Revista Colombiana de Estadística - Theoretical statistics
July 2023, volume 46, issue 2, pp. 19–53
http://doi.org/10.15446/rce.v46n2.103163

The Type II Exponentiated Half Logistic-Marshall-Olkin-G Family of Distributions with Applications

La familia de distribuciones tipo II exponenciada media logística-Marshall-Olkin-G con aplicaciones

Broderick Oluyede[a], Morongwa Gabanakgosi[b]

Department of Mathematics and Statistical Sciences, Faculty of Sciences, Botswana International University of Science and Technology, Palapye, Botswana

Abstract

A new generalized family of distributions called the type II exponentiated half logistic-Marshall-Olkin-G distribution is developed. Some special cases of the new model are presented. We explore some statistical properties of the new family of distributions. The statistical properties studied include expansion of the density function, hazard rate and quantile functions, moments, moment generating functions, probability weighted moments, stochastic ordering, distribution of order statistics and Rényi entropy. The maximum likelihood, ordinary and weighted least-squares techniques for the estimation of model parameters are presented, and Monte Carlo simulations for the new family of distributions are conducted. The importance of the new family of distributions is examined by means of applications to two real data sets.

***Key words*:** Marshall-Olkin-G distribution; Maximum likelihood estimation; Simulations; Type II exponentiated half logistic distribution.

Resumen

Se desarrolla una nueva familia generalizada de distribuciones denominada distribución media exponenciada tipo II-Marshall-Olkin-G logística. Se presentan algunos casos especiales del nuevo modelo. Exploramos algunas propiedades estadísticas de la nueva familia de distribuciones. Las propiedades estadísticas estudiadas incluyen la expansión de la función de densidad, la tasa de riesgo y las funciones de cuantiles, momentos, funciones generadoras de momentos, momentos ponderados de probabilidad,

[a]Ph.D. E-mail: oluyedeo@biust.ac.bw
[b]Ph.D. E-mail: morongwa.gabanakgosi@studentmail.biust.ac.bw

ordenamiento estocástico, distribución de estadísticas de orden y entropía Rényi. Se presentan las técnicas de máxima verosimilitud, mínimos cuadrados ordinarios y ponderados para la estimación de los parámetros del modelo, y se realizan simulaciones Monte Carlo para la nueva familia de distribuciones. Se examina la importancia de la nueva familia de distribuciones mediante aplicaciones a dos conjuntos de datos reales.

***Palabras clave*:** Distribución Marshall-Olkin-G; Estimación de máxima verosimilitud; Simulaciones; Tipo II distribución semilogística exponencial.

1. Introduction

There are well known methods of generating new families of distributions in the literature due to the demand for more generalized distributions that can fit data well in reliability, economics, finance, applied sciences and engineering among others. In response to this, scholars have shifted their attention to generating new distributions that can provide greater flexibility when modelling data in practice.

Examples of generated distributions include, the type II half-logistic-G (Hassan et al., 2017), Weibull-G by Bourguignon et al. (2014), odd Dagum-G by Afify & Alizadeh (2020), Kumaraswamy-G by Cordeiro & de Castro (2011), Marshall-Olkin-G by Marshall & Olkin (1997), gamma-G by Zografos & Balakrishnan (2009), Topp-Leone Marshall-Olkin-G by Chipepa et al. (2020), beta Marshall-Olkin-G (BMO-G) by Alizadeh et al. (2015), Marshall-Olkin Extended Weibull (MOEW) distribution by Ghitany et al. (2005), Marshall-Olkin Weibull distribution by Cui et al. (2020) and Marshall-Olkin type II Topp-Leone-G (MO-TII-TL-G) by Rannona et al. (2022).

Additional examples of generators in the literature include: T-X family method of generating continuous distributions by Alzaatreh et al. (2013), the exponentiated half logistic-G (EHL-G) family of distribution by Cordeiro et al. (2014), gamma generator by Ristić & Balakrishnan (2012), Topp-Leone-G by Al-Shomrani et al. (2016) and Marshall-Olkin-G by Marshall & Olkin (1997).

Al-Mofleh et al. (2020) using the gamma generator (Ristić & Balakrishnan (2012)) obtained the type II exponentiated half logistic-G (TIIEHL-G) family of distributions which has the cumulative distribution function (cdf) of the form

$$\begin{aligned} F_{_{TIIEHL-G}}(x;\gamma,a,\zeta) &= 1-\int_0^{-\log(G(x;\zeta))}\frac{2a\gamma e^{-\gamma t}(1+e^{-\gamma t})^{a-1}}{(1-e^{-\gamma t})^{a+1}}dt \\ &= 1-\left[\frac{1-[G(x;\zeta)]^{\gamma}}{1+[G(x;\zeta)]^{\gamma}}\right]^{a}, \end{aligned} \tag{1}$$

for $\gamma, a > 0$, where $G(x;\zeta)$ is the baseline cdf with parameter vector ζ. The corresponding probability density function (pdf) is given by

$$f_{_{TIIEHL-G}}(x;\gamma,a,\zeta) = \frac{2a\gamma(1-[G(x;\zeta)]^{\gamma})^{a-1}[G(x;\zeta)]^{\gamma-1}g(x;\zeta)}{(1+[G(x;\zeta)]^{\gamma})^{a+1}}. \tag{2}$$

Furthermore, the Marshall-Olkin-G family of distributions has the cdf and pdf given by

$$F_{_{MO-G}}(x;\delta,\zeta) = 1 - \frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)} \tag{3}$$

and

$$f_{_{MO-G}}(x;\delta,\zeta) = \frac{\delta g(x;\zeta)}{[1-\bar{\delta}\bar{G}(x;\zeta)]^2}, \tag{4}$$

respectively, where δ is the tilt parameter, $\bar{\delta} = 1-\delta$ and $G(x;\zeta)$ is the baseline cdf which depends on the parameter vector ζ.

In this note, we develop the new type II exponentiated half logistic-Marshall-Olkin-G (TIIEHL-MO-G) family of distributions by inserting equation (3) into equation (1), with $\gamma = 1$. The motivations for developing this new family of distributions are:

- The new family of distribution has greater flexibility when fitted to real-life data as compared to models with the same number of parameters.
- The hazard rate function has shapes that are monotonic and non-monotonic.
- The family of distributions can model data that are heavy-tailed.
- We considered the generalization of the Weibull distribution when taken as the baseline cdf, given that Weibull distribution has some limitations, in order to obtain more flexibility.

The results of this paper are outlined as follows: In Section 2, we present the new family of distributions and its sub-families. Mathematical and statistical properties of the new model, including expansion of the probability density function, quantile function, moments, generating function, stochastic orders, probability weighted moments, order statistics and Rényi entropy are presented in Section 3. Estimation methods and observed information matrix are given in Section 4. Some special cases of the new family of distributions are given in Section 5. A Monte Carlo simulation study to examine the average bias and mean square error of the maximum likelihood estimates of the special case of the type II exponentiated half logistic-Marshall-Olkin-Weibull distribution are presented in Section 6. Section 7 contain applications of the new model to actual data sets. Lastly, we give concluding remarks in Section 8.

2. The Model

Equation (1) (with $\gamma = 1$) and equation (3) can be combined to obtain a new family of distributions called the type II exponentiated half logistic-Marshall-Olkin-G (TIIEHL-MO-G) distribution. The cdf and pdf of the new family of distributions are given by

$$F_{TIIEHL-MO-G}(x;a,\delta,\zeta)=1-\left[\frac{\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}}{1+\left(1-\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}\right)}\right]^{a}, \tag{5}$$

and

$$f_{TIIEHL-MO-G}(x;a,\delta,\zeta)=2a\delta\frac{\left[\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}\right]^{a-1}}{\left[1+\left(1-\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}\right)\right]^{a+1}}\frac{g(x;\zeta)}{\left[1-\bar{\delta}\bar{G}(x;\zeta)\right]^{2}}, \tag{6}$$

respectively, for $x,a,\delta>0$, $\bar{\delta}=1-\delta$ and parameter vector ζ. The hazard rate function (hrf) is given by

$$\begin{aligned}h_{TIIEHL-MO-G}(x;a,\delta,\zeta) &= 2a\delta\frac{\left[\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}\right]^{a-1}}{\left[1+\left(1-\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}\right)\right]^{a+1}}\frac{g(x;\zeta)}{\left[1-\bar{\delta}\bar{G}(x;\zeta)\right]^{2}}\\ &\quad\times\left[\frac{\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}}{1+\left(1-\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}\right)}\right]^{-a},\end{aligned} \tag{7}$$

for $x,a,\delta>0$, $\bar{\delta}=1-\delta$ and parameter vector ζ.

The shapes of the pdf and hrf of the new family of distribution are described below. The critical points of the TIIEHL-MO-G family of distributions are the roots of the equation

$$\begin{aligned}\frac{d\log f(x;a,\delta,\zeta)}{dx} &= -(a-1)\frac{\delta g(x;\zeta)}{\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}}+\frac{g'(x;\zeta)}{g(x;\zeta)}-(a+1)\frac{1}{1+\left(1-\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}\right)}\\ &\quad\times\frac{\delta g(x;\zeta)}{[1-\bar{\delta}\bar{G}(x;\zeta)]^{2}}-2\frac{g(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}=0,\end{aligned} \tag{8}$$

and the second derivative is

$$\begin{aligned}\frac{d^{2}\log f(x;a,\delta,\zeta)}{dx^{2}} =&\ -(a-1)\frac{\delta g(x;\zeta)\left[\frac{\delta G(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}\right]+\frac{\delta^{2}g(x;\zeta)^{2}}{[1-\bar{\delta}\bar{G}(x;\zeta)]^{2}}}{\left[\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}\right]^{2}}\\ &+\frac{g''(x;\zeta)g(x;\zeta)-g'(x;\zeta)g'(x;\zeta)}{g(x;\zeta)^{2}}-(a+1)\Bigg[\delta g'(x;\zeta)\\ &\times\left(1+\left(1-\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}\right)\right)[1-\bar{\delta}\bar{G}(x;\zeta)]^{2}\\ &-\left(\delta g(x;\zeta)+2[1-\bar{\delta}\bar{G}(x;\zeta)]\bar{\delta}g(x;\zeta)\left(1+\left(1-\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}\right)\right)\right)\Bigg]\\ &\times\left[1+\left(1-\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}\right)[1-\bar{\delta}\bar{G}(x;\zeta)]^{2}\right]^{-2}\\ &-2\frac{g'(x;\zeta)[1-\bar{\delta}G(x;\zeta)]-g(x;\zeta)^{2}}{[1-\bar{\delta}\bar{G}(x;\zeta)]^{2}},\end{aligned} \tag{9}$$

where $g'(x;\zeta)=\frac{dg(x;\zeta)}{dx}$, and $g''(x;\zeta)=\frac{d^{2}g(x;\zeta)}{dx^{2}}$.

If $x = x_0$ is a root of equation (8), then it is in accordance with a local minimum (maximum) if $\frac{d^2 \log f(x;a,\delta,\zeta)}{dx^2} > 0(< 0)$. It gives points of inflection if either $\frac{d^2 \log f(x;a,\delta,\zeta)}{dx^2} > 0$ for all $x \neq x_0$ or $\frac{d^2 \log f(x;a,\delta,\zeta)}{dx^2} < 0$ for all $x \neq x_0$.

The critical points of the hrf are obtained by the equation

$$\begin{aligned}\frac{d \log h(x;a,\delta,\zeta)}{dx} =& -(a-1)\frac{\delta g(x;\zeta)}{\frac{\delta \bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}} + \frac{g'(x;\zeta)}{g(x;\zeta)} - (a+1)\frac{1}{1+\left(1-\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}\right)} \\ &\times \frac{\delta g(x;\zeta)}{[1-\bar{\delta}\bar{G}(x;\zeta)]^2} - 2\frac{g(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)} + a\Big(-\frac{\delta g(x;\zeta)}{\delta\bar{G}(x;\zeta)} \\ &+ \frac{1}{1+\left(1-\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}\right)}\frac{\delta g(x;\zeta)}{[1-\bar{\delta}\bar{G}(x;\zeta)]^2}\Big) = 0, \end{aligned} \tag{10}$$

and the second derivative of $\log h(x;a,\delta,\zeta)$ is

$$\begin{aligned}\frac{d^2 \log h(x;a,\delta,\zeta)}{dx^2} =& -(a-1)\frac{\delta g(x;\zeta)\left[\frac{\delta G(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}\right] + \frac{\delta^2 g(x;\zeta)^2}{[1-\bar{\delta}\bar{G}(x;\zeta)]^2}}{\left[\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}\right]^2} \\ &+ \frac{g''(x;\zeta)g(x;\zeta) - g'(x;\zeta)g'(x;\zeta)}{g(x;\zeta)^2} - (a+1)\Bigg[\delta g'(x;\zeta) \\ &\times \left(1+\left(1-\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}\right)\right)[1-\bar{\delta}\bar{G}(x;\zeta)]^2 \\ &- \left(\delta g(x;\zeta) + 2[1-\bar{\delta}\bar{G}(x;\zeta)]\bar{\delta}g(x;\zeta)\left(1+\left(1-\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}\right)\right)\right)\Bigg] \\ &\times \left[1+\left(1-\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}\right)[1-\bar{\delta}\bar{G}(x;\zeta)]^2\right]^{-2} \\ &- 2\frac{g'(x;\zeta)[1-\bar{\delta}G(x;\zeta)] - g(x;\zeta)^2}{[1-\bar{\delta}\bar{G}(x;\zeta)]^2} - a\frac{\delta^2 g'(x;\zeta)\bar{G}(x;\zeta) - \delta^2 g(x;\zeta)^2}{[\delta\bar{G}(x;\zeta)]^2} \\ &+ a\frac{\left(\delta g(x;\zeta) + 2[1-\bar{\delta}\bar{G}(x;\zeta)]\bar{\delta}g(x;\zeta)\left(1+\left(1-\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}\right)\right)\right)}{\left[1+\left(1-\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}\right)[1-\bar{\delta}\bar{G}(x;\zeta)]^2\right]^2}. \end{aligned} \tag{11}$$

If $x = x_0$ is a root of equation (10), then it is in accordance with a local minimum (maximum) if $\frac{d^2 \log h(x;a,\delta,\zeta)}{dx^2} > 0(< 0)$. It gives points of inflection if either $\frac{d^2 \log h(x;a,\delta,\zeta)}{dx^2} > 0$ for all $x \neq x_0$ or $\frac{d^2 \log h(x;a,\delta,\zeta)}{dx^2} < 0$ for all $x \neq x_0$.

To test for identifiability of the new family of distributions, let $\rho_1 = (a_1, \delta_1)$ and $\rho_2 = (a_2, \delta_2)$, such that,

$$f_{\rho_1} = 2a_1\delta_1\frac{\left[\frac{\delta_1\bar{G}(x;\zeta)}{1-\bar{\delta}_1\bar{G}(x;\zeta)}\right]^{a_1-1}}{\left[1+\left(1-\frac{\delta_1\bar{G}(x;\zeta)}{1-\bar{\delta}_1\bar{G}(x;\zeta)}\right)\right]^{a_1+1}}\frac{g(x;\zeta)}{\left[1-\bar{\delta}_1\bar{G}(x;\zeta)\right]^2},$$

and

$$f_{\rho_2} = 2a_2\delta_2 \frac{\left[\frac{\delta_2 \bar{G}(x;\zeta)}{1-\bar{\delta}_2\bar{G}(x;\zeta)}\right]^{a_2-1}}{\left[1+\left(1-\frac{\delta_2\bar{G}(x;\zeta)}{1-\bar{\delta}_2\bar{G}(x;\zeta)}\right)\right]^{a_1+1}} \frac{g(x;\zeta)}{\left[1-\bar{\delta}_2\bar{G}(x;\zeta)\right]^2}.$$

Then

$$f_{\rho_1} = f_{\rho_2} \Leftrightarrow \Phi_1 - \Phi_2 = 0, \tag{12}$$

where

$$\Phi_1 = 2a_1\delta_1 \frac{\left[\frac{\delta_1 \bar{G}(x;\zeta)}{1-\bar{\delta}_1\bar{G}(x;\zeta)}\right]^{a_1-1}}{\left[1+\left(1-\frac{\delta_1\bar{G}(x;\zeta)}{1-\bar{\delta}_1\bar{G}(x;\zeta)}\right)\right]^{a_1+1}} \frac{1}{\left[1-\bar{\delta}_1\bar{G}(x;\zeta)\right]^2},$$

and

$$\Phi_2 = 2a_2\delta_2 \frac{\left[\frac{\delta_2 \bar{G}(x;\zeta)}{1-\bar{\delta}_2\bar{G}(x;\zeta)}\right]^{a_2-1}}{\left[1+\left(1-\frac{\delta_2\bar{G}(x;\zeta)}{1-\bar{\delta}_2\bar{G}(x;\zeta)}\right)\right]^{a_2+1}} \frac{1}{\left[1-\bar{\delta}_2\bar{G}(x;\zeta)\right]^2},$$

respectively. Therefore, equation (12) is zero for all $G(x;\zeta)$ when its coefficients are equal to zero, and that is possible when $a_1 = a_2$, $\delta_1 = \delta_2$. For this model all parameters are greater than zero, hence, we conclude that the new family of distributions is identifiable: $f_{\rho_1} = f_{\rho_2} \Leftrightarrow \rho_1 = \rho_2$.

2.1. Sub-Models

- When $a = 1$, we obtain a new type II half logistic-Marshall-Olkin-G (TIIHL-MO-G) family of distributions with the cdf

  $$F(x;\delta,\zeta) = 1 - \left[\frac{\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}}{1+\left(1-\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}\right)}\right],$$

  for $\delta > 0$ and parameter vector ζ.

- If we let $\delta = 1$, we get the type II exponentiated half logistic-G (TIIEHL-G) (Al-Mofleh et al., 2020) family of distributions with the cdf given by

  $$F(x;a,\zeta) = 1 - \left[\frac{\bar{G}(x;\zeta)}{1+G(x;\zeta)}\right]^a$$

  for $a > 0$ and parameter vector ζ.

- By letting $a = \delta = 1$, we get the type II half logistic-G (TIIHL-G) (Soliman et al., 2017) family of distributions with the cdf

  $$F(x;\zeta) = \frac{2G(x;\zeta)}{1+G(x;\zeta)},$$

  for the parameter vector ζ.

3. Some Statistical Properties

Some statistical properties of the TIIEHL-MO-G family of distributions are presented in this section. The statistical properties considered include: expansion of the density, quantile function, moments, generating function, stochastic orders, probability weighted moments, distribution of order statistics and Rényi entropy.

3.1. Linear Representation of Density Function

In this subsection, we present series expansion for the pdf of the TIIEHL-MO-G family of distributions using the following generalized binomial series expansions

$$(1+z)^{-(k+1)} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma((k+1)+j)}{\Gamma(k+1) j!} z^j,$$

and

$$(1-z)^b = \sum_{j=0}^{\infty} \binom{b}{j} (-1)^j z^j, \quad \text{for} \quad |z| < 1.$$

The pdf of the TIIEHL-MO-G family of distributions can be expressed as

$$\begin{aligned} f(x; a, \delta, \zeta) = 2a \sum_{j,i,k,l=0}^{\infty} & \delta^{i+a} \bar{\delta}^k \frac{(-1)^{j+i+l}}{(l+1)} \binom{-(a+1)}{j} \binom{j}{i} \binom{-(i+a+1)}{k} \\ & \times \binom{k+i+a-1}{l} (l+1) g(x;\zeta) [G(x;\zeta)]^l \\ & = \sum_{l=0}^{\infty} t_{l+1} h_{l+1}(x;\zeta), \end{aligned} \tag{13}$$

where $h_{l+1}(x;\zeta) = (l+1)[G(x;\zeta)]^l g(x;\zeta)$ is the exponentiated-G (Exp-G) density with power parameter $l+1$ and parameter vector ζ, and

$$\begin{aligned} t_{l+1} = 2a \sum_{j,i,k=0}^{\infty} & \delta^{i+a} \bar{\delta}^k \frac{(-1)^{j+i+l}}{(l+1)} \binom{-(a+1)}{j} \binom{j}{i} \binom{-(i+a+1)}{k} \\ & \times \binom{k+i+a-1}{l}. \end{aligned} \tag{14}$$

Details of derivations are given in the Appendix.

3.2. Quantile Function

We obtain the quantile function of the TIIEHL-MO-G family of distributions by inverting the non-linear equation

$$F(x;a,\delta,\zeta)=1-\left[\frac{\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}}{1+\left(1-\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}\right)}\right]^{a}=u \quad \text{for} \quad 0\leq u\leq 1.$$

Note that,

$$1-u=\left[\frac{\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}}{1+\left(1-\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}\right)}\right]^{a},$$

so that

$$\frac{2(1-u)^{\frac{1}{a}}}{1+(1-u)^{\frac{1}{a}}}=\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)},$$

which simplifies to

$$\bar{G}(x;\zeta)=\frac{\frac{2(1-u)^{\frac{1}{a}}}{1+(1-u)^{\frac{1}{a}}}}{\delta+\bar{\delta}\left(\frac{2(1-u)^{\frac{1}{a}}}{1+(1-u)^{\frac{1}{a}}}\right)}.$$

Finally, the quantile function of the TIIEHL-MO-G family of distributions reduces to

$$Q_G(u;a,\delta,\zeta)=G^{-1}\left(1-\left[\frac{\frac{2(1-u)^{\frac{1}{a}}}{1+(1-u)^{\frac{1}{a}}}}{\delta+\bar{\delta}\left(\frac{2(1-u)^{\frac{1}{a}}}{1+(1-u)^{\frac{1}{a}}}\right)}\right]\right). \tag{15}$$

Consequently, random numbers can be generated from the TIIEHL-MO-G family of distributions via equation (15) for specified baseline cdf G.

3.3. Moments, Generating Function and Probability Weighted Moments

In this subsection, we present the moments, generating function and probability weighted moments (PWMs) of the TIIEHL-MO-G family of distributions. The r^{th} moment of the TIIEHL-MO-G family of distributions is given by

$$E(X^r)=\int_{-\infty}^{\infty}x^r f_{TIIEHL-MO-G}(x;a,\delta,\zeta)dx=\sum_{l=0}^{\infty}t_{l+1}E(Y_{l+1}^r), \tag{16}$$

where $E(Y_{l+1}^r)$ is the r^{th} moment of Y_{l+1} which follows the Exp-G distribution with power parameter $l+1$ and t_{l+1} is defined in equation (14). The moment generating function (mgf) is obtained as

$$M_X(h)=E\left(e^{hX}\right)=\sum_{l=0}^{\infty}t_{l+1}E\left(e^{hY_{l+1}}\right),$$

where $E\left(e^{hY_{l+1}}\right)$ is the mgf of the exponentiated-G (Exp-G) family of distributions with power parameter $l+1$ and t_{l+1} is defined in equation (14).

The PWMs of a random variable X are defined by

$$\omega_{a,r} = E(X^a[F(X)]^r) = \int_{-\infty}^{\infty} x^a [F(x)]^r f(x) dx.$$

Considering the generalized binomial series expansions given in the Appendix, we obtain

$$\begin{aligned} f(x)[F(x)]^r = 2a \sum_{s,j,i,k,q=0}^{\infty} & \delta^{a(s+1)+i} \bar{\delta}^k \frac{(-1)^{s+j+i+q}}{q+1} \binom{r}{s} \binom{-(a(s+1)+1)}{j} \\ & \times \binom{j}{i} \binom{-(a(s+1)+i+1)}{k} \binom{a(s+1)+i+k-1}{q} \\ & \times (q+1) g(x;\zeta)[G(x;\zeta)]^q \\ & = \sum_{q=0}^{\infty} V_{q+1} h_{q+1}(x;\zeta), \end{aligned}$$

where $h_{q+1}(x;\zeta) = (q+1)[G(x;\zeta)]^q g(x;\zeta)$ is the Exp-G density with the power parameter $q+1$ and parameter vector ζ, and

$$\begin{aligned} V_{q+1} = 2a \sum_{s,j,i,k=0}^{\infty} & \delta^{a(s+1)+i} \bar{\delta}^k \frac{(-1)^{s+j+i+q}}{q+1} \binom{r}{s} \binom{-(a(s+1)+1)}{j} \\ & \times \binom{j}{i} \binom{-(a(s+1)+i+1)}{k} \binom{a(s+1)+i+k-1}{q}. \end{aligned}$$

Consequently, the PWMs of the TIIEHL-MO-G family of distributions is given by

$$\omega_{a,r} = \sum_{q=0}^{\infty} V_{q+1} \int_{-\infty}^{\infty} x^a h_{q+1}(x;\zeta) dx. \tag{17}$$

3.4. Stochastic Ordering

In this subsection, we present stochastic orders for the TIIEHL-MO-G family of distributions. Suppose we have two random variables Z and T with distribution functions $F_Z(r)$ and $F_T(r)$, respectively, and $\overline{F}_Z(r) = 1 - F_Z(r)$ is the survival function. Note that Z is stochastically smaller than T if $\overline{F}_Z(r) \leq \overline{F}_T(r)$ for all r or $F_Z(r) \geq F_T(r)$ for all r. This is denoted by $Z <_s T$. Hazard rate order and likelihood ratio order are stronger, and are given by $Z <_{hr} T$ if $h_Z(r) \geq h_T(r)$ for all r, and $Z <_{lr} T$ if $\frac{f_Z(r)}{f_T(r)}$ is decreasing in r, (Shaked & Shanthikumar, 2007). We know that $Z <_{lr} T \Rightarrow Z <_{hr} T \Rightarrow Z <_s T$.

Theorem 1. *Suppose X_1 and X_2 are two independent random variables following TIIEHL-MO-G(a_1, δ, ζ) and TIIEHL-MO-G (a_2, δ, ζ) distributions, respectively, then $X_1 <_s X_2$.*

Proof. Let

$$f_{_1}(x;a_1,\delta,\zeta)=2a_1\delta\frac{\left[\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}\right]^{a_1-1}}{\left[1+\left(1-\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}\right)\right]^{a_1+1}}\frac{g(x)}{\left[1-\bar{\delta}\bar{G}(x;\zeta)\right]^2},$$

and

$$f_{_2}(x;a_2,\delta,\zeta)=2a_2\delta\frac{\left[\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}\right]^{a_2-1}}{\left[1+\left(1-\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}\right)\right]^{a_2+1}}\frac{g(x)}{\left[1-\bar{\delta}\bar{G}(x;\zeta)\right]^2}.$$

Note that

$$\frac{f_1(x;a_1,\delta,\zeta)}{f_2(x;a_2,\delta,\zeta)}=\frac{a_1}{a_2}\left[\frac{\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}}{1+\left(1-\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}\right)}\right]^{a_1-a_2}. \tag{18}$$

Upon differentiating equation (18) with respect to x, we obtain

$$\frac{d}{dx}\left(\frac{f_1(x;a_1,\delta,\zeta)}{f_2(x;a_2,\delta,\zeta)}\right)=\frac{a_1}{a_2}(a_1-a_2)\left[\frac{\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}}{1+\left(1-\frac{\delta\bar{G}(x;\zeta)}{1-\bar{\delta}\bar{G}(x;\zeta)}\right)}\right]^{a_1-a_2-1}\frac{2\delta g(x;\zeta)}{[1-\bar{\delta}\bar{G}(x;\zeta)]^2},$$

which is ≤ 0 if $a_1\leq a_2$. Therefore, $X_1<_{lr}X_2, X_1<_{hr}X_2$ and $X_1<_s X_2$, and the random variables X_1 and X_2 are stochastically ordered. □

3.5. Distribution of Order Statistics

In this subsection, we present the distribution of the i^{th} order statistics for the TIIEHL-MO-G family of distributions. The pdf of the i^{th} order statistics is given by

$$f_{_{i:n}}(x)=\frac{n!f(x)}{(i-1)!(n-i)!}\sum_{r=0}^{n-i}(-1)^r\binom{n-i}{r}[F(x)]^{r+i-1}.$$

Applying the generalized binomial series expansions outlined in the Appendix, we can write

$$\begin{aligned}f(x)[F(x)]^{r+i-1}&=2a\sum_{s,j,m,k,q=0}^{\infty}\delta^{a(s+1)+i}\bar{\delta}^k\frac{(-1)^{s+j+m+q}}{q+1}\binom{r+i-1}{s}\binom{j}{m}\\&\times\binom{-(a(s+1)+1)}{j}\binom{-(a(s+1)+m+1)}{k}\\&\times\binom{a(s+1)+m+k-1}{q}(q+1)g(x;\zeta)[G(x;\zeta)]^q\\&=\sum_{q=0}^{\infty}W_{q+1}h_{q+1}(x;\zeta),\end{aligned}$$

where $h_{q+1}(x;\zeta) = (q+1)[G(x;\zeta)]^q g(x;\zeta)$ is the Exp-G density with power parameter $q+1$ and parameter vector ζ, and

$$\begin{aligned} W_{q+1} = \; & 2a \sum_{s,j,m,k=0}^{\infty} \delta^{a(s+1)+i} \bar{\delta}^k \frac{(-1)^{s+j+m+q}}{q+1} \binom{r+i-1}{s} \binom{-(a(s+1)+1)}{j} \\ & \times \binom{j}{m} \binom{-(a(s+1)+m+1)}{k} \binom{a(s+1)+m+k-1}{q}. \end{aligned}$$

Therefore, the pdf of the i^{th} order statistic from the TIIEHL-MO-G family of distributions is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{r=0}^{n-i} \sum_{q=0}^{\infty} (-1)^r \binom{n-i}{r} W_{q+1} h_{q+1}(x;\zeta).$$

3.6. Rényi Entropy

Rényi entropy of the proposed family of distributions is given in this subsection. Rényi entropy (Rényi, 1960) is a measure of variation of uncertainty and is defined by

$$I_R(\nu) = (1-\nu)^{-1} \log \left[\int_{-\infty}^{\infty} f^{\nu}(x) dx \right] \quad \text{for} \quad \nu > 0 \quad \text{and} \quad \nu \neq 1.$$

Rényi entropy of the TIIEHL-MO-G family of distributions can be expressed as

$$\begin{aligned} I_R(\nu) = \; & \frac{1}{1-\nu} \log \left[(2a)^{\nu} \sum_{j,i,k,l=0}^{\infty} \delta^{i+\nu a} \bar{\delta}^k (-1)^{j+i+l} \binom{-\nu(a+1)}{j} \right. \\ & \times \binom{j}{i} \binom{-(i+\nu(a+1))}{k} \binom{k+i+\nu(a-1)}{l} \\ & \left. \times \frac{1}{(\frac{l}{\nu}+1)^{\nu}} \int_0^{\infty} \left(\left[\left(\frac{l}{\nu}+1 \right) [G(x;\zeta)]^{\frac{l}{\nu}} g(x;\zeta) \right]^{\nu} dx \right) \right] \\ = \; & \frac{1}{1-\nu} \log \left[\sum_{l=0}^{\infty} d_l^* \exp\left((1-\nu) I_{REG}\right) \right], \quad \nu > 0 \quad \text{and} \quad \nu \neq 1, \end{aligned}$$

where

$$I_{REG} = \frac{1}{1-\nu} \log \left[\int_0^{\infty} \left(\left[\left(\frac{l}{\nu}+1 \right) [G(x;\zeta)]^{\frac{l}{\nu}} g(x;\zeta) \right]^{\nu} dx \right) \right]$$

is the Rényi entropy of the Exp-G density with power parameter $(\frac{l}{\nu}+1)$ and

$$d_l^* = (2a)^\nu \sum_{j,i,k=0}^{\infty} \delta^{i+\nu a}\bar{\delta}^k (-1)^{j+i+l} \binom{-\nu(a+1)}{j}\binom{j}{i} \times \binom{-(i+\nu(a+1))}{k}\binom{k+i+\nu(a-1)}{l}\frac{1}{(\frac{l}{\nu}+1)^\nu}.$$

More details on derivations follow in the appendix.

4. Estimation Methods

In this section, we discuss three estimation methods including maximum likelihood estimation (MLE), ordinary and weighted least squares.

4.1. Maximum Likelihood Estimation

The log-likelihood function (ℓ_n) based on a random sample of size n from the TIIEHL-MO-G family of distributions is given by

$$\ell_n(\Delta) = n\log(2a\delta) + (a-1)\sum_{i=0}^{n}\log\left[\frac{\delta\bar{G}(x_i;\zeta)}{1-\bar{\delta}\bar{G}(x_i;\zeta)}\right] + \sum_{i=0}^{n}\log[g(x_i;\zeta)] - (a+1)\sum_{i=0}^{n}\log\left[1+\left(1-\frac{\delta\bar{G}(x_i;\zeta)}{1-\bar{\delta}\bar{G}(x_i;\zeta)}\right)\right] - 2\sum_{i=0}^{n}\log[1-\bar{G}(x_i;\zeta)].$$

The maximum likelihood estimates of the parameters, denoted by $\hat{\mathbf{\Delta}}$ is obtained by solving the non-linear equation $(\frac{\partial\ell_n}{\partial a}, \frac{\partial\ell_n}{\partial\delta}, \frac{\partial\ell_n}{\partial\zeta_k})^T = \mathbf{0}$, using a numerical method such as Newton-Raphson procedure. The Fisher information matrix is given by $\mathbf{I}(\mathbf{\Delta}) = [\mathbf{I}_{\theta_i,\theta_j}]_{(p+2)X(p+2)} = E(-\frac{\partial^2\ell_n}{\partial\theta_i\partial\theta_j})$, $i,j=1,2,\ldots,(p+2)$, and can be numerically obtained by MATLAB or NLMIXED in SAS or R software. The total Fisher information matrix $n\mathbf{I}(\mathbf{\Delta})$ can be approximated by

$$\mathbf{J}_n(\hat{\mathbf{\Delta}}) \approx \left[-\frac{\partial^2\ell_n}{\partial\theta_i\partial\theta_j}\bigg|_{\mathbf{\Delta}=\hat{\mathbf{\Delta}}}\right]_{(p+2)X(p+2)}, \quad i,j=1,2,\ldots,(p+2). \tag{19}$$

The elements of the score vector are given in the appendix.

4.2. Ordinary and Weighted Least-Squares

The ordinary least-squares estimates (OLSEs) of the parameters of the TIIEHL-MO-G family of distributions are derived by minimizing the function

$$K(a,\delta,\zeta) = \sum_{i=0}^{n}\left[F(x_i,a,\delta,\zeta) - \frac{i}{n+1}\right]^2,$$

with respect to the parameters a, δ and parameter vector ζ. The OLSE can be obtained by solving the non-linear equations

$$\sum_{i=0}^{n}\left[F(x_i,a,\delta,\zeta)-\frac{i}{n+1}\right]\Delta_r(x_i,a,\delta,\zeta)=0, \quad r=1,2,3,$$

where

$$\Delta_1(x_i,a,\delta,\zeta)=\frac{\partial}{\partial a}F(x_i,a,\delta,\zeta), \quad \Delta_2(x_i,a,\delta,\zeta)=\frac{\partial}{\partial \delta}F(x_i,a,\delta,\zeta),$$

$$\text{and} \qquad \Delta_3(x_i,a,\delta,\zeta)=\frac{\partial}{\partial \zeta_s}F(x_i,a,\delta,\zeta). \tag{20}$$

The weighted least-squares estimates (WLSEs) of the parameters of the TIIEHL-MO-G family of distributions are obtained by minimizing equation (21) with respect to the parameters a, δ and parameter vector ζ, where

$$W(a,\delta,\zeta)=\sum_{i=0}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_i,a,\delta,\zeta)-\frac{i}{n+1}\right]^2. \tag{21}$$

The WLSE can now be obtained by solving the non-linear equations

$$\sum_{i=0}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_i,a,\delta,\zeta)-\frac{i}{n+1}\right]\Delta_r(x_i,a,\delta,\zeta)=0, \quad r=1,2,3,$$

where $\Delta_1(x_i,a,\delta,\zeta)$, $\Delta_2(x_i,a,\delta,\zeta)$ and $\Delta_3(x_i,a,\delta,\zeta)$ are given in equation (20).

5. Some Special Models

Some special models of the TIIEHL-MO-G family of distributions are presented in this section. The baseline distributions considered are Weibull, Burr XII and Burr III distributions, respectively.

5.1. Type II Exponentiated Half Logistic-Marshall-Olkin-Weibull (TIIEHL-MO-W) Distribution

Suppose we take the baseline distribution to be Weibull distribution with the cdf and pdf $G(x;\lambda)=1-e^{-x^{\lambda}}$, and $g(x;\lambda)=\lambda x^{\lambda-1}e^{-x^{\lambda}}$, respectively, for $x>0$ and $\lambda>0$, then the cdf and pdf of the TIIEHL-MO-W distribution are given by

$$F_{TIIEHL-MO-W}(x;a,\delta,\lambda)=1-\left[\frac{\frac{\delta e^{-x^{\lambda}}}{1-\bar{\delta}e^{-x^{\lambda}}}}{1+\left(1-\frac{\delta e^{-x^{\lambda}}}{1-\bar{\delta}e^{-x^{\lambda}}}\right)}\right]^a$$

and

$$f_{TIIEHL-MO-W}(x;a,\delta,\lambda) \quad = \quad 2a\delta\frac{\left[\frac{\delta e^{-x^{\lambda}}}{1-\bar{\delta}e^{-x^{\lambda}}}\right]^{a-1}}{\left[1+\left(1-\frac{\delta e^{-x^{\lambda}}}{1-\bar{\delta}e^{-x^{\lambda}}}\right)\right]^{a+1}}\frac{\lambda x^{\lambda-1}e^{-x^{\lambda}}}{[1-\bar{\delta}e^{-x^{\lambda}}]^2},$$

respectively, for $x, a, \delta, \lambda > 0$. Plots of the pdf and hrf for selected parameter values are given in Figure 1.

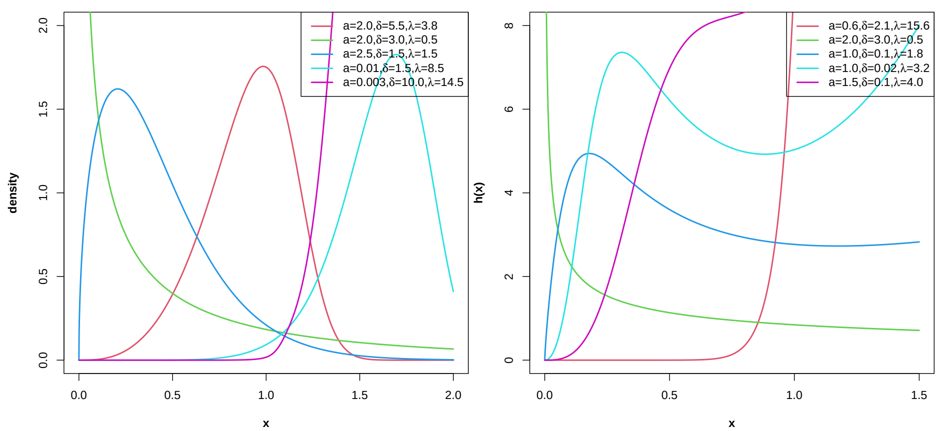

Figure 1: Plots of the pdf and hrf for the TIIEHL-MO-W distribution

The pdf of the TIIEHL-MO-W distribution can take several shapes including left-skewed, right-skewed, almost symmetric, reverse-J and J shapes, whereas the hrf displays upside-down bathtub, upside-down bathtub followed by bathtub, increasing and decreasing shapes. Table 1 gives the table of quantiles for TIIEHL-MO-W distribution for selected parameter values.

Table 1: Some quantiles for TIIEHL-MO-W distribution

	(a, δ, λ)				
u	$(1, 1.5, 1.3)$	$(0.7, 1, 1.5)$	$(0.4, 1, 1)$	$(2.1, 0.5, 1.9)$	$(1.5, 1, 1.2)$
0.1	0.1431	0.1825	0.1403	0.1004	0.0623
0.2	0.2580	0.3094	0.3175	0.1513	0.1181
0.3	0.3743	0.4349	0.5419	0.1961	0.1779
0.4	0.4992	0.5698	0.8299	0.2403	0.2449
0.5	0.6397	0.7217	1.2025	0.2869	0.3228
0.6	0.8047	0.9008	1.6939	0.3392	0.4183
0.7	1.0089	1.1239	2.3649	0.4018	0.5422
0.8	1.2857	1.4254	3.3482	0.4852	0.7196
0.9	1.7356	1.9066	5.0667	0.6209	1.0308

Figures 2 and 3 shows the plots of skewness and kurtosis for TIIEHL-MO-W distribution. The plots shows that:

- When we fix the parameter λ, skewness and kurtosis of the TIIEHL-MO-W distribution increases as a and δ increase.
- When we fix the parameter a, skewness and kurtosis of the TIIEHL-MO-W distribution decreases as δ and λ increase.

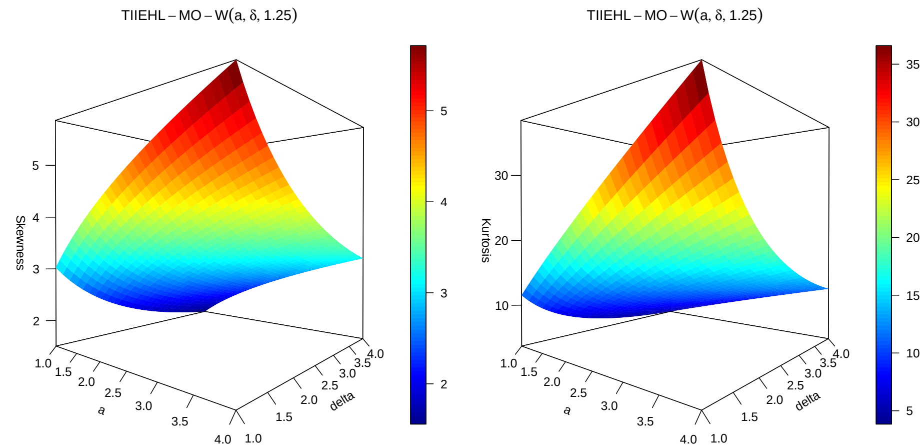

FIGURE 2: 3D Plots of Skewness and Kurtosis for TIIEHL-MO-W distribution

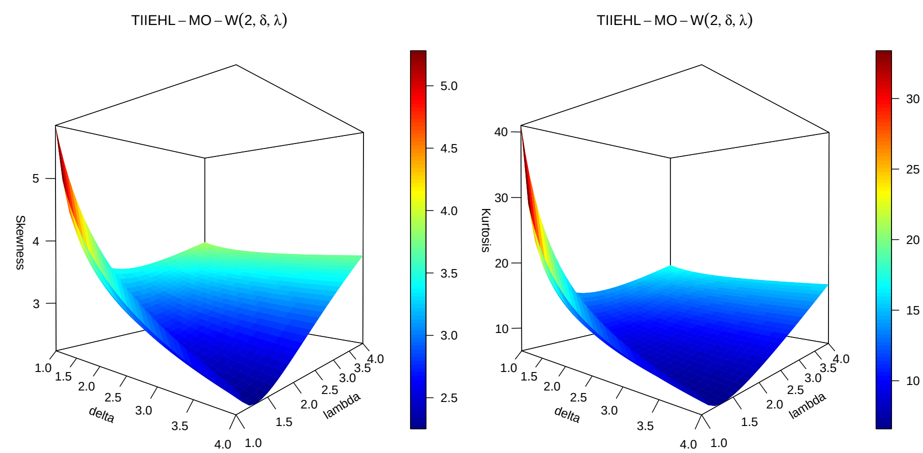

FIGURE 3: 3D plots of Skewness and Kurtosis for TIIEHL-MO-W distribution

5.2. Type II Exponentiated Half Logistic-Marshall-Olkin-Burr XII (TIIEHL-MO-BXII) Distribution

If we let the baseline distribution to be Burr XII distribution with the cdf and pdf given by $G(x; c, k) = 1 - (1 + x^c)^{-k}$, and $g(x; c, k) = kcx^{c-1}(1 + x^c)^{-k-1}$, respectively, for $x > 0$ and $c, k > 0$, then, we obtain the cdf and pdf of the TIIEHL-MO-BXII distribution as

$$F_{TIIEHL-MO-BXII}(x;a,\delta,c,k)=1-\left[\frac{\frac{\delta(1+x^c)^{-k}}{1-\bar{\delta}(1+x^c)^{-k}}}{1+\left(1-\frac{\delta(1+x^c)^{-k}}{1-\bar{\delta}(1+x^c)^{-k}}\right)}\right]^{a},$$

and

$$f_{TIIEHL-MO-BXII}(x;a,\delta,c,k)=2a\delta\frac{\left[\frac{\delta(1+x^c)^{-k}}{1-\bar{\delta}(1+x^c)^{-k}}\right]^{a-1}}{\left[1+\left(1-\frac{\delta(1+x^c)^{-k}}{1-\bar{\delta}(1+x^c)^{-k}}\right)\right]^{a+1}}\frac{kcx^{c-1}(1+x^c)^{-k-1}}{[1-\bar{\delta}(1+x^c)^{-k}]^2},$$

respectively, for $x>0$ and $a,\delta,c,k>0$.

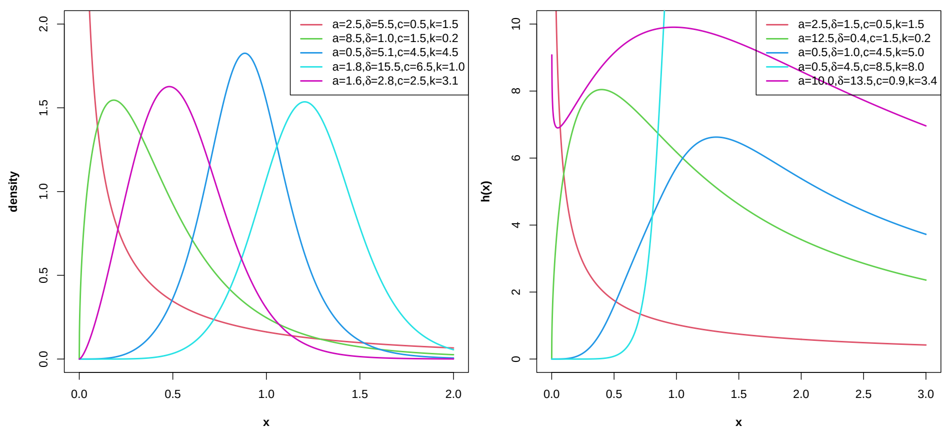

Figure 4: Plots of the pdf and hrf for the TIIEHL-MO-BXII distribution

Table 2: Some quantiles for TIIEHL-MO-BXII distribution

	(a,δ,c,k)				
u	$(1,1.5,3.3,0.6)$	$(0.7,0.3,1,1.5)$	$(0.4,1,1,2.5)$	$(1,0.5,2.2,1.5)$	$(0.5,1,2.3,1.6)$
0.1	0.5544	0.0162	0.0577	0.1628	0.3179
0.2	0.7157	0.0372	0.1351	0.2347	0.4599
0.3	0.8526	0.0656	0.2420	0.2991	0.5919
0.4	0.9895	0.1048	0.3937	0.3639	0.7321
0.5	1.1400	0.1629	0.6177	0.4352	0.8942
0.6	1.3220	0.2546	0.9691	0.5194	1.0967
0.7	1.5665	0.4175	1.5755	0.6272	1.3756
0.8	1.9512	0.7652	2.8163	0.7849	1.8206
0.9	2.7843	1.8747	6.5880	1.0839	2.7914

Plots of the pdf and hrf for the TIIEHL-MO-BXII distribution are given in Figure 4. The pdf exhibit right-skewed, left-skewed, almost symmetric and reverse-J shapes. The hrf of the TIIEHL-MO-BXII distribution on the other hand displays uni-modal, upside-down bathtub, bathtub followed by upside-down bathtub, decreasing and increasing shapes. Quantiles for the TIIEHL-MO-BXII distribution for selected parameter values are given in Table 2.

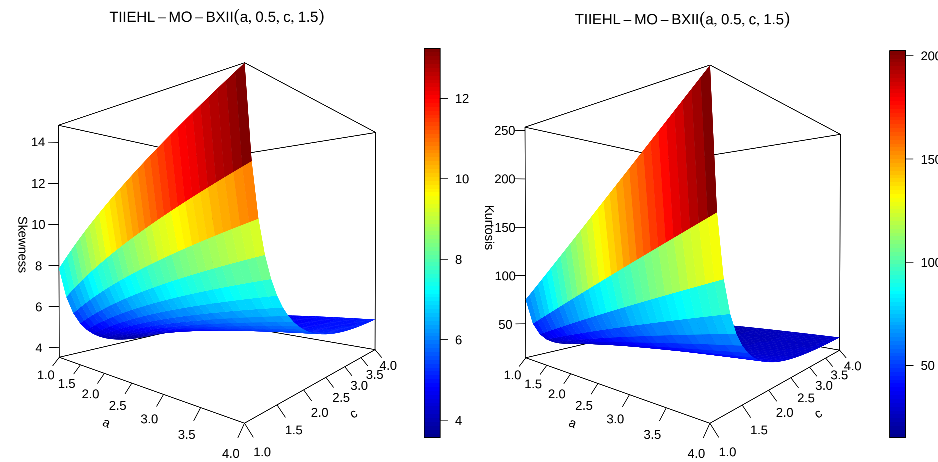

Figure 5: 3D plots of Skewness and Kurtosis for TIIEHL-MO-BXII distribution.

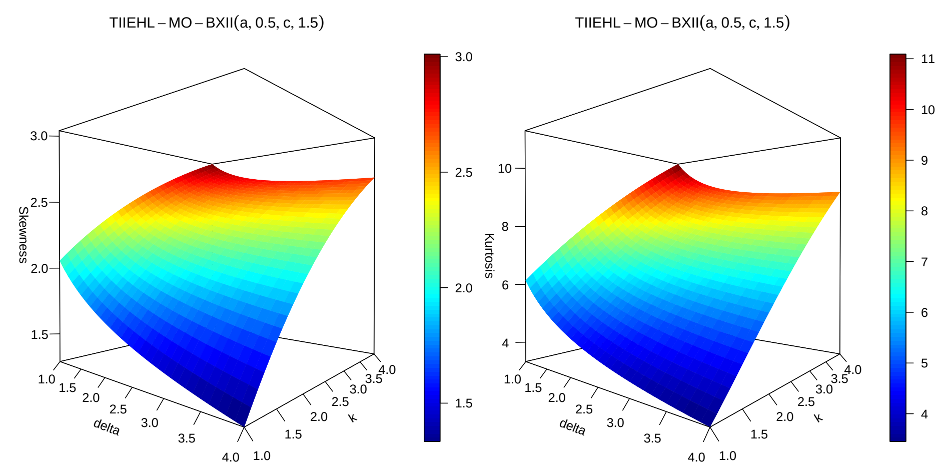

Figure 6: 3D Plots of Skewness and Kurtosis for TIIEHL-MO-BXII distribution.

The 3D plots of skewness and kurtosis for the TIIEHL-MO-BXII distribution are given in Figures 5 and 6. From the plots, we can see that the TIIEHL-MO-BXII distribution can model datasets with different levels of skewness and kurtosis.

5.3. Type II Exponentiated Half Logistic-Marshall-Olkin-Burr III (TIIEHL-MO-BIII) Distribution

When we take the baseline distribution to be Burr III distribution with the cdf and pdf given by $G(x;\alpha,\beta) = (1+x^{-\beta})^{-\alpha}$, and $g(x;\alpha,\beta) = \alpha\beta x^{-\beta-1}(1+x^{-\beta})^{-\alpha-1}$, respectively, for $\alpha,\beta > 0$ and $x > 0$, then the cdf and pdf of the TIIEHL-MO-BIII distribution are

$$F_{TIIEHL-MO-BIII}(x;a,\delta,\alpha,\beta) = 1 - \left[\frac{\frac{\delta(1-(1+x^{-\beta})^{-\alpha})}{1-\bar{\delta}(1-(1+x^{-\beta})^{-\alpha})}}{1+\left(1-\frac{\delta(1-(1+x^{-\beta})^{-\alpha})}{1-\bar{\delta}(1-(1+x^{-\beta})^{-\alpha})}\right)}\right]^{a}$$

and

$$f_{TIIEHL-MO-BII}(x;a,\delta,\alpha,\beta) = \quad 2a\delta\frac{\left[\frac{\delta(1-(1+x^{-\beta})^{-\alpha})}{1-\bar{\delta}(1-(1+x^{-\beta})^{-\alpha})}\right]^{a-1}}{\left[1+\left(1-\frac{\delta(1-(1+x^{-\beta})^{-\alpha})}{1-\bar{\delta}(1-(1+x^{-\beta})^{-\alpha})}\right)\right]^{a+1}} \times\frac{\alpha\beta x^{-\beta-1}(1+x^{-\beta})^{-\alpha-1}}{[1-\bar{\delta}(1-(1+x^{-\beta})^{-\alpha})]^{2}},$$

respectively, for $x > 0$ and $a,\delta,\alpha,\beta > 0$. Plots of the pdf and hrf are given in Figure 7.

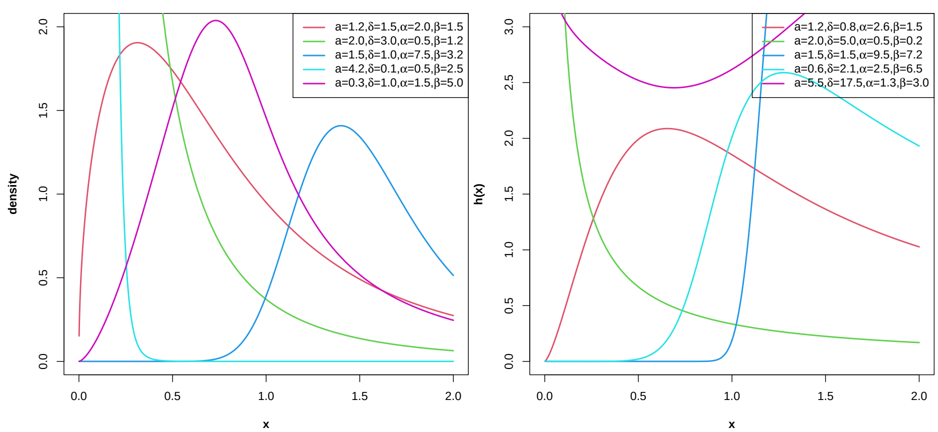

Figure 7: Plots of the pdf and hrf for the TIIEHL-MO-BIII distribution

In Figure 7, the pdf of the TIIEHL-MO-BIII distribution displays right-skewed, left-skewed, almost symmetric, and reverse-J shapes, whereas the hrf depicts upside-down bathtub, bathtub, decreasing and increasing shapes. Table 3 gives quantiles for the TIIEHL-MO-BIII distribution for selected parameter values.

TABLE 3: Some quantiles for TIIEHL-MO-BIII distribution

	$(a, \delta, \alpha, \beta)$				
u	$(1, 1.5, 0.3, 1.1)$	$(0.7, 1, 1.5, 2)$	$(0.4, 1, 1, 2.5)$	$(2.1, 0.5, 1.9, 0.5)$	$(1.5, 0.3, 1.2, 0.4)$
0.1	0.0007	0.4651	0.4690	0.0125	0.0000
0.2	0.0039	0.6427	0.6743	0.0313	0.0006
0.3	0.0140	0.8101	0.8767	0.0579	0.0012
0.4	0.0368	0.9924	1.1083	0.0956	0.0034
0.5	0.0812	1.2105	1.4023	0.1513	0.0076
0.6	0.1636	1.4962	1.8154	0.2383	0.0165
0.7	0.3192	1.9139	2.4757	0.3881	0.0379
0.8	0.6489	2.6385	3.7620	0.6945	0.1033
0.9	1.6283	4.4289	7.5689	1.6232	0.4470

6. Simulation Study

In this section, we present simulation results for the TIIEHL-MO-W distribution. We conducted various simulations for different sample sizes and for different parameter values. Equation (15) was used to generate random samples from TIIEHL-MO-W distribution via the R package. The simulation was repeated $N = 1000$ times each with sample sizes $n = 50, 100, 200, 400, 800, 1000$ as given in Tables 4 and Table 5. For simulations we considered only the MLE method.

The estimated mean, average bias (ABIAS) and root mean square errors (RMSEs) of the parameter say, $\hat{\zeta}$, are computed as $Mean = \frac{\sum_{i=1}^{N} \hat{\zeta}_i}{N}$,
ABIAS $(\hat{\zeta}) = \frac{\sum_{i=1}^{N} (\hat{\zeta}_i - \zeta)}{N}$, and $RMSE = \sqrt{\frac{\sum_{i=1}^{N} (\hat{\zeta}_i - \zeta)^2}{N}}$, respectively.

Tables 4 and 5 list the mean MLEs of the parameters along with the respective RMSEs and average bias. From the results, one can see that as the sample size n increases, the mean estimates of the parameters approximate the true parameter values while the RMSEs decreases and average bias decay towards zero. Therefore, we can conclude that this model gives consistent estimates.

Table 4: Simulation Results for TIIEHL-MO-W Distribution: Mean, RMSE and ABIAS

		(0.1, 2.0, 1.4)			(0.1, 1.2, 2.8)			(0.2, 1.0, 1.5)		
Parameter	n	Mean	RMSE	ABIAS	Mean	RMSE	ABIAS	Mean	RMSE	ABIAS
a	50	0.1517	0.1558	0.0517	0.1489	0.1603	0.0489	0.4719	4.1657	0.2719
	100	0.1403	0.1353	0.0403	0.1343	0.1211	0.0343	0.3567	1.1772	0.1567
	200	0.1232	0.0891	0.0232	0.1208	0.0955	0.0208	0.2541	0.2059	0.0541
	400	0.1108	0.0539	0.0108	0.1065	0.0391	0.0065	0.2232	0.1068	0.0232
	800	0.1039	0.0265	0.0039	0.1023	0.0205	0.0023	0.2059	0.0440	0.0059
	1000	0.1030	0.0229	0.0030	0.1013	0.0144	0.0013	0.2028	0.0344	0.0028
δ	50	4.1836	5.4966	2.1836	2.7128	3.9507	1.5129	4.1178	4.6294	3.1178
	100	3.6748	4.7746	1.6748	2.2519	3.3128	1.0519	2.9222	3.3242	1.9222
	200	2.9136	3.1937	0.9136	1.8661	2.5319	0.6661	1.7158	2.5589	0.7158
	400	2.4350	2.0052	0.4350	1.4230	1.1435	0.2230	1.3120	1.3349	0.3120
	800	2.1648	1.0374	0.1648	1.2804	0.5951	0.0804	1.0830	0.5144	0.0830
	1000	2.1132	0.8924	0.1132	1.2411	0.3793	0.0411	1.0393	0.3736	0.0393
λ	50	1.3530	0.2390	-0.0469	2.6948	0.4449	-0.1052	1.4579	0.2837	-0.0421
	100	1.3633	0.2105	-0.0367	2.7342	0.4008	-0.0658	1.4404	0.2698	-0.0595
	200	1.3716	0.1620	-0.0284	2.7559	0.2989	-0.0441	1.4708	0.2071	-0.0292
	400	1.3853	0.1176	-0.0147	2.7848	0.1985	-0.0152	1.4816	0.1481	-0.0184
	800	1.3952	0.0788	-0.0048	2.7979	0.1359	-0.0021	1.4977	0.0944	-0.0023
	1000	1.3958	0.0695	-0.0042	2.7996	0.1188	-0.0004	1.5015	0.0832	0.0015

TABLE 5: Simulation Results for TIIEHL-MO-W Distribution: Mean, RMSE and ABIAS

		(0.1, 0.1, 2.5)			(0.1, 1.0, 1.3)			(0.1, 1.5, 2.5)		
Parameter	n	Mean	RMSE	ABIAS	Mean	RMSE	ABIAS	Mean	RMSE	ABIAS
a	50	0.1525	0.2619	0.0525	0.1585	0.2359	0.0585	0.1565	0.1827	0.0565
	100	0.1169	0.1314	0.0160	0.1366	0.1260	0.0366	0.1381	0.1322	0.0381
	200	0.1039	0.0219	0.0039	0.1218	0.1035	0.0218	0.1193	0.0845	0.0193
	400	0.1014	0.0147	0.0014	0.1075	0.0419	0.0075	0.1079	0.0449	0.0079
	800	0.1005	0.0101	0.0005	0.1025	0.0211	0.0025	0.1023	0.0177	0.0023
	1000	0.1001	0.0092	0.0001	0.1014	0.0137	0.0014	0.1012	0.0154	0.0012
δ	50	0.5914	2.9731	0.4914	2.6365	4.9654	1.6365	3.5298	5.4091	2.0298
	100	0.2283	1.1952	0.1283	2.0025	3.1718	1.0025	2.7852	3.9801	1.2852
	200	0.1269	0.0894	0.0269	1.5965	2.4814	0.5965	2.2289	2.5404	0.7289
	400	0.1110	0.0553	0.0110	1.1997	1.0812	0.1997	1.7970	1.4367	0.2970
	800	0.1042	0.0344	0.0042	1.0666	0.5457	0.0666	1.5965	0.5874	0.0965
	1000	0.1017	0.0289	0.0017	1.0203	0.3035	0.0203	1.5448	0.4836	0.0448
λ	50	2.4074	0.3614	-0.0926	1.2472	0.2169	-0.0528	2.3956	0.4177	-0.1044
	100	2.4726	0.2552	-0.0274	1.2619	0.1854	-0.0380	2.4399	0.3723	-0.0600
	200	2.4971	0.1689	-0.0029	1.2753	0.1369	-0.0247	2.4637	0.2731	-0.0363
	400	2.5007	0.1231	0.0007	1.2881	0.0925	-0.0119	2.4833	0.1919	-0.0167
	800	2.5031	0.0864	0.0031	1.2969	0.0606	-0.0031	2.4967	0.1228	-0.0033
	1000	2.5038	0.0783	0.0038	1.2981	0.0524	-0.0019	2.5009	0.1112	0.0009

7. Applications

We give three examples of applications to show the flexibility and applicability of the TIIEHL-MO-G family of distributions in this section. The TIIEHL-MO-W distribution is compared with other existing models including the type II exponentiated half logistic-Weibull (TIIEHLW) distribution by Al-Mofleh et al. (2020), the Marshall-Olkin extended inverse Weibull (MOIW) distribution by Pakungwati et al. (2018), Marshall-Olkin extended Fréchet (MOEFr) and Marshall-Olkin extended generalized exponential (MOEGE) distributions by Barreto-Souza et al. (2013), the new Marshall-Olkin Weibull (NMOW) distribution by Cui et al. (2020), Marshall-Olkin type II Topp-Leone Weibull (MOTIITLW) distribution by Rannona et al. (2022) and Weibull Exponential (WE) distribution (Oguntunde et al., 2015). The pdfs of the above distributions are

$$f_{TIIEHLW}(x;\lambda,a,\gamma)=2a\lambda\gamma x^{\gamma-1}e^{-x^{\gamma}}(1-e^{-x^{\gamma}})^{\lambda-1}\frac{\left[1-(1-e^{-x^{\gamma}})^{\lambda}\right]^{a-1}}{\left[1+(1-e^{-x^{\gamma}})\right]^{a+1}},$$

for $x>0,\lambda,a,\gamma>0$,

$$f_{MOIW}(x;\alpha,\lambda,\theta)=\frac{\alpha\lambda\theta^{-\lambda}x^{-\lambda-1}e^{-(\theta x)^{-\lambda}}}{[\alpha-(\alpha-1)e^{-(\theta x)^{-\lambda}}]^{2}},$$

for $x>0,\alpha,\lambda,\theta>0$,

$$f_{MOEFr}(x;\alpha,\delta,\lambda)=\frac{\alpha\lambda\delta^{\lambda}x^{-(\lambda+1)}e^{-(\frac{\delta}{x})^{\lambda}}}{\{1-\bar{\alpha}[1-e^{-(\frac{\delta}{x})^{\lambda}}]\}^{2}},$$

for $x>0,\alpha,\delta,\lambda>0$,

$$f_{MOEGE}(x;\alpha,\gamma,\lambda)=\frac{\alpha\gamma\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\gamma-1}}{\{1-\bar{\alpha}(1-e^{-\lambda x})^{\gamma}\}^{2}},$$

for $x>0,\alpha,\gamma,\lambda>0$,

$$f_{NMOW}(x;\beta,\lambda,\theta)=\frac{\theta\lambda\beta x^{\beta-1}e^{-\lambda x^{\beta}}}{(\theta+(1-\theta)e^{-\lambda x^{\beta}})^{2}},$$

for $x>0,\delta,b,\lambda>0$,

$$f_{MOTIITLW}(x;\delta,b,\lambda)=\frac{2\delta b\lambda x^{\lambda-1}e^{-x^{\lambda}}(1-e^{-x^{\lambda}})[1-(1-e^{-x^{\lambda}})^{2}]^{b-1}}{(1-\bar{\delta}[1-(1-e^{-x^{\lambda}})^{2}]^{b})^{2}},$$

for $x>0,\delta,\lambda,\theta>0$, and

$$f_{WE}(x;\alpha,\beta,\lambda)=\alpha\beta(\lambda e^{-\lambda x})\left[\frac{(1-e^{-\lambda x})^{\beta-1}}{(e^{-\lambda x})^{\beta+1}}\right]\exp\left\{-\alpha\left(\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right)^{\beta}\right\},$$

for $x>0,\alpha,\beta,\lambda>0$.

The maximum likelihood estimates (MLE's) of the parameters of the TIIEHL-MO-W distribution with standard errors in parentheses are given in Tables 6 and 7, respectively. The goodness-of-fit statistics including -2log-likelihood statistic (-2log(L)), Akaike Information Criterion (AIC) by Akaike (1974), Consistent Akaike Information Criterion (AICC) (Bozdogan, 1987), Bayesian Information Criterion (BIC) (Schwarz, 1978), Cramér-von Mises (W^*) and Anderson-Darling (A^*) statistic by Chen & Balakrishnan (1995), Kolmogorov-Smirnov (K-S) by Chakravarti et al. (1967) statistic and its p-value are also presented in the tables.

Plots of the fitted densities, the histogram of the data, probability plots, estimated cdf, Kaplan-Meier (K-M) survival plots, Total Time on Test (TTT) and estimated hazard rate function (hrf) plots are given in Figures 8, 9, 10, 11, 12, 13, 14, 15 and 16, respectively. For the probability plots, we plotted $F(x_{(j)}; \hat{a}, \hat{\delta}, \hat{\lambda})$ against $\dfrac{j-0.375}{n+0.25}$, $j = 1, 2, \ldots, n$, where $x_{(j)}$ are the ordered values of the observed data. The measure of closeness to the diagonal line is given by the sum of squares (SS):

$$SS = \sum_{j=1}^{n} \left[F(x_{(j)}; \hat{a}, \hat{\delta}, \hat{\lambda}) - \left(\frac{j-0.375}{n+0.25} \right) \right]^2,$$

(see Chambers et al. 1983 for additional details).

7.1. Remission Times for Cancer Patients

The first data consists of remission times for 128 cancer patients Lee & Wang (2003). The data are: 0.08, 4.98, 25.74, 3.7, 10.06, 2.69, 7.62, 1.26, 7.87, 4.4, 2.02, 21.73, 2.09, 6.97, 0.5, 5.17, 14.77, 4.18, 10.75, 2.83, 11.64, 5.85, 3.31, 2.07, 3.48, 9.02, 2.46, 7.28, 32.15, 5.34, 16.62, 4.33, 17.36, 8.26, 4.51, 3.36, 4.87, 13.29, 3.64, 9.74, 2.64, 7.59, 43.01, 5.49, 1.4, 11.98, 6.54, 6.93, 6.94, 0.4, 5.09, 14.76, 3.88, 10.66, 1.19, 7.66, 3.02, 19.13, 8.53, 8.65, 8.66, 2.26, 7.26, 26.31, 5.32, 15.96, 2.75, 11.25, 4.34, 1.76, 12.03, 12.63, 13.11, 3.57, 9.47, 0.81, 7.39, 36.66, 4.26, 17.14, 5.71, 3.25, 20.28, 22.69, 23.63, 5.06, 14.24, 2.62, 10.34, 1.05, 5.41, 79.05, 7.93, 4.5, 2.02, 0.2, 7.09, 25.82, 3.82, 14.83, 2.69, 7.63, 1.35, 11.79, 6.25, 3.36, 2.23, 9.22, 0.51, 5.32, 34.26, 4.23, 17.12, 2.87, 18.1, 8.37, 6.76, 3.52, 13.8, 2.54, 7.32, 0.9, 5.41, 46.12, 5.62, 1.46, 12.02, 12.07.

The estimated variance-covariance matrix of the TIIEHL-MO-W distribution for remission times data is given by

$$\begin{bmatrix} 0.0151 & 0.5659 & -0.0131 \\ 0.5659 & 31.3974 & -0.4617 \\ -0.0131 & -0.4617 & 0.0121 \end{bmatrix}.$$

Table 6: Estimates of models for remission times of cancer patients

Model	Estimates			Statistics								
	a	δ	λ	$-2\log L$	AIC	AICC	BIC	W^*	A^*	K-S	P-value	SS
TIIEHL-MO-W	0.2664	11.8913	0.7787	820.51	826.51	826.70	835.07	0.0357	0.2230	0.0462	0.9474	0.0403
	(0.1231)	(5.6163)	(0.1098)									
	λ	a	γ									
TIIEHLW	4.2347	0.2311	0.7508	831.51	837.51	837.71	846.07	0.0608	0.3947	0.4979	2.2000×10^{-16}	14.8749
	(1.9754)	(0.1451)	(0.1865)									
	α	λ	θ									
MOIW	$8.4603\times10^{+03}$	$1.7246\times10^{+00}$	$3.1093\times10^{+01}$	822.91	828.91	829.10	837.46	9.4500	44.4472	0.9999	2.2000×10^{-16}	42.6094
	(2.6331×10^{-02})	(1.2759×10^{-01})	$(1.2429\times10^{+01})$									
	α	δ	λ									
MOEFr	2.2466	1.6906	0.8885	880.39	886.39	886.59	894.95	0.6045	3.7641	0.1429	0.0108	0.9307
	(3.5348)	(2.0515)	(0.2825)									
	α	γ	λ									
MOEGE	4.5374	1.6511	0.0708	820.69	826.69	826.88	835.24	0.1873	1.1287	0.6684	2.2000×10^{-16}	25.0748
	(3.6057)	(0.2108)	(0.0269)									
	β	λ	θ									
NMOW	1.0558	0.0904	1.0308	828.06	834.06	834.25	842.61	0.1296	0.7769	0.0703	0.5521	0.1495
	(0.2268)	(0.0981)	(0.9473)									
	δ	b	λ									
MOTIITLW	2.1394	0.3389	0.7487	824.13	830.13	830.33	838.69	0.0867	0.5229	0.0644	0.6639	0.0865
	(4.2745)	(0.7062)	(0.4213)									
	α	β	λ									
WE	$1.8110\times10^{+03}$	$1.0468\times10^{+00}$	8.0600×10^{-05}	828.21	1834.21	834.41	842.77	0.1316	0.7877	0.0706	0.5453	0.1524
	(8841×10^{-13})	(3.5088×10^{-09})	(6.8000×10^{-06})									

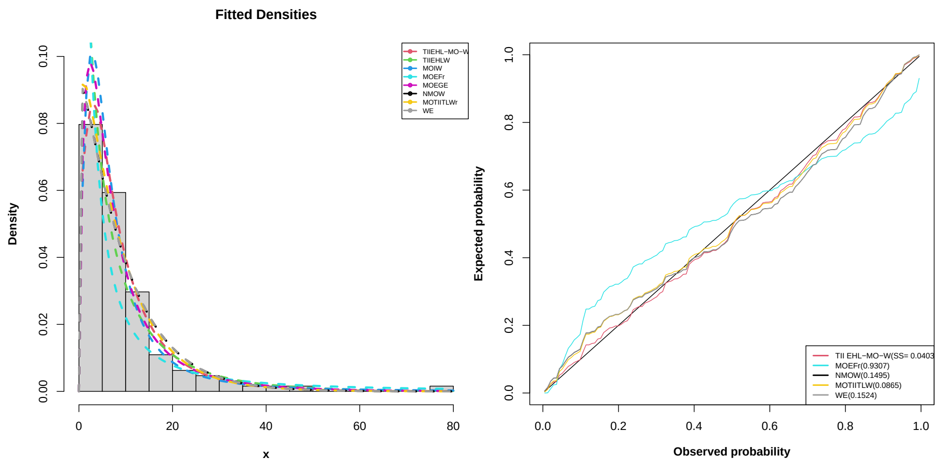

FIGURE 8: Fitted densities and probability plots for remission times data

In Table 6 above, the values of the goodness-of-fit statistics: AIC, AICC and BIC are smallest for the TIIEHL-MO-W distribution as compared to the models listed in the table. Also, the value of SS from the probability plots and the values of the goodness-of-fit statistics: A^*, W^* and K-S are smallest for the TIIEHL-MO-W distribution, which indeed showed that this model fits remission times of cancer patients data well. The value of K-S p-value for the TIIEHL-MO-W distribution is largest compared to the distributions.

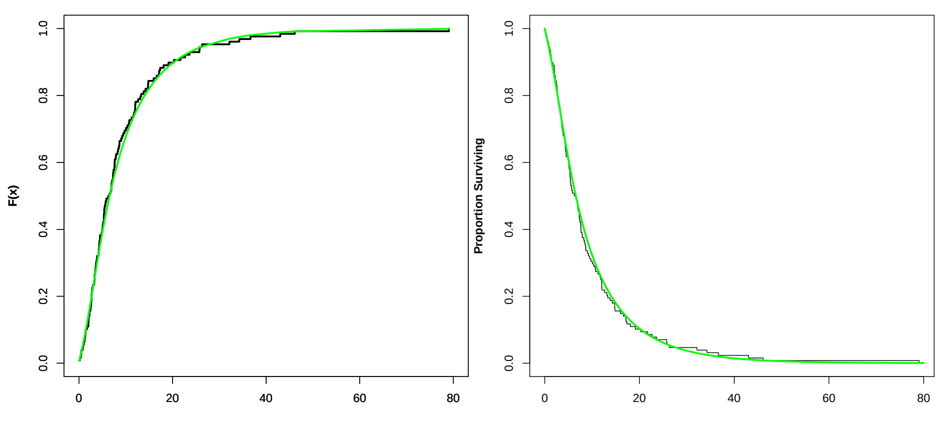

FIGURE 9: Estimated cdf and K-M survival plots for remission times data

Figures 9 and 10 shows the estimated cdf, K-M survival, TTT-transform and estimated hrf plots for remission times data set. The cdf line for TIIEHL-MO-W distribution is very close to the empirical cdf while the survival function for

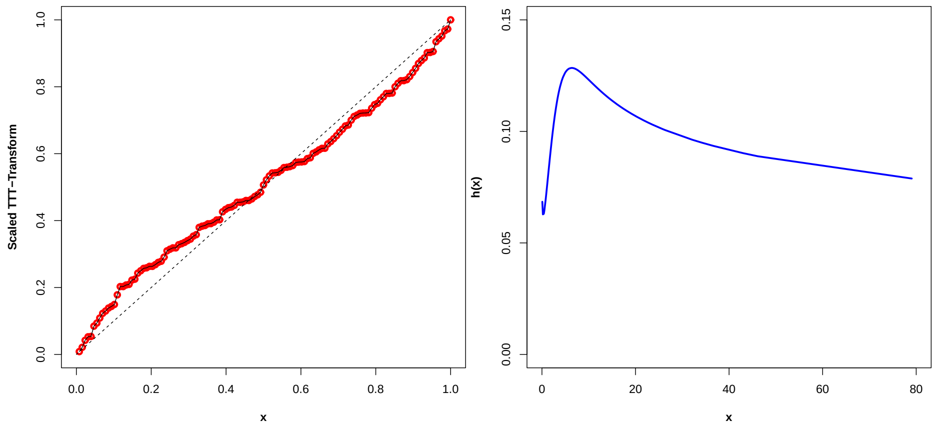

Figure 10: TTT-transform and estimated hrf plots for remission times data

TIIEHL-MO-W distribution is also closer to the K-M curve which shows that indeed our model is the best in explaining remission times data. TTT-transform plot indicates a uni-modal hazard rate function for the remission times data.

7.2. Time to Failure (in Hours) of 59 Test Conductors of 400 Micrometer Length

The second data presents hours to failure of 59 test conductors of 400 micrometer length, by Schafft et al. (1987). The hours to failure are: 6.545, 9.289, 7.543, 6.956, 6.492, 5.459, 8.120, 4.706, 8.687, 2.997, 8.591, 6.129, 11.038, 5.381, 6.958, 4.288, 6.522, 4.137, 7.459, 7.495, 6.573, 6.538, 5.589, 6.087, 5.807, 6.725, 8.532, 9.663, 6.369, 7.024, 8.336, 9.218, 7.945, 6.869, 6.352, 4.700, 6.948, 9.254, 5.009, 7.489, 7.398, 6.033, 10.092, 7.496, 4.531,7.974, 8.799, 7.683, 7.224, 7.365, 6.923, 5.640, 5.434, 7.937, 6.515, 6.476, 6.071, 10.491, 5.923.

The estimated variance-covariance matrix of TIIEHL-MO-W distribution for time to failure of test conductors data is given by

$$\begin{bmatrix} 9.3767 \times 10^{-02} & 6.2245 \times 10^{-06} & -1.5292 \times 10^{-02} \\ 6.2245 \times 10^{-06} & 4.1335 \times 10^{-10} & -1.0210 \times 10^{-06} \\ -1.5292 \times 10^{-02} & -1.0210 \times 10^{-06} & 2.7401 \times 10^{-03} \end{bmatrix}.$$

TABLE 7: Estimates of models for time to failure of 59 test conductors

Model	Estimates			Statistics								
	a	δ	λ	$-2\log L$	AIC	AICC	BIC	W^*	A^*	K-S	P-value	SS
TIIEHL-MO-W	7.0507×10^{-01}	$4.0874\times10^{+03}$	$1.0852\times10^{+00}$	222.75	228.75	229.18	234.98	0.0199	0.1284	0.0506	0.9963	0.0186
	(3.0613×10^{-01})	(2.0317×10^{-05})	(5.2340×10^{-02})									
	λ	a	γ									
TIIEHLW	0.7276	0.0664	1.3602	321.78	327.78	328.21	334.01	0.0352	0.2006	0.9919	7.7720×10^{-16}	19.6038
	(0.4275)	(0.0245)	(0.1716)									
	α	λ	θ									
MOIW	$1.4673\times10^{+03}$	$7.5117\times10^{+00}$	3.8466×10^{-01}	224.12	230.12	230.56	236.36	4.0942	19.6988	0.9993	7.7720×10^{-16}	19.5837
	(1.0752×10^{-05})	(8.1356×10^{-01})	(4.2213×10^{-02})									
	α	δ	λ									
MOEFr	0.0004	68.3500	0.8874	231.47	237.47	237.90	243.70	0.1155	0.7377	0.07950	0.8211	0.0638
	(0.0003)	(0.0001)	(0.0323)									
	α	γ	λ									
MOEGE	0.0005	1.0359	1.1078	223.09	229.09	229.52	235.32	0.0353	0.1990	0.9999	7.7720×10^{-16}	19.6238
	(0.0004)	(0.0002)	(0.1219)									
	β	λ	θ									
NMOW	$4.6541\times10^{+00}$	8.0659×10^{-05}	9.6584×10^{-01}	225.07	231.07	231.51	237.31	0.0832	0.4678	0.0957	0.6178	0.0811
	(1.7883×10^{-09})	(0.0981×10^{-05})	(4.3754×10^{-10})									
	δ	b	λ									
MOTIITLW	296.3100	0.5294	1.2569	223.62	229.62	230.06	235.85	0.0411	0.2408	0.0608	0.9717	0.0300
	(0.0004)	(0.1892)	(0.1654)									
	α	β	λ									
WE	$5.4349\times10^{+02}$	$4.2393\times10^{+00}$	2.6702×10^{-02}	225.62	231.62	232.04	237.84	0.0912	0.5139	0.1022	0.5356	0.0951
	(5.5751×10^{-05})	(3.6954×10^{-01})	(2.9392×10^{-03})									

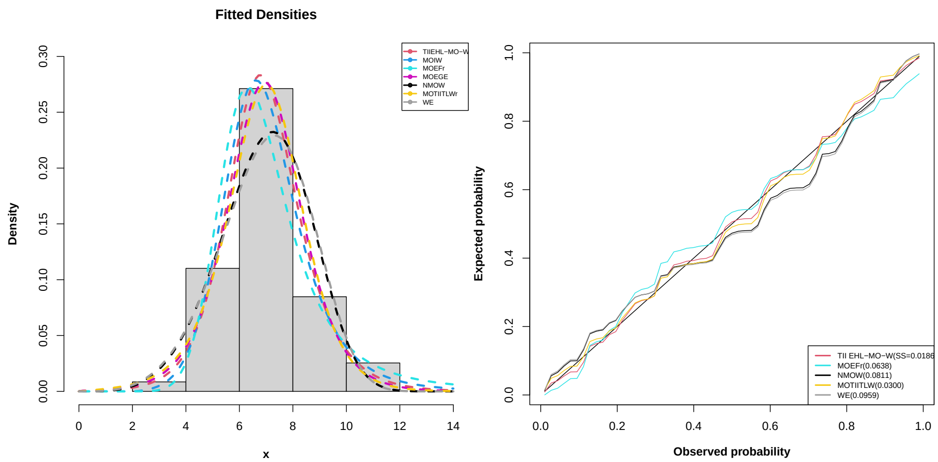

FIGURE 11: Fitted densities and probability plots of test conductors data

Table 7, shows that the values of the goodness-of-fit statistics: AIC, AICC and BIC are the smallest for the TIIEHL-MO-W distribution as compared to the models listed in the table. The value of SS from the probability plots and the values of the goodness-of-fit statistics: A^*, W^* and K-S are smallest for the TIIEHL-MO-W distribution, and the K-S p-value is closer to 1, hence the TIIEHL-MO-W distribution is a better fit for failure times of 59 test conductors data.

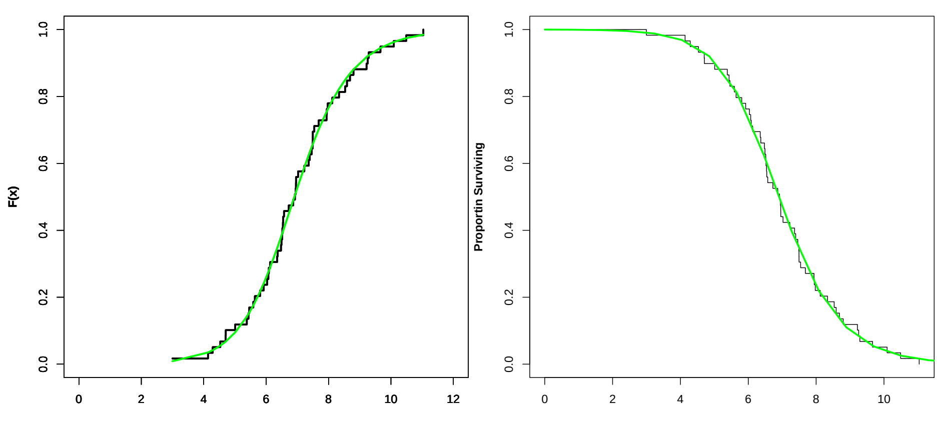

FIGURE 12: Estimated cdf and K-M survival plots for test conductors data

Figures 12 and 13 above gives the estimated cdf, K-M survival plot, TTT-transform and estimated hrf plots for time to failure of test conductors data set. The cdf line for TIIEHL-MO-W distribution is close to the empirical cdf, while the survival function for TIIEHL-MO-W distribution is also closer to the K-M curve which

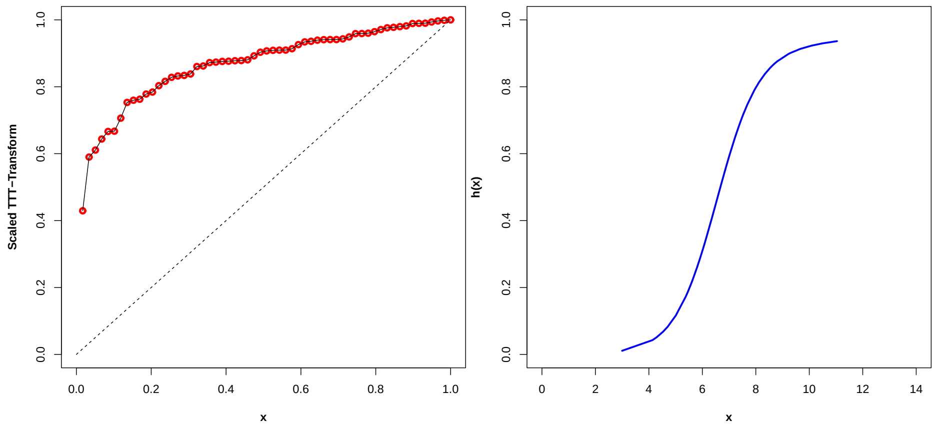

FIGURE 13: TTT-transform and estimated hrf plots for test conductors data

shows that our model is better in explaining time to failure of test conductors data. TTT-transform plot is concave indicating an increasing hrf for time to failure of test conductors data.

7.3. COVID-19 Data in Canada

This data consists of deaths due to COVID-19 in Canada for 36 days from April 10, 2020 to May 15, 2020, (see `https://covid19.who.int/` for details). The observations are: 3.1091, 3.3825, 3.1444, 3.2135, 2.4946, 3.5146, 4.9274, 3.3769, 6.8686, 3.0914, 4.9378, 3.1091, 3.2823, 3.8594, 4.0480, 4.1685, 3.6426, 3.2110, 2.8636, 3.2218, 2.9078, 3.6346, 2.7957, 4.2781, 4.2202, 1.5157, 2.6029, 3.3592, 2.8349, 3.1348, 2.5261, 1.5806, 2.7704, 2.1901, 2.4141, 1.9048.

The estimated variance-covariance matrix of TIIEHL-MO-W distribution for Europe COVID-19 data is given by

$$\begin{bmatrix} 0.0491 & 1.0651 & -0.0383 \\ 1.06511 & 2966.7550 & 1.8663 \\ -0.0383 & 1.8663 & 0.0355 \end{bmatrix}.$$

Table 8: Estimates of models for Canada COVID-19 data

Model	Estimates			Statistics								
	a	δ	λ	$-2\log$L	AIC	AICC	BIC	W^*	A^*	K-S	P-value	SS
TIIEHL-MO-W	0.4071	115.0155	1.4980	95.80	101.80	102.55	106.55	0.0640	0.3698	0.1163	0.7147	0.0896
	(0.2215)	(54.4679)	(0.1885)									
	λ	a	γ									
TIIEHLW	25.7059	0.7364	1.0943	99.02	105.02	105.78	109.78	0.1293	0.7749	0.2030	0.1029	0.2942
	(6.0515)	(0.3688)	(0.1711)									
	α	λ	θ									
MOIW	2.6234×10^{-05}	0.5908	0.0059	96.38	102.38	103.14	107.14	0.1214	0.7279	0.9987	2.2000×10^{-16}	11.9509
	(4.0461×10^{-05})	(8.3759×10^{-05})	(0.0014)									
	α	δ	λ									
MOEFr	0.0001	7.5237	0.6863	97.01	103.01	103.77	107.77	0.1035	0.6372	0.1169	0.7088	0.0968
	(0.0001)	(0.0001)	(0.0310)									
	α	γ	λ									
MOEGE	43.7075	12.3057	0.4208	96.47	102.47	103.22	107.22	0.1125	0.6718	0.9781	2.2000×10^{-16}	11.8601
	(75.0656)	(4.9945)	(0.2163)									
	β	λ	θ									
NMOW	1.7131	0.4040	0.0514	100.43	106.43	107.18	111.18	0.1450	0.8179	0.1319	0.5584	0.1155
	(0.3487)	(0.2864)	(0.0550)									
	δ	b	λ									
MOTIITLW	17.9001	0.4802	1.6389	99.15	105.15	105.90	109.90	0.1303	0.7319	0.1250	0.6268	0.0944
	(27.3796)	(0.6084)	(0.6107)									
	α	β	λ									
WE	24830.0000	3.2218	0.0116	103.24	109.24	109.99	113.99	0.1757	1.0082	0.1517	0.3791	0.1952
	(4.6583×10^{-07})	(0.3415)	(0.0036)									

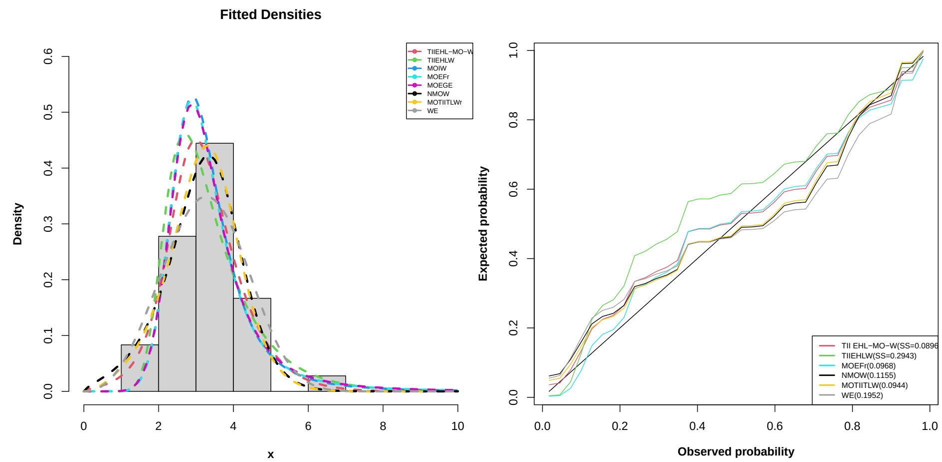

FIGURE 14: Fitted densities and probability plots for Canada COVID-19 data

The values of the goodness-of-fit statistics: AIC, AICC and BIC are the least for TIIEHL-MO-W distribution in Table 8 as compared to the other models listed. The value of SS from the probability plots and the values of the goodness-of-fit statistics: A^*, W^* and K-S are smallest for TIIEHL-MO-W distribution, and the K-S p-value is largest for the new distribution. Therefore, the TIIEHL-MO-W distribution is a good fit for Canada COVID-19 data.

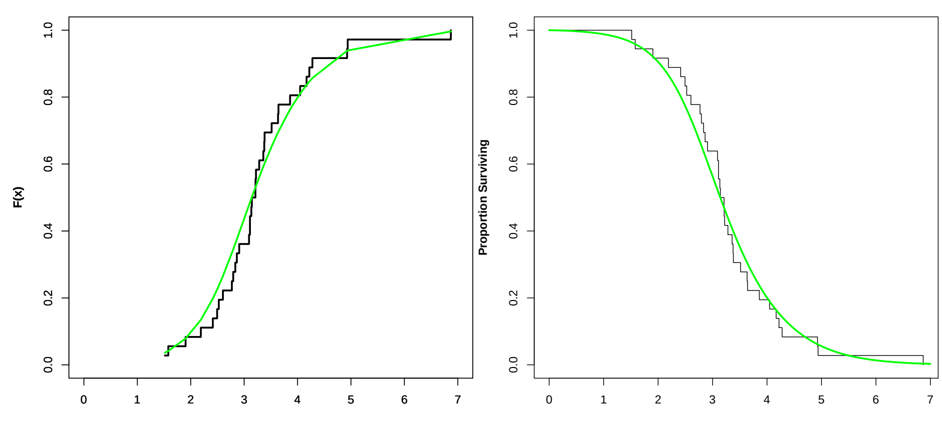

FIGURE 15: Estimated cdf and K-M survival plots for Canada COVID-19 data

The cdf line for TIIEHL-MO-W distribution is superimposed on the empirical cdf, while the survival function for TIIEHL-MO-W distribution is also closer to the K-M curve which shows that our model is the better in explaining Canada COVID-19 data. TTT-transform plot indicates an increasing hrf for Canada COVID-19

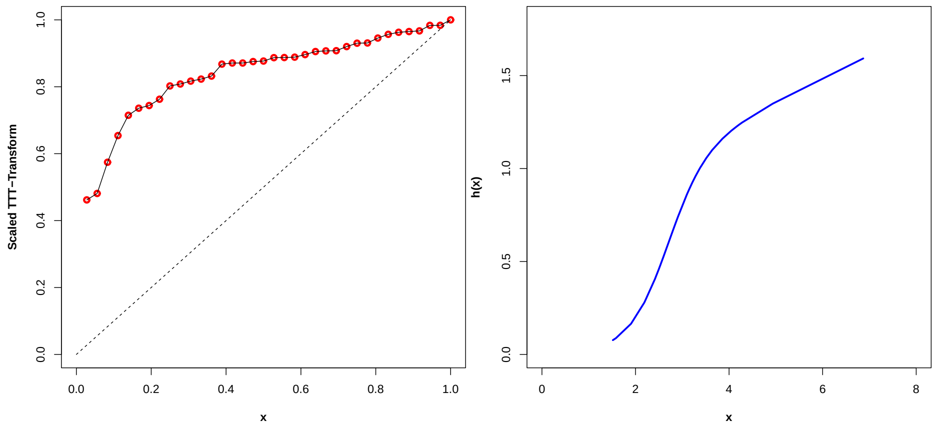

FIGURE 16: TTT-transform and estimated hrf plots for Canada COVID-19 data

data.

8. Concluding Remarks

A new family of distributions referred to as the type II Exponentiated Half-Logistic-Marshall-Olkin-G distribution was introduced. Some statistical properties of the new family of distributions were presented. Three methods of estimation namely, maximum likelihood, ordinary and weighted least squares were used to estimate model parameters. The method of maximum likelihood was implemented in full. We also studied special cases of the new family of distributions including the Type II Exponentiated Half Logistic-Marshall-Olkin-Weibull, the Type II Exponentiated Half Logistic-Marshall-Olkin-Burr XII and the Type II Exponentiated Half Logistic-Marshall-Olkin-Burr III distributions. The new family of distributions can model data with heavy tails with different levels of skewness and kurtosis. The usefulness of the new family of distributions was examined by means of applications to three real data sets.

In future research, we will also like to consider other estimation techniques including Bayesian method. Bayesian techniques are of interest because they are easily dealt with for almost all parametric techniques and are increasingly used in global health modeling. This method can also be used for estimating the parameters of the proposed family of distributions.

Acknowledgements

The authors would like to express their gratitude to the reviewers for their insightful remarks and comments which greatly improved the content of this paper.

[Received: June 2022 — Accepted: April 2023]

References

Afify, A. Z. & Alizadeh, M. (2020), 'The odd Dagum family of distributions: properties and applications', *Journal of Applied Probability and Statistics* **15**(1), 45–72.

Akaike, H. (1974), 'A new look at the statistical model identification', *IEEE transactions on automatic control* **19**(6), 716–723.

Al-Mofleh, H., Elgarhy, M., Afify, A. & Zannon, M. (2020), 'Type II exponentiated half logistic generated family of distributions with applications', *Electronic Journal of Applied Statistical Analysis* **13**(2), 536–561.

Al-Shomrani, A., Arif, O., Shawky, A., Hanif, S. & Shahbaz, M. Q. (2016), 'Topp-Leone family of distributions: Some properties and application', *Pakistan Journal of Statistics and Operation Research* pp. 443–451.

Alizadeh, M., Cordeiro, G. M., Brito, E. & Demétrio, C. G. (2015), 'The beta Marshall-Olkin family of distributions', *Journal of Statistical Distributions and Applications* **2**(1), 1–18.

Alzaatreh, A., Lee, C. & Famoye, F. (2013), 'A new method for generating families of continuous distributions', *Metron* **71**(1), 63–79.

Barreto-Souza, W., Lemonte, A. J. & Cordeiro, G. M. (2013), 'General results for the Marshall and Olkin's family of distributions', *Anais da Academia Brasileira de Ciências* **85**, 3–21.

Bourguignon, M., Silva, R. B. & Cordeiro, G. M. (2014), 'The Weibull-G family of probability distributions', *Journal of Data Science* **12**(1), 53–68.

Bozdogan, H. (1987), 'Model selection and Akaike's information criterion (AIC): The general theory and its analytical extensions', *Psychometrika* **52**(3), 345–370.

Chakravarti, I. M., Laha, R. G. & Roy, J. (1967), 'Handbook of methods of applied statistics', *Wiley Series in Probability and Mathematical Statistics* .

Chambers, J., Cleveland, W., Kleiner, B. & Tukey, J. (1983), *Graphical Methods for Data Analysis*, Chapman and Hall, London.

Chen, G. & Balakrishnan, N. (1995), 'A general purpose approximate goodness-of-fit test', *Journal of Quality Technology* **27**(2), 154–161.

Chipepa, F., Oluyede, B. & Makubate, B. (2020), 'The Topp-Leone Marshall-Olkin-G family of distributions with applications', *International Journal of Statistics and Probability* **9**(4), 15–32.

Cordeiro, G. M., Alizadeh, M. & Ortega, E. M. (2014), 'The exponentiated half-logistic family of distributions: Properties and applications', *Journal of Probability and Statistics* **2014**.

Cordeiro, G. M. & de Castro, M. (2011), 'A new family of generalized distributions', *Journal of Statistical Computation and Simulation* **81**(7), 883–898.

Cui, W., Yan, Z. & Peng, X. (2020), 'A new Marshall-Olkin Weibull Distribution', *Engineering Letters* **28**(1).

Ghitany, M. E., Al-Hussaini, E. K. & Al-Jarallah, R. A. (2005), 'Marshall-Olkin extended weibull distribution and its application to censored data', *Journal of Applied Statistics* **32**(10), 1025–1034.

Hassan, A., Elgarhy, M. A. E. & Shakil, M. (2017), 'Type II half logistic family of distributions with applications', *Pakistan Journal of Statistics and Operation Research* **13**(2), 245–264.

Lee, E. T. & Wang, J. (2003), *Statistical Methods for Survival Data Analysis*, Vol. 476, John Wiley & Sons.

Marshall, A. W. & Olkin, I. (1997), 'A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families', *Biometrika* **84**(3), 641–652.

Oguntunde, P. E., Balogun, O. S., Okagbue, H. I. & Bishop, S. A. (2015), 'The Weibull-Exponential distribution: Its properties and applications', *Journal of Applied Sciences* **15**(11), 1305–1311.

Pakungwati, R., Widyaningsih, Y. & Lestari, D. (2018), 'Marshall-Olkin extended inverse Weibull distribution and its application', *Journal of physics: conference series* **1108**(1), 012114.

Rannona, K., Oluyede, B., Chipepa, F. & Makubate, B. (2022), 'The Marshall-Olkin-Type II-Topp-Leone-G Family of Distributions: Model, Properties and Applications', *Journal of Probability and Statistical Science* **20**(1), 127–149.

Rényi, A. (1960), On measures of entropy and information, *in* 'Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics', Vol. 4, University of California Press, pp. 547–562.

Ristić, M. M. & Balakrishnan, N. (2012), 'The gamma-exponentiated exponential distribution', *Journal of statistical computation and simulation* **82**(8), 1191–1206.

Schafft, H. A., Staton, T. C., Mandel, J. & Shott, J. D. (1987), 'Reproducibility of electromigration measurements', *IEEE Transactions on Electron Devices* **34**(3), 673–681.

Schwarz, G. (1978), 'Estimating the dimension of a model', *The annals of statistics* pp. 461–464.

Shaked, M. & Shanthikumar, J. G. (2007), *Stochastic Orders*, Springer, New York.

Soliman, A. H., Elgarhy, M. A. E. & Shakil, M. (2017), 'Type II half logistic family of distributions with applications', *Pakistan Journal of Statistics and Operation Research* pp. 245–264.

Zografos, K. & Balakrishnan, N. (2009), 'On families of beta and generalized gamma-generated distribution and associated inference', *Statistical Methodology* **6**(4), 344–362.

Appendix

`https://drive.google.com/file/d/1OEFbhtqNfXTIs5iSnhk_Wq9OnxeuzFpK/view?usp=sharing`