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WIENER MEASURE ON $P_0(G)$

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ABSTRACT Nonstandard methods allow a flat integral representation of de Wiener measure on $P_0(R)$. A representation of the Wiener measure on $P_0(R^d)$ is given, allowing us to give a nonstandard representation of the Wiener measure on $P_0(G)$ by using Ito map.

0. PRELIMINARIES

For a good introduction of nonstandard analysis we can see (Albeverio, S. (1986)).

The main features that we need in our work are the following.

We assume the existence of a set $^*\mathbb{R} \supset \mathbb{R}$, called the set of the nonstandard real numbers and a mapping $*: V(\mathbb{R}) \rightarrow V(^*\mathbb{R})$, (where $V_1(S) = S$, $V_{n+1}(S) = V_n(S) \cup \mathcal{P}(V_n(S))$ and $V(S) = \bigcup_{n \in \mathbb{N}} V_n(S)$) with three basic properties. To state the properties we give the following notions.

An elementary statement is a statement $\Phi$ built up from $"=", "\in",$ relations: $u = v$, $u \in v$, the connectives "and", "or", "not", and "implies", bounded quantifiers $(\forall u \in v), (\exists u \in v)$.

An internal object $A$ is an element of $V(^*\mathbb{R})$ such that $A =^* S, S \in V(\mathbb{R})$. A set in $V(^*\mathbb{R})$ which is not internal is called external.

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(1) **Extension Principle.** \(*R\) is a proper extension of \(R\) and \(* : V(R) \to V(*R)\) is an embedding such that \(*r = r\) for all \(r \in R\).

(2) **The Saturation Property:** Let \(\{R_n : n \in N\}\) be a sequence of internal objects and \(\{S_m : m \in N\}\) be a sequence of internal sets. If for each \(m \in N\) there is an \(N_m \in N\) such that for all \(n \geq N_m\), \(R_n \subseteq S_m\), then \(\{R_n : n \in N\}\) can be extended to an internal sequence \(\{R_\eta : \eta \in *N\}\) such that \(R_\eta \subseteq \cap_m S_m\) for every \(\eta \in *N - N\).

(2') **General Saturation Principle:** Let \(\kappa\) be an infinite cardinal. A nonstandard extension is called \(\kappa\)-saturated if for every family \(\{X_i\}_{i \in I} : \text{card}(I) < \kappa\), with the infinite intersection property, the intersection \(\cap_i X_i\) is nonempty, i.e. this intersection contains some internal object.

(3) **Transfer Principle:** Let \(\Phi(X_1, ..., X_m, x_1, ..., x_n)\) be an elementary statement in \(V(R)\). Then, for any \(A_1, ..., A_m \subseteq R\) and \(r_1, ..., r_n \in R\),

\[
\Phi(A_1, ..., A_m, r_1, ..., r_n)
\]

is true in \(V(R)\) if and only if

\[
\Phi(*A_1, ..., *A_m, *r_1, ..., *r_n)
\]

is true in \(V(*R)\).

\((*R, +, *, \leq)\) extends \(R\) as an ordered field, in general we will omit the * for the operation and the order relation.

In \(R\) we can distinguish three kinds for numbers:
(a) \( x \in \ast \mathbb{R} \) is infinitesimal, if \( |x| < r \) for each \( r \in \mathbb{R}^+ \).

(b) \( x \in \ast \mathbb{R} \) is finite, if there is a real number \( r \in \mathbb{R}^+ \) such that \( |x| < r \).

(c) \( x \in \ast \mathbb{R} \) is infinite number, if \( |x| > r \) for each \( r \in \mathbb{R}^+ \).

For each finite number \( x \in \ast \mathbb{R} \) we can associate a unique real \( r := st(x) := \ast x \), such that \( x = r + \epsilon \), where \( \epsilon \) is infinitesimal. We say that \( x \) is infinitely closed to \( y \), denoted by \( x \approx y \) if and only if \( x - y \) is infinitesimal.

In general we use capital letters \( H, F, X \), etc. for internal functions and processes, while \( h, f, X \) etc. are used for standard ones. For stopping times we will always use capital letters, and specify whether standard or nonstandard is meant.

For given set \( A \), \( \ast A \) stands for the elementary extension of \( A \), and \( ns(\ast A) \) denotes the nearstandard points in \( \ast A \). If \( s \) is an element in \( ns(\ast A) \), the standard part of \( s \) is written as \( st(s) \), or \( \ast s \). For given function \( f \), \( \ast f \) means the elementary extension of \( f \).

We say that the set \( T \) is \( S \)-dense if \( \{ \ast t : t \in T, \ast t < \infty \} = [0, \infty) \), and \( ns(T) = \{ t \in T : \ast t < \infty \} \). With \( T \) we denote an internal \( S \)-dense subset of \( \ast [0, \infty) \). The elements of \( T \), or more generally, of \( \ast [0, \infty) \), are denoted with \( s, t, u \), etc. The real numbers in \( [0, \infty) \) are denoted by \( s, t, u \), etc. We will work with different sets \( T \), so will always specify the definition of such \( T \).

With \( \mathbb{N} \) we denote the set of nonzero natural numbers \( \{1, 2, 3, \ldots\} \), and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). Elements of \( \mathbb{N}_0 \) are denoted with \( n, m, l \), etc. while, elements in \( \ast \mathbb{N} - \mathbb{N} \) will be denoted with \( \eta, \mathbb{N}, \) etc.

When we say that \( F : A \rightarrow B \) is an internal function, mean that the domain, the range and the graph of the function are internal concepts.
1. **Definition.** A subset $A \subseteq \ast \mathbb{R}$ which is internal and for which there exists $N \in \ast \mathbb{N}$ and an internal bijection $F : A \rightarrow \{0,1,2,\ldots,N-1\}$ is called hyperfinite set. In such case $A$ is said to have hyperfinite internal cardinality $N$, and we write $|A| = N$.

Hyperfinite sets are to the nonstandard universe what the finite sets are to the standard one.

2. **Proposition.** Let $A$ and $B$ be hyperfinite sets with internal cardinalities $H$ and $N$, respectively. Then:

i) $A \times B$ is hyperfinite, with $|A \times B| = HN$

ii) $A^B = \{F : B \rightarrow A : F$ is an internal function$\}$ is a hyperfinite set and its cardinality is $H^N$.

iii) $A \cup B$, $A \cap B$ are hyperfinite.

iv) If $A$ is hyperfinite and $C \subseteq A$ is an internal set, also $C$ is hyperfinite.

Let $\ast \mathbb{R}^+ = \ast \mathbb{R} \cup \{0, \infty\}$ be the extended nonnegative hyperreals. An internal finitely additive measure on the internal algebra $\mathcal{U}$ is an internal set function $\mu : \mathcal{U} \rightarrow \ast \mathbb{R}^+$, such that

(i) $\mu(\emptyset) = 0$

(ii) For $A, B \in \mathcal{U}$ with $A \cap B = \emptyset$, $\mu(A \cup B) = \mu(A) + \mu(B)$.

Since $\mu$ is internal, the finite additivity extends to hyperfinite unions. Let $\Omega$ be a hyperfinite set and let $\mathcal{U}$ be the class of all internal subsets of $\Omega$. Let us define a finitely additive measure $^\circ \mu : \mathcal{U} \rightarrow \ast \mathbb{R}^+$ by $^\circ \mu(A) = ^\circ (\mu(A))$, where $^\circ r = \infty$ when $r$ is an infinitely large element of $\ast \mathbb{R}^+$.
A countable union of sets can be written as a countable disjoint union of sets of the same kind. As have seen in Corollary A2.8 (Muñoz de Özak, M. (1995)), a countable union of disjoint internal sets is not internal. Then, $\sigma$ is a $\sigma$-additive measure on the algebra of internal hyperfinite subsets of $\Omega$. The Loeb measure is basically the extension $v$ of $\sigma$ to the $\sigma$-algebra generated by $U$ by means of the Carathéodory's Extension Theorem.

3. Theorem (Loeb). The extended real valued function $v = L(\mu)$ has a standard $\sigma$-additive extension to the smallest (external) $\sigma$-algebra $M$ on $\Omega$ containing $U$. For each $B \in M$, the value of this extension is given by $v(B) = \inf_{A \in U, B \subseteq A} \sigma(A)$. This extension is unique if $\mu(\Omega) < +\infty$, in which case, for each $B \in M$, $v(B) = \sup_{A \in U, B \subseteq A} \sigma(A)$ and there is $A \in U$ with $v(B\Delta A) = v((B - A) \cup (A - B)) = 0$.

For the proof see (Loeb, P. (1975)).

We say that $A$ is Loeb measurable if

$$P_{ex}(B) = \inf_{A \in U, B \subseteq A} \sigma(A) = \sup_{A \in U, B \subseteq A} \sigma(A) = P_{in}(B),$$

and we denote this common value by $L(\mu)$. The collection of all measurable sets is denoted with $L(\Omega)$. The collection of all measurable sets is denoted with $L(\Omega)$.

4. Theorem. $(\Omega, L(\Omega), L(\mu))$ is a complete probability space which extends $(\Omega, U, \mu)$. It is called the Loeb space associated with $(\Omega, U, \mu)$.

For the proof see A3.2 in the appendix in (Muñoz de Özak, M (1995)).

5. Theorem. (Fubini type) Let $(\Omega_1, U_1, P_1)$ and $(\Omega_2, U_2, P_2)$ be hyperfinite probability spaces and let $F : \Omega_1 \times \Omega_2 \rightarrow R$ be a Loeb integrable function. Then:

(i) $f(w_1, \cdot)$ is Loeb integrable for almost all $w_1 \in \Omega_1$. 

(ii) $g(w_1) = \int f(w_1, w_2) \, dL(P_2)$ is Loeb integrable on $\Omega_1$.

(iii) $\int f(w_1, w_2) \, dL(P_1 \times P_2) = \int (\int f(w_1, w_2) \, dL(P_2)) \, dL(P_1)$.

The proof is due to Keisler. See (Keisler, H.J. (1984)), Theorem 1.14.b)

1. INTRODUCTION

We extend the one dimensional definition of $N$. Cutland (1990) of the Wiener measure on $P_0(\mathbb{R})$ to $P_0(\mathbb{R}^d)$. This allows to give a nonstandard definition of Wiener measure on Lie algebras. Then by means of Ito’s map, we obtain the notion of a nonstandard representation of the Wiener measure on $P_c(G)$, where $G$ is a Lie group.

2. WIENER MEASURE ON $P_c(G)$

Let

$$P_0(\mathbb{R}) = \{x : [0,1] \rightarrow \mathbb{R} \mid x \text{ is continuous and } x_0 = 0\}$$

and let $\mathcal{C}$ the Borel $\sigma$-algebra on $P_0(\mathbb{R})$ ($P_0(\mathbb{R})$ is given with the uniform convergence norm). The Wiener measure $\mu_0$ over $(P_0(\mathbb{R}), \mathcal{C})$ is a probability measure such that, for $0 = t_0 < t_1 < \ldots < t_n = 1$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$,

$$\mu_0(x_t, \leq \alpha_i, 1 \leq i \leq n) = \int \prod_{i=0}^{n-1} \left(2\pi (t_{i+1} - t_i)\right)^{-1/2} \exp \left(-\frac{(y_{i+1} - y_i)^2}{2(t_{i+1} - t_i)}\right) dy$$

where $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, $y_0 = 0$ and $dy$ the Lebesgue measure on $\mathbb{R}^n$. $\mu_0$ can be also described as a probability on $(P_0(\mathbb{R}), \mathcal{C})$ making the increments $(X_{t_{i+1}} - X_{t_i})_{0 \leq i \leq n-1}$ independent and $N(0, t_{i+1} - t_i)$ distributed. The canonical continuous process given by $\mu_0$ is a Brownian motion.

Let $T = \{0, \Delta t, 2\Delta t, \ldots, 1\}$ be the hyperfinite unit interval. Following Cutland
we can make a nonstandard construction of the Brownian motion that gives us an adequate definition of the Wiener measure on \((P_\circ (\mathbb{R}), \mathcal{C})\) as follows:

Fix an internal probability space \((\Omega, \mathcal{U}, \mathcal{P})\) carrying independent \(N(0, t)\) random variables \((\eta_t)_{t \in T}\). Define a process \(B : T \times \Omega \rightarrow \ast \mathbb{R}\) by

\[
B(0, w) = 0
\]

\[
\Delta B(t, w) = B(t, w) - B(t - \Delta t, w) = \eta_t, \quad t \in T.
\]

Let \(P = L(P)\). Cutland obtains the following result:

(i) For \(P\)-a.a. \(w\), \(B(\cdot, w)\) is \(S\)-continuous.

(ii) The process \(b(\cdot, w) = \circ B(\cdot, w)\) is a brownian motion.

Cutland also shows that this construction of \(b\) gives rise to a construction of the Wiener measure that can be expressed as follows: Let \(\Gamma\) be the internal measure on \(\ast \mathbb{R}^T\) induced by \(B\), i.e., for \(A \in \mathcal{D}\), where \(\mathcal{D}\) is the Borel \(\sigma\)-algebra in \(\ast \mathbb{R}^T\),

\[
\Gamma(A) = \mathcal{P}(B(\cdot, w) \in A)
\]

\[
= (2\pi \Delta t)^{-N/2} \int_A \prod_{i \in T} \exp \left( -\frac{(X_{i - 1} - X_{i - \Delta t})^2}{2\Delta t} \right) dX_{\Delta t} dX_{2\Delta t} \cdots dX_1
\]

with \(dX_i\) denoting the \(\ast\)Lebesgue measure over \(\ast \mathbb{R}\). Writing \(dX\) for the \(\ast\)Lebesgue measure on \(\ast \mathbb{R}^T\), and

\[
\dot{X}_t = \frac{X_{t - 1} - X_{t - \Delta t}}{\Delta t} = \frac{\Delta X_{t - 1}}{\Delta t},
\]

we have

\[
\Gamma(A) = (2\pi \Delta t)^{-N/2} \int_A \exp \left( -\frac{1}{2} \sum_{i \in T} \dot{X}_t^2 \Delta t \right) dX
\]
and is follows that, with respect to $L(\Gamma)$, $X$ is $S$-continuous for almost all $X \in \mathcal{R}^T$.

and the Wiener measure on $(P_0(\mathcal{R}), \mathcal{C})$ is given by

$$
\mu_0(D) = L(\Gamma)(st^{-1}(D)), \quad D \in \mathcal{C},
$$

where $st^{-1}(D) = \{X \in \mathcal{R}^T : {}^0X \in D\}$.

Now consider

$$
P_0(\mathcal{R}^d) = \{x : [0,1] \to \mathcal{R}^d | x \text{ continuous and } x_0 = 0\}
$$

and denoted with $\mathcal{C}^d$ the Borel $\sigma$-algebra on $P_0(\mathcal{R}^d)$. The Wiener measure on $(P_0(\mathcal{R}^d), \mathcal{C}^d)$ is defined by

$$
\mu_0(x_t, \in A_i, 1 \leq i \leq n) =
\int_{A_1} \cdots \int_{A_n} \prod_{i=0}^{n-1} (2\pi(t_{i+1} - t_i))^{-d/2} \exp \left( -\frac{||y_{i+1} - y_i||^2}{2(t_{i+1} - t_i)} \right) dy_1 \cdots dy_n
$$

where $\{t_i : 1 \leq i \leq n\}$ is a partition of $[0,1]$, $A_i \in \mathcal{B}(\mathcal{R}^d)$, $||\alpha||$ is the length of $\alpha$ and $dy_i$ is the Lebesgue measure on $\mathcal{R}^d$.

Generalizing Cutland's constructions for the Brownian motion, we can construct $d$ independent $B^i(\cdot, w)$ processes such that $b^i(\cdot, w) = {}^0B^i(\cdot, w)$. Then

$$
{}^0B(\cdot, w) = (b^1(\cdot, w), \ldots, b^d(\cdot, w))
$$

is an $\mathcal{R}^d$ valued Brownian motion. Similarly as for the one dimensional Brownian
motion, we can construct a Wiener measure that can be expressed as follows:

\[
\Gamma^d (D) = \tilde{\mathcal{P}} (B (\cdot, w) \in D)
\]

\[
= (2\pi\Delta t)^{-Nd/2} \int_D \exp \left( -\frac{1}{2} \sum_{t \in T} \left\| \dot{X}_t \right\|^2 \Delta t \right) dX_{\Delta t} dX_{2\Delta t} \ldots dX_1
\]

Where \( D \in D \times \cdots \times D \) (d-times), \( dX_t \) denotes the *Lebesgue measure over \(*R^d*, and

\[
\dot{X}_t = \frac{\Delta X_t}{\Delta t} \in *R^T.
\]

Now let \( D = D_1 \times \cdots \times D_d \), where \( D_i \) is an internal Borel set in \(*R^T*. For

\( i = 1, \ldots, d \). This class of sets generates \( D^d \). For \( X \in (R^d)^T, X = (X^1, \ldots, X^d) \),

with \( X_i \in *R^T, i = 1, \ldots, d \). Applying Theorem 5. (Keisler-Fubini Theorem) we

have

\[
\Gamma (D_1) \cdots \Gamma (D_d) = (2\pi\Delta t)^{-Nd/2} \left[ \int_{D_1} \exp \left( -\frac{1}{2} \sum_{t \in T} \left( \dot{X}_t^1 \right)^2 \Delta t \right) dX_{\Delta t}^1 dX_{2\Delta t}^1 \ldots dX_1^1 \right] \cdots
\]

\[
= (2\pi\Delta t)^{-Nd/2} \left[ \int_{D_1} \ldots \int_{D_d} \exp \left( -\frac{1}{2} \sum_{t \in T} \left( \dot{X}_t^d \right)^2 \Delta t \right) dX_{\Delta t}^d dX_{2\Delta t}^d \ldots dX_1^d \right]
\]

\[
= (2\pi\Delta t)^{-Nd/2} \left[ \int_{D_1} \ldots \int_{D_d} \exp \left( -\frac{1}{2} \sum_{t \in T} \left\| \dot{X}_t \right\|^2 \Delta t \right) \right] dX_{\Delta t}^1 \cdots dX_1^1
\]

so that for \( D = D_1 \times \cdots \times D_d, D_i \in D \),

\[
\Gamma^d (D) = \Gamma (D_1) \cdots \Gamma (D_d)
\]
and for $A = A_1 \times \cdots \times A_d$, with $A_i \in C$, $i = 1, 2, \ldots, d$,

$$\mu_0^d (A) = \mu_0 (A_1) \cdots \mu_0 (A_d) = L (\Gamma) (s^{-1} (A_1)) \cdots L (\Gamma) (s^{-1} (A_d))$$

Since the sets $A = A_1 \times \cdots \times A_d$, with $A_i \in C$, $i = 1, 2, \ldots, d$, generate the Borel $\sigma$-algebra $C^d$, we can extend the definition of $\mu_0^d$ to $C^d$.

Let $G$ be a compact, connected Lie group, and let $g$ be the corresponding Lie algebra. Let us take an Euclidean metric on $g$ which is $Ad(g)$ invariant. This metric induces a Riemannian metric on $G$. Suppose $\dim G = d$. Using and orthonormal basis,

$$P_0 (g) = \{ x : [0, 1] \rightarrow g | x \text{ is continuous and } x_0 = 0 \}$$

is isomorphic to $P_0 (\mathbb{R}^d)$. Let $P_e (G)$ be the set of $x : [0, 1] \rightarrow G$ which are continuous, $x_0 = e$ and $x_t$ is invertible with respect to the group operation for all $t \in [0, 1]$. From Wiener's Theorem we can assume the existence of a Wiener measure on $(P_e (G), \mathcal{B} (P_e (G)))$, where $\mathcal{B} (P_e (G))$ is the Borel $\sigma$-algebra on $P_e (G)$, we want to give a nonstandard construction of this Wiener measure.

Following P. Malliavin and M. Malliavin (1990), given $x \in P_0 (g)$ and a partition $S = \{ t_0, \ldots, t_n \}$ of $[0, 1]$, we define $exp_s (x) = \gamma$ as follows:

$$\gamma (0) = e$$

$$\gamma (t) = \gamma (t_{j-1}) \exp \left( \left( \frac{t-t_{j-1}}{t_j-t_{j-1}} \right) (x (t_j) - x (t_{j-1})) \right), \quad t \in [t_{j-1}, t_j]$$

It is known that when the mesh of $S$ tends to zero $\mu_0^d$ a.e., then, the following limit
exists in the metric space $P_e(G)$:

$$\lim \exp_p(x) = I(x)$$

The map $x \to I(x)$ is called the Ito map and is a measurable map.

Now consider the space $^*g^T$. We know that the nearstandard elements of this space are the $S$-continuous functions, and also that with respect to $L(\Gamma^\mathcal{G})$, $X$ is $S$-continuous for almost all $X \in ^*g^T$. With no loss of generality we can assume that for all $X \in ^*g^T$, $X$ is $S$-continuous.

For $X \in ^*g^T$ define the internal function $Y \in ^*G$ as follows:

$$Y(0) = e$$

$$Y(t) = \prod_{j=0}^{k-1} \exp \left( X_{t_{j+1}} - X_{t_j} \right)$$

where, $t = t_k = k\delta t$, $t \in T_\eta = T$. Considering $^*\gamma$, the elementary extension of $\gamma$, defined above, we see that $^*\gamma|_T = Y$; and since $^*\gamma$ is $S$-continuous, then $Y$ is $S$-continuous and so $Y \in ^*G^T$. Thus, $Y$ is nearstandard in $^*G^T$. Also $Y(t)$ is invertible for all $t \in T$, and we can define a map $\overline{I} : ^*g^T \to ^*G^T$, such that $\overline{I}(X) = Y$.

From the above nonstandard construction of the Wiener measure on $P_e(\mathbb{R}^d)$ and the $\mathbb{R}^d$ valued Brownian motion, we have that

$$^*\overline{I}(B(\cdot, w)) = \mathcal{E}(^*B(\cdot, w)) = I(b(\cdot, w)),$$

where $\mathcal{E}$ is the stochastic exponential function defined in Theorem 1.3.8 in (Muñoz de Ózak, M. (1995)). Since $I$ is a measurable map, $\overline{I}$ is a $^*$Borel measurable map. We
can define an internal measure on \((^*G^T, B (^*G^T))\) by

\[
\nu(A) = \Gamma^d \left( \overline{T}^{-1}(A) \right)
\]

for \(A\) Borel subset of \(^*G^T\).

6. **Theorem.** For a Borel set \(B\) in \(P_e(G)\), we can define the Wiener measure \(\mu_{P_e(G)}(B)\) as

\[
\mu_{P_e(G)}(B) = L(\nu)(st^{-1}(B)).
\]

**proof.** For \(B\) a Borel set in \(P_e(G)\) we have

\[
st^{-1}(I^{-1}(B)) = \{X \in ^*g^T : \circ X \in I^{-1}(B)\}
\]

\[
= \{X \in ^*g^T : I(\circ X) \in B\}
\]

and

\[
\overline{T}^{-1}(st^{-1}(B)) = \overline{T}^{-1}(\{Y \in ^*G^T : \circ Y \in B\})
\]

\[
= \{X \in ^*g^T : \circ \overline{T}(X) \in B\}
\]

\[
= \{X \in ^*g^T : I(\circ X) \in B\}
\]

so that, \(st^{-1}(I^{-1}(B)) = \overline{T}^{-1}(st^{-1}(B))\). Since \(\mu_{P_e(G)}(B) = \mu_0^d(I^{-1}(B))\) from the nonstandard definition of \(\mu_0^d\), we then have

\[
\mu_{P_e(G)}(B) = \mu_0^d(I^{-1}(B)) = L(\Gamma^d)(st^{-1}(I^{-1}(B)))
\]

\[
= L(\Gamma^d)\left(\overline{T}^{-1}(st^{-1}(B))\right) = L(\nu)(st^{-1}(B))
\]

\(\square\)
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