}$$

$$(12) \quad v(1) = \frac{(\mu a_0 + r v_0) \alpha}{\alpha(r - \lambda \pi \sum_{i=1}^{\infty} ic_i) + \beta(\mu - \lambda \sum_{i=1}^{\infty} ic_i)}$$

Using (11) and (12), the normalizing condition  $a(1) + v(1) = 1$  yields,

$$(13) \quad \mu a_0 + r v_0 = \frac{\alpha(r - \lambda \pi \sum_{i=1}^{\infty} ic_i) + \beta(\mu - \lambda \sum_{i=1}^{\infty} ic_i)}{(\alpha + \beta)}$$

Now, in view of the fact that in the steady state the overall utilization of the M/M/1 system is  $\frac{\lambda}{\mu}$ , it follows that the probability that there is no customer in the system irrespective of whether the server is available or not is given by

$$(14) \quad a_0 + v_0 = 1 - \frac{\lambda}{\mu}$$

From (13) and (14), we have, on simplifying

$$(15) \quad v_0 = \frac{(\mu - \lambda)(\alpha + \beta) + \alpha(\lambda \pi \sum_{i=1}^{\infty} ic_i - r) + \beta(\lambda \sum_{i=1}^{\infty} ic_i - \mu)}{(\alpha + \beta)(\mu - r)}$$

This is the probability that the system is empty i.e. there is no customer and also the server is not available.

From (14) and (15), we then have

$$(16) \quad a_0 = \frac{\beta \mu^2 + r(\lambda \alpha + \lambda \beta - \mu \beta) - \mu(\pi \alpha + \beta) \lambda \sum_{i=1}^{\infty} ic_i}{\mu(\alpha + \beta)(\mu - r)}$$

This is the probability that there is no customer in the system although the server is available but idle.

Having thus determined  $v_0$  and  $a_0$ , we can substitute their values given by (15) and (16) in (8) and (9) to enable us to completely and explicitly determine  $a(z)$  and  $v(z)$  from which one can determine  $v_n$  and  $a_n$  for  $n \geq 1$  either by expanding (if

possible) and picking up the coefficients of  $z^n$  or by successive differentiations. Thus we have

$$(17) \quad a(z) = \frac{(z-1)[\beta\mu^2 + r(\lambda\alpha + \lambda\beta - \mu\beta) - \mu(\pi\alpha + \beta)\lambda \sum_{i=1}^{\infty} ic_i]k_2(z)}{(\alpha + \beta)(\mu - r)[k_1(z)k_2(z) - \alpha\beta z^2]} + \frac{(z-1)[(\mu - \lambda)(\alpha + \beta) + \alpha(\lambda\pi \sum_{i=1}^{\infty} ic_i - r) + \beta(\lambda \sum_{i=1}^{\infty} ic_i - \mu)]r\beta z}{(\alpha + \beta)(\mu - r)[k_1(z)k_2(z) - \alpha\beta z^2]}$$

$$(18) \quad v(z) = \frac{(z-1)[(\mu - \lambda)(\alpha + \beta) + \alpha(\lambda\pi \sum_{i=1}^{\infty} ic_i - r) + \beta(\lambda \sum_{i=1}^{\infty} ic_i - \mu)]rk_1(z)}{(\alpha + \beta)(\mu - r)[k_1(z)k_2(z) - \alpha\beta z^2]} + \frac{(z-1)[\beta\mu^2 + r(\lambda\alpha + \lambda\beta - \mu\beta) - \mu(\pi\alpha + \beta)\lambda \sum_{i=1}^{\infty} ic_i]\alpha z}{(\alpha + \beta)(\mu - r)[k_1(z)k_2(z) - \alpha\beta z^2]}$$

It should be noted that in view of equations (11), (12), (15) and (16) the stability conditions are given by

$$(19) \quad \lambda \sum_{i=1}^{\infty} ic_i(\alpha\pi + \beta) < \alpha r + \beta\mu \text{ and } r < \mu$$

## PARTICULAR CASES.

### (I) BALKING BUT NO RENEGING DURING VACATIONS

Letting  $r = 0$  equations (15) and (16) yield

$$(20) \quad v_0 = \frac{-\lambda(\alpha + \beta) + \mu\alpha + (\alpha\pi + \beta)\lambda \sum_{i=1}^{\infty} ic_i}{\mu(\alpha + \beta)}$$

$$(21) \quad a_0 = \frac{\beta\mu^2 - \mu(\alpha\pi + \beta)\lambda \sum_{i=1}^{\infty} ic_i}{\mu^2(\alpha + \beta)}$$

with the values of  $v_0$  and  $a_0$  from (20) and (21), the corresponding expressions for  $a(z)$  and  $v(z)$  are determined from equations (17) and (18):

$$(22) \quad a(z) = \frac{(z-1)[\beta\mu^2 - \mu(\pi\alpha + \beta)\lambda \sum_{i=1}^{\infty} ic_i]k_2(z)}{\mu(\alpha + \beta)[k_1(z)k_2(z) - \alpha\beta z^2]}$$

$$(23) \quad v(z) = \frac{(z-1)[\beta\mu^2 - \mu(\pi\alpha + \beta)\lambda \sum_{i=1}^{\infty} ic_i]\alpha z}{\mu(\alpha + \beta)[k_1(z)k_2(z) - \alpha\beta z^2]}$$

where

$$(24) \quad \begin{aligned} k_1(z) &= -\lambda z c(z) + (\lambda + \alpha + \mu)z - \mu \\ k_2(z) &= -\lambda \pi z c(z) + (\lambda \pi + \beta)z \end{aligned}$$

The stability condition in this case would be

$$(25) \quad \lambda \sum_{i=1}^{\infty} i c_i (\alpha \pi + \beta) < \beta \mu$$

## (II) RENEGING BUT NO BALKING DURING VACATIONS

If there is only renegeing but no balking during vacations, then with  $\pi = 1$  equations (15) and (16) yield

$$(26) \quad v_0 = \frac{(\mu - \lambda)(\alpha + \beta) - \alpha r + (\alpha + \beta)\lambda \sum_{i=1}^{\infty} i c_i - \beta \mu}{(\alpha + \beta)(\mu - r)}$$

$$(27) \quad a_0 = \frac{\beta \mu^2 + r(\lambda \alpha + \lambda \beta - \mu \beta) - \mu(\alpha + \beta)\lambda \sum_{i=1}^{\infty} i c_i}{\mu(\alpha + \beta)(\mu - r)}$$

The corresponding expressions for  $a(z)$  and  $v(z)$  determined from (17) and (18) are

$$(28) \quad a(z) = \frac{(z-1)[\beta \mu^2 + r(\lambda \alpha + \lambda \beta - \mu \beta) - \mu(\alpha + \beta)\lambda \sum_{i=1}^{\infty} i c_i]k_2(z)}{(\alpha + \beta)(\mu - r)[k_1(z)k_2(z) - \alpha \beta z^2]} + \frac{(z-1)[(\mu - \lambda)(\alpha + \beta) + \alpha(\lambda \sum_{i=1}^{\infty} i c_i - r) + \beta(\lambda \sum_{i=1}^{\infty} i c_i - \mu)]r \beta z}{(\alpha + \beta)(\mu - r)[k_1(z)k_2(z) - \alpha \beta z^2]}$$

$$(29) \quad v(z) = \frac{(z-1)[(\mu - \lambda)(\alpha + \beta) + \alpha(\lambda \sum_{i=1}^{\infty} i c_i - r) + \beta(\lambda \sum_{i=1}^{\infty} i c_i - \mu)]r k_1(z)}{(\alpha + \beta)(\mu - r)[k_1(z)k_2(z) - \alpha \beta z^2]} + \frac{(z-1)[\beta \mu^2 + r(\lambda \alpha + \lambda \beta - \mu \beta) - \mu(\alpha + \beta)\lambda \sum_{i=1}^{\infty} i c_i]\alpha z}{(\alpha + \beta)(\mu - r)[k_1(z)k_2(z) - \alpha \beta z^2]}$$

where

$$(30) \quad \begin{aligned} k_1(z) &= -\lambda z c(z) + (\lambda + \alpha + \mu)z - \mu \\ k_2(z) &= -\lambda z c(z) + (\lambda \pi + \beta)z - r \end{aligned}$$

The stability conditions in this case are given by

$$(31) \quad \lambda \sum_{i=1}^{\infty} i c_i (\alpha + \beta) < \alpha r + \beta \mu \text{ and } r < \mu$$

**(III) NO RENEGING AND NO BALKING DURING VACATIONS**

In this case, we let  $r = 0$  and  $\pi = 1$  in equations (15) to (18). We thus have

$$(32) \quad v_0 = \frac{\mu\alpha + (\lambda \sum_{i=1}^{\infty} ic_i - \lambda)(\alpha + \beta)}{(\alpha + \beta)\mu}$$

$$(33) \quad a_0 = \frac{\beta\mu - (\alpha + \beta)\lambda \sum_{i=1}^{\infty} ic_i}{(\alpha + \beta)\mu}$$

$$(34) \quad a(z) = \frac{(z-1)[\beta\mu - (\alpha + \beta)\lambda \sum_{i=1}^{\infty} ic_i]k_2(z)}{(\alpha + \beta)[k_1(z)k_2(z) - \alpha\beta z^2]}$$

$$(35) \quad v(z) = \frac{(z-1)[\beta\mu^2 - \mu(\alpha + \beta)\lambda \sum_{i=1}^{\infty} ic_i]\alpha z}{\mu(\alpha + \beta)[k_1(z)k_2(z) - \alpha\beta z^2]}$$

where

$$(36) \quad \begin{aligned} k_1(z) &= -\lambda z c(z) + (\lambda + \alpha + \mu)z - \mu \\ k_2(z) &= -\lambda z c(z) + (\lambda + \beta)z \end{aligned}$$

The stability condition in this case is given by

$$(37) \quad \lambda \sum_{i=1}^{\infty} ic_i(\alpha + \beta) < \beta\mu$$

**(IV) SINGLE ARRIVAL, NO RENEGING AND NO BALKING DURING VACATIONS**

In this case, we have  $c_i = 1$  for  $i = 1$  and  $c_i = 0$  for  $i \neq 1$ . Therefore,  $c(z) = z$ . In addition,  $r = 0$  and  $\pi = 1$ . Thus equations (15) to (18) would yield

$$(38) \quad v_0 = \frac{\alpha}{(\alpha + \beta)}$$

$$(39) \quad a_0 = \frac{\beta\mu - (\alpha + \beta)\lambda}{(\alpha + \beta)\mu}$$

$$(40) \quad a(z) = \frac{(z-1)[\beta\mu - (\alpha + \beta)\lambda]k_2(z)}{(\alpha + \beta)[k_1(z)k_2(z) - \alpha\beta z^2]}$$

$$(41) \quad v(z) = \frac{(z-1)[\beta\mu - (\alpha + \beta)\lambda]\alpha z}{(\alpha + \beta)[k_1(z)k_2(z) - \alpha\beta z^2]}$$

where

$$(42) \quad \begin{aligned} k_1(z) &= -\lambda z^2 + (\lambda + \alpha + \mu)z - \mu \\ k_2(z) &= -\lambda z^2 + (\lambda + \beta)z \end{aligned}$$

The stability condition in this case is given by

$$(43) \quad \lambda(\alpha + \beta) < \beta\mu$$

### (V) SINGLE ARRIVALS AND NO SERVER VACATIONS

In this case,  $\alpha = 0$ ,  $v_0 = 0$  and, for that matter,  $v(z) = 0$ . thus from case (IV) we have, on simplifying

$$(44) \quad a_0 = 1 - \frac{\lambda}{\mu}$$

$$(45) \quad a(z) = \frac{(z-1)(\mu - \lambda)}{-\lambda z^2 + (\lambda + \mu)z - \mu}$$

which can be further simplified to

$$(46) \quad a(z) = \frac{1 - \frac{\lambda}{\mu}}{1 - \frac{\lambda}{\mu}z}$$

The stability condition in this case is reduced to

$$(47) \quad \lambda < \mu$$

The results in (44), (46) and (47) are well-known results. (see Gross and Harris [4], page 67, equations 2.14).

### 6. THE AVERAGE NUMBER IN THE SYSTEM.

Let  $L$  denote the average number in the system. Then we have, on adding (8) and (9),

$$(48) \quad \begin{aligned} p(z) &= a(z) + v(z) \\ &= \frac{(z-1)[\mu a_0(\alpha z + k_2(z)) + r v_0(\beta z + k_1(z))]}{k_1(z)k_2(z) - \alpha\beta z^2} = \frac{n(z)}{d(z)} \quad (\text{say}) \end{aligned}$$

where  $k_1(z)$  and  $k_2(z)$  are given by (10).

Then the mean number in the system is given by  $L = p'(z)$  at  $z = 1$ . It can be easily verified that  $p(z)$  in (48) is indeterminate of the form  $\frac{0}{0}$  at  $z = 1$ , since  $n(1) = d(1) = 0$ . Consequently it can be show (see Kashyap and Chaudhry [7], page 49, and Madan [8]) that on twice using L'Hopital rule

$$(49) \quad p'(1) \rightarrow \frac{d'(1)n''(1) - n'(1)d''(1)}{2[d'(1)]^2} \text{ as } z \rightarrow 1$$

where the primes stand for the differentiation w.r.t.  $z$ .

From (5),  $c(1) = 1$ ,  $c'(1) = \sum_{i=1}^{\infty} ic_i$  and  $c''(1) = \sum_{i=1}^{\infty} i(i-1)c_i$  and with these values equations (10) yields

$$\begin{aligned} k_1(1) &= \alpha, & k_2(1) &= \beta, \\ k_1'(1) &= \alpha + \mu - \lambda \sum_{i=1}^{\infty} ic_i, & k_2'(1) &= \beta + r - \lambda\pi \sum_{i=1}^{\infty} ic_i, \\ k_1''(1) &= -\lambda \sum_{i=1}^{\infty} i(i-1)c_i - 2\lambda \sum_{i=1}^{\infty} ic_i, & k_2''(1) &= -\lambda\pi \sum_{i=1}^{\infty} i(i-1)c_i - 2\lambda\pi \sum_{i=1}^{\infty} ic_i. \end{aligned}$$

Carrying out the derivatives of  $n(z)$  and  $d(z)$  in (48) and using the above values of  $k_1(1)$ , etc. we have, on simplifying

$$\begin{aligned} n'(1) &= (\alpha + \beta)(\mu a_0 + r v_0) \\ n''(1) &= 2[\mu a_0(\alpha + r + \beta - \lambda\pi \sum_{i=1}^{\infty} ic_i) + r v_0(\alpha + \beta + \mu - \lambda \sum_{i=1}^{\infty} ic_i)] \\ d'(1) &= \alpha(r - \lambda\pi \sum_{i=1}^{\infty} ic_i) + \beta(\mu - \lambda \sum_{i=1}^{\infty} ic_i) \\ d''(1) &= 2(\alpha + \mu - \lambda \sum_{i=1}^{\infty} ic_i)(\beta + \rho - \lambda\pi \sum_{i=1}^{\infty} ic_i) - (\alpha\pi + \beta)\lambda(\sum_{i=1}^{\infty} i^2 c_i + \sum_{i=1}^{\infty} ic_i) \end{aligned}$$

With these values, equation (49) finally yields

$$(50) \quad L = \frac{\left\{ \left[ \alpha(r - \lambda\pi \sum_{i=1}^{\infty} ic_i) + \beta(\mu - \lambda \sum_{i=1}^{\infty} ic_i) \right] \left[ 2[\mu a_0(\alpha + r + \beta - \lambda\pi \sum_{i=1}^{\infty} ic_i) + r v_0(\alpha + \beta + \mu - \lambda \sum_{i=1}^{\infty} ic_i)] \right] - \left[ (\alpha + \beta)(\mu a_0 + r v_0) \right] \left[ 2(\alpha + \mu - \lambda \sum_{i=1}^{\infty} ic_i)(\beta + r - \lambda\pi \sum_{i=1}^{\infty} ic_i) - (\alpha\pi + \beta)\lambda(\sum_{i=1}^{\infty} i^2 c_i + \sum_{i=1}^{\infty} ic_i) \right] \right\}}{2 \left[ \alpha(r - \pi\lambda \sum_{i=1}^{\infty} ic_i) + \beta(\mu - \lambda \sum_{i=1}^{\infty} ic_i) \right]^2}$$

The average number in the system for various particular cases obtained from (50) is as follows:

**(I) BALKING BUT NO RENEGING DURING VACATIONS ( $r = 0$ )**

$$(51) \quad L = \frac{\left\{ \left[ -\lambda\alpha\pi \sum_{i=1}^{\infty} ic_i + \beta(\mu - \lambda \sum_{i=1}^{\infty} ic_i) \right] \left[ 2[\mu a_0(\alpha + \beta - \lambda\pi \sum_{i=1}^{\infty} ic_i)] \right] - \mu a_0(\alpha + \beta) \left[ 2(\alpha + \mu - \lambda \sum_{i=1}^{\infty} ic_i) (\beta - \lambda\pi \sum_{i=1}^{\infty} ic_i) - (\alpha\pi + \beta)\lambda \left( \sum_{i=1}^{\infty} i^2 c_i + \sum_{i=1}^{\infty} ic_i \right) \right] \right\}}{2 \left[ -\lambda\alpha\pi \sum_{i=1}^{\infty} ic_i + \beta(\mu - \lambda \sum_{i=1}^{\infty} ic_i) \right]^2}$$

**(II) RENEGING BUT NO BALKING DURING VACATIONS ( $\pi = 1$ )**

$$(52) \quad L = \frac{\left\{ \left[ \alpha(r - \lambda \sum_{i=1}^{\infty} ic_i + \beta(\mu - \lambda \sum_{i=1}^{\infty} ic_i)) \right] \left[ 2[\mu a_0(\alpha + r + \beta - \lambda \sum_{i=1}^{\infty} ic_i) + rv_0(\alpha + \beta + \mu - \lambda \sum_{i=1}^{\infty} ic_i)] \right] - \left[ (\alpha + \beta) (\mu a_0 + rv_0) \right] \left[ 2(\alpha + \mu - \lambda \sum_{i=1}^{\infty} ic_i)(\beta + r - \lambda \sum_{i=1}^{\infty} ic_i) - (\alpha + \beta)\lambda \left( \sum_{i=1}^{\infty} i^2 c_i + \sum_{i=1}^{\infty} ic_i \right) \right] \right\}}{2 \left[ \alpha(r - \lambda \sum_{i=1}^{\infty} ic_i) + \beta(\mu - \lambda \sum_{i=1}^{\infty} ic_i) \right]^2}$$

**(III) NO RENEGING AND NO BALKING DURING VACATIONS ( $r = 0, \pi = 1$ )**

$$(53) \quad L = \frac{\left\{ \left[ -\lambda\alpha \sum_{i=1}^{\infty} ic_i + \beta(\mu - \lambda \sum_{i=1}^{\infty} ic_i) \right] \left[ 2[\mu a_0(\alpha + \beta - \lambda \sum_{i=1}^{\infty} ic_i)] \right] - \mu a_0(\alpha + \beta) \left[ 2(\alpha + \mu - \lambda \sum_{i=1}^{\infty} ic_i) (\beta - \lambda \sum_{i=1}^{\infty} ic_i) - (\alpha + \beta)\lambda \left( \sum_{i=1}^{\infty} i^2 c_i + \sum_{i=1}^{\infty} ic_i \right) \right] \right\}}{2 \left[ -\lambda\alpha \sum_{i=1}^{\infty} ic_i + \beta(\mu - \lambda \sum_{i=1}^{\infty} ic_i) \right]^2}$$

**(IV) SINGLE ARRIVALS, NO RENEGING AND NO BALKING DURING VACATIONS ( $r = 0$  and  $\pi = 1$ )**

$c_i = 1$  for  $i = 1$  and  $c_i = 0$  for  $i \neq 1$

$$(54) \quad L = \frac{\left\{ \left[ -\lambda\alpha + \beta(\mu - \lambda) \right] \left[ \mu a_0(\alpha + \beta - \lambda) \right] - \mu a_0(\alpha + \beta) \left[ (\alpha + \mu - \lambda)(\beta - \lambda) - (\alpha + \beta)\lambda \right] \right\}}{\left[ -\lambda\alpha + \beta(\mu - \lambda) \right]^2}$$

SINGLE ARRIVALS AND NO SERVER VACATIONS ( $\alpha = 0$ )

$$(55) \quad L = \frac{[\beta(\mu - \lambda)] [\mu a_0(\beta - \lambda)] - \mu a_0 \beta [(\mu - \lambda)(\beta - \lambda) - \beta \lambda]}{[\beta(\mu - \lambda)]^2}$$

where  $a_0 = 1 - \frac{\lambda}{\mu}$

On letting  $\frac{1}{\beta} \rightarrow 0$  and further simplifying, (55) becomes

$$(56) \quad L = \frac{\frac{\lambda}{\mu}}{1 - \frac{\lambda}{\mu}}$$

which agrees with the known result. (see Gross and Harris [4])

## REFERENCES

- [1] Barrer, D. Y. (1957), *Queueing with Impatient Customers and Indifferent Clerks*, Oper. Res. 5, 644-649.
- [2] Doshi, B. T. (1986), *Queueing Systems with Vacation- a Survey*, Queueing System 1, 29-66.
- [3] Gaver, D. P (1962), *A waiting line with interrupted service including Priorities*, J. Roy. Statist. B 24, 73-90.
- [4] Gross, D and Harris, C. M. (1985), *The fundamentals of Queueing Theory*, second edition, Jons Wiley and Sons.
- [5] Haight, F. A. (1957), *Queueing with Balking*, Biometrika 46, 360-369.
- [6] Haight, F.A. (1959), *Queueing with Reneging*, Metrika 2, 186-197.
- [7] Kashyap, B.R.K. and Chaudhry, M.L (1988), *An introduction to Queueing Theory*, A.& A. Publications, Kingston, Ontario, Canada.
- [8] Madan, K.C (1991), *On a  $M^{[x]}/M^{[b]}/1$  System with general vacations times*, International Journal of Information and Management Sciences 2, 51-60.
- [9] Manoharan, M. and Krishnamoorthy, A (1989), *A finite capacity M/G/1 Queue with vacation*, Tamkang Journal of Management Sciences 10 No. 2, 37-45.
- [10] Shantlukumar, J.G. (1988), *On Stochastic Descomposition in the M/G/1 type queues with generalized vacations*, Oper. Res 36, 566-569.