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The Gamma Odd Weibull Generalized-G Family of Distributions: Properties and Applications

La familia de distribuciones Gamma Odd Weibull Generalized-G: propiedades y aplicaciones

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Abstract

A new generalized family of models called the Gamma Odd Weibull Generalized-G (GOWG-G) family of distributions is proposed and studied. Properties of the new family of distributions including moments, conditional moments, distribution of the order statistics and Rényi entropy are derived. Maximum likelihood estimation technique is used to estimate the model parameters. Four special cases of the GOWG-G family of distributions are considered. A simulation study was carried out to examine the accuracy of the Maximum Likelihood Estimates (MLE) of the parameters.

***Key words*:** Generalized-G distribution; Generalized transformation; Generalized distribution; Maximum likelihood estimation.

Resumen

Una nueva familia generalizada de modelos llamada Gamma Odd Weibull Se propone y estudia la familia de distribuciones Generalized-G (GOWG-G). Propiedades de la nueva familia de distribuciones incluyendo momentos, condicional Se derivan momentos, estadísticas de distribución del orden y entropía de Rényi. La técnica de estimación de máxima verosimilitud se utiliza para estimar los parámetros del modelo. Cuatro casos especiales de la familia de distribuciones GOWG-G son considerado. Un estudio de simulación para examinar el sesgo y el error cuadrático medio de los estimadores de máxima verosimilitud y aplicaciones a conjuntos de datos reales para ilustra la utilidad de la distribución generalizada.

***Palabras clave*:** Distribución G generalizada; Transformación generalizada; Distribución generalizada; Estimación de máxima verosimilitud.

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1. Introduction

There are several extensions and modifications of the Weibull distribution in the literature meant to accomodate both monotone and non-monotone hazard rate functions. Extended and generalized distributions for use in the modeling of monotone and non-monotone hazard rate functions, particularly in reliability and survival analysis are very crucial. Generalized distributions including modified Weibull distributions (see Oluyede et al. (2015) for details) play very prominent role in the modelling of monotone and non-monotone hazard rate functions, exploring tail variation and improved goodness-of-fit tests.

Over the years, several new and very useful families of distributions have been developed and new generalized distributions obtained by adding one or more parameters to existing distributions in the statistical literature including the Transform-transformer (T-X) by Alzaghal et al. (2013), Weibull-G by Bourguignon et al. (2014), beta-G by Eugene et al. (2002), McDonald-G (Mc-G) by Alexander et al. (2012) and Lomax generator by Cordeiro et al. (2014), Kumaraswamy odd log-logistic family by Alizadeh, Emadi, Doostparast, Cordeiro, Ortega & Pescim (2015), Kumaraswamy Marshall-Olkin family by Alizadeh, Tahir, Cordeiro, Mansoor, Zubair & Hamedani (2015). Peter et al. (2021) developed and studied the gamma odd Burr III family of distributions. Cordeiro et al. (2013) introduced a class of distributions called the exponentiated generalized (EG) class of distributions. The process of finding an adequate model to make inferences is a very important problem in statistical modeling. Barreto-Souza et al. (2013) stated that adding parameters to an established baseline distribution is a well established technique for obtaining more flexible new families of distributions.

The generalized-G (G-G) class of distributions extended several well know distributions including distributions in the literature such as the exponential, Weibull, gamma, Fréchet and Gumbel distributions. An attractive feature about the model is that the extra parameter that is introduced can control both tail weight and possibly adding entropy to the density function, depending on the resulting distribution.

Let X be a random variable with cumulative distribution function (cdf) $G(x;\xi)$ depending on the parameter vector ξ. The cdf and probability density function (pdf) of the generalized-G (G-G) are given by

$$K(x;\xi) = 1-(1-G(x;\xi))^{\alpha} \quad \text{and} \quad k(x;\xi) = \alpha g(x;\xi)(1-G(x;\xi))^{\alpha-1}, \tag{1}$$

for $\alpha > 0$, $x \in \mathcal{R}$. The corresponding hazard rate function (hrf) is given by

$$h_{_K}(x;\xi) = \alpha g(x;\xi)(1-G(x;\xi))^{-1}. \tag{2}$$

Alzaghal et al. (2013) defined the $T-X$ family of distributions given by

$$F(x) = \int_a^{W(G(x))} r(t)dt, \tag{3}$$

where $W(G(x))$ satisfies the following conditions: (1) $W(G(x)) \in [a,b]$, for $-\infty \leq a < x < b < \infty$, (2) $W(G(x))$ is diifferentiable and monotonically non-decreasing, and (3) $W(G(x)) \to a$ as $x \to -\infty$ and $W(G(x)) \to b$ as $x \to \infty$. In this note, we consider the transformation $W(G(x;\xi)) = \frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}$ for the baseline cumulative distribution function (cdf) $G(x;\xi)$. In fact this paper is concerned with the development and extension of the gamma operator via the odd Weibull generalized-G transformation by introducing additional shape parameters to obtain a new generalized distribution called the gamma odd Weibull generalized-G (GOWG-G) family of distributions. A comparison of special cases of the new members of the family of distributions with several other distributions in the literature via several goodness-of-fit statistics is conducted in order to establish the effectiveness and usefulness of this new distribution.

The new distribution allows for modeling of asymmetric, heavy and long tailed, left skewed, reverse-J shapes, as well for special models that provide better fits than the baseline distribution. The general motivations for the construction and development of GOWG-G family of distributions in practice are:

- to generalize the odd Weibull-G and odd Weibull generalized-G families of distributions;
- to develop distributions that can readily deal with or handle monotone and non-monotone hazard rate functions;
- to obtain very flexible distributions that take into consideration not only shape but also skewness, kurtosis and tail variation, as well as to improve goodness-of-fit to real data with wider applications in several areas including environmental sciences, reliability, and actuarial sciences.

The results in this note are organized in the following manner. Section 2 contain the new GOWG-G family of distributions and its sub-models, hazard rate function and the quantile function. In Section 3, moments and generating function, conditional and incomplete moments are presented. In Section 4, the distribution of order statistics and Rényi entropy are presented. Section 5 contain the estimation of the parameters of the GOWG-G family of distributions via the method of maximum likelihood, followed by a Monte Carlo simulation study to examine the bias and mean square error of the maximum likelihood estimators in Section 6. Some applications to real data sets are given in Section 7, followed by concluding remarks in Section 8, and future work in Section 8.

2. The Model, Sub-Models, Hazard Rate and Quantile Functions

The derivation of some of the statistical properties of the gamma odd Weibull generalized-G (GOWG-G) family of distributions including sub-models, expansion of the density, hazard rate function, and quantile function are presented in this section.

2.1. The Model

Let us first consider the cdf and pdf of the odd Weibull generalized-G (OWG-G) family of distributions given by

$$\begin{aligned} F(x;\beta,\alpha,\xi) &= \int_0^{\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}} \beta t^{\beta-1} e^{-t^{\beta}} dt \\ &= 1-\exp\left(-\left(\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}\right)^{\beta}\right) \end{aligned} \tag{4}$$

and

$$f(x;\beta,\alpha,\xi) = \alpha\beta \frac{g(x;\xi)}{\left[\overline{G}(x;\xi)\right]^{\alpha+1}} \left(\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}\right)^{\beta-1} e^{\left(-\left[\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}\right]^{\beta}\right)}, \tag{5}$$

respectively, for $\alpha, \beta > 0$ and parameter vector ξ. If a random variable X has the OWG-G family of distributions, we write $X \sim OWG-G(\alpha,\beta,\xi)$. Note that if T is a random variable denoting the lifetime of a system with distribution function $K(x;\xi) = 1-\overline{G}^{\alpha}(x;\xi)$ and the random variable X is the odds ratio, then the risk that the system with lifetime T will fail at time x is given by $\frac{K(x;\xi)}{1-K(x;\xi)} = \frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}$. Now, suppose we model the randomness of the odds ratio via the Weibull distribution with shape parameter $\beta > 0$, that is, $r(t;\beta) = \beta t^{\beta-1} e^{-t^{\beta}}$, for $t > 0$, then the cdf of the random variable X is the new $OWG-G(\beta,\alpha,\xi)$, that is,

$$P(X \leq x) = R\left(\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}\right) = F(x;\beta,\alpha,\xi). \tag{6}$$

Note that when $\alpha = 1$, we obtain the odd Weibull-G (OW-G) family of distributions (Bourguignon et al., 2014) with the pdf

$$f(x;\beta,\xi) = \beta \frac{g(x;\xi)}{\left[\overline{G}(x;\xi)\right]^{2}} \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\beta-1} \exp\left(-\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{\beta}\right), \tag{7}$$

for $\beta > 0$, and parameter vector ξ.

The cdf and pdf of the proposed gamma odd Weibull generalized-G (GOWG-G) family of distributions are given by

$$\begin{aligned} F(x;\alpha,\beta,\delta,\xi) &= 1-\frac{1}{\Gamma(\delta)}\int_0^{-\log\left(1-e^{-\left(\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}\right)^{\beta}}\right)} t^{\delta-1}e^{-t}dt \\ &= 1-\frac{\gamma\left(-\log\left[1-e^{-\left(\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}\right)^{\beta}}\right],\delta\right)}{\Gamma(\delta)} \end{aligned} \tag{8}$$

and

$$f(x;\alpha,\beta,\delta,\xi) = \frac{\alpha\beta}{\Gamma(\delta)}\frac{g(x;\xi)}{\left[\overline{G}(x;\xi)\right]^{\alpha+1}}\left(-\log\left[1-e^{-\left(\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}\right)^{\beta}}\right]\right)^{\delta-1} \times \left[\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}\right]^{\beta-1}\exp\left(-\left[\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}\right]^{\beta}\right), \quad (9)$$

respectively, for $\alpha,\beta,\delta>0$ and parameter vector ξ. If a random variable X has the GOWG-G family of distributions, we write $X\sim GOWG-G(\beta,\alpha,\delta,\xi)$.

2.2. Sub-models of the GOWG-G Family of Distributions

In this subsection, some sub-families of the GOWG-G family of distributions are presented.

- When $\delta=1$, we obtain the odd Weibull generalized-G (OWG-G) family of distributions with pdf

$$f(x;\alpha,\beta,\xi) = \frac{\alpha\beta g(x;\xi)}{\left[\overline{G}(x;\xi)\right]^{\alpha+1}}\left(\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}\right)^{\beta-1} e^{\left(-\left[\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}\right]^{\beta}\right)} \quad (10)$$

  for $\alpha,\beta>0$, and parameter vector ξ. This is a new family of distributions.

- When $\beta=1$, we obtain the gamma odd exponential generalized-G (GOEG-G) family of distributions wth pdf

$$f(x;\alpha,\delta,\xi) = \frac{\alpha}{\Gamma(\delta)}\left(-\log\left[1-e^{-\left(\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}\right)}\right]\right)^{\delta-1}\frac{g(x;\xi)}{\left[\overline{G}(x;\xi)\right]^{\alpha+1}} \times \exp\left(-\left[\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}\right]\right) \quad (11)$$

  for $\alpha,\delta>0$ and parameter vector ξ. This is a new family of distributions.

- When $\beta=2$, we obtain the gamma odd Rayleigh generalized-G (GORG-G) family of distributions with the pdf

$$f(x;\alpha,\delta,\xi) = \frac{2\alpha}{\Gamma(\delta)}\frac{g(x;\xi)}{\left[\overline{G}(x;\xi)\right]^{\alpha+1}}\left(-\log\left[1-e^{-\left(\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}\right)^{2}}\right]\right)^{\delta-1} \times \left(\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}\right)\exp\left(-\left[\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}\right]^{2}\right) \quad (12)$$

  for $\alpha,\delta>0$ and parameter vector ξ. This is a new family of distributions.

- When $\alpha = \beta = 1$, we obtain the gamma odd exponential-G (GOE-G) family of distributions with the pdf

$$\begin{aligned} f(x;\delta,\xi) &= \frac{1}{\Gamma(\delta)}\frac{g(x;\xi)}{\left[\overline{G}(x;\xi)\right]^2}\left(-\log\left[1-e^{-\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)}\right]\right)^{\delta-1} \\ &\times \exp\left(-\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]\right) \end{aligned} \tag{13}$$

for $\delta > 0$ and parameter vector ξ. This is a new family of distributions.

- When $\alpha = \delta = 1$, we obtain the odd Weibull-G (OW-G) family of distributions (Bourguignon et al., 2014) with the pdf

$$f(x;\beta,\xi) = \beta\frac{g(x;\xi)}{\left[\overline{G}(x;\xi)\right]^2}\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\beta-1}\exp\left(-\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{\beta}\right) \tag{14}$$

for $\beta > 0$ and parameter vector ξ.

- If $\alpha = \delta = 1$, and $\beta = 1$, we obtain the odd exponential-G (OE-G) family of distributions with the pdf

$$f(x;\xi) = \frac{g(x;\xi)}{[\overline{G}(x;\xi)]^2}\exp\left(-\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]\right), \tag{15}$$

for the parameter vector ξ.

- When $\alpha = \delta = 1$, and $\beta = 2$, we obtain the odd Rayleigh-G (OR-G) family of distributions with the pdf

$$f(x;\xi) = 2\frac{g(x;\xi)}{\left[\overline{G}(x;\xi)\right]^2}\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]\exp\left(-\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^2\right) \tag{16}$$

for the parameter vector ξ.

- When $\alpha = 1$ and $\beta = 2$, we have gamma odd Rayleigh-G (GOR-G) family of distributions with pdf

$$\begin{aligned} f(x;\delta,\xi) &= \frac{2}{\Gamma(\delta)}\frac{g(x;\xi)}{\left[\overline{G}(x;\xi)\right]^2}\left(-\log\left[1-e^{-\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\beta}}\right]\right)^{\delta-1} \\ &\times \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)\exp\left(-\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^2\right) \end{aligned} \tag{17}$$

for $\delta > 0$ and parameter vector ξ.

- If $\delta = 1$, and $\beta = 2$, we have the odd Rayleigh generalized-G (ORG-G) family of distributions with pdf

$$f(x;\alpha,\xi) = 2\alpha\left(\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}\right)\frac{g(x;\xi)}{[\overline{G}(x;\xi)]^{\alpha+1}}e^{-\left(\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}\right)^2} \tag{18}$$

for $\alpha > 0$, and parameter ξ. This is a new family of distributions.

- If $\alpha = 1$, we obtain the gamma odd Weibull-G (GOW-G) family of distributions with pdf

$$\begin{aligned} f(x;\beta,\delta,\xi) &= \frac{\beta}{\Gamma(\delta)} \frac{g(x;\xi)}{\left[\overline{G}(x;\xi)\right]^2} \left(-\log\left[1-e^{-\left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\beta}}\right]\right)^{\delta-1} \\ &\times \left(\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right)^{\beta-1} \exp\left(-\left[\frac{G(x;\xi)}{\overline{G}(x;\xi)}\right]^{\beta}\right) \end{aligned} \tag{19}$$

for $\beta, \delta > 0$ and parameter vector ξ. This is a new family of distributions.

We also note that different baseline cdf $G(x;\xi)$ lead to several new gamma generalized distributions as sub-models. Furthermore, a number of sub-models are possible by changing the parameter vector ξ, and these special cases as well as corresponding parameters are presented in Table 1.

Table 1: Distributions and corresponding $G(x;\xi)/\bar{G}(x;\xi)$ functions.

Distribution	$G(x;\xi)/\bar{G}(x;\xi)$	$\Psi = (\alpha,\beta,\delta,\xi)$	GOWG-G
Uniform	$x/(\theta - x)$	$(\alpha,\beta,\delta,\theta)$	GOWG-Uniform
Exponential	$e^{\lambda x} - 1$	$(\alpha,\beta,\delta,\lambda)$	GOWG-Exponential
Weibull	$e^{\lambda x^{\gamma}} - 1$	$(\alpha,\beta,\delta,\lambda,\gamma)$	GOWG-Weibull
Fréchet	$\left(e^{\lambda x^{\gamma}} - 1\right)^{-1}$	$(\alpha,\beta,\delta,\lambda,\gamma)$	GOWG-Fréchet
Half-logistic	$(e^{x} - 1)/2$	(α,β,δ)	GOWG-Half-logistic
Power function	$\left[(\theta x)^{-k} - 1\right]^{-1}$	$(\alpha,\beta,\delta,\theta,k)$	GOWG-Power function
Pareto	$(x/\theta)^{k} - 1$	$(\alpha,\beta,\delta,\theta,k)$	GOWG-Pareto
Burr XII	$[1 - x^{c}]^{k} - 1$	$(\alpha,\beta,\delta,c,k)$	GOWG-Burr XII
Log-logistic	$[1 - x^{c}] - 1$	(α,β,δ,c)	GOWG-Log-logistic
Lomax	$[1 - x]^{k} - 1$	(α,β,δ,k)	GOWG-Lomax
Gumbel	$\left[e^{e^{-\frac{x-\mu}{\sigma}}} - 1\right]^{-1}$	$(\alpha,\beta,\delta,\mu,\sigma)$	GOWG-Gumbel
Normal	$\frac{\Phi((x-\mu)/\sigma)}{[1-\Phi((x-\mu)/\sigma)]}$	$(\alpha,\beta,\delta,\mu,\sigma)$	GOWG-Normal
Kumaraswamy	$(1 - x^{a})^{-b} - 1$	$(\alpha,\beta,\delta,a,b)$	GOWG-Kumaraswamy
Modified Exponential	$e^{\theta x e^{\lambda x}} - 1$	$(\alpha,\beta,\delta,\theta,\lambda)$	GOWG-Exponential

2.3. Hazard Rate and Quantile Functions

In this section, we present the hazard rate and quantile functions of the GOWG-G family of distributions. The hazard rate function (hrf) of the GOWG-G family of distributions is given by

$$h_F(x;\alpha,\beta,\delta,\xi) = \frac{\alpha\beta\frac{g(x;\xi)}{[\overline{G}(x;\xi)]^{\alpha+1}}\left(\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}(x;\xi)}\right)^{\beta-1} e^{\left(-\left[\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}\right]^{\beta}\right)}}{\gamma\left(1-e^{-\left(\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}\right)^{\beta}},\delta\right)} \times \left(-\log\left[1-e^{-\left(\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}\right)^{\beta}}\right]\right)^{\delta-1}. \quad (20)$$

The quantile function of the GOWG-G family of distributions is obtained by solving the non-linear equation:

$$F(Q(u)) = 1 - \frac{\gamma\left(-\log\left[1-e^{-\left(\frac{1-\overline{G}^{\alpha}(Q(u))}{\overline{G}^{\alpha}(Q(u))}\right)^{\beta}}\right],\delta\right)}{\Gamma(\delta)} = u,$$

for $0 \leq u \leq 1$, that is,

$$-\log\left[1-e^{-\left(\frac{1-\overline{G}^{\alpha}(Q(u))}{\overline{G}^{\alpha}(Q(u))}\right)^{\beta}}\right] = \gamma^{-1}\left((1-u)\,\Gamma(\delta),\delta\right).$$

Equivalently,

$$e^{-\left(\overline{G}^{-\alpha}(Q(u))-1\right)^{\beta}} = 1-e^{-\gamma^{-1}((1-u)\Gamma(\delta),\delta)}.$$

Now,

$$\overline{G}^{-\alpha}(Q(u)) = 1+\left[-\log\left(1-e^{-\gamma^{-1}((1-u)\Gamma(\delta),\delta)}\right)\right]^{1/\beta},$$

and

$$\overline{G}(Q(u)) = \left(1+\left[-\log\left(1-e^{-\gamma^{-1}((1-u)\Gamma(\delta),\delta)}\right)\right]^{1/\beta}\right)^{-1/\alpha}.$$

Consequently, the quantile function for the GOWG-G family of distributions is given by

$$Q_G(u;\alpha,\beta,\delta,\xi) = G^{-1}\left[1-\left(1+\left[-\log\left(1-e^{-\gamma^{-1}((1-u)\Gamma(\delta),\delta)}\right)\right]^{1/\beta}\right)^{-1/\alpha}\right]. \quad (21)$$

It follows therefore that random numbers can be generated from the GOWG-G family of distributions based on equation (21), for specified baseline cdf G.

2.4. Some Special Cases

In this section, we consider some special cases of the GOWG-G family of distributions, specifically when the baseline distribution function $G(x;\xi)$ are uniform, log-logistic, logistic and Weibull distributions, respectively. There are several new sub-models that can be readily obtained from these special cases for selected values of the model parameters. Plots of the density functions as well as the hazard rate functions for these special cases are presented.

2.4.1. GOWG-Uniform Distribution

Suppose the cdf and pdf of the baseline distribution are given by $G(x;\theta) = \frac{x}{\theta}$ and $g(x;\theta) = \frac{1}{\theta}$ for $0 < x < \theta < \infty$. Then, the new GOWG-Uniform distribution has the cdf and pdf given by

$$F(x;\alpha,\beta,\delta,\theta) = 1 - \frac{\gamma\left(-\log\left(1 - e^{-((1-\frac{x}{\theta})^{-\alpha}-1)^{\beta}}\right), \delta\right)}{\Gamma(\delta)}, \tag{22}$$

and

$$\begin{aligned} f(x;\alpha,\beta,\delta,\theta) &= \frac{\alpha\beta}{\theta\Gamma(\delta)} \frac{\left(-\log\left(1 - e^{-((1-\frac{x}{\theta})^{-\alpha}-1)^{\beta}}\right)\right)^{\delta-1} \left((1-\frac{x}{\theta})^{\alpha}-1\right)^{\beta-1}}{(1-\frac{x}{\theta})^{\alpha+1}} \\ &\times \exp\left(-\left(\left(1-\frac{x}{\theta}\right)^{-\alpha}-1\right)^{\beta}\right), \end{aligned} \tag{23}$$

for $\alpha, \beta, \delta > 0$, and $0 < x < \theta < \infty$. The survival function and hrf are given by

$$S(x;\alpha,\beta,\delta,\theta) = \frac{\gamma\left(-\log\left(1 - e^{-((1-\frac{x}{\theta})^{-\alpha}-1)^{\beta}}\right), \delta\right)}{\Gamma(\delta)}, \tag{24}$$

and

$$h_F(x) = \frac{\frac{\alpha\beta}{\theta} \frac{\left(-\log\left(1-e^{-((1-\frac{x}{\theta})^{-\alpha}-1)^{\beta}}\right)\right)^{\delta-1}\left((1-\frac{x}{\theta})^{\alpha}-1\right)^{\beta-1}}{(1-\frac{x}{\theta})^{\alpha+1}} e^{-((1-\frac{x}{\theta})^{-\alpha}-1)^{\beta}}}{\gamma\left(-\log\left(1-e^{-((1-\frac{x}{\theta})^{-\alpha}-1)^{\beta}}\right), \delta\right)}. \tag{25}$$

Figure 1 represents the plots of density function and hrf of the GOWG-Uniform distribution for several combinations of the parameters α, β, δ, and θ.

The pdfs of GOWG-Uniform distribution takes on various shapes, including uni-modal, right-skewed and reverse-J. Further, hazard rate functions of GOWG-Uniform distribution exhibit decreasing, increasing, bathtub and upside-down bathtub followed by bathtub shapes.

2.4.2. GOWG-Log-logistic Distribution

Suppose the cdf and pdf of the baseline distribution are given by $G(x;c) = 1 - (1+x^c)^{-1}$ and $g(x;c) = cx^{c-1}(1+x^c)^{-2}$ for $c > 0$ and $x > 0$. The new GOWG-Log-logistic distribution has cdf and pdf given by

$$F(x;\alpha,\beta,\delta,c) = 1 - \frac{\gamma\left(-\log\left(1 - e^{-[(1+x^c)^{\alpha}-1]^{\beta}}\right), \delta\right)}{\Gamma(\delta)}, \tag{26}$$

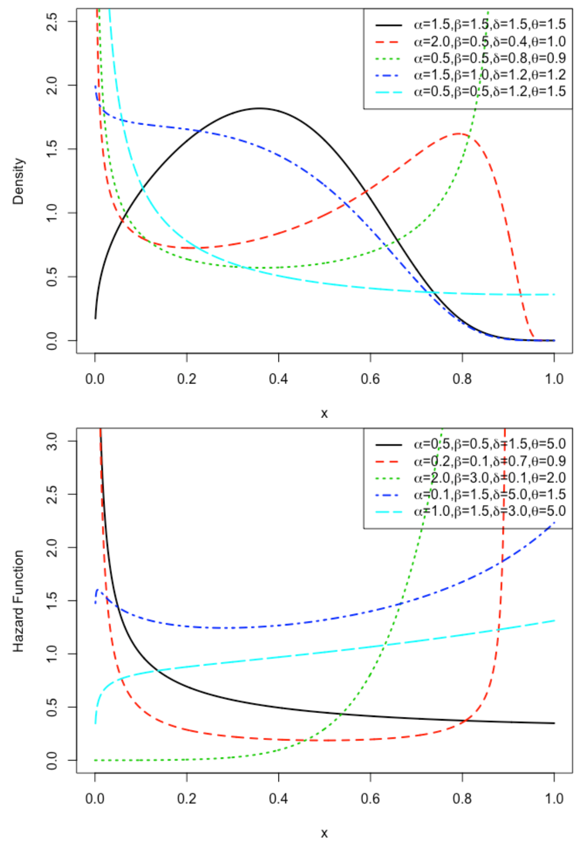


FIGURE 1: Plots of the pdf and hrf for the GOWG-Uniform distribution

and

$$\begin{aligned} f(x;\alpha,\beta,\delta,c) &= \frac{\alpha\beta c x^{c-1}(1+x^c)^{\alpha-1}}{\Gamma(\delta)}\left(-\log\left(1-e^{-[(1+x^c)^\alpha-1]^\beta}\right)\right)^{\delta-1} \\ &\times \left((1+x^c)^\alpha-1\right)^{\beta-1} e^{\left(-[(1+x^c)^\alpha-1]^\beta\right)}, \end{aligned} \tag{27}$$

for $\alpha, \beta, \delta, c > 0$, and $x > 0$. The GOWG-Log-logistic survival function and hrf are given by

$$S(x;\alpha,\beta,\delta,c) = \frac{\gamma\left(-\log\left(1-e^{-[(1+x^c)^\alpha-1]^\beta}\right),\delta\right)}{\Gamma(\delta)}, \tag{28}$$

and

$$\begin{aligned} h_F(x) &= \alpha\beta c x^{c-1}(1+x^c)^{\alpha-1}\left(-\log\left(1-e^{-[(1+x^c)^\alpha-1]^\beta}\right)\right)^{\delta-1} \\ &\times \left((1+x^c)^\alpha-1\right)^{\beta-1} e^{\left(-[(1+x^c)^\alpha-1]^\beta\right)} \\ &\times \left(\gamma\left(-\log\left(1-e^{-[(1+x^c)^\alpha-1]^\beta}\right),\delta\right)\right)^{-1}. \end{aligned} \tag{29}$$

Figure 2 illustrates the graphs of the density function and `hrf` of the GOWG-Log-logistic distribution using several combinations of the parameters α, β, δ, and c.

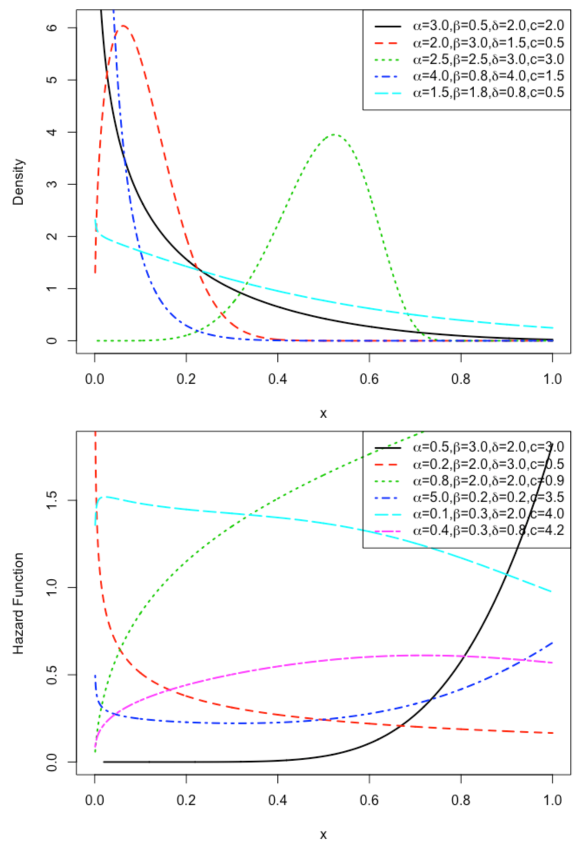


FIGURE 2: Plots of the pdf and hrf for the GOWG-Log-logistic distribution

In the GOWG-Log-logistic distribution, the pdf appears to be in various shapes, such as unimodal, reverse-J and left- or right-skewed. The plots of the GOWG-Log-logistic distribution hrf shows decreasing, increasing, bathtub and upside-down bathtub shapes.

2.4.3. GOWG-Logistic Distribution

Suppose the cdf and pdf of the baseline distribution are given by $G(x;\sigma) = (1+\exp(-x/\sigma))^{-1}$ and $g(x;\sigma) = \frac{\exp(-x/\sigma)}{\sigma}\left(1+\exp(-x/\sigma)\right)^{-2}$ for $\sigma > 0$. Then the new GOWG-Logistic distribution has cdf and pdf given by

$$F(x;\alpha,\beta,\delta,\sigma) = 1 - \frac{\gamma\left(-\log\left(1 - e^{-\left(\left[1-(1+\exp(-x/\sigma))^{-1}\right]^{-\alpha}-1\right)^{\beta}}\right), \delta\right)}{\Gamma(\delta)}, \tag{30}$$

and

$$\begin{aligned} f(x;\alpha,\beta,\delta,\sigma) &= \frac{\alpha\beta e^{(-x/\sigma)}\left(1+e^{(-x/\sigma)}\right)^{-2}}{\sigma\Gamma(\delta)} \\ &\times \left(-\log\left(1-e^{-\left([1-(1+\exp(-x/\sigma))^{-1}]^{-\alpha}-1\right)^{\beta}}\right)\right)^{\delta-1} \\ &\times (1-(1+e^{(-x/\sigma)})^{-1})^{-\alpha-1}e^{\left(-\left([1-(1+e^{(-x/\sigma)})^{-1}]^{-\alpha}-1\right)^{\beta}\right)}, \end{aligned} \tag{31}$$

for $\alpha,\beta,\delta,\sigma > 0$. The survival function and hrf are given by

$$S(x;\alpha,\beta,\delta,\sigma) = \frac{\gamma\left(-\log\left(1-e^{-\left([1-(1+\exp(-x/\sigma))^{-1}]^{-\alpha}-1\right)^{\beta}}\right),\delta\right)}{\Gamma(\delta)}, \tag{32}$$

and

$$\begin{aligned} h_F(x) &= \frac{\frac{\alpha\beta}{\sigma}e^{(-x/\sigma)}\left(1+e^{(-x/\sigma)}\right)^{-2}\left(-\log\left(1-e^{-\left([1-(1+\exp(-x/\sigma))^{-1}]^{-\alpha}-1\right)^{\beta}}\right)\right)^{\delta-1}}{\gamma\left(-\log\left(1-e^{-([1-(1+\exp(-x/\sigma))^{-1}]^{-\alpha}-1)^{\beta}}\right),\delta\right)} \\ &\times (1-(1+e^{(-x/\sigma)})^{-1})^{-\alpha-1}e^{\left(-\left([1-(1+e^{(-x/\sigma)})^{-1}]^{-\alpha}-1\right)^{\beta}\right)}. \end{aligned} \tag{33}$$

Figure 3 shows graphs of density function and hrf for different combinations of GOWG-Logistic parameters.

It appears that the pdf of the GOWG-Logistic distribution can take various shapes, such as unimodal, increasing and left- or right-skewed. Plots of the GOWG-Logistic distribution hrf show decreasing, increasing, upside-down bathtub and upside-down bathtub followed by bathtub shapes.

2.4.4. GOWG-Weibull Distribution

Suppose the cdf andf pdf of the baseline distribution are given by $G(x;c) = 1-\exp(-x^c)$ and $g(x;c) = cx^{c-1}\exp(-x^c)$ for $c > 0$ and $x > 0$. Then the new GOWG-Weibull distribution has cdf and pdf given by

$$F(x;\alpha,\beta,\delta,c) = 1 - \frac{\gamma\left(-\log\left(1-e^{-\left(e^{\alpha x^c}-1\right)^{\beta}}\right),\delta\right)}{\Gamma(\delta)}, \tag{34}$$

and

$$\begin{aligned} f(x;\alpha,\beta,\delta,c) &= \frac{\alpha\beta c x^{c-1}\exp(\alpha x^c)}{\Gamma(\delta)}\left(e^{\alpha x^c}-1\right)^{\beta-1} \\ &\times \left(-\log\left(1-e^{-\left(e^{\alpha x^c}-1\right)^{\beta}}\right)\right)^{\delta-1}\exp\left(-\left(e^{\alpha x^c}-1\right)^{\beta}\right), \end{aligned} \tag{35}$$

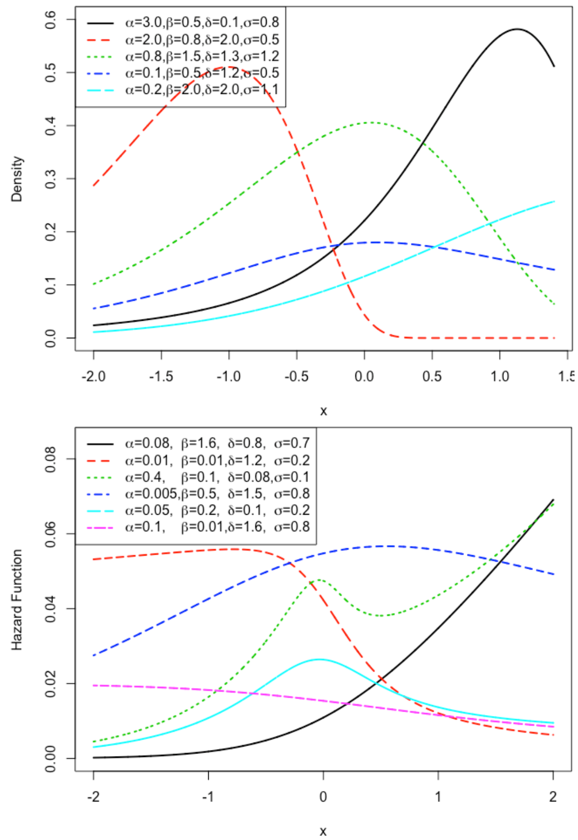


FIGURE 3: Plots of the pdf and hrf for the GOWG-Logistic distribution

for $\alpha, \beta, \delta, c > 0$, and $x > 0$. When $c = 1$ and $c = 2$, we obtain the GOWG-exponential and GOWG-Rayleigh (GOWG-R) distributions, respectively. The GOWG-Weibull survival function and hrf are given by

$$S(x; \alpha, \beta, \delta, c) = \frac{\gamma\left(-\log\left(1 - e^{-\left(e^{\alpha x^c}-1\right)^{\beta}}\right), \delta\right)}{\Gamma(\delta)}, \tag{36}$$

and

$$\begin{aligned} h_F(x) \quad &= \quad \frac{\alpha\beta c x^{c-1}\exp(\alpha x^c)\left(e^{\alpha x^c}-1\right)^{\beta-1}}{\gamma\left(-\log\left(1-e^{-\left(e^{\alpha x^c}-1\right)^{\beta}}\right), \delta\right)} \\ &\times \quad \left(-\log\left(1-e^{-\left(e^{\alpha x^c}-1\right)^{\beta}}\right)\right)^{\delta-1} \exp\left(-\left(e^{\alpha x^c}-1\right)^{\beta}\right), \end{aligned} \tag{37}$$

for $\alpha, \beta, \delta, c > 0$, and $x > 0$. Figure 4 illustrates the graphs of the density function and hrf of the GOWG-Weibull distribution using several combinations of the parameters α, β, δ, and c.

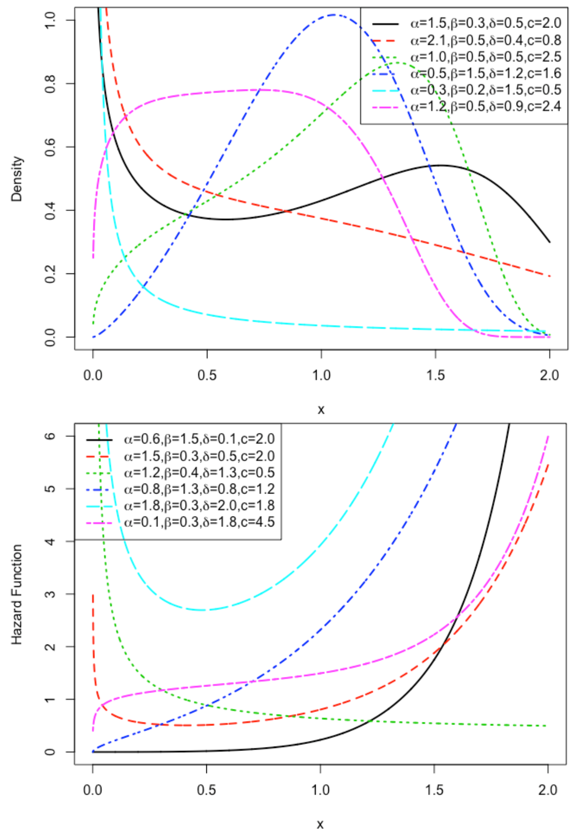


FIGURE 4: Plots of the pdf and hrf for the GOWG-Weibull distribution

It appears that the pdf of the GOWG-Weibull distribution can take various shapes, such as unimodal, reverse-J and left- or right-skewed. Plots of the GOWG-Weibull distribution hrf show decreasing, increasing, and bathtub shapes.

2.5. Series Expansion of Density Function

In this section, we present the series expansion of the GOWG-G family of density functions.

Let $y = e^{-\left(\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}\right)^{\beta}}$, then using the series representation $-\log(1-y) = \sum_{i=0}^{\infty} \frac{y^{i+1}}{i+1}$, we have

$$\Big[-\log(1-y)\Big]^{\delta-1} = y^{\delta-1}\left[\sum_{m=0}^{\infty}\binom{\delta-1}{m} y^{m}\left(\sum_{s=0}^{\infty}\frac{y^{s}}{s+2}\right)^{m}\right],$$

and applying the result on power series raised to a positive integer, with $a_s =$

$(s+2)^{-1}$, that is,

$$\left(\sum_{s=0}^{\infty} a_s y^s\right)^m = \sum_{s=0}^{\infty} b_{s,m} y^s, \tag{38}$$

where $b_{s,m} = (sa_0)^{-1}\sum_{l=1}^{s}[m(l+1)-s]a_l b_{s-l,m}$, and $b_{0,m} = a_0^m$, (Gradshteyn & Ryzhik, 2000), and applying the exponential series expansion, as well as the generalized binomial theorem:

$$\begin{aligned}\exp\left(-\left[\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}\right]^{\beta}\right) &= \sum_{k,s,t=0}^{\infty}\frac{(-1)^{k+s+t}}{k!}\binom{k\beta}{s}\binom{s\alpha}{t}\\ &\times \left[G(x;\xi)\right]^t\left[\overline{G}(x;\xi)\right]^{-k\alpha\beta},\end{aligned} \tag{39}$$

and

$$(1-(\overline{G}(x;\xi))^{\alpha})^c = \sum_{p=0}^{\infty}\binom{c}{p}(-1)^p(\overline{G}(x;\xi))^{\alpha p}, \tag{40}$$

we have

$$f(x;\alpha,\beta,\delta,\xi) = \sum_{q,t=0}^{\infty}\omega(q,t)g^{*}_{q+t+1}(x,\xi), \tag{41}$$

where $g^{*}_{q+t+1}(x;\xi) = (q+t+1)[G(x;\xi)]^{q+t+1-1}g(x;\xi)$ is the exponentiated-G (exp-G) pdf with the exponentiation parameter $q+t+1>0$ and parameter vector ξ, where $\omega(q,t)$ is given by

$$\begin{aligned}\omega(q,t) &= \sum_{m,s,k,p=0}^{\infty}\binom{\delta-1}{m}\binom{k\beta+\beta-1}{p}\binom{p\alpha}{q}\binom{-\alpha\beta(k+1)-1}{t}\\ &\times \frac{b_{s,m}(-1)^{k+p+q+t}(m+s+\delta)^k}{(q+t+1)k!}\frac{\alpha\beta}{\Gamma(\delta)}.\end{aligned} \tag{42}$$

Thus, the pdf of the GOWG-G family of distributions is a mixture of exp-G densities. Consequently, the mathematical and statistical properties of the GOWG-G family of distributions follow directly from those of the exp-G distribution. Please see appendix A for details of the series expansion.

Note that using the fact that

$$\begin{aligned}[G(x;\xi)]^{q+t} &= [1-(1-G(x;\xi))]^{q+t}\\ &= \sum_{m=0}^{\infty}(-1)^r\binom{q+t}{m}(1-G(x;\xi))^m\\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{m}(-1)^{m+k}\binom{q+t}{m}\binom{m}{k}(G(x;\xi))^k\\ &= \sum_{k=0}^{\infty}\sum_{m=k}^{\infty}(-1)^{m+k}\binom{q+t}{m}\binom{m}{k}(G(x;\xi))^k,\end{aligned} \tag{43}$$

the cdf of the GOWG-G family of distributions can be written as

$$\begin{aligned} F(x;\alpha,\beta,\delta,\xi) &= \sum_{q,t=0}^{\infty} \omega(q,t) G^{*}_{q+t}(x;\xi) \\ &= \sum_{q,t=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=k}^{\infty} \omega(q,t)(-1)^{m+k}\binom{q+t}{m}\binom{m}{k}(G(x;\xi))^{k} \\ &= \sum_{k=0}^{\infty} C_k (G(x;\xi))^{k}, \end{aligned} \tag{44}$$

where

$$C_k = \sum_{q,t=0}^{\infty}\sum_{m=k}^{\infty} \omega(q,t)(-1)^{m+k}\binom{q+t}{m}\binom{m}{k}, \tag{45}$$

and $\omega(q,t)$ is given by equation (42). The corresponding GOWG-G pdf is given by

$$f(x;\alpha,\beta,\delta,\xi) = \sum_{k=0}^{\infty} C_{k+1} g^{*}_{k+1}(x;\xi). \tag{46}$$

3. Moments, Conditional and Incomplete Moments

In this section, moments, moment generating function, conditional moments for the GOWG-G family of distributions are presented. Moments are very useful in the study of important features and characteristics of a distribution (e.g., central tendency, dispersion, skewness and kurtosis). These measures (moments, moment generating function, incomplete moments) can be readily obtained for the sub-models given in section 2.

3.1. Moments and Generating Function

Let $Y_{k+1} \sim Exponentiated - G(k+1,\xi)$, then the r^{th} raw moment, μ'_r of the GOWG-G family of distributions is given by:

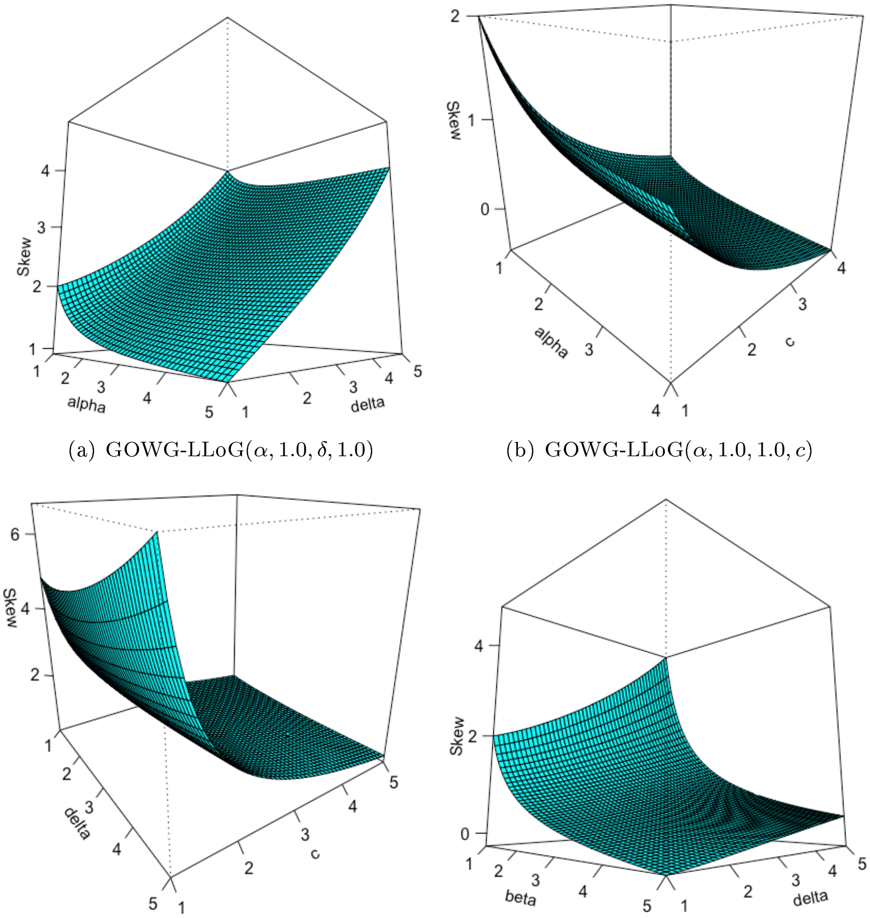
$$\begin{aligned} \mu'_r = E(X^r) &= \sum_{k=0}^{\infty} C_{k+1} E(Y_{k+1}^{r}) = \sum_{k=0}^{\infty} C_{k+1}\int_0^{\infty} y^r g^{*}_{k+1}(y;\xi)dy \\ &= \sum_{k=0}^{\infty} C_{k+1}(k+1)\int_0^1 [Q_G(u;\xi)]^r u^k du. \end{aligned} \tag{47}$$

The moment generating function (MGF), for $|a| < 1$, is given by:

$$M_X(a) = \sum_{k=0}^{\infty} C_{k+1} M_Y(a) = \sum_{k=0}^{\infty}\sum_{i=0}^{\infty} C_{k+1}\frac{a^i}{i!}E(Y^i).$$

The coefficients of variation (CV), Skewness (CS) and Kurtosis (CK) can be readily obtained. The variance (σ^2), standard deviation (SD=σ), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) can be readily obtained for specified baseline distribution. Also, note that the r^{th} cumulant of the random variable X can be readily obtained from the recursive relationship: $\kappa_r = \mu'_r - \sum_{s=1}^{r-1} \binom{r-1}{s-1} \mu'_{r-s} \kappa_s$, where $\mu'_r = E(X - \mu'_1)^r$, so that the CS and CK are given by $\gamma_1 = \frac{\kappa_3}{\kappa_2^{3/2}}$ and $\gamma_2 = \frac{\kappa_4}{\kappa_2^2}$.

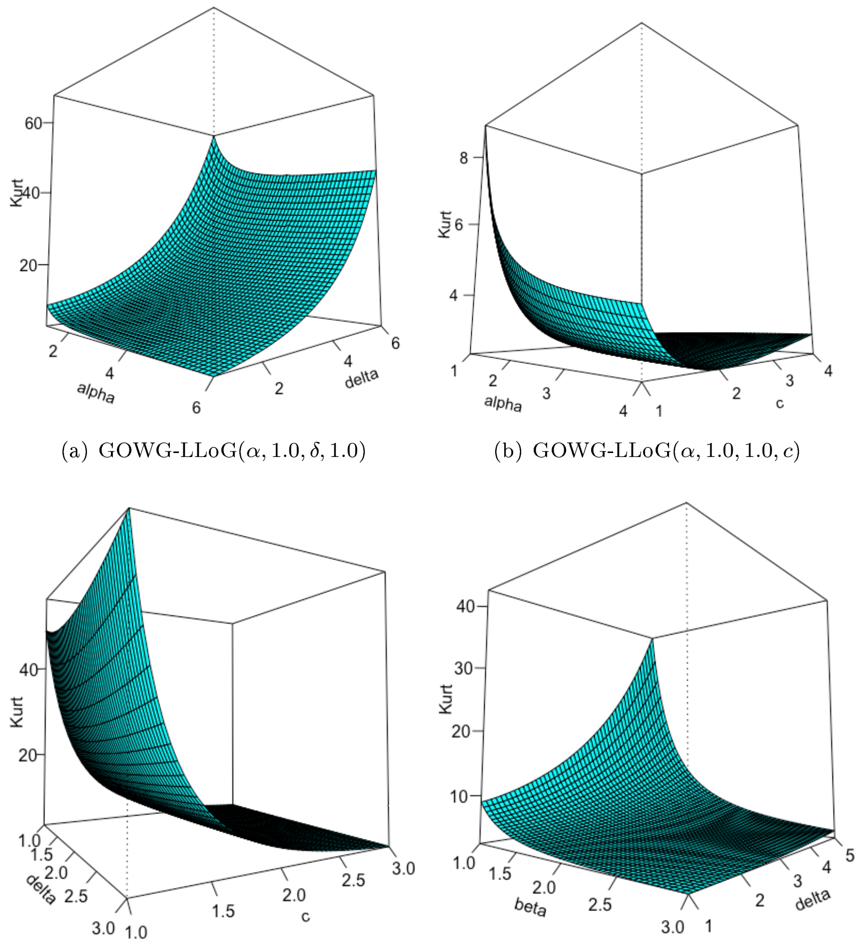
We present in Figures 5 and 6, 3D plots of skewness and kurtosis of the GOWG-Log-logistic distribution.



(a) GOWG-LLoG($\alpha, 1.0, \delta, 1.0$) (b) GOWG-LLoG($\alpha, 1.0, 1.0, c$)

(c) GOWG-LLoG($0.5, 1.0, \delta, c$) (d) GOWG-LLoG($1.0, \beta, \delta, 1.0$)

Figure 5: 3D plots of the skewness for the GOWG-Log-logistic distribution for some selected parameter values.



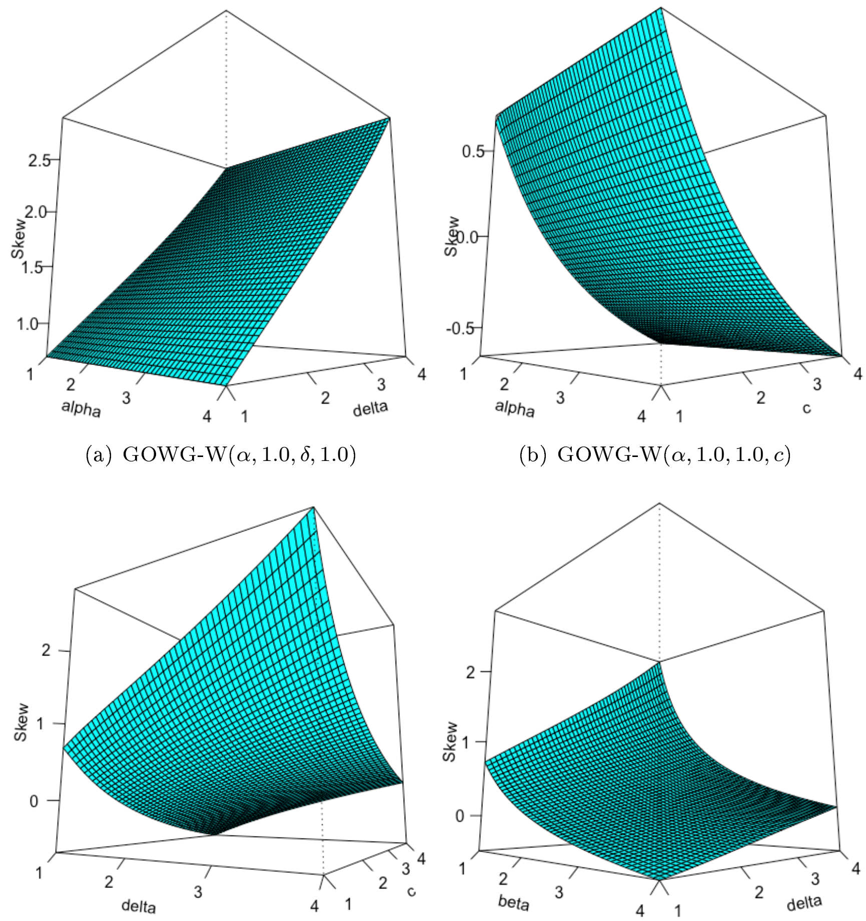
(a) GOWG-LLoG($\alpha, 1.0, \delta, 1.0$)
(b) GOWG-LLoG($\alpha, 1.0, 1.0, c$)
(c) GOWG-LLoG($0.5, 1.0, \delta, c$)
(d) GOWG-LLoG($1.0, \beta, \delta, 1.0$)

FIGURE 6: 3D plots of the kurtosis (Kurt) for the GOWG-Log-logistic distribution for some selected parameter values.

It can be observed that

- GOWG-LLoG skewness and kurtosis increase as δ increases when α, β, and c are all fixed.
- GOWG-LLoG skewness and kurtosis decrease as α increases when β, δ and c are all fixed.
- GOWG-LLoG skewness and kurtosis decrease as c increases when α, β and δ are all fixed.
- GOWG-LLoG skewness and kurtosis decrease as β increases when α, δ and c are all fixed.

In Figures 7 and 8, 3D plots of skewness and kurtosis of the GOWG-Weibull distribution are presented.

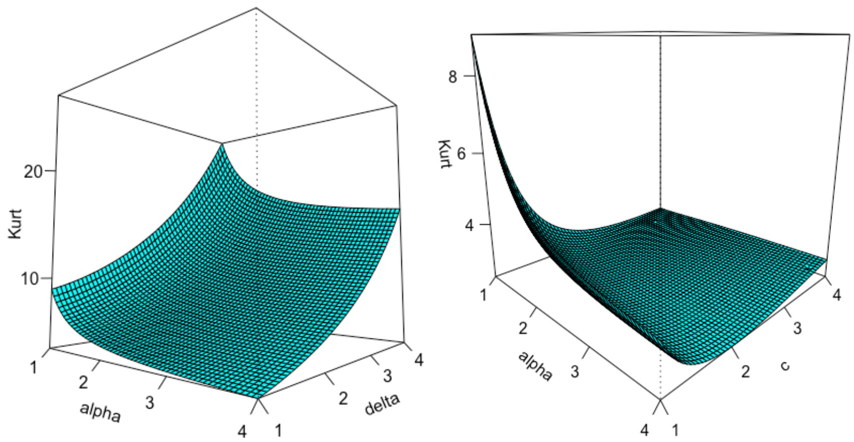


(a) GOWG-W$(\alpha, 1.0, \delta, 1.0)$
(b) GOWG-W$(\alpha, 1.0, 1.0, c)$
(c) GOWG-W$(0.5, 1.0, \delta, c)$
(d) GOWG-W$(1.0, \beta, \delta, 1.0)$

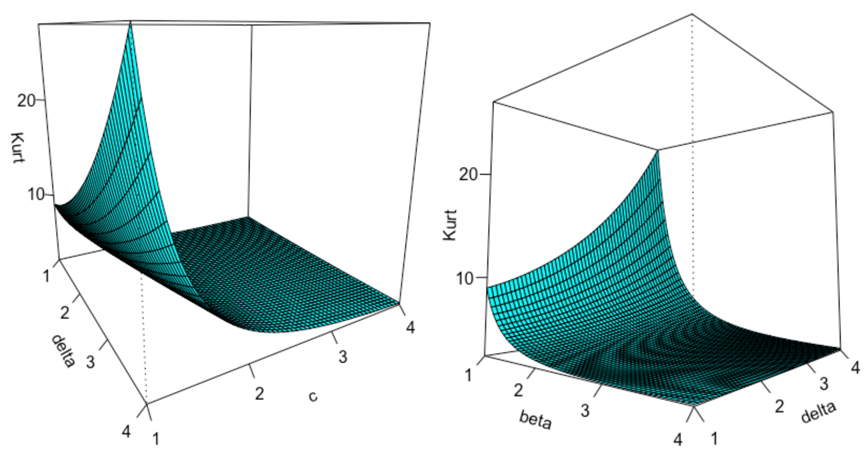
FIGURE 7: 3D plots of the skewness for the GOWG-Weibull distribution for some selected parameter values.

We can observe that

- GOWG-W skewness and kurtosis increase as δ increases when α, β, and c are all fixed.
- GOWG-W skewness decreases as c increases when α, β and δ are all fixed.
- GOWG-W skewness and kurtosis decrease as β increases when α, δ and c are all fixed.



(a) GOWG-W($\alpha, 1.0, \delta, 1.0$) (b) GOWG-W($\alpha, 1.0, 1.0, c$)



(c) GOWG-W($0.5, 1.0, \delta, c$) (d) GOWG-W($1.0, \beta, \delta, 1.0$)

FIGURE 8: 3D plots of the kurtosis (Kurt) for the GOWG-Weibull distribution for some selected parameter values

3.2. Conditional Moments

For lifetime models and measures of inequality, it is of particular interest to find the conditional and incomplete moments. The r^{th} conditional moments of the GOWG-G family of distributions is given by

$$\begin{aligned} E(X^r|X \geq a) &= \frac{1}{\overline{F}(a;\alpha,\beta,\delta,\xi)} \int_t^{\infty} x^r f(x;\alpha,\beta,\delta,\xi)dx \\ &= \frac{1}{\overline{F}(a;\alpha,\beta,\delta,\xi)} \sum_{k=0}^{\infty} C_{k+1} E\left(Y_{k+1}^r I_{\{Y_{k+1}^r \geq t\}}\right), \end{aligned} \tag{48}$$

where

$$E\left(Y_{k+1}^{r} I_{\{Y_{k+1}^{r} \geq t\}}\right) = \int_{t}^{\infty} y^{r} g_{k+1}^{*}(y;\xi) dy = (k+1) \int_{G(u;\xi)}^{1} [Q_{G}(u;\xi)]^{r} u^{k} du, \quad (49)$$

for $\alpha, \beta, \delta > 0$, and parameter vector ξ. The mean residual life function is given by $E(X-a|X>a) = E(X|X>a) - a = V_{F}(a) - a$, where $V_{F}(a)$ is referred to as the vitalilty function of the distribution function F. The mean deviations, Bonferroni and Lorenz curves can be readily obtained from the conditional and incomplete moments.

4. Order Statistics and Rényi Entropy

Order statistics and entropy play important roles in probability and statistics, particularly in reliability, lifetime data analysis and information theory. In this section, we present the distribution of the i^{th} order statistics and Rényi entropy for the GOWG-G family of distributions.

4.1. Order Statistics

In this subsection, the pdf of the i^{th} order statistic is presented. Let $X_1, X_2, \ldots,$ X_n be independent and identically distributed GOWG-G random variables. Using the binomial expansion

$$(1-F(x))^{n-i} = \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^{j} [F(x)]^{j},$$

the pdf of the i^{th} order statistic from the GOWG-G pdf $f(x;\alpha,\beta,\delta,\xi) = f(x)$ can be written as

$$f_{i:n}(x) = \sum_{m=0}^{n-i} \omega(i,m) f_{m+i}(x;\alpha,\beta,\delta,\xi), \quad (50)$$

where $f_{m+i}(x;\alpha,\beta,\delta,\xi)$ is the exponentiated gamma odd generalized-G pdf with exponentiated parameter $m+i>0$ and weights $\omega(i,m)$ given by

$$\begin{aligned} \omega(i,m) &= \frac{1}{B(i,n-i+1)} \frac{(-1)^{m}}{m+i} \binom{n-i}{m} \\ &= (-1)^{m} \binom{n}{m+i} \binom{m+i-1}{m}. \end{aligned} \quad (51)$$

Note that from the results given by Hosseini et al. (2018), we can also write the pdf of the i^{th} order statistic as

$$\begin{aligned} f_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}[F(x)]^{j+i-1}f(x) \\ &= \sum_{r,k=0}^{\infty} m_{r,k}g^*_{r+k+1}(x), \end{aligned} \tag{52}$$

where $g^*_{r+k+1}(x)$ is the exp-G density function with power parameter $r+k+1$,

$$m_{r,k} = \frac{n!(r+1)(i-1)!C_{r+1}}{r+k+1}\sum_{j=0}^{n-i}\frac{(-1)^j f_{j+i-1,k}}{(n-i-j)!j!}, \tag{53}$$

and C_r is defined by the equation (45). The quantities $f_{j+i-1,k}$ are given by the recursively by

$$f_{j+i-1,k} = (kC_0)^{-1}\sum_{m=1}^{k}[m(j+i)-k]C_m f_{j+i-1,k-m}. \tag{54}$$

We can also obtain the distribution of the i^{th} order statistic as follows, by applying the result of Gradshteyn & Ryzhik (2000) of a power series raised to a positive integer:

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}\sum_{k=0}^{\infty}d_{j+i-1,k}(G(x;\xi))^k f(x), \tag{55}$$

where $d_{j+i-1,0} = C_0^{j+i-1}$, and for $k \geq 1$,

$$d_{j+i-1,k} = (kC_0)^{-1}\sum_{l=1}^{k}[l(j+i)-k]C_l d_{j+i-1,k-l}. \tag{56}$$

Replacing the GOWG-G pdf $f(x)$ by equation (46) and applying the result on product of two series (Gradshteyn & Ryzhik, 2000), we have

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}\sum_{k=0}^{\infty}a_k(G(x;\xi))^k g(x;\xi), \tag{57}$$

where $b_r = C_{r+1}(r+1)$, and $a_k = \sum_{l=0}^{k} d_{j+i-1,l}b_{k-l}$.

Consequently, the pdf of the i^{th} order statistic from the GOWG-G family of distributions can be written as

$$\begin{aligned} f_{i:n}(x) &= \sum_{k=0}^{\infty}\frac{n!(k+1)}{(i-1)!(n-i)!(k+1)}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}a_k(G(x;\xi))^k g(x;\xi) \\ &= \sum_{k=0}^{\infty} v_{k+1}g^*_{k+1}(x;\xi), \end{aligned} \tag{58}$$

where

$$v_{k+1}=\frac{n!}{(i-1)!(n-i)!(k+1)}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}a_k, \tag{59}$$

and $g^*_{k+1}(x;\xi)=(k+1)(G(x;\xi))^k g(x;\xi)$ is the exp-G density function with power parameter $k+1$. Thus, the pdf of the GOWG-G distribution is an infinite mixture of exp-G densities. The structural properties such as moments, incomplete moments of the distribution of the i^{th} order statistic and other measures of the GOWG-G distribution follows or can be readily obtained from those of the exp-G distribution.

4.2. Rényi Entropy

Rényi entropy (Rényi, 1961) is an extension of Shannon entropy. Let $f(x;\alpha,\beta,\delta,\xi)=f(x)$. Rényi entropy is defined to be

$$I_R(v)=\frac{1}{1-v}\log\left(\int_0^\infty [f(x)]^v dx\right), v\neq 1, v>0. \tag{60}$$

Rényi entropy tends to Shannon entropy as $v\to 1$. Note that

$$\begin{aligned}\int_0^\infty f^v(x)dx &= \frac{(\alpha\beta)^v}{(\Gamma(\delta))^v}\sum_{m,s,k,p,q,t=0}^{\infty}\binom{v\delta-v}{m}\binom{\beta(k+v)-v}{p}\binom{p\alpha}{q}\\ &\times \binom{-\alpha\beta(k+v)-v}{t}\frac{b_{s,m}(-1)^{k+p+q+t}(m+s+v\delta)^k}{k!}\\ &\times \int_0^\infty [G(x;\xi)]^{q+t}(g(x;\xi))^v dx. \end{aligned} \tag{61}$$

Consequently, Rényi entropy for the GOWG-G family of distributions is given by

$$\begin{aligned} I_R(v) &= \frac{1}{1-v}\log\left[\frac{(\alpha\beta)^v}{(\Gamma(\delta))^v}\sum_{m,s,k,p,q,t=0}^{\infty}\binom{v\delta-v}{m}\binom{\beta(k+v)-v}{p}\binom{p\alpha}{q}\right.\\ &\times \binom{-\alpha\beta(k+v)-v}{t}\frac{b_{s,m}(-1)^{k+p+q+t}(m+s+v\delta)^k}{k!}\\ &\times \left.\int_0^\infty [G(x;\xi)]^{q+t}(g(x;\xi))^v dx\right], \end{aligned} \tag{62}$$

for $v>0$, $v\neq 1$, where the integral $\int_0^\infty [G(x;\xi)]^{q+t}(g(x;\xi))^v dx$ can be numerically evaluated. See appendix A for details.

5. Maximum Likelihood Estimation

Let $X \sim GOWG-G(\alpha,\beta,\delta,\xi)$ and $\mathbf{\Delta}=(\alpha,\beta,\delta,\xi)^T$ be the vector of model parameters. The log-likelihood function $\ell_n=\ell_n(\mathbf{\Delta})$ based on a random sample of size n from the GOWG-G family of distributions is given by

$$\begin{aligned}\ell_n(\mathbf{\Delta}) &= -n\ln(\Gamma(\delta))+(\delta-1)\sum_{i=1}^{n}\ln\left(-\ln\left[1-e^{-\left(\overline{G}^{-\alpha}(x_i;\xi)-1\right)^{\beta}}\right]\right)\\ &+ n\ln(\alpha)+n\ln(\beta)+(\beta-1)\sum_{i=1}^{n}\ln\left(\overline{G}^{-\alpha}(x_i;\xi)-1\right)+\sum_{i=1}^{n}\ln(g(x_i;\xi))\\ &- (\alpha+1)\sum_{i=1}^{n}\ln(\overline{G}(x_i;\xi))-\sum_{i=1}^{n}(\overline{G}^{-\alpha}(x_i;\xi)-1)^{\beta}. \end{aligned} \quad (63)$$

The elements of the score vector $U(\mathbf{\Delta})$ are given in appendix B. The maximum likelihood estimates of the parameters, denoted by $\hat{\mathbf{\Delta}}$ is obtained by solving the nonlinear equation $(\frac{\partial\ell_n}{\partial\alpha},\frac{\partial\ell_n}{\partial\beta},\frac{\partial\ell_n}{\partial\delta},\frac{\partial\ell_n}{\partial\xi_k})^T=\mathbf{0}$, using a numerical method such as Newton-Raphson procedure. The multivariate normal distribution $N_{q+3}(\underline{0},J(\hat{\mathbf{\Delta}})^{-1})$, where the mean vector $\underline{\mathbf{0}}=(0,0,0,\underline{0})^T$ and $J(\hat{\mathbf{\Delta}})^{-1}$ is the observed Fisher information matrix evaluated at $\hat{\mathbf{\Delta}}$, can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions.

6. Simulation Study

The performance of the GOWG-LLoG is examined by conducting various simulations for different sizes (n=25, 50, 75, 100, 200, 400, 800, 1200) via the R package. We simulate $N=1000$ samples for the true parameters values given in Table 2. Additional simulation results are available upon request or in the appendix C. The tables list the mean MLEs of the model parameters along with the respective average bias and root mean squared errors (RMSEs). The average bias and RMSE for the estimated parameter, say, $\hat{\theta}$, say, are given by:

$$AverageBias(\hat{\theta})=\frac{\sum_{i=1}^{N}\hat{\theta}_i}{N}-\theta,\quad \text{and}\quad RMSE(\hat{\theta})=\sqrt{\frac{\sum_{i=1}^{N}(\hat{\theta}_i-\theta)^2}{N}},$$

respectively. As we can see from the results, RMSE decreases as the sample size n increases, so the mean estimates of parameter values are closer to the true parameter values.

Table 2: Monte Carlo simulation results for GOWG-LLoG distribution: mean, average bias and RMSE.

		$\alpha = 0.5, \beta = 1.5, \delta = 2.0, c = 2.5$			$\alpha = 1.5, \beta = 0.5, \delta = 0.5, c = 0.8$		
Parameter	n	Mean	Average Bias	RMSE	Mean	Average Bias	RMSE
α	25	0.860410	0.360410	0.839061	1.650463	0.150463	0.983916
	50	0.683464	0.183464	0.467255	1.545448	0.045448	0.746975
	75	0.652738	0.152738	0.394617	1.542869	0.042869	0.613463
	100	0.649833	0.149833	0.356858	1.534506	0.034506	0.588553
	200	0.601665	0.101665	0.265918	1.523633	0.023633	0.411651
	400	0.562980	0.062980	0.180220	1.520182	0.020182	0.294348
	800	0.540081	0.040081	0.128405	1.513901	0.013901	0.246615
	1200	0.532535	0.032535	0.113986	1.509011	0.009011	0.202548
β	25	5.701472	4.201472	6.037640	1.451168	0.951168	2.727758
	50	5.227618	3.727618	5.511201	1.188359	0.688359	2.294176
	75	4.966545	3.466545	5.296061	0.924755	0.424755	1.822518
	100	4.602688	3.102688	5.002049	0.831008	0.331008	1.569851
	200	4.047373	2.547373	4.385498	0.603453	0.103453	0.804342
	400	3.053328	1.553328	3.332436	0.526761	0.026761	0.333675
	800	2.430686	0.930686	2.436491	0.509658	0.009658	0.229784
	1200	2.238897	0.738898	2.055682	0.495428	-0.004572	0.162081
δ	25	4.590924	2.590924	3.984176	1.326580	0.826580	2.320062
	50	4.352548	2.352548	3.688997	1.061772	0.561772	1.829322
	75	4.152000	2.152000	3.476014	0.805933	0.305933	1.317430
	100	3.830906	1.830906	3.185761	0.732953	0.232953	1.073860
	200	3.560011	1.560011	2.829737	0.551097	0.051097	0.504603
	400	3.016958	1.016958	2.169986	0.488514	-0.011486	0.201255
	800	2.664622	0.664622	1.670941	0.485043	-0.014957	0.139945
	1200	2.536260	0.536260	1.378154	0.489757	-0.010244	0.105082
c	25	2.341661	-0.158339	2.876671	1.180272	0.380272	0.976142
	50	2.438771	-0.061229	2.809694	1.075027	0.275027	0.774389
	75	2.464829	-0.035171	2.632967	1.058375	0.258375	0.673186
	100	2.549123	0.049123	2.498856	1.045742	0.245742	0.651721
	200	2.397428	-0.102572	2.033538	1.003916	0.203916	0.546761
	400	2.608542	0.108542	1.823549	0.964802	0.164802	0.438886
	800	2.606980	0.106980	1.477629	0.942231	0.142231	0.397450
	1200	2.539176	0.039176	1.308549	0.930722	0.130722	0.354788

7. Applications

In this section, we present examples to illustrate the flexibility and usefulness of the GOWG-LLoG and GOWG-W distributions for data modeling. The GOWG-LLoG distrbution is fitted to the data set in subsection 7.1 while the GOWG-W distribution is fitted to the data sets in subsections 7.2 and 7.3. GOWG-LLoG fitted in subsection 7.1, and GOWG-W in subsections 7.2 and 7.3. These fits are compared to the fits of several competing non-nested distributions with equal number of parameters. The GOWG-LLoG distribution is compared with exponentiated power generalized Weibull (EPGW) (Péna-Ramirez et al., 2018), burr XII Poisson (BXIIP) (da Silva et al., 2015), Topp-Leone-Marshall-Olkin-Weibull (TLMO-W) (Chipepa et al., 2020), beta Weibull (BW) (Lee et al., 2007), the exponential Lindley odd loglogistic Weibull (ELOLLW) (Korkmaz et al., 2018) and Kumaraswamy-Weibull (KwW) (Cordeiro et al., 2010).

The GOWG-W distribution is compared with EPGW (Péna-Ramirez et al., 2018), beta generalized Lindley (BGL) (Oluyede & Yang, 2015), BW (Lee et al., 2007), Kumaraswamy odd Lindley-Log logistic (KOL-LLoG) (Chipepa et al., 2019), new modified Weibull (NMW) (Doostmoradi et al., 2014) and odd log-logistic exponentiated Weibull (OLLEW) (Afify et al., 2018) distributions. The pdfs of the EPGW, BXIIP, BGL, BW, KOL-LLoG, KwW, TLMO-W, ELLOLW, NMW and OLLEW distributions are given in appendix D.

We used the NLmixed in SAS to estimate the model parameters and the package AdequacyModel in R software for goodness-of-fit test. The estimated values of the parameters (standard error in parenthesis), -2log-likelihood statistic $(-2\ln(L))$, Akaike Information Criterion ($AIC = 2p - 2\ln(L)$), Bayesian Information Criterion ($BIC = p\ln(n) - 2\ln(L)$) and Consistent Akaike Information Criterion $\left(AICC = AIC + 2\frac{p(p+1)}{n-p-1}\right)$, where $L = L(\hat{\boldsymbol{\Delta}})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters are presented.

We also obtain the following goodness-of-fit statistics: Crameŕ-von Mises (W^*) and Anderson-Darling Statistics (A^*) described by Chen & Balakrishnan (1995), as well as Kolmogorov-Smirnov (K-S) statistic and its P-value. Note that for the value of the log-likelihood function at its maximum (ℓ_n), larger value is good and preferred, and for AIC, AICC, BIC, and the goodness-of-fit statistics W^*, A^* and $K-S$, smaller values are preferred. The results are shown in Tables 3, 4 and 5.

Plots of the fitted densities, the histogram of the data and probability plots (Chambers et al., 1983) are given for each example to show how well our model fits the observed data sets. For the probability plot, we plotted $F_{GOWG-G}(x_{(j)}; \hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{c})$ against $\dfrac{j-0.375}{n+0.25}$, $j = 1, 2, \cdots, n$, where $x_{(j)}$ are the ordered values of the observed data. The measures of closeness are given by the sum of squares $SS = \sum_{j=1}^{n}\left[F_{GOWG-G}(x_{(j)}; \hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{c}) - \left(\dfrac{j-0.375}{n+0.25}\right)\right]^2$. These plots are shown in Figures 9, 11 and 13.

7.1. Failure Times Data

As a first example, we looked at a set of data reported by Murthy et al. (2004), which represents failure rates (per 1000h) for 50 components. The observations are as follows: 0.036, 0.148, 0.590, 3.076, 6.816, 0.058, 0.183, 0.618, 3.147, 7.896, 0.061, 0.192, 0.645, 3.625, 7.904, 0.074, 0.254, 0.961, 3.704, 8.022, 0.078, 0.262, 1.228, 3.931, 9.337, 0.086, 0.379, 1.600, 4.073, 10.940, 0.102, 0.381, 2.006, 4.393, 11.020, 0.103, 0.538, 2.054, 4.534, 13.880, 0.114, 0.570, 2.804, 4.893, 14.73, 0.116, 0.574, 3.058, 6.274, 15.08.

The estimated variance-covariance matrix for GOWG-LLoG model on failure times data set is given by

$$\begin{bmatrix} 0.7172 & 24.8507 & 0.7058 & -0.4153 \\ 24.8507 & 931.0600 & 61.4167 & -14.9263 \\ 0.7058 & 61.4167 & 20.3219 & -0.6934 \\ -0.4153 & -14.9263 & -0.6934 & 0.2448 \end{bmatrix}$$

and the 95% two-sided asymptotic confidence intervals for α, β, δ and c are given by 0.66626 ± 1.6599, 9.3350 ± 59.8060, 4.9922 ± 8.8356, and 0.1466 ± 0.9698, respectively.

Table 3: Parameter estimates and goodness-of-fit statistics for various models fitted for failure times data.

	Estimates				Statistics							
Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{c}$	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	$K-S$	p-value
GOWG-LLoG	0.6626	9.3350	4.9922	0.1466	203.2	211.2	212.1	218.9	0.1444	0.9009	0.1162	0.4738
	(0.8469)	(30.5133)	(4.5080)	(0.4947)								
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\lambda}$								
EPGW	0.4361	4.4646	1.3477	0.1280	204.4	212.4	213.3	220.1	0.1463	0.9292	0.1370	0.2788
	(2.1571)	(64.2876)	(7.8066)	(1.3921)								
	$\hat{c}$	$\hat{k}$	$\hat{s}$	$\hat{\lambda}$								
BXIIP	0.6708	42.0493	110395	31.5131	205.0	213.0	213.9	220.6	0.1547	0.9677	0.1261	0.3727
	(0.0787)	(101.6300)	(0.02999)	(63.8581)								
	$\hat{b}$	$\hat{\delta}$	$\hat{\lambda}$	$\hat{\gamma}$								
TLMO-W	0.6761	1.3029	0.1886	0.8232	204.7	212.7	213.5	220.3	0.1469	0.9372	0.1403	0.2538
	(1.2480)	(1.4028)	(0.5680)	(1.0324)								
	$\hat{k}$	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$								
BW	0.5250	0.0111	1.4514	9.9529	204.7	212.7	213.6	220.3	0.1550	0.9657	0.1197	0.4374
	(0.7545)	(0.0052)	(3.4572)	(0.0651)								
	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\gamma}$								
ELOLLW	1.1051	0.0423	5.0895	0.6521	204.7	212.7	213.6	220.3	0.1512	0.9494	0.1285	0.3512
	(3.0861)	(0.0295)	(0.6697)	(0.0861)								
	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$								
KwW	3.6048	110.40	0.0017	0.2114	204.8	212.8	213.7	220.5	0.1555	0.9699	0.1208	0.4253
	(0.0010)	(2.1385×10^{-5})	(0.0010)	(0.0211)								

Based on Table 3, GOWG-LLoG distribution has the highest p-value for the K-S statistic and the lowest goodness-of-fit statistics compared to other non-nested models. Thus, we conclude that the GOWG-LLoG model performs better with failure times data than non-nested EPGW, BXIIP, TLMO-W, BW, ELOLLW, and KwW models. Moreover, Figure 9 shows that our model has the lowest SS value from the probability plots compared to the competing non-nested models on the failure times data.

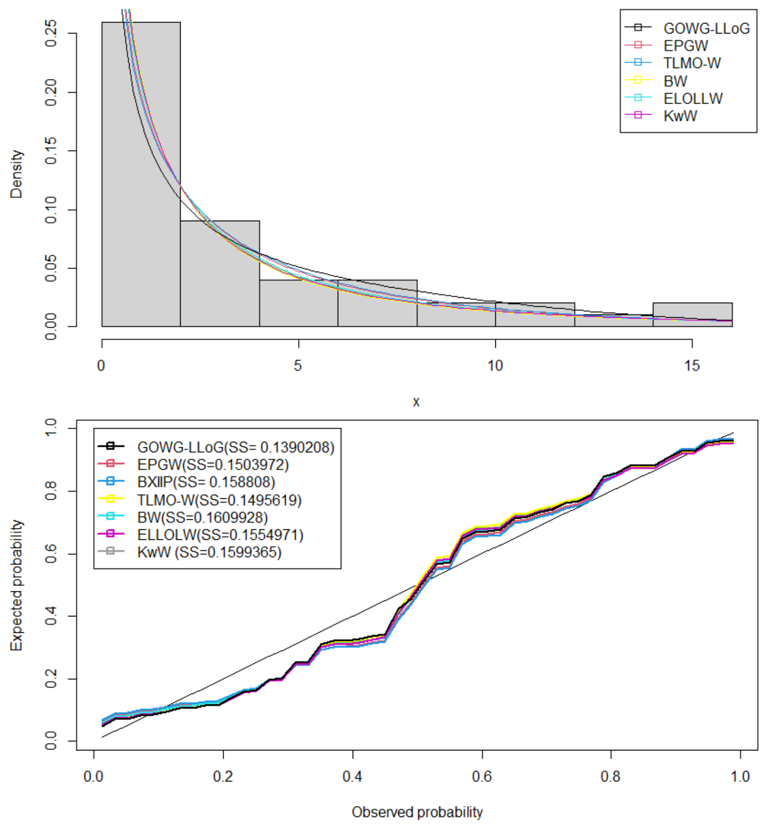


FIGURE 9: Histogram, fitted density and probability plots for failure times data

In Figure 10, we see that the cdf line for the GOWG-LLoG distribution indicated by the blue line is closer to the empirical cdf while the survival function in blue is also close to the Kaplan-Meier(K-M) curve which indicate that our model is the best in explaining the failure times data. The TTT plot for failure times data indicates a decreasing hazard rate function, hence the failure times data can be fitted to our model.

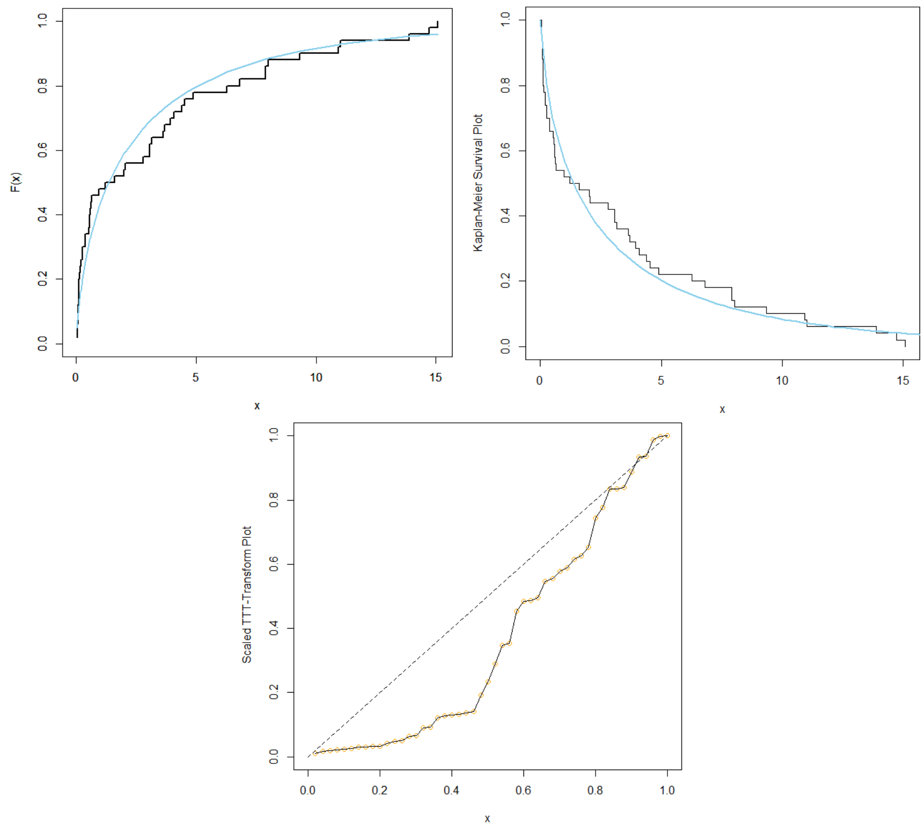


FIGURE 10: Estimated cdf, Kaplan-Meier survival and scaled TTT-Transform plots for the GOWG-LLoG distribution for failure times data

7.2. Aircraft Windshield Data

The second data set consists of failure times of 84 aircraft windshields given in Table 16.11 of Murthy et al. (2004). The observations are as follows: 0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.823, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663.

The estimated variance-covariance matrix for GOWG-W model on aircraft windshield data set is given by

$$\begin{bmatrix} 0.02850 & -0.02396 & -0.00608 & -0.00565 \\ -0.02396 & 0.18970 & -0.00190 & -0.08932 \\ -0.00608 & -0.00190 & 0.00240 & 0.00403 \\ -0.00565 & -0.08932 & 0.00403 & 0.05532 \end{bmatrix}$$

and the 95% two-sided asymptotic confidence intervals for α, β, δ and c are given by 0.6989 ± 0.3309, 1.0575 ± 0.8537, 0.1455 ± 0.4339, and 0.9722 ± 0.9505, respectively.

Table 4: Parameter estimates and goodness-of-fit statistics for various models fitted for aircraft windshield data.

	Estimates				Statistics							
Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{c}$	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	$K-S$	p-value
GOWG-W	0.6989	1.0575	0.1455	0.9722	253.8	261.8	262.3	271.6	0.0963	0.6524	0.0855	0.5733
	(0.1688)	(0.4355)	(0.0490)	(0.2352)								
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\lambda}$								
EPGW	3.0449	9.0767	0.5253	0.0016	254.9	262.9	263.4	272.6	0.1384	0.8753	0.0868	0.5515
	(8.6642)	(100.29)	(1.6474)	(0.0005)								
	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$								
BGL	39.8830	0.6822	0.0487	157.8600	255.1	263.1	263.6	272.8	0.0995	0.6935	0.1026	0.3396
	(26.7480)	(0.1831)	(0.0446)	(29.0346)								
	$\hat{k}$	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$								
BW	5.9092	0.2653	0.2715	0.7834	255.2	263.2	263.7	272.9	0.1330	0.8514	0.0862	0.5605
	(0.5880)	(0.0571)	(0.0474)	(0.7805)								
	$\hat{a}$	$\hat{b}$	$\hat{\lambda}$	$\hat{c}$								
KOL-LLoG	0.2015	3.0578	0.0003	5.2752	256.8	264.8	265.3	274.5	0.1036	0.6650	0.1016	0.3510
	(0.0501)	(4.7351)	(0.0010)	(0.2933)								
	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\lambda}$	$\hat{\beta}$								
NMW	0.0938	1.8987	2.8630	0.0027	255.8	267.8	268.3	277.5	0.1150	0.7573	0.0999	0.3707
	(0.0223)	(0.4130)	(0.6011)	(0.0161)								
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\theta}$								
OLLEW	4.1570	7.5298	0.1589	1.3303	262.3	270.3	270.8	280.0	0.1379	0.8582	0.1104	0.2580
	(0.5285)	(0.2947)	(0.0377)	(0.3700)								

Table 4 indicates that GOWG-W distribution has the highest p-value for the K-S statistic and the lowest values for all goodness-of-fit statistics. Thus, we conclude that the GOWG-W model performs better on aircraft windshield data than non-nested EPGW, BGL, BW, KOL-LLoG, NMW, and OLLEW models. Furthermore, Figure 11 shows that our model has the lowest SS value from the probability plots compared to the competing non-nested models on the aircraft windshield data.

In Figure 12, we see that the cdf line for the GOWG-W distribution indicated by the blue line is closer to the empirical cdf while the survival function in blue is also close to the Kaplan-Meier(K-M) curve which indicate that our model is the best in explaining the aircraft windshield data. The TTT plot for aircraft windshield data indicates a increasing hazard rate function, hence the aircraft windshield data can be fitted to our model.

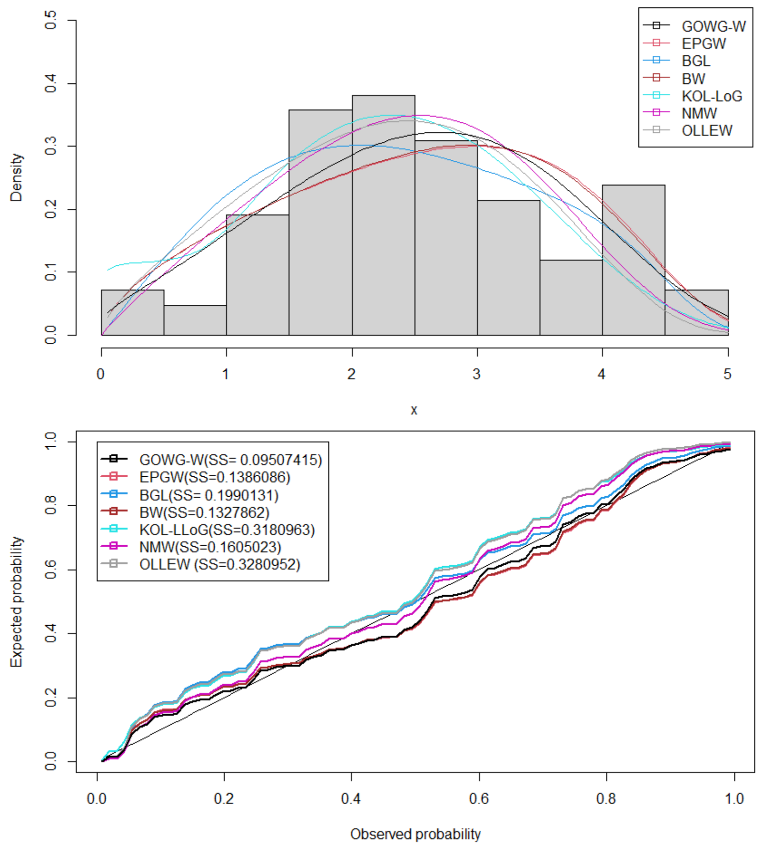


FIGURE 11: Histogram, fitted density and probability plots for aircraft windshield data

7.3. Level of Mercury Data

The third data set refers to the level of mercury in 34 albacore caught in the Eastern Mediterranean obtained from Mol et al. (2012). The observations are as follows: 1.007, 1.447, 0.763, 2.010, 1.346, 1.243, 1.586, 0.821, 1.735, 1.396, 1.109, 0.993, 2.007, 1.373, 2.242, 1.647, 1.350, 0.948, 1.501, 1.907, 1.952, 0.996, 1.433, 0.866, 1.049, 1.665, 2.139, 0.534, 1.027, 1.678, 1.214, 0.905, 1.525, 0.763. The estimated variance-covariance matrix for GOWG-W model on level of mercury data set is given by

$$\begin{bmatrix} 0.2617 & 32.4102 & -2.6103 & -0.4064 \\ 32.4102 & 4777.92 & -51.4461 & -53.4280 \\ -2.6103 & -51.4461 & 123.14 & 2.9475 \\ -0.4064 & -53.4280 & 2.9475 & 0.6444 \end{bmatrix}$$

and the 95% two-sided asymptotic confidence intervals for α, β, δ and c are given by $0.5099 \pm 1.0027, 24.7336 \pm 135.48, 8.0039 \pm 21.7498$, and 0.2565 ± 1.5734, respectively.

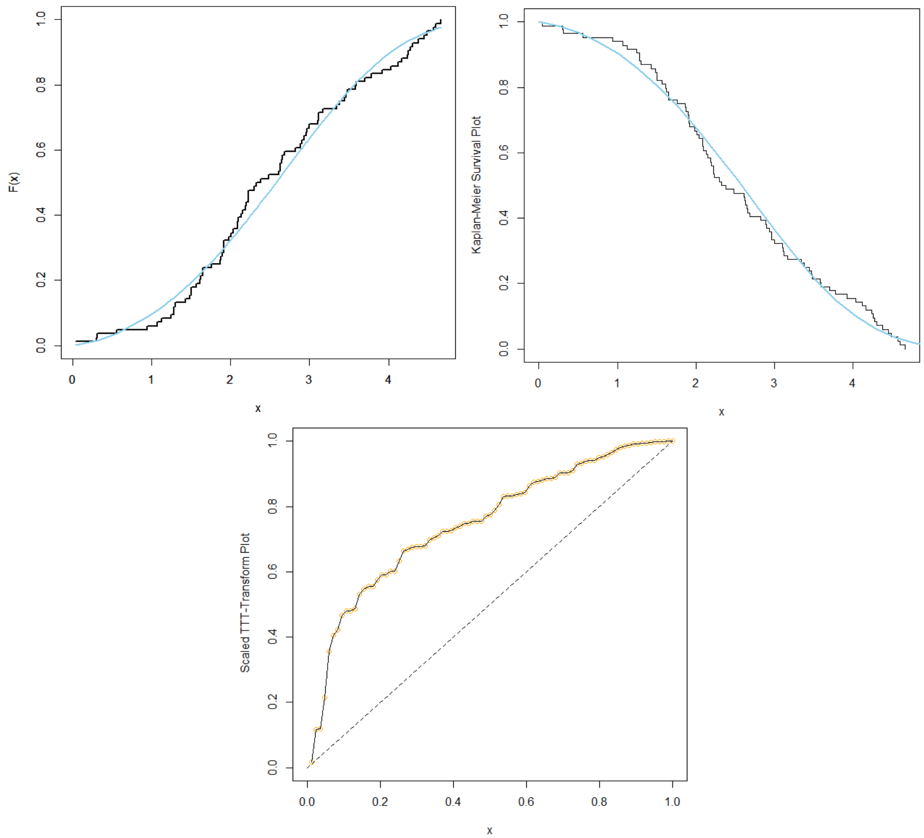


FIGURE 12: Estimated cdf, Kaplan-Meier survival and scaled TTT-Transform plots for the GOWG-W distribution for aircraft windshield data

Table 5 indicates that GOWG-W has the highest p-value for the K-S statistic and the lowest values of all goodness-of-fit statistics. The GOWG-W model therefore works better with level of mercury data than EPGW, BGL, BW, KOL-LLoG, NMW, and OLLEW models. Moreover, Figure 13 shows that our model has the lowest SS value from the probability plots compared to the competing non-nested models on the level of mercury data.

In Figure 14, we see that the cdf line for the GOWG-W distribution indicated by the blue line is closer to the empirical cdf while the survival function in blue is also close to the Kaplan-Meier(K-M) curve which indicate that our model is the best in explaining the level of mercury data. The TTT plot for level of mercury data indicates a increasing hazard rate function, hence the level of mercury data can be fitted to our model.

Table 5: Parameter estimates and goodness-of-fit statistics for various models fitted for level of mercury data.

	Estimates				Statistics							
Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{c}$	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	$K-S$	p-value
GOWG-W	0.5099	24.7336	8.0039	0.2565	38.3	46.3	47.6	52.4	0.0354	0.2406	0.1038	0.8569
	(0.5115)	(69.1225)	(11.0969)	(0.8027)								
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\lambda}$								
EPGW	0.1881	6.5552	4.3600	0.1262	39.0	47.0	48.4	53.1	0.0377	0.2539	0.1134	0.7742
	(6.0550)	(228.9800)	(23.5801)	(4.2732)								
	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$								
BGL	1.2010	0.2210	4.3360	83.7324	38.9	46.9	48.3	53.0	0.0379	0.2491	0.1173	0.7377
	(0.2000)	(0.0527)	(0.3952)	(2.6204)								
	$\hat{k}$	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$								
BW	3.0086	1.9538	3.0473	0.0507	38.9	46.9	48.3	53.0	1.0154	5.6303	0.2059	0.1120
	(0.6774)	(0.7491)	(2.5778)	(0.0892)								
	$\hat{a}$	$\hat{b}$	$\hat{\lambda}$	$\hat{c}$								
KOL-LLoG	1.3856	0.0929	3.1200	3.0736	39.4	47.4	48.8	53.5	0.0373	0.2558	0.1333	0.5814
	(1.4032)	(0.0170)	(0.6820)	(0.2290)								
	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\lambda}$	$\hat{\beta}$								
NMW	0.0410	4.8836	4.4038	0.2485	42.2	50.2	51.6	56.3	0.0506	0.3491	0.1061	0.8385
	(0.0419)	(1.8504)	(0.9638)	(0.1033)								
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\theta}$								
OLLEW	1.1767	3.5727	7.0916	0.2889	43.2	51.2	52.6	57.4	0.0443	0.2482	0.1892	0.1754
	(0.2109)	(1.2799)	(7.0347)	(0.2212)								

8. Concluding Remarks

A new generalized distribution called the gamma odd Weibull generalized-G (GOWG-G) family of distributions is developed and presented. The GOWG-G distribution has several new and known distributions as special cases or sub-models. The behaviour of the hazard rate function of the GOWG-G family of distributions is flexible. We also obtain closed form expressions for the moments, mean and median deviations, distribution of order statistics and entropy. Maximum likelihood estimation technique is used to estimate the model parameters. The performance of the special case of the GOWG-G family of distributions was examined by conducting various simulations for different sizes. Finally, two special cases of the GOWG-G family of distributions are fitted to real data sets to illustrate the applicability and usefulness of the distributions.

Future Work

We hope that the new GOWG-G family of distributions will contribute valuable information toward generalizing odd-Weibull-G as well as odd-Weibull generalized-G families of distributions in terms of flexibility and versatility. Moreover, from the standpoint of practical applications, we believe that we should estimate the pa-

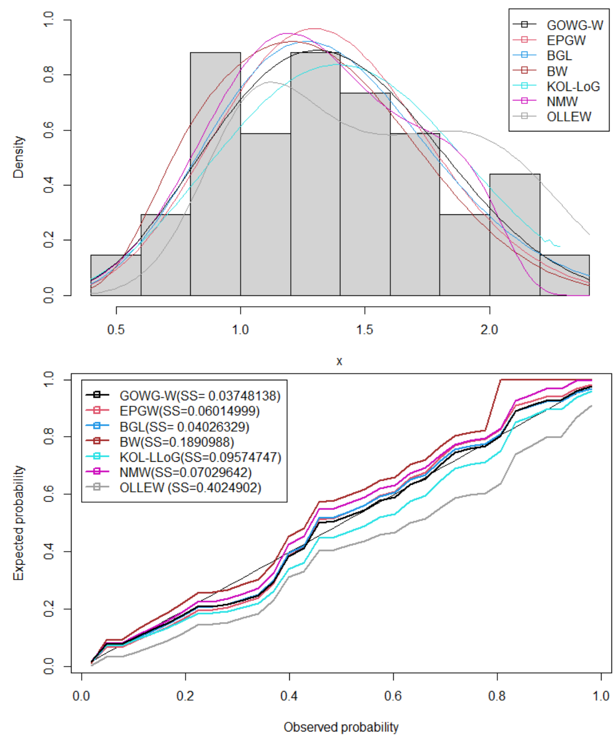


FIGURE 13: Histogram, fitted density and probability plots for level of mercury data.

rameters of the GOWG-G family of distributions by using a variety of estimation methods such as Bayesian estimation, ordinary least-squares estimation, weighted least squares estimation, Anderson-Darling estimation, Cramer-von Mises estimation, and maximum product of spacings estimations. Here, we will refer to works by Teamah et al. (2021), Al-Mofleh et al. (2020), Afify, Alizadeh, Zayed, Ramires & Louzada (2020) and Afify, Nassar, Cordeiro & Kumar (2020). In a future paper, we will continue investigating this aspect, and we hope that our study will serve as a reference for future research in this area.

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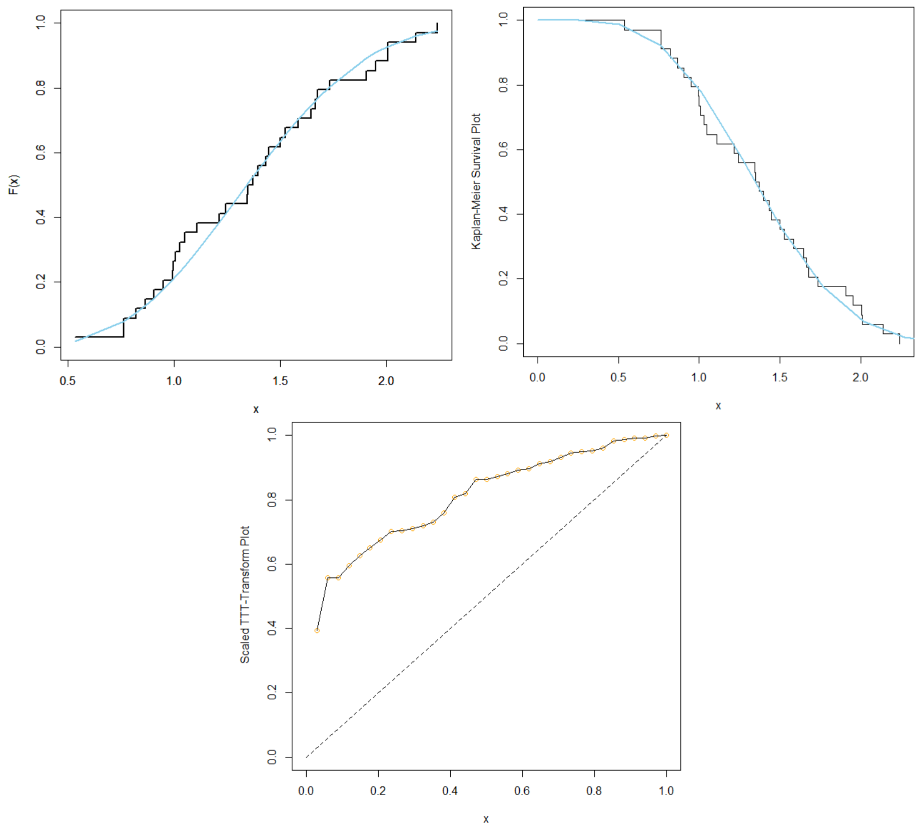

FIGURE 14: Estimated cdf, Kaplan-Meier survival and scaled TTT-Transform plots for the GOWG-W distribution for level of mercury data.

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Appendix A. Details of Some Useful Expansions

Details of the derivation of equation (41)

$$\begin{aligned}
f(x;\alpha,\beta,\delta,\xi) &= \frac{\alpha\beta}{\Gamma(\delta)}\frac{g(x;\xi)}{\left[\overline{G}(x;\xi)\right]^{\alpha+1}}\left(-\log\left[1-e^{-\left(\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}\right)^{\beta}}\right]\right)^{\delta-1}\\
&\times \left[\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}\right]^{\beta-1}\exp\left(-\left[\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}\right]^{\beta}\right)\\
&= \frac{\alpha\beta}{\Gamma(\delta)}\sum_{m,s=0}^{\infty}\binom{\delta-1}{m}b_{s,m}\exp\left(-(m+s+\delta)\left[\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}\right]^{\beta}\right)\\
&\times \left[\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}\right]^{\beta-1}\frac{g(x;\xi)}{\left[\overline{G}(x;\xi)\right]^{\alpha+1}}\\
&= \frac{\alpha\beta}{\Gamma(\delta)}\sum_{m,s=0}^{\infty}\binom{\delta-1}{m}b_{s,m}\sum_{k=0}^{\infty}\frac{(-1)^{k}(m+s+\delta)^{k}}{k!}\left(1-\overline{G}^{\alpha}(x;\xi)\right)^{k\beta}\\
&\times (\overline{G}(x;\xi))^{-k\alpha\beta}\left(1-\overline{G}^{\alpha}(x;\xi)\right)^{\beta-1}(\overline{G}(x;\xi))^{-\alpha(\beta-1)-\alpha-1}g(x;\xi)\\
&= \frac{\alpha\beta}{\Gamma(\delta)}\sum_{m,s=0}^{\infty}\binom{\delta-1}{m}b_{s,m}\sum_{k=0}^{\infty}\frac{(-1)^{k+p}(m+s+\delta)^{k}}{k!}\\
&\times \sum_{p=0}^{\infty}\binom{k\beta+\beta-1}{p}(\overline{G}(x;\xi))^{p\alpha}[\overline{G}(x;\xi)]^{-k\alpha\beta-\alpha\beta-1}g(x;\xi)\\
&= \frac{\alpha\beta}{\Gamma(\delta)}\sum_{m,s,k,p=0}^{\infty}\binom{\delta-1}{m}b_{s,m}\frac{(-1)^{k+p}(m+s+\delta)^{k}}{k!}\binom{k\beta+\beta-1}{p}\\
&\times \sum_{q=0}^{\infty}\binom{p\alpha}{q}(-1)^{q}[G(x;\xi)]^{q}\sum_{t=0}^{\infty}\binom{-\alpha\beta(k+1)-1}{t}(-1)^{t}[G(x;\xi)]^{t}g(x;\xi)\\
&= \sum_{q,t=0}^{\infty}\left[\sum_{m,s,k,p=0}^{\infty}\binom{\delta-1}{m}\binom{k\beta+\beta-1}{p}\binom{p\alpha}{q}\binom{-\alpha\beta(k+1)-1}{t}b_{s,m}\right.\\
&\times \left.\frac{(-1)^{k+p+q+t}(m+s+\delta)^{k}}{(q+t+1)k!}\frac{\alpha\beta}{\Gamma(\delta)}\right](q+t+1)[G(x;\xi)]^{q+t}g(x;\xi).
\end{aligned}$$

Details of the derivation of equation (50)

$$\begin{aligned}
f_{i:n}(x) &= \frac{n!f(x)}{(i-1)!(n-i)!}[F(x)]^{i-1}[1-F(x)]^{n-i}\\
&= \frac{n!f(x)}{(i-1)!(n-i)!}\sum_{m=0}^{n-i}(-1)^m\binom{n-i}{m}[F(x)]^{m+i-1}\\
&= \sum_{m=0}^{n-i}\frac{n!}{(i-1)!(n-i)!(m+i)}(-1)^m\binom{n-i}{m}(m+i)[F(x)]^{m+i-1}f(x).
\end{aligned}$$

Details of the derivation of equation (55)

$$\begin{aligned}
f_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}[F(x)]^{j+i-1}f(x)\\
&= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}\left(\sum_{k=0}^{\infty}C_k(G(x;\xi))^k\right)^{j+i-1}f(x).
\end{aligned}$$

Details of the derivation of equation (57)

$$\begin{aligned}
f_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}\left(\sum_{k=0}^{\infty}d_{j+i-1,k}(G(x;\xi))^k\right)\\
&\times \left(\sum_{r=0}^{\infty}C_{r+1}(r+1)[G(x;\xi)]^r g(x;\xi)\right)\\
&= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}\left(\sum_{k=0}^{\infty}d_{j+i-1,k}(G(x;\xi))^k\right)\\
&\times \left(\sum_{r=0}^{\infty}b_r[G(x;\xi)]^r g(x;\xi)\right).
\end{aligned}$$

Details of the derivation of equation (61)

$$
\begin{aligned}
(f(x))^v &= \frac{(\alpha\beta)^v}{(\Gamma(\delta))^v}\frac{(g(x;\xi))^v}{\left[\overline{G}(x;\xi)\right]^{v(\alpha+1)}}\left(-\log\left[1-e^{-\left(\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}\right)^{\beta}}\right]\right)^{v\delta-v}\\
&\times \left[\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}\right]^{v\beta-v} e^{\left(-v\left[\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}\right]^{\beta}\right)}\\
&= \frac{(\alpha\beta)^v}{(\Gamma(\delta))^v}\frac{(g(x;\xi))^v}{\left[\overline{G}(x;\xi)\right]^{v(\alpha+1)}}\sum_{m,s=0}^{\infty}\binom{v\delta-v}{m}b_{s,m}\left[\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}\right]^{v\beta-v}\\
&\times e^{\left(-(m+s+v\delta)\left[\frac{1-\overline{G}^{\alpha}(x;\xi)}{\overline{G}^{\alpha}(x;\xi)}\right]^{\beta}\right)}\\
&= \frac{(\alpha\beta)^v}{(\Gamma(\delta))^v}\sum_{m,s=0}^{\infty}\binom{v\delta-v}{m}b_{s,m}\sum_{k=0}^{\infty}\frac{(-1)^k(m+s+v\delta)^k}{k!}\left(1-\overline{G}^{\alpha}(x;\xi)\right)^{k\beta}\\
&\times [\overline{G}(x;\xi)]^{-k\alpha\beta}\left(1-\overline{G}^{\alpha}(x;\xi)\right)^{v\beta-v}\left[\overline{G}(x;\xi)\right]^{-\alpha(v\beta-v)-v\alpha-v}(g(x;\xi))^v\\
&= \frac{(\alpha\beta)^v}{(\Gamma(\delta))^v}\sum_{m,s,k,p,q,t=0}^{\infty}\binom{v\delta-v}{m}\binom{\beta(k+v)-v}{p}\binom{p\alpha}{q}\binom{-\alpha\beta(k+v)-v}{t}\\
&\times \frac{b_{s,m}(-1)^{k+p+q+t}(m+s+v\delta)^k}{k!}[G(x;\xi)]^{q+t}(g(x;\xi))^v.
\end{aligned}
$$

Appendix B. Elements of the Score Vector

The first derivative of the log-likelihood function with respect to each component of the parameter vector $\mathbf{\Delta}=(\alpha,\beta,\delta,\xi)^T$, that is, elements of the score vector $U(\mathbf{\Delta})$ are given by

$$
\begin{aligned}
\frac{\partial \ell_n}{\partial \alpha} &= \frac{n}{\alpha}+(\delta-1)\sum_{i=1}^{n}\frac{e^{-\left(\overline{G}^{-\alpha}(x_i;\xi)-1\right)^{\beta}}\beta\left(\overline{G}^{-\alpha}(x_i;\xi)-1\right)^{\beta-1}(\overline{G}^{-\alpha}(x;\xi))\ln(\overline{G}^{-\alpha}(x;\xi))}{\left[1-e^{-\left(\overline{G}^{-\alpha}(x_i;\xi)-1\right)^{\beta}}\right]\ln\left[1-e^{-\left(\overline{G}^{-\alpha}(x_i;\xi)-1\right)^{\beta}}\right]}\\
&- (\beta-1)\sum_{i=1}^{n}\frac{(\overline{G}^{-\alpha}(x_i;\xi))\ln(\overline{G}(x_i;\xi))}{\left(\overline{G}^{-\alpha}(x_i;\xi)-1\right)}-\sum_{i=1}^{n}\ln(\overline{G}(x_i;\xi))\\
&+ \beta\sum_{i=1}^{n}\left(\overline{G}^{-\alpha}(x_i;\xi)-1\right)^{\beta-1}\overline{G}^{-\alpha}(x_i;\xi)\ln(\overline{G}(x_i;\xi)),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \ell_n}{\partial \beta} &= \frac{n}{\beta}+(\delta-1)\sum_{i=1}^{n}\frac{e^{-\left(\overline{G}^{-\alpha}(x_i;\xi)-1\right)^{\beta}}\left(\overline{G}^{-\alpha}(x_i;\xi)-1\right)^{\beta}(\overline{G}^{\alpha}(x_i;\xi)-1)^{\beta}\ln(\overline{G}(x;\xi)-1)}{\left[1-e^{-\left(\overline{G}^{-\alpha}(x_i;\xi)-1\right)^{\beta}}\right]\ln\left[1-e^{-\left(\overline{G}^{-\alpha}(x_i;\xi)-1\right)^{\beta}}\right]}\\
&+ \sum_{i=1}^{n}\ln(\overline{G}^{-\alpha}(x;\xi)-1)-\sum_{i=1}^{n}\left(\overline{G}^{-\alpha}(x_i;\xi)-1\right)^{\beta}\ln(\overline{G}^{-\alpha}(x;\xi)-1),
\end{aligned}
$$

$$\frac{\partial \ell_n}{\partial \delta} = \frac{n\Gamma'(\delta)}{\Gamma(\delta)} + \sum_{i=1}^{n} \ln\left(-\ln\left[1-e^{-\left(\overline{G}^{-\alpha}(x_i;\xi)-1\right)^{\beta}}\right]\right),$$

and

$$\begin{aligned}\frac{\partial \ell_n}{\partial \xi_k} &= \frac{n}{\alpha} - (\delta-1)\sum_{i=1}^{n} \frac{e^{-\left(\overline{G}^{-\alpha}(x_i;\xi)-1\right)^{\beta}}\beta\left(\overline{G}^{-\alpha}(x_i;\xi)-1\right)^{\beta-1}\alpha\overline{G}^{\alpha-1}(x_i;\xi)\frac{\partial \overline{G}(x_i;\xi)}{\partial \xi_k}}{\left[1-e^{-\left(\overline{G}^{-\alpha}(x_i;\xi)-1\right)^{\beta}}\right]\ln\left[1-e^{-\left(\overline{G}^{-\alpha}(x_i;\xi)-1\right)^{\beta}}\right]}\\ &+ (\beta-1)\sum_{i=1}^{n}\frac{\alpha\overline{G}^{\alpha-1}(x_i;\xi)\frac{\partial \overline{G}(x_i;\xi)}{\partial \xi_k}}{\left(\overline{G}^{-\alpha}(x_i;\xi)-1\right)} + \sum_{i=1}^{n}\frac{\frac{\partial g(x_i;\xi)}{\partial \xi_k}}{g(x_i;\xi)} - (\alpha+1)\sum_{i=1}^{n}\frac{\frac{\partial \overline{G}(x_i;\xi)}{\partial \xi_k}}{\overline{G}(x_i;\xi)}\\ &+ \alpha\beta\sum_{i=1}^{n}(\overline{G}^{-\alpha}(x_i;\xi)-1)^{\beta-1}[\overline{G}(x;\xi)]^{-\alpha-1}\frac{\partial \overline{G}(x_i;\xi)}{\partial \xi_k}.\end{aligned}$$

Appendix C. Additional Simulation Results

The simulations results for GOWG-Weibull distribution are given in Table A1.

Appendix D. Pdfs of the Non-nested Models

The pdfs of the EPGW, BXIIP, BGL, BW, KOL-LLoG, KwW, TLMO-W, ELLOLW, NMW and OLLEW distributions are given by

$$\begin{aligned} f_{EPGW}(x;\lambda,\alpha,\beta,\delta) &= \lambda\alpha\beta\delta x^{\alpha-1}\left(1+\lambda x^{\alpha}\right)^{\beta-1}\exp\left(1-[1+\lambda x^{\alpha}]^{\beta}\right)\\ &\times \left[1-\exp\left(1-[1+\lambda x^{\alpha}]^{\beta}\right)\right]^{\delta-1},\end{aligned}$$

for $x > 0$, $\lambda > 0$, $\alpha > 0$, $\beta > 0$, $\delta > 0$, (when $\delta = 1$, we have the power generalized Weibull (PGW) distribution Nikulin & Haghighi, 2009),

$$f_{BXIIP}(x) = \frac{cks^{-c}\lambda}{1-e^{-\lambda}}x^{c-1}\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-k-1}\exp\left\{-\lambda\left[1-\left(1+\left(\frac{x}{s}\right)^{c}\right)^{-k}\right]\right\}$$

for $x > 0$,$c > 0$,$k > 0$,$s > 0$,$\lambda > 0$,

$$\begin{aligned} f_{BGL}(x;\alpha,\lambda,a,b) &= \frac{\alpha\lambda^2(1+x)\exp(-\lambda x)}{(1+\lambda)B(a,b)}\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{a\alpha-1}\\ &\times \left\{1-\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\exp(-\lambda x)\right]^{\alpha}\right\}^{b-1},\end{aligned}$$

for $x > 0$, $\alpha > 0$, $\lambda > 0$, $a > 0$, $b > 0$,

$$f_{BW}(x;k,\lambda,a,b) = \frac{k\lambda^k}{B(a,b)}x^{k-1}\exp(-b(\lambda x)^k)[1-\exp(-(\lambda x)^k)]^{a-1},$$

Table A1: Monte Carlo simulation results for GOWG-Weibull distribution: mean, average bias and RMSE.

		$\alpha=1.2,\beta=0.8,\delta=1.5,c=0.5$			$\alpha=1.5,\beta=0.5,\delta=1.0,c=0.5$		
Parameter	n	Mean	Average Bias	RMSE	Mean	Average Bias	RMSE
α	25	3.122411	1.922411	3.544828	2.129375	0.629375	1.570994
	50	2.152399	0.952399	1.727887	1.872956	0.3729558	1.050039
	75	1.873433	0.673433	1.301698	1.762973	0.2629729	0.8415205
	100	1.717492	0.517492	0.994711	1.732748	0.232748	0.745626
	200	1.493035	0.293035	0.642358	1.579199	0.079199	0.525466
	400	1.350209	0.150209	0.413595	1.553359	0.053359	0.389172
	800	1.260209	0.060209	0.280101	1.533316	0.033316	0.266274
	1200	1.231763	0.031763	0.209236	1.501225	0.001225	0.209826
β	25	0.990956	0.190956	1.577928	0.774041	0.274041	2.158992
	50	0.947908	0.147908	1.001243	0.553332	0.053332	0.349348
	75	0.879050	0.079050	0.929388	0.517447	0.017447	0.232108
	100	0.854699	0.054699	0.828695	0.516342	0.016342	0.187327
	200	0.838846	0.038846	0.519968	0.501066	0.001066	0.131012
	400	0.779080	-0.020920	0.277986	0.499466	-0.000534	0.094957
	800	0.789028	-0.010972	0.214357	0.496550	-0.003450	0.072669
	1200	0.790375	-0.009625	0.172370	0.496826	-0.003174	0.063956
δ	25	1.443850	-0.056150	1.679883	0.972987	-0.027013	0.847135
	50	1.476072	-0.023928	1.434908	0.980633	-0.019367	0.672120
	75	1.448622	-0.051378	1.085246	0.943850	-0.056150	0.416816
	100	1.388070	-0.111930	0.810976	0.936671	-0.063329	0.384338
	200	1.401039	-0.098961	0.581400	0.982344	-0.017656	0.312805
	400	1.457458	-0.042542	0.448835	0.979786	-0.020214	0.249331
	800	1.492793	-0.007207	0.340613	0.983442	-0.016558	0.182321
	1200	1.501346	0.001346	0.275575	0.998282	-0.001718	0.154646
c	25	0.749957	0.249957	0.733894	0.797384	0.297384	0.562149
	50	0.652671	0.152671	0.518274	0.675324	0.175324	0.403464
	75	0.639326	0.139326	0.432635	0.617211	0.117211	0.340435
	100	0.623616	0.123616	0.379463	0.576961	0.076961	0.270992
	200	0.585558	0.085558	0.314170	0.564146	0.064146	0.225285
	400	0.580980	0.080980	0.260408	0.534028	0.034028	0.163997
	800	0.552340	0.052340	0.194071	0.520952	0.020952	0.117372
	1200	0.537484	0.037484	0.157240	0.521251	0.021251	0.097642

for $x>0$, $k>0$, $\lambda>0$, $a>0$, $b>0$, (when $k=1$, the beta exponential pdf is obtained, Nadarajah & Kotz, 2006),

$$\begin{aligned} f_{KOL-LLoG}(x;a,b,\lambda,c) &= ab\left[\frac{\lambda^2}{(1+\lambda)}\frac{cx^{c-1}}{(1+x^c)^{-1}}\exp\left\{-\lambda\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right\}\right] \\ &\times \left[1-\frac{\lambda+(1+x^c)^{-1}}{(1+\lambda)(1+x^c)^{-1}}\exp\left\{-\lambda\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right\}\right]^{a-1} \\ &\times \left(1-\left[1-\frac{\lambda+(1+x^c)^{-1}}{(1+\lambda)(1+x^c)^{-1}}\exp\left\{-\lambda\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right\}\right]^{a}\right)^{b-1}, \end{aligned}$$

for $x > 0$, $a > 0$, $b > 0$, $\alpha > 0$, $\beta > 0$,

$$f_{KwW}(x; a, b, \alpha, \beta) = ab\alpha^{\beta} x^{\beta-1} e^{-(\alpha x)^{\beta}} \left(1 - e^{-(\alpha x)^{\beta}}\right)^{a-1} \left(1 - \left(1 - e^{-(\alpha x)^{\beta}}\right)^{a}\right)^{b-1},$$

for $x > 0$, $a > 0$, $b > 0$, $\alpha > 0$, $\beta > 0$,

$$f_{TLMO-W}(x; b, \delta, \lambda, \gamma) = \frac{2b\delta^2 \lambda\gamma x^{\gamma-1} e^{-2\lambda x^{\gamma}}}{\left(1 - \bar{\delta} e^{-\lambda x^{\gamma}}\right)^3} \left[1 - \frac{\delta^2 e^{-2\lambda x^{\gamma}}}{\left(1 - \bar{\delta} e^{-\lambda x^{\gamma}}\right)^2}\right]^{b-1},$$

for $x > 0$, $b > 0$, $\delta > 0$, $\lambda > 0$, $\gamma > 0$,

$$\begin{aligned} f_{ELOLLW}(x; \alpha, \beta, \gamma, \theta, \lambda) = & \frac{\alpha\theta^2 \gamma \lambda^{\gamma} x^{\gamma-1} e^{-(\lambda x)^{\gamma}} \left(e^{-(\lambda x)}\right)^{\alpha\theta-1} \left(1 - e^{-(\lambda x)^{\gamma}}\right)^{\alpha-1}}{(\theta+\beta)\left(\left(1-e^{-(\lambda x)^{\gamma}}\right)^{\alpha} + e^{-\alpha(\lambda x)^{\gamma}}\right)^{\theta-1}} \\ & \times \left(1 - \beta \log\left[\frac{e^{-(\lambda x)^{\gamma}}}{\left(1-e^{-(\lambda x)^{\gamma}}\right)^{\alpha} + e^{-\alpha(\lambda x)^{\gamma}}}\right]\right), \end{aligned}$$

for $x > 0$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\theta > 0$, $\lambda > 0$,

$$f_{NMW}(x; \alpha, \gamma, \lambda, \beta) = \left(\alpha\gamma x^{\gamma-1} e^{\alpha x^{\gamma}} + \lambda\beta x^{\lambda-1} e^{-\beta x^{\lambda}}\right) e^{-e^{\alpha x^{\gamma}} + e^{-\beta x^{\lambda}}},$$

for $x > 0$, $\alpha > 0$, $\gamma > 0$, $\lambda > 0$, $\beta > 0$, and

$$f_{OLLEW}(x; \alpha, \beta, \gamma, \theta) = \frac{\theta\beta\gamma x^{\beta-1} e^{-(\frac{x}{\alpha})^{\beta}} \left[1 - e^{-(\frac{x}{\alpha})^{\beta}}\right]^{\gamma\theta-1} \left\{1 - \left[1 - e^{-(\frac{x}{\alpha})^{\beta}}\right]^{\gamma}\right\}^{\theta-1}}{\alpha^{\beta} \left\{\left[1 - e^{-(\frac{x}{\alpha})^{\beta}}\right]^{\gamma\theta} + \left\{1 - \left[1 - e^{-(\frac{x}{\alpha})^{\beta}}\right]^{\gamma}\right\}^{\theta}\right\}^2},$$

for $x > 0$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\theta > 0$. For the ELOLLW distribution we considered the case when $\alpha = 1$.