The Type II Exponentiated Half Logistic-Marshall-Olkin-G Family of Distributions with Applications

La familia de distribuciones tipo II exponenciada media logística-Marshall-Olkin-G con aplicaciones

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Abstract

A new generalized family of distributions called the type II exponentiated half logistic-Marshall-Olkin-G distribution is developed. Some special cases of the new model are presented. We explore some statistical properties of the new family of distributions. The statistical properties studied include expansion of the density function, hazard rate and quantile functions, moments, moment generating functions, probability weighted moments, stochastic ordering, distribution of order statistics and Rényi entropy. The maximum likelihood, ordinary and weighted least-squares techniques for the estimation of model parameters are presented, and Monte Carlo simulations for the new family of distributions are conducted. The importance of the new family of distributions is examined by means of applications to two real data sets.

Key words: Marshall-Olkin-G distribution; Maximum likelihood estimation; Simulations; Type II exponentiated half logistic distribution.

Resumen

Se desarrolla una nueva familia generalizada de distribuciones denominada distribución media exponenciada tipo II-Marshall-Olkin-G logística. Se presentan algunos casos especiales del nuevo modelo. Exploramos algunas propiedades estadísticas de la nueva familia de distribuciones. Las propiedades estadísticas estudiadas incluyen la expansión de la función de densidad, la tasa de riesgo y las funciones de cuantiles, momentos, funciones generadoras de momentos, momentos ponderados de probabilidad,

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ordenamiento estocástico, distribución de estadísticas de orden y entropía Rényi. Se presentan las técnicas de máxima verosimilitud, mínimos cuadrados ordinarios y ponderados para la estimación de los parámetros del modelo, y se realizan simulaciones Monte Carlo para la nueva familia de distribuciones. Se examina la importancia de la nueva familia de distribuciones mediante aplicaciones a dos conjuntos de datos reales.

Palabras clave: Distribución Marshall-Olkin-G; Estimación de máxima verosimilitud; Simulaciones; Tipo II distribución semilogística exponencial.

1. Introduction

There are well known methods of generating new families of distributions in the literature due to the demand for more generalized distributions that can fit data well in reliability, economics, finance, applied sciences and engineering among others. In response to this, scholars have shifted their attention to generating new distributions that can provide greater flexibility when modelling data in practice.


Al-Mofleh et al. (2020) using the gamma generator (Ristić & Balakrishnan (2012)) obtained the type II exponentiated half logistic-G (TIIEHL-G) family of distributions which has the cumulative distribution function (cdf) of the form

\[ F_{TIIEHL-G}(x; \gamma, a, \zeta) = 1 - \int_0^{-\log(G(x; \zeta))} \frac{2a\gamma e^{-\gamma t}(1 + e^{-\gamma t})^{a-1}}{(1 - e^{-\gamma t})^{a+1}} dt \]

\[ = 1 - \left[ \frac{1 - [G(x; \zeta)]^{\gamma}}{1 + [G(x; \zeta)]^{\gamma}} \right]^{a} \cdot \quad (1) \]

for \( \gamma, a > 0 \), where \( G(x; \zeta) \) is the baseline cdf with parameter vector \( \zeta \). The corresponding probability density function (pdf) is given by

\[ f_{TIIEHL-G}(x; \gamma, a, \zeta) = \frac{2a\gamma(1 - [G(x; \zeta)]^{\gamma})^{a-1}[G(x; \zeta)]^{\gamma-1}g(x; \zeta)}{(1 + [G(x; \zeta)]^{\gamma})^{a+1}}. \quad (2) \]
Furthermore, the Marshall-Olkin-G family of distributions has the cdf and pdf given by

\[ F_{MO-G}(x; \delta, \zeta) = 1 - \frac{\delta \tilde{G}(x; \zeta)}{1 - \delta \tilde{G}(x; \zeta)} \]  

and

\[ f_{MO-G}(x; \delta, \zeta) = \frac{\delta g(x; \zeta)}{[1 - \delta \tilde{G}(x; \zeta)]^2}, \]  

respectively, where \( \delta \) is the tilt parameter, \( \tilde{\delta} = 1 - \delta \) and \( G(x; \zeta) \) is the baseline cdf which depends on the parameter vector \( \zeta \).

In this note, we develop the new type II exponentiated half logistic-Marshall-Olkin-G (TIIEHL-MO-G) family of distributions by inserting equation (3) into equation (1), with \( \gamma = 1 \). The motivations for developing this new family of distributions are:

- The new family of distribution has greater flexibility when fitted to real-life data as compared to models with the same number of parameters.
- The hazard rate function has shapes that are monotonic and non-monotonic.
- The family of distributions can model data that are heavy-tailed.
- We considered the generalization of the Weibull distribution when taken as the baseline cdf, given that Weibull distribution has some limitations, in order to obtain more flexibility.

The results of this paper are outlined as follows: In Section 2, we present the new family of distributions and its sub-families. Mathematical and statistical properties of the new model, including expansion of the probability density function, quantile function, moments, generating function, stochastic orders, probability weighted moments, order statistics and Rényi entropy are presented in Section 3. Estimation methods and observed information matrix are given in Section 4. Some special cases of the new family of distributions are given in Section 5. A Monte Carlo simulation study to examine the average bias and mean square error of the maximum likelihood estimates of the special case of the type II exponentiated half logistic-Marshall-Olkin-Weibull distribution are presented in Section 6. Section 7 contain applications of the new model to actual data sets. Lastly, we give concluding remarks in Section 8.

2. The Model

Equation (1) (with \( \gamma = 1 \)) and equation (3) can be combined to obtain a new family of distributions called the type II exponentiated half logistic-Marshall-Olkin-G (TIIEHL-MO-G) distribution. The cdf and pdf of the new family of distributions are given by
and the second derivative is

$$F_{TIEHL-MO-G}(x; a, \delta, \zeta) = 1 - \left[ \frac{\delta g(x; \zeta)}{1 - \delta G(x; \zeta)} \right]^a,$$

and

$$f_{TIEHL-MO-G}(x; a, \delta, \zeta) = 2a\delta \frac{\left[ \frac{\delta g(x; \zeta)}{1 - \delta G(x; \zeta)} \right]^{a-1}}{\left[ 1 + \left( \frac{1 - \delta G(x; \zeta)}{1 - \delta G(x; \zeta)} \right) \right]^{a+1}} g(x; \zeta),$$

respectively, for \( x, a, \delta > 0 \), \( \delta = 1 - \delta \) and parameter vector \( \zeta \). The hazard rate function (hrf) is given by

$$h_{TIEHL-MO-G}(x; a, \delta, \zeta) = 2a\delta \frac{\left[ \frac{\delta g(x; \zeta)}{1 - \delta G(x; \zeta)} \right]^{a-1}}{\left[ 1 + \left( \frac{1 - \delta G(x; \zeta)}{1 - \delta G(x; \zeta)} \right) \right]^{a+1}} \frac{g(x; \zeta)}{\left[ 1 - \delta G(x; \zeta) \right]^{2a}},$$

for \( x, a, \delta > 0 \), \( \delta = 1 - \delta \) and parameter vector \( \zeta \).

The shapes of the pdf and hrf of the new family of distribution are described below. The critical points of the TIEHL-MO-G family of distributions are the roots of the equation

$$\frac{d \log f(x; a, \delta, \zeta)}{dx} = -(a - 1) \frac{\delta g(x; \zeta)}{1 - \delta G(x; \zeta)} + \left( \frac{g'(x; \zeta)}{g(x; \zeta)} \right) - (a + 1) \frac{1}{1 + \left( \frac{1 - \delta G(x; \zeta)}{1 - \delta G(x; \zeta)} \right)} \times \frac{\delta g(x; \zeta)}{\left[ 1 - \delta G(x; \zeta) \right]^{2a}} = 0,$$

and the second derivative is

$$\frac{d^2 \log f(x; a, \delta, \zeta)}{dx^2} = -(a - 1) \frac{\delta g(x; \zeta)}{1 - \delta G(x; \zeta)} \left[ \frac{\delta G(x; \zeta)}{1 - \delta G(x; \zeta)} \right]^2 + \frac{\delta^2 g(x; \zeta)^2}{\left[ 1 - \delta G(x; \zeta) \right]^2}$$

$$+ g''(x; \zeta) g(x; \zeta) - \frac{g'(x; \zeta) g'(x; \zeta)}{g(x; \zeta)} - (a + 1) \left[ \delta g'(x; \zeta) \right]$$

$$\times \left( 1 + \left( \frac{1 - \delta G(x; \zeta)}{1 - \delta G(x; \zeta)} \right) \right) \left[ 1 - \delta G(x; \zeta) \right]^2$$

$$- \left( \delta g(x; \zeta) + 2 \left[ \delta G(x; \zeta) \right] \delta g(x; \zeta) \left( 1 + \left( \frac{1 - \delta G(x; \zeta)}{1 - \delta G(x; \zeta)} \right) \right) \right)$$

$$\times \left[ 1 + \left( \frac{1 - \delta G(x; \zeta)}{1 - \delta G(x; \zeta)} \right) \right] \left[ 1 - \delta G(x; \zeta) \right]^2$$

$$- 2 g'(x; \zeta) \left[ 1 - \delta G(x; \zeta) \right] - g(x; \zeta)^2,$$

where \( g'(x; \zeta) = \frac{dg(x; \zeta)}{dx} \), and \( g''(x; \zeta) = \frac{d^2 g(x; \zeta)}{dx^2} \).
If \( x = x_0 \) is a root of equation (8), then it is in accordance with a local minimum (maximum) if \( \frac{d^2 \log f(x;a,\delta,\zeta)}{dx^2} > 0(<0) \). It gives points of inflection if either \( \frac{d^2 \log f(x;a,\delta,\zeta)}{dx^2} > 0 \) for all \( x \neq x_0 \) or \( \frac{d^2 \log f(x;a,\delta,\zeta)}{dx^2} < 0 \) for all \( x \neq x_0 \).

The critical points of the hfr are obtained by the equation

\[
\frac{d \log h(x;a,\delta,\zeta)}{dx} = -(a-1)\frac{\delta g(x;\zeta)}{\delta G(x;\zeta)} + \frac{g'(x;\zeta)}{g(x;\zeta)} = (a+1)\frac{1}{1 - \delta G(x;\zeta)} \left( \frac{\delta g(x;\zeta)}{\delta G(x;\zeta)} \right)^2 + \frac{\delta g(x;\zeta)}{\delta G(x;\zeta)} \left( \frac{1}{1 - \delta G(x;\zeta)} \right)^2 - \delta g(x;\zeta)
\]

**and the second derivative of \( \log h(x;a,\delta,\zeta) \) is**

\[
\frac{d^2 \log h(x;a,\delta,\zeta)}{dx^2} = -(a-1)\frac{\delta g(x;\zeta)}{\delta G(x;\zeta)} + \frac{g''(x;\zeta)g(x;\zeta) - g'(x;\zeta)g'(x;\zeta)}{g(x;\zeta)^2} - (a+1) \left\{ \delta g(x;\zeta) \left[ \left( 1 - \frac{\delta G(x;\zeta)}{\delta G(x;\zeta)} \right) \left( 1 - \delta G(x;\zeta) \right) \right] \right. \\
\times \left[ \left( 1 - \frac{\delta G(x;\zeta)}{\delta G(x;\zeta)} \right) \left( 1 - \delta G(x;\zeta) \right) \right]^{-2} \\
- \left( \delta g(x;\zeta) + 2\left( 1 - \delta G(x;\zeta) \right) \delta g(x;\zeta) \left( 1 + \left( 1 - \frac{\delta G(x;\zeta)}{\delta G(x;\zeta)} \right) \right) \right) \left[ \left( 1 - \frac{\delta G(x;\zeta)}{\delta G(x;\zeta)} \right) \left( 1 - \delta G(x;\zeta) \right) \right]^{-2} \\
+ a\left( \delta g(x;\zeta) + 2\left( 1 - \delta G(x;\zeta) \right) \delta g(x;\zeta) \left( 1 + \left( 1 - \frac{\delta G(x;\zeta)}{\delta G(x;\zeta)} \right) \right) \right) \left[ \left( 1 - \frac{\delta G(x;\zeta)}{\delta G(x;\zeta)} \right) \left( 1 - \delta G(x;\zeta) \right) \right]^{-2}
\]

(11)

If \( x = x_0 \) is a root of equation (10), then it is in accordance with a local minimum (maximum) if \( \frac{d^2 \log h(x;a,\delta,\zeta)}{dx^2} > 0(<0) \). It gives points of inflection if either \( \frac{d^2 \log h(x;a,\delta,\zeta)}{dx^2} > 0 \) for all \( x \neq x_0 \) or \( \frac{d^2 \log h(x;a,\delta,\zeta)}{dx^2} < 0 \) for all \( x \neq x_0 \).

To test for identifiability of the new family of distributions, let \( \rho_1 = (a_1,\delta_1) \) and \( \rho_2 = (a_2,\delta_2) \), such that,

\[
f_{\rho_1} = 2a_1\delta_1\left( \frac{\delta_1 G(x;\zeta)}{1 - \delta_1 G(x;\zeta)} \right)^{a_1-1} \frac{g(x;\zeta)}{1 + \left( 1 - \frac{\delta_1 G(x;\zeta)}{1 - \delta_1 G(x;\zeta)} \right)^{a_1+1}} \left[ 1 - \delta_1 G(x;\zeta) \right]^{2}.
\]
and

\[ f_{\varphi_2} = 2a_2 \delta_2 \frac{\left( \frac{\delta_2 \bar{G}(x; \zeta)}{1 - \delta_2 \bar{G}(x; \zeta)} \right)^{a_2 - 1}}{\left[ 1 + \left( 1 - \frac{\delta_2 \bar{G}(x; \zeta)}{1 - \delta_2 \bar{G}(x; \zeta)} \right)^{a_1 + 1} \right]} \frac{g(x; \zeta)}{\left( 1 - \delta_2 \bar{G}(x; \zeta) \right)^2}. \]

Then

\[ f_{\varphi_1} = f_{\varphi_2} \iff \Phi_1 - \Phi_2 = 0, \] (12)

where

\[ \Phi_1 = 2a_1 \delta_1 \frac{\left( \frac{\delta_1 \bar{G}(x; \zeta)}{1 - \delta_1 \bar{G}(x; \zeta)} \right)^{a_1 - 1}}{\left[ 1 + \left( 1 - \frac{\delta_1 \bar{G}(x; \zeta)}{1 - \delta_1 \bar{G}(x; \zeta)} \right)^{a_1 + 1} \right]} \frac{1}{\left( 1 - \delta_1 \bar{G}(x; \zeta) \right)^2}, \]

and

\[ \Phi_2 = 2a_2 \delta_2 \frac{\left( \frac{\delta_2 \bar{G}(x; \zeta)}{1 - \delta_2 \bar{G}(x; \zeta)} \right)^{a_2 - 1}}{\left[ 1 + \left( 1 - \frac{\delta_2 \bar{G}(x; \zeta)}{1 - \delta_2 \bar{G}(x; \zeta)} \right)^{a_2 + 1} \right]} \frac{1}{\left( 1 - \delta_2 \bar{G}(x; \zeta) \right)^2}, \]

respectively. Therefore, equation (12) is zero for all \( G(x; \zeta) \) when its coefficients are equal to zero, and that is possible when \( a_1 = a_2, \delta_1 = \delta_2 \). For this model all parameters are greater than zero, hence, we conclude that the new family of distributions is identifiable: \( f_{\varphi_1} = f_{\varphi_2} \iff \rho_1 = \rho_2 \).

### 2.1. Sub-Models

- When \( a = 1 \), we obtain a new type II half logistic-Marshall-Olkin-G (TIHL-MO-G) family of distributions with the cdf

\[ F(x; \delta, \zeta) = 1 - \left[ \frac{\delta \bar{G}(x; \zeta)}{1 - \delta \bar{G}(x; \zeta)} \right] \left[ 1 + \left( 1 - \frac{\delta \bar{G}(x; \zeta)}{1 - \delta \bar{G}(x; \zeta)} \right) \right], \]

for \( \delta > 0 \) and parameter vector \( \zeta \).

- If we let \( \delta = 1 \), we get the type II exponentiated half logistic-G (TIIEHL-G) (Al-Mofleh et al., 2020) family of distributions with the cdf given by

\[ F(x; a, \zeta) = 1 - \left[ \frac{\bar{G}(x; \zeta)}{1 + \bar{G}(x; \zeta)} \right]^a \]

for \( a > 0 \) and parameter vector \( \zeta \).

- By letting \( a = \delta = 1 \), we get the type II half logistic-G (TIHL-G) (Soliman et al., 2017) family of distributions with the cdf

\[ F(x; \zeta) = \frac{2G(x; \zeta)}{1 + G(x; \zeta)}, \]

for the parameter vector \( \zeta \).
3. Some Statistical Properties

Some statistical properties of the Type II Exponentiated Half Logistic-Marshall-Olkin-G family of distributions are presented in this section. The statistical properties considered include: expansion of the density, quantile function, moments, generating function, stochastic orders, probability weighted moments, distribution of order statistics and Rényi entropy.

3.1. Linear Representation of Density Function

In this subsection, we present series expansion for the pdf of the Type II Exponentiated Half Logistic-Marshall-Olkin-G family of distributions using the following generalized binomial series expansions

\[(1 + z)^{-(k+1)} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma((k + 1) + j)}{\Gamma(k + 1)!} z^j,\]

and

\[(1 - z)^b = \sum_{j=0}^{\infty} \binom{b}{j} (-1)^j z^j, \quad \text{for } |z| < 1.
\]

The pdf of the Type II Exponentiated Half Logistic-Marshall-Olkin-G family of distributions can be expressed as

\[f(x; a, \delta, \zeta) = 2a \sum_{j,i,k,l=0}^{\infty} \delta^i + a \delta^k \frac{(-1)^{j+i+l}}{(l+1)} \binom{j}{j} \binom{-(i+a+1)}{k} \]
\[\times \left( \frac{k + i + a - 1}{l} \right) (l+1) g(x; \zeta) [G(x; \zeta)]^l \]
\[= \sum_{l=0}^{\infty} t_{l+1} h_{l+1}(x; \zeta), \quad (13)\]

where \(h_{l+1}(x; \zeta) = (l+1) [G(x; \zeta)]^l g(x; \zeta)\) is the exponentiated-G (Exp-G) density with power parameter \(l+1\) and parameter vector \(\zeta\), and

\[t_{l+1} = 2a \sum_{j,i,k=0}^{\infty} \delta^i + a \delta^k \frac{(-1)^{j+i+l}}{(l+1)} \binom{j}{j} \binom{-(i+a+1)}{k} \]
\[\times \left( \frac{k + i + a - 1}{l} \right). \quad (14)\]

Details of derivations are given in the Appendix.

3.2. Quantile Function

We obtain the quantile function of the Type II Exponentiated Half Logistic-Marshall-Olkin-G family of distributions by inverting the non-linear equation...
\[
F(x; a, \delta, \zeta) = 1 - \left[ \frac{\delta \tilde{G}(x; \zeta)}{1 - \delta \tilde{G}(x; \zeta)} \right]^{\frac{1}{1 - \delta \tilde{G}(x; \zeta)}} = u \quad \text{for} \quad 0 \leq u \leq 1.
\]

Note that,
\[
1 - u = \left[ \frac{\delta \tilde{G}(x; \zeta)}{1 - \delta \tilde{G}(x; \zeta)} \right]^{\frac{1}{a}},
\]
so that
\[
\frac{2(1 - u)^{\frac{1}{a}}}{1 + (1 - u)^{\frac{1}{a}}} = \frac{\delta \tilde{G}(x; \zeta)}{1 - \delta \tilde{G}(x; \zeta)},
\]
which simplifies to
\[
\tilde{G}(x; \zeta) = \frac{2(1-u)^{\frac{1}{a}}}{1+2(1-u)^{\frac{1}{a}}}.
\]

Finally, the quantile function of the TIIEHL-MO-G family of distributions reduces to
\[
Q_G(u; a, \delta, \zeta) = G^{-1} \left( 1 - \left[ \frac{2(1-u)^{\frac{1}{a}}}{1+2(1-u)^{\frac{1}{a}}} \right] \right), \quad (15)
\]

Consequently, random numbers can be generated from the TIIEHL-MO-G family of distributions via equation (15) for specified baseline cdf G.

3.3. Moments, Generating Function and Probability Weighted Moments

In this subsection, we present the moments, generating function and probability weighted moments (PWMs) of the TIIEHL-MO-G family of distributions. The \( r \)th moment of the TIIEHL-MO-G family of distributions is given by
\[
E(X^r) = \int_{-\infty}^{\infty} x^r f_{TIEHL-MO-G}(x; a, \delta, \zeta) dx = \sum_{l=0}^{\infty} t_{l+1} E(Y_{l+1}^r), \quad (16)
\]
where \( E(Y_{l+1}^r) \) is the \( r \)th moment of \( Y_{l+1} \) which follows the Exp-G distribution with power parameter \( l + 1 \) and \( t_{l+1} \) is defined in equation (14). The moment generating function (mgf) is obtained as
\[
M_X(h) = E(e^{hX}) = \sum_{l=0}^{\infty} t_{l+1} E(e^{hY_{l+1}}),
\]
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where $E \left( e^{Y_{l+1}} \right)$ is the mgf of the exponentiated-G (Exp-G) family of distributions with power parameter $l + 1$ and $t_{l+1}$ is defined in equation (14).

The PWMs of a random variable $X$ are defined by

$$
\omega_{a,r} = E(X^a[F(X)]^r) = \int_{-\infty}^{\infty} x^a[F(x)]^r f(x) dx.
$$

Considering the generalized binomial series expansions given in the Appendix, we obtain

$$
f(x)[F(x)]^r = 2a \sum_{s,j,i,k,q=0} \delta^{a(s+1)+i} \bar{\delta}^k \left( \frac{-1}{q+1} \right) \left( \begin{array}{c} r \vspace{0.2cm} \cr \begin{array}{c} s \cr j \end{array} \end{array} \right) \left( \begin{array}{c} q \vspace{0.2cm} \cr \begin{array}{c} a(s+1) + i + q \cr j \end{array} \end{array} \right)
\times \binom{a(s+1) + i + k - 1}{q} x^{a(s+1) + i + k - 1} x
\times (q+1)g(x; \zeta)[G(x; \zeta)]^q
= \sum_{q=0}^{\infty} V_{q+1} h_{q+1}(x; \zeta),
$$

where $h_{q+1}(x; \zeta) = (q+1)[G(x; \zeta)]^q g(x; \zeta)$ is the Exp-G density with the power parameter $q + 1$ and parameter vector $\zeta$, and

$$
V_{q+1} = 2a \sum_{s,j,i,k,q=0} \delta^{a(s+1)+i} \bar{\delta}^k \left( \frac{-1}{q+1} \right) \left( \begin{array}{c} r \vspace{0.2cm} \cr \begin{array}{c} s \cr j \end{array} \end{array} \right) \left( \begin{array}{c} q \vspace{0.2cm} \cr \begin{array}{c} a(s+1) + i + k - 1 \cr j \end{array} \end{array} \right).
$$

Consequently, the PWMs of the TIEHL-MO-G family of distributions is given by

$$
\omega_{a,r} = \sum_{q=0}^{\infty} V_{q+1} \int_{-\infty}^{\infty} x^a h_{q+1}(x; \zeta) dx.
$$

3.4. Stochastic Ordering

In this subsection, we present stochastic orders for the TIEHL-MO-G family of distributions. Suppose we have two random variables $Z$ and $T$ with distribution functions $F_Z(r)$ and $F_T(r)$, respectively, and $F_Z(r) = 1 - F_T(r)$ is the survival function. Note that $Z$ is stochastically smaller than $T$ if $F_Z(r) \leq F_T(r)$ for all $r$ or $F_Z(r) \geq F_T(r)$ for all $r$. This is denoted by $Z <_s T$. Hazard rate order and likelihood ratio order are stronger, and are given by $Z <_{hr} T$ if $h_Z(r) \geq h_T(r)$ for all $r$, and $Z <_{lr} T$ if $f_Z(r) / f_T(r)$ is decreasing in $r$ (Shaked & Shanthikumar, 2007). We know that $Z <_{lr} T \Rightarrow Z <_{hr} T \Rightarrow Z <_s T$.

**Theorem 1.** Suppose $X_1$ and $X_2$ are two independent random variables following TIEHL-MO-G $(a_1, \delta, \zeta)$ and TIEHL-MO-G $(a_2, \delta, \zeta)$ distributions, respectively, then $X_1 <_s X_2$. 

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Proof. Let
\[
f_1(x; a_1, \delta, \zeta) = 2a_1\delta \frac{\left[\frac{\delta \hat{G}(x; \zeta)}{1 - \delta \hat{G}(x; \zeta)}\right]^{a_1-1}}{1 + \left(1 - \frac{\delta \hat{G}(x; \zeta)}{1 - \delta \hat{G}(x; \zeta)}\right)^{a_1+1}} g(x) \left[1 - \delta \hat{G}(x; \zeta)\right]^2,
\]
and
\[
f_2(x; a_2, \delta, \zeta) = 2a_2\delta \frac{\left[\frac{\delta \hat{G}(x; \zeta)}{1 - \delta \hat{G}(x; \zeta)}\right]^{a_2-1}}{1 + \left(1 - \frac{\delta \hat{G}(x; \zeta)}{1 - \delta \hat{G}(x; \zeta)}\right)^{a_2+1}} g(x) \left[1 - \delta \hat{G}(x; \zeta)\right]^2.
\]
Note that
\[
\frac{f_1(x; a_1, \delta, \zeta)}{f_2(x; a_2, \delta, \zeta)} = \frac{a_1}{a_2} \left[\frac{\delta \hat{G}(x; \zeta)}{1 - \delta \hat{G}(x; \zeta)}\right]^{a_1-a_2} \left[1 + \left(1 - \frac{\delta \hat{G}(x; \zeta)}{1 - \delta \hat{G}(x; \zeta)}\right)^{a_1-a_2+1}\right]^2 \left[1 - \delta \hat{G}(x; \zeta)\right]^2.
\]
Upon differentiating equation (18) with respect to \(x\), we obtain
\[
\frac{d}{dx} \left(\frac{f_1(x; a_1, \delta, \zeta)}{f_2(x; a_2, \delta, \zeta)}\right) = \frac{a_1}{a_2} \left(1 - \frac{\delta \hat{G}(x; \zeta)}{1 - \delta \hat{G}(x; \zeta)}\right) \left[1 + \left(1 - \frac{\delta \hat{G}(x; \zeta)}{1 - \delta \hat{G}(x; \zeta)}\right)^{a_1-a_2+1}\right]^2 \frac{2\delta g(x; \zeta)}{1 - \delta \hat{G}(x; \zeta)}
\]
which is \(\leq 0\) if \(a_1 \leq a_2\). Therefore, \(X_1 <_{hr} X_2, X_1 <_{hr} X_2\) and \(X_1 <_{s} X_2\), and the random variables \(X_1\) and \(X_2\) are stochastically ordered.

3.5. Distribution of Order Statistics

In this subsection, we present the distribution of the \(i^{th}\) order statistics for the TIEHL-MO-G family of distributions. The pdf of the \(i^{th}\) order statistics is given by
\[
f_{i,n}(x) = \frac{n!f(x)}{(i-1)!((n-i))!} \sum_{r=0}^{n-i} (-1)^r \binom{n-i}{r} \left(F(x)\right)^{r+i-1}.
\]
Applying the generalized binomial series expansions outlined in the Appendix, we can write
\[
f(x)[F(x)]^{r+i-1} = 2a \sum_{s,j,m,k,q=0}^{\infty} \delta^{a(s+1)+i} \left(\sum_{j=0}^{(r+i-1)} \binom{r+i-1}{j} \binom{q+1}{s} \binom{j}{m}\right) \left((-a(s+1) + 1)\right)^{j} \left((-a(s+1) + m + 1)\right)^{k} \left(a(s+1) + m + k - 1\right)^{q} \left(q+1\right) \left(G(x; \zeta)\right)^{q}
\]
\[
= \sum_{q=0}^{\infty} W_{q+1} h_{q+1}(x; \zeta),
\]

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where \( h_{q+1}(x; \zeta) = (q + 1)[G(x; \zeta)]^q g(x; \zeta) \) is the Exp-G density with power parameter \( q + 1 \) and parameter vector \( \zeta \), and

\[
W_{q+1} = 2a \sum_{s,j,m,k=0}^{\infty} \delta^a(s+1+i)\delta^k(-1)^{s+j+m+q} \frac{r + i - 1}{s} \binom{r}{j} (-a(s + 1) + 1) \binom{a(s + 1) + k}{m} \binom{a(s + 1) + m + k - 1}{q}.
\]

Therefore, the pdf of the \( i^{th} \) order statistic from the TIIEHL-MO-G family of distributions is given by

\[
f_{i,n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{r=0}^{n-i} \sum_{q=0}^{\infty} (-1)^r \binom{n-i}{r} W_{q+1} h_{q+1}(x; \zeta).
\]

### 3.6. Rényi Entropy

Rényi entropy of the proposed family of distributions is given in this subsection. Rényi entropy (Rényi, 1960) is a measure of variation of uncertainty and is defined by

\[
I_R(\nu) = (1 - \nu)^{-1} \log \left[ \int_{-\infty}^{\infty} f^\nu(x) dx \right] \quad \text{for } \nu > 0 \text{ and } \nu \neq 1.
\]

Rényi entropy of the TIIEHL-MO-G family of distributions can be expressed as

\[
I_R(\nu) = \frac{1}{1 - \nu} \log \left[ (2a)^\nu \sum_{j,i,k,l=0}^{\infty} \delta^i+j\delta^k (-1)^{i+l} \binom{-\nu(a + 1)}{j} \binom{k + i + \nu(a - 1)}{l} \right]
\]

\[
\times \binom{j}{i} \binom{-i + \nu(a + 1)}{k} \binom{k + i + \nu(a - 1)}{l}
\]

\[
\times \frac{1}{(\nu + 1)^\nu} \int_{0}^{\infty} \left[ \left( \int_{0}^{\infty} G(x; \zeta)[G(x; \zeta)]^{\frac{1}{\nu}} g(x; \zeta) dx \right)^\nu \right]
\]

\[
= \frac{1}{1 - \nu} \log \left[ \sum_{l=0}^{\infty} d^l \exp ((1 - \nu)I_{REG}) \right], \quad \nu > 0 \text{ and } \nu \neq 1,
\]

where

\[
I_{REG} = \frac{1}{1 - \nu} \log \left[ \int_{0}^{\infty} \left[ \left( \int_{0}^{\infty} G(x; \zeta)[G(x; \zeta)]^{\frac{1}{\nu}} g(x; \zeta) dx \right)^\nu \right] dx \right]
\]
is the Rényi entropy of the Exp-G density with power parameter \((\frac{1}{\nu} + 1)\) and

\[
d_r^i = (2a)^\nu \sum_{j,i,k=0}^{\infty} \delta^i \nu a \delta^j (1)^{j+i+1} \binom{-\nu(a+1)}{j} \binom{\nu(a+1) - 1}{i} \frac{1}{\nu^i}.
\]

More details on derivations follow in the appendix.

4. Estimation Methods

In this section, we discuss three estimation methods including maximum likelihood estimation (MLE), ordinary and weighted least squares.

4.1. Maximum Likelihood Estimation

The log-likelihood function \(\ell_n\) based on a random sample of size \(n\) from the TIIEHL-MO-G family of distributions is given by

\[
\ell_n(\Delta) = n \log(2a\delta) + (a - 1) \sum_{i=0}^{n} \log \left( \frac{\delta \bar{G}(x_i; \zeta)}{1 - \delta \bar{G}(x_i; \zeta)} \right) + \sum_{i=0}^{n} \log[g(x_i; \zeta)] - (a + 1) \sum_{i=0}^{n} \log \left[ 1 + \left( 1 - \frac{\delta \bar{G}(x_i; \zeta)}{1 - \delta \bar{G}(x_i; \zeta)} \right) \right] - 2 \sum_{i=0}^{n} \log[1 - \bar{G}(x_i; \zeta)].
\]

The maximum likelihood estimates of the parameters, denoted by \(\hat{\Delta}\), is obtained by solving the non-linear equation

\[
\left( \frac{\partial \ell_n}{\partial a}, \frac{\partial \ell_n}{\partial \delta}, \frac{\partial \ell_n}{\partial \zeta} \right)^T = 0,
\]

using a numerical method such as Newton-Raphson procedure. The Fisher information matrix is given by

\[
I(\Delta) = \begin{bmatrix} I_{\theta_1, \theta_2} & I_{\theta_1, \theta_3} & I_{\theta_2, \theta_3} \end{bmatrix} (p+2) \times (p+2), \quad i, j = 1, 2, \ldots, (p + 2),
\]

and can be numerically obtained by MATLAB or NLMIXED in SAS or R software. The total Fisher information matrix \(nI(\Delta)\) can be approximated by

\[
J_n(\Delta) \approx \left[ -\frac{\partial^2 \ell_n}{\partial \theta_i \partial \theta_j} \right]_{\Delta = \hat{\Delta}} (p+2) \times (p+2), \quad i, j = 1, 2, \ldots, (p + 2).
\]

The elements of the score vector are given in the appendix.

4.2. Ordinary and Weighted Least-Squares

The ordinary least-squares estimates (OLSEs) of the parameters of the TIIEHL-MO-G family of distributions are derived by minimizing the function

\[
K(a, \delta, \zeta) = \sum_{i=0}^{n} \left[ F(x_i, a, \delta, \zeta) - \frac{i}{n+1} \right]^2,
\]

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with respect to the parameters \(a, \delta\) and parameter vector \(\zeta\). The OLSE can be obtained by solving the non-linear equations

\[
\sum_{i=0}^{n} \left[F(x_i, a, \delta, \zeta) - \frac{i}{n+1}\right] \Delta_r(x_i, a, \delta, \zeta) = 0, \quad r = 1, 2, 3,
\]

where

\[
\Delta_1(x_i, a, \delta, \zeta) = \frac{\partial}{\partial a} F(x_i, a, \delta, \zeta), \quad \Delta_2(x_i, a, \delta, \zeta) = \frac{\partial}{\partial \delta} F(x_i, a, \delta, \zeta),
\]

and

\[
\Delta_3(x_i, a, \delta, \zeta) = \frac{\partial}{\partial \zeta} F(x_i, a, \delta, \zeta).
\] (20)

The weighted least-squares estimates (WLSs) of the parameters of the TIIEHL-MO-G family of distributions are obtained by minimizing equation (21) with respect to the parameters \(a, \delta\) and parameter vector \(\zeta\), where

\[
W(a, \delta, \zeta) = \sum_{i=0}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_i, a, \delta, \zeta) - \frac{i}{n+1}\right]^2.
\] (21)

The WLS can now be obtained by solving the non-linear equations

\[
\sum_{i=0}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_i, a, \delta, \zeta) - \frac{i}{n+1}\right] \Delta_r(x_i, a, \delta, \zeta) = 0, \quad r = 1, 2, 3,
\]

where \(\Delta_1(x_i, a, \delta, \zeta), \Delta_2(x_i, a, \delta, \zeta)\) and \(\Delta_3(x_i, a, \delta, \zeta)\) are given in equation (20).

\section{5. Some Special Models}

Some special models of the TIIEHL-MO-G family of distributions are presented in this section. The baseline distributions considered are Weibull, Burr XII and Burr III distributions, respectively.

\subsection*{5.1. Type II Exponentiated Half Logistic-Marshall-Olkin-Weibull (TIIEHL-MO-W) Distribution}

Suppose we take the baseline distribution to be Weibull distribution with the cdf and pdf \(G(x; \lambda) = 1 - e^{-x^\lambda}\), and \(g(x; \lambda) = \lambda x^{\lambda-1} e^{-x^\lambda}\), respectively, for \(x > 0\) and \(\lambda > 0\), then the cdf and pdf of the TIIEHL-MO-W distribution are given by

\[
F_{\text{TIIEHL-MO-W}}(x; a, \delta, \lambda) = 1 - \left[\frac{\delta e^{-x^\lambda}}{1-\delta e^{-x^\lambda}}\right]^a
\]

and

\[
f_{\text{TIIEHL-MO-W}}(x; a, \delta, \lambda) = 2a\delta \left[\frac{\delta e^{-x^\lambda}}{1-\delta e^{-x^\lambda}}\right]^{a-1} \lambda x^{\lambda-1} e^{-x^\lambda} \left[1-\delta e^{-x^\lambda}\right]^{a+1}.
\]
respectively, for \( x, a, \delta, \lambda > 0 \). Plots of the pdf and hrf for selected parameter values are given in Figure 1.

![Plots of the pdf and hrf for the TIEHLMO-W distribution](image)

**Figure 1:** Plots of the pdf and hrf for the TIEHLMO-W distribution

The pdf of the TIEHLMO-W distribution can take several shapes including left-skewed, right-skewed, almost symmetric, reverse-J and J shapes, whereas the hrf displays upside-down bathtub, upside-down bathtub followed by bathtub, increasing and decreasing shapes. Table 1 gives the table of percentiles for TIEHLMO-W distribution for selected parameter values.

**Table 1:** Some percentiles for TIEHLMO-W distribution

<table>
<thead>
<tr>
<th>( (\alpha, \delta, \lambda) )</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u ) \quad ( (1.1, 1.3) )</td>
<td>0.1431</td>
<td>0.2580</td>
<td>0.3743</td>
<td>0.4992</td>
<td>0.6397</td>
<td>0.8047</td>
<td>1.0089</td>
<td>1.2857</td>
<td>1.7356</td>
</tr>
<tr>
<td>( (0.7, 1.1, 1.5) )</td>
<td>0.3925</td>
<td>0.4968</td>
<td>0.5460</td>
<td>0.5938</td>
<td>0.7217</td>
<td>0.9137</td>
<td>1.3129</td>
<td>2.0059</td>
<td>2.8059</td>
</tr>
<tr>
<td>( (0.4, 1, 1) )</td>
<td>0.3175</td>
<td>0.3759</td>
<td>0.4169</td>
<td>0.4537</td>
<td>0.5296</td>
<td>0.6326</td>
<td>0.8496</td>
<td>1.2926</td>
<td>1.8926</td>
</tr>
<tr>
<td>( (2.1, 0.5, 1.9) )</td>
<td>0.1403</td>
<td>0.1513</td>
<td>0.1961</td>
<td>0.2403</td>
<td>0.2869</td>
<td>0.3392</td>
<td>0.4018</td>
<td>0.4852</td>
<td>0.6209</td>
</tr>
<tr>
<td>( (1.5, 1, 1.2) )</td>
<td>0.1004</td>
<td>0.1181</td>
<td>0.1779</td>
<td>0.2449</td>
<td>0.3228</td>
<td>0.4183</td>
<td>0.5422</td>
<td>0.7196</td>
<td>1.0398</td>
</tr>
</tbody>
</table>

Figures 2 and 3 shows the plots of skewness and kurtosis for TIEHLMO-W distribution. The plots shows that:

- When we fix the parameter \( \lambda \), skewness and kurtosis of the TIEHLMO-W distribution increases as \( a \) and \( \delta \) increase.

- When we fix the parameter \( a \), skewness and kurtosis of the TIEHLMO-W distribution decreases as \( \delta \) and \( \lambda \) increase.
5.2. Type II Exponentiated Half Logistic-Marshall-Olkin-Burr XII (TIEHL-MO-BXII) Distribution

If we let the baseline distribution to be Burr XII distribution with the cdf and pdf given by $G(x; c, k) = 1 - (1 + x^c)^{-k}$, and $g(x; c, k) = kcx^{c-1}(1 + x^c)^{-k-1}$, respectively, for $x > 0$ and $c, k > 0$, then, we obtain the cdf and pdf of the TIEHL-MO-BXII distribution as
\[ F_{TIEHL-MO-BXII}(x; a, \delta, c, k) = 1 - \left[ \frac{\delta(1+x^c)^{-k}}{1 - \delta(1+x^c)^{-k}} \right] \left[ \frac{1 - \delta(1+x^c)^{-k}}{1 - \delta(1+x^c)^{-k}} \right], \]

and

\[ f_{TIEHL-MO-BXII}(x; a, \delta, c, k) = 2a\delta \left[ \frac{\delta(1+x^c)^{-k}}{1 - \delta(1+x^c)^{-k}} \right] \left[ \frac{1 - \delta(1+x^c)^{-k}}{1 - \delta(1+x^c)^{-k}} \right]^{a+1} kC_{x}^{c-1}(1+x^c)^{-k-1}, \]

respectively, for \( x > 0 \) and \( a, \delta, c, k > 0 \).

![Plots of the pdf and hrf for the TIEHL-MO-BXII distribution](image)

**Figure 4:** Plots of the pdf and hrf for the TIEHL-MO-BXII distribution

**Table 2:** Some quantiles for TIEHL-MO-BXII distribution

<table>
<thead>
<tr>
<th>(a, ( \delta, c, k ))</th>
<th>(1, 1.5, 3.3, 0.6)</th>
<th>(0.7, 0.3, 1, 1.5)</th>
<th>(0.4, 1, 1.2, 5)</th>
<th>(1, 0.5, 2, 2, 5)</th>
<th>(0.5, 1, 2, 3, 1.6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>u</td>
<td>0.0544</td>
<td>0.0162</td>
<td>0.0577</td>
<td>0.1628</td>
<td>0.3179</td>
</tr>
<tr>
<td></td>
<td>0.7157</td>
<td>0.0372</td>
<td>0.1351</td>
<td>0.2347</td>
<td>0.4599</td>
</tr>
<tr>
<td></td>
<td>0.8526</td>
<td>0.0656</td>
<td>0.2420</td>
<td>0.2991</td>
<td>0.5919</td>
</tr>
<tr>
<td></td>
<td>0.9895</td>
<td>0.1048</td>
<td>0.3937</td>
<td>0.3639</td>
<td>0.7321</td>
</tr>
<tr>
<td></td>
<td>1.1400</td>
<td>0.1629</td>
<td>0.6177</td>
<td>0.4352</td>
<td>0.8942</td>
</tr>
<tr>
<td></td>
<td>1.3220</td>
<td>0.2546</td>
<td>0.9691</td>
<td>0.5194</td>
<td>1.0967</td>
</tr>
<tr>
<td></td>
<td>1.5665</td>
<td>0.4175</td>
<td>1.5755</td>
<td>0.6272</td>
<td>1.3756</td>
</tr>
<tr>
<td></td>
<td>1.9512</td>
<td>0.7652</td>
<td>2.8163</td>
<td>0.7849</td>
<td>1.8206</td>
</tr>
<tr>
<td></td>
<td>2.7843</td>
<td>1.8747</td>
<td>6.5880</td>
<td>1.0839</td>
<td>2.7914</td>
</tr>
</tbody>
</table>

Plots of the pdf and hrf for the TIEHL-MO-BXII distribution are given in Figure 4. The pdf exhibit right-skewed, left-skewed, almost symmetric and reverse-J shapes. The hrf of the TIEHL-MO-BXII distribution on the other hand displays
uni-modal, upside-down bathtub, bathtub followed by upside-down bathtub, decreasing and increasing shapes. Quantiles for the TIIEHL-MO-BXII distribution for selected parameter values are given in Table 2.

The 3D plots of skewness and kurtosis for the TIIEHL-MO-BXII distribution are given in Figures 5 and 6. From the plots, we can see that the TIIEHL-MO-BXII distribution can model datasets with different levels of skewness and kurtosis.
5.3. Type II Exponentiated Half Logistic-Marshall-Olkin-Burr III (TIIHEHL-MO-BIII) Distribution

When we take the baseline distribution to be Burr III distribution with the cdf and pdf given by
\[ G(x; \alpha, \beta) = (1 + x^{-\beta})^{-\alpha}, \]
and
\[ g(x; \alpha, \beta) = \alpha \beta x^{-\beta-1}(1 + x^{-\beta})^{-\alpha-1}, \]
respectively, for \( \alpha, \beta > 0 \) and \( x > 0 \), then the cdf and pdf of the TIIHEHL-MO-BIII distribution are
\[
F_{\text{TIIHEHL-MO-BIII}}(x; a, \delta, \alpha, \beta) = 1 - \left[ \frac{\delta (1 - (1+x^{-\beta})^{-\alpha})}{1 - \delta (1 - (1+x^{-\beta})^{-\alpha})} \right]^\alpha
\]
and
\[
f_{\text{TIIHEHL-MO-BIII}}(x; a, \delta, \alpha, \beta) = 2 \alpha \delta \left[ \frac{\delta (1 - (1+x^{-\beta})^{-\alpha})}{1 - \delta (1 - (1+x^{-\beta})^{-\alpha})} \right]^{a-1} \times \frac{x^{-\beta-1}(1 + x^{-\beta})^{-\alpha-1}}{[1 - \delta (1 - (1+x^{-\beta})^{-\alpha})]^2},
\]
respectively, for \( x > 0 \) and \( a, \delta, \alpha, \beta > 0 \). Plots of the pdf and hrf are given in Figure 7.

In Figure 7, the pdf of the TIIHEHL-MO-BIII distribution displays right-skewed, left-skewed, almost symmetric, and reverse-J shapes, whereas the hrf depicts upside-down bathtub, bathtub, decreasing and increasing shapes. Table 3 gives quantiles for the TIIHEHL-MO-BIII distribution for selected parameter values.
Table 3: Some quantiles for TIIEHL-MO-BIII distribution

<table>
<thead>
<tr>
<th>( u )</th>
<th>((1, 1.5, 0.3, 1.1))</th>
<th>((0.7, 1, 1.5, 2))</th>
<th>((0.4, 1, 1.25))</th>
<th>((2.1, 0.5, 1.9, 0.5))</th>
<th>((1.5, 0.3, 1.2, 0.4))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.0007</td>
<td>0.4651</td>
<td>0.4690</td>
<td>0.0125</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.2</td>
<td>0.0039</td>
<td>0.6427</td>
<td>0.6743</td>
<td>0.0313</td>
<td>0.0006</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0140</td>
<td>0.8101</td>
<td>0.8767</td>
<td>0.0579</td>
<td>0.0012</td>
</tr>
<tr>
<td>0.4</td>
<td>0.0368</td>
<td>0.9924</td>
<td>1.1083</td>
<td>0.0956</td>
<td>0.0034</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0812</td>
<td>1.2105</td>
<td>1.4023</td>
<td>0.1513</td>
<td>0.0076</td>
</tr>
<tr>
<td>0.6</td>
<td>0.1636</td>
<td>1.4962</td>
<td>1.8154</td>
<td>0.2383</td>
<td>0.0165</td>
</tr>
<tr>
<td>0.7</td>
<td>0.3192</td>
<td>1.9139</td>
<td>2.4757</td>
<td>0.3881</td>
<td>0.0379</td>
</tr>
<tr>
<td>0.8</td>
<td>0.6489</td>
<td>2.6385</td>
<td>3.7620</td>
<td>0.6945</td>
<td>0.1033</td>
</tr>
<tr>
<td>0.9</td>
<td>1.6283</td>
<td>4.4289</td>
<td>7.5689</td>
<td>1.6232</td>
<td>0.4470</td>
</tr>
</tbody>
</table>

6. Simulation Study

In this section, we present simulation results for the TIIEHL-MO-W distribution. We conducted various simulations for different sample sizes and for different parameter values. Equation (15) was used to generate random samples from TIIEHL-MO-W distribution via the R package. The simulation was repeated \( N = 1000 \) times each with sample sizes \( n = 50, 100, 200, 400, 800, 1000 \) as given in Tables 4 and Table 5. For simulations we considered only the MLE method.

The estimated mean, average bias (ABIAS) and root mean square errors (RMSEs) of the parameter say, \( \hat{\zeta} \), are computed as \( \text{Mean} = \frac{\sum_{i=1}^{N} \hat{\zeta}_{i}}{N} \), \( \text{ABIAS} (\hat{\zeta}) = \frac{\sum_{i=1}^{N} (\hat{\zeta}_{i} - \zeta)}{N} \), and \( \text{RMSE} = \sqrt{\frac{\sum_{i=1}^{N} (\hat{\zeta}_{i} - \zeta)^2}{N}} \), respectively.

Tables 4 and 5 list the mean MLEs of the parameters along with the respective RMSEs and average bias. From the results, one can see that as the sample size \( n \) increases, the mean estimates of the parameters approximate the true parameter values while the RMSEs decreases and average bias decay towards zero. Therefore, we can conclude that this model gives consistent estimates.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>n</th>
<th>(0.1, 1.4)</th>
<th>(0.1, 2.0, 1.4)</th>
<th>(0.1, 1.2, 2.8)</th>
<th>(0.2, 1.0, 1.5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>50</td>
<td>0.1517</td>
<td>0.1558</td>
<td>0.1489</td>
<td>0.1463</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>0.1403</td>
<td>0.1353</td>
<td>0.1343</td>
<td>0.1211</td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>0.1232</td>
<td>0.0891</td>
<td>0.1208</td>
<td>0.0955</td>
</tr>
<tr>
<td></td>
<td>400</td>
<td>0.1108</td>
<td>0.0539</td>
<td>0.1065</td>
<td>0.0391</td>
</tr>
<tr>
<td></td>
<td>800</td>
<td>0.1039</td>
<td>0.0265</td>
<td>0.1023</td>
<td>0.0205</td>
</tr>
<tr>
<td></td>
<td>1000</td>
<td>0.1030</td>
<td>0.0229</td>
<td>0.1013</td>
<td>0.0144</td>
</tr>
<tr>
<td>δ</td>
<td>50</td>
<td>4.1836</td>
<td>5.4966</td>
<td>2.1836</td>
<td>2.7128</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>3.6748</td>
<td>4.7746</td>
<td>1.6748</td>
<td>2.2519</td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>2.9136</td>
<td>3.1937</td>
<td>0.9136</td>
<td>1.8661</td>
</tr>
<tr>
<td></td>
<td>400</td>
<td>2.4350</td>
<td>2.0052</td>
<td>0.4350</td>
<td>1.4230</td>
</tr>
<tr>
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<td>2.1648</td>
<td>1.0374</td>
<td>0.1648</td>
<td>1.2804</td>
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<td>0.8924</td>
<td>0.1132</td>
<td>1.2411</td>
</tr>
<tr>
<td>λ</td>
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<td>0.2390</td>
<td>-0.0469</td>
<td>2.6948</td>
</tr>
<tr>
<td></td>
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<td>-0.0367</td>
<td>2.7342</td>
</tr>
<tr>
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<td>2.7559</td>
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<tr>
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<td>2.7848</td>
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<td>1000</td>
<td>1.3958</td>
<td>0.0695</td>
<td>-0.0042</td>
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</tr>
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</table>
Table 5: Simulation Results for TIEHL-MO-W Distribution: Mean, RMSE and ABIAS

<table>
<thead>
<tr>
<th>Parameter</th>
<th>(a)</th>
<th>(\delta)</th>
<th>(\lambda)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(n)</td>
<td>Mean</td>
<td>RMSE</td>
<td>ABIAS</td>
</tr>
<tr>
<td>50</td>
<td>0.1525</td>
<td>0.2619</td>
<td>0.0525</td>
</tr>
<tr>
<td>100</td>
<td>0.1169</td>
<td>0.1314</td>
<td>0.0160</td>
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<tr>
<td>200</td>
<td>0.1039</td>
<td>0.0219</td>
<td>0.0039</td>
</tr>
<tr>
<td>400</td>
<td>0.1014</td>
<td>0.0147</td>
<td>0.0014</td>
</tr>
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<td>800</td>
<td>0.1005</td>
<td>0.0101</td>
<td>0.0005</td>
</tr>
<tr>
<td>1000</td>
<td>0.1001</td>
<td>0.0092</td>
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</tr>
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</table>

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7. Applications

We give three examples of applications to show the flexibility and applicability of the TIEHL-MO-G family of distributions in this section. The TIEHL-MO-W distribution is compared with other existing models including the type II exponentiated half logistic-Weibull (TIEHLW) distribution by Al-Mofleh et al. (2020), the Marshall-Olkin extended inverse Weibull (MOIW) distribution by Pakungwati et al. (2018), Marshall-Olkin extended Fréchet (MOEFr) and Marshall-Olkin extended generalized exponential (MOEGE) distributions by Barreto-Souza et al. (2013), the new Marshall-Olkin Weibull (NMOW) distribution by Cui et al. (2020), Marshall-Olkin type II Topp-Leone Weibull (MOTIITLW) distribution by Ran-non et al. (2022) and Weibull Exponential (WE) distribution (Oguntunde et al., 2015). The pdfs of the above distributions are

\[
f_{\text{TIEHLW}}(x; \lambda, a, \gamma) = 2a\lambda\gamma x^{\gamma - 1}e^{-x^\gamma}(1 - e^{-x^\gamma})^{\lambda - 1}\left[1 - (1 - e^{-x^\gamma})^{\lambda}\right]^{a - 1}\left[1 + (1 - e^{-x^\gamma})\right]^{a + 1},
\]
for \(x > 0, \lambda, a, \gamma > 0\),

\[
f_{\text{MOIW}}(x; \alpha, \lambda, \theta) = \frac{\alpha\lambda\theta^{\lambda}x^{-\lambda}e^{-(\theta x)^\lambda}}{[\alpha - (\alpha - 1)e^{-(\theta x)^\lambda}]^2},
\]
for \(x > 0, \alpha, \lambda, \theta > 0\),

\[
f_{\text{MOEFr}}(x; \alpha, \delta, \lambda) = \frac{\alpha\delta^{\lambda}x^{-(\lambda + 1)}e^{-(\frac{\lambda}{\delta})^\lambda}}{\left[1 - \delta [1 - e^{-(\frac{\lambda}{\delta})^\lambda}]\right]^2},
\]
for \(x > 0, \alpha, \delta, \lambda > 0\),

\[
f_{\text{MOEGE}}(x; \alpha, \gamma, \lambda) = \frac{\alpha\gamma\lambda e^{-\lambda x}(1 - e^{-\lambda x})^{\gamma - 1}}{\left[1 - \alpha(1 - e^{-\lambda x})\gamma\right]^2},
\]
for \(x > 0, \alpha, \gamma, \lambda > 0\),

\[
f_{\text{NMOW}}(x; \beta, \lambda, \theta) = \frac{\theta\lambda\beta x^{\beta - 1}e^{-\lambda x^\beta}}{(\theta + (1 - \theta)e^{-\lambda x^\beta})^2},
\]
for \(x > 0, \delta, b, \lambda > 0\),

\[
f_{\text{MOTIITLW}}(x; \delta, b, \lambda) = \frac{2\delta b\lambda x^{\lambda - 1}e^{-x^\lambda}(1 - e^{-x^\lambda})[1 - (1 - e^{-x^\lambda})^2]^b - 1}{(1 - \delta [1 - (1 - e^{-x^\lambda})^2]^b)^2},
\]
for \(x > 0, \delta, \lambda, \theta > 0\), and

\[
f_{\text{WE}}(x; \alpha, \beta, \lambda) = \alpha\beta(\lambda e^{-\lambda x})^{\beta - 1}\left[\frac{(1 - e^{-\lambda x})^{\beta - 1}}{(e^{-\lambda x})^{\beta + 1}}\right] \exp\left\{-\alpha \left(\frac{1 - e^{-\lambda x}}{e^{-\lambda x}}\right)\beta\right\},
\]
for \(x > 0, \alpha, \beta, \lambda > 0\).
The maximum likelihood estimates (MLE’s) of the parameters of the TIIEHL-MO-W distribution with standard errors in parentheses are given in Tables 6 and 7, respectively. The goodness-of-fit statistics including -2log-likelihood statistic (-2log(L)), Akaike Information Criterion (AIC) by Akaike (1974), Consistent Akaike Information Criterion (AICC) (Bozdogan, 1987), Bayesian Information Criterion (BIC) (Schwarz, 1978), Cramér-von Mises ($W^*$) and Anderson-Darling ($A^*$) statistic by Chen & Balakrishnan (1995), Kolmogorov-Smirnov (K-S) by Chakravarti et al. (1967) statistic and its p-value are also presented in the tables.

Plots of the fitted densities, the histogram of the data, probability plots, estimated cdf, Kaplan-Meier (K-M) survival plots, Total Time on Test (TTT) and estimated hazard rate function (hrf) plots are given in Figures 8, 9, 10, 11, 12, 13, 14, 15 and 16, respectively. For the probability plots, we plotted $F(x_{(j)}; \hat{a}, \hat{\delta}, \hat{\lambda})$ against $j - 0.375 \frac{j - 1}{n + 0.25}$, $j = 1, 2, \ldots, n$, where $x_{(j)}$ are the ordered values of the observed data. The measure of closeness to the diagonal line is given by the sum of squares (SS):

$$SS = \sum_{j=1}^{n} \left[ F(x_{(j)}; \hat{a}, \hat{\delta}, \hat{\lambda}) - \left( \frac{j - 0.375}{n + 0.25} \right) \right]^2,$$

(see Chambers et al. 1983 for additional details).

7.1. Remission Times for Cancer Patients

The first data consists of remission times for 128 cancer patients Lee & Wang (2003). The data are: 0.08, 4.98, 25.74, 3.7, 10.06, 2.69, 7.62, 1.26, 7.87, 4.4, 2.02, 21.73, 2.09, 6.97, 0.5, 5.17, 14.77, 4.18, 10.75, 2.83, 11.64, 5.85, 3.31, 2.07, 3.48, 9.02, 2.46, 7.28, 32.15, 3.15, 16.62, 4.33, 17.36, 8.26, 4.51, 3.36, 4.87, 13.29, 3.64, 9.74, 2.64, 7.59, 43.01, 5.49, 1.4, 11.98, 6.54, 6.93, 6.94, 0.4, 5.09, 14.76, 3.88, 10.66, 1.19, 7.66, 3.02, 19.13, 8.53, 8.65, 8.66, 2.26, 7.26, 26.31, 5.32, 15.96, 2.75, 11.25, 4.34, 1.76, 12.03, 12.63, 13.11, 3.57, 9.47, 0.81, 7.39, 36.66, 4.26, 17.14, 5.71, 3.25, 20.28, 22.69, 23.63, 5.06, 14.24, 2.62, 10.34, 1.05, 5.41, 79.05, 7.93, 4.5, 2.02, 0.2, 7.09, 25.82, 3.82, 14.83, 2.69, 7.63, 1.35, 11.79, 6.25, 3.36, 2.23, 9.22, 0.51, 5.32, 34.26, 4.23, 17.12, 2.87, 18.1, 8.37, 6.76, 3.52, 13.8, 2.54, 7.32, 0.9, 5.41, 46.12, 5.62, 1.46, 12.02, 12.07.

The estimated variance-covariance matrix of the TIIEHL-MO-W distribution for remission times data is given by

$$
\begin{pmatrix}
0.0151 & 0.5659 & -0.0131 \\
0.5659 & 31.3974 & -0.4617 \\
-0.0131 & -0.4617 & 0.0121
\end{pmatrix}.
$$

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Table 6: Estimates of models for remission times of cancer patients

<table>
<thead>
<tr>
<th>Model</th>
<th>( a )</th>
<th>( \delta )</th>
<th>( \lambda )</th>
<th>(-2 \log L)</th>
<th>( AIC )</th>
<th>( AICC )</th>
<th>( BIC )</th>
<th>( W^* )</th>
<th>( A^* )</th>
<th>( K-S )</th>
<th>( P\text{-value} )</th>
<th>( SS )</th>
</tr>
</thead>
<tbody>
<tr>
<td>THEHLMO-W</td>
<td>0.2664</td>
<td>11.8913</td>
<td>0.7787</td>
<td>820.51</td>
<td>826.51</td>
<td>826.70</td>
<td>835.07</td>
<td>0.0357</td>
<td>0.2230</td>
<td>0.0462</td>
<td>0.9474</td>
<td>0.0403</td>
</tr>
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<td></td>
<td>(0.1231)</td>
<td>(5.6163)</td>
<td>(0.1098)</td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>THEHLW</td>
<td>4.2347</td>
<td>0.2311</td>
<td>0.7508</td>
<td>831.51</td>
<td>837.51</td>
<td>837.71</td>
<td>846.07</td>
<td>0.0608</td>
<td>0.3947</td>
<td>0.4979</td>
<td>2.2000 \times 10^{-16}</td>
<td>14.8749</td>
</tr>
<tr>
<td></td>
<td>(1.9754)</td>
<td>(0.1451)</td>
<td>(0.1865)</td>
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<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>MOIW</td>
<td>8.4603 \times 10^{+03}</td>
<td>1.7246 \times 10^{+00}</td>
<td>3.1093 \times 10^{+01}</td>
<td>822.91</td>
<td>828.91</td>
<td>829.10</td>
<td>837.46</td>
<td>9.4500</td>
<td>44.4472</td>
<td>0.9999</td>
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</tr>
<tr>
<td></td>
<td>(2.6331 \times 10^{-02})</td>
<td>(1.2759 \times 10^{-01})</td>
<td>(1.2429 \times 10^{+01})</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MOEFr</td>
<td>2.2466</td>
<td>1.6906</td>
<td>0.8855</td>
<td>880.39</td>
<td>886.39</td>
<td>886.59</td>
<td>894.95</td>
<td>0.6045</td>
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</tr>
<tr>
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<td>(3.5348)</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MOEGE</td>
<td>4.5374</td>
<td>1.6511</td>
<td>0.0708</td>
<td>820.69</td>
<td>826.69</td>
<td>826.88</td>
<td>835.24</td>
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<td>0.6684</td>
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<td>(3.6057)</td>
<td>(0.2108)</td>
<td>(0.0269)</td>
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<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>NMOW</td>
<td>1.0558</td>
<td>0.0904</td>
<td>1.0308</td>
<td>828.06</td>
<td>834.06</td>
<td>834.25</td>
<td>842.61</td>
<td>0.1296</td>
<td>0.7769</td>
<td>0.0703</td>
<td>0.5521</td>
<td>0.1495</td>
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<tr>
<td></td>
<td>(0.2268)</td>
<td>(0.0981)</td>
<td>(0.9473)</td>
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<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>MOITTLW</td>
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<td>0.3389</td>
<td>0.7487</td>
<td>824.13</td>
<td>830.13</td>
<td>830.33</td>
<td>838.69</td>
<td>0.0867</td>
<td>0.5229</td>
<td>0.0644</td>
<td>0.6639</td>
<td>0.0865</td>
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<tr>
<td></td>
<td>(4.2745)</td>
<td>(0.7062)</td>
<td>(0.1213)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>WE</td>
<td>1.8100 \times 10^{+03}</td>
<td>1.0468 \times 10^{+00}</td>
<td>8.0600 \times 10^{-05}</td>
<td>828.21</td>
<td>1834.21</td>
<td>834.41</td>
<td>842.77</td>
<td>0.1316</td>
<td>0.7877</td>
<td>0.0706</td>
<td>0.5453</td>
<td>0.1524</td>
</tr>
<tr>
<td></td>
<td>(8.841 \times 10^{-13})</td>
<td>(3.5088 \times 10^{-09})</td>
<td>(6.8000 \times 10^{-06})</td>
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</tbody>
</table>
In Table 6 above, the values of the goodness-of-fit statistics: AIC, AICC and BIC are smallest for the TIIEHL-MO-W distribution as compared to the models listed in the table. Also, the value of SS from the probability plots and the values of the goodness-of-fit statistics: $A^*$, $W^*$ and K-S are smallest for the TIIEHL-MO-W distribution, which indeed showed that this model fits remission times of cancer patients data well. The value of K-S p-value for the TIIEHL-MO-W distribution is largest compared to the distributions.

Figures 9 and 10 shows the estimated cdf, K-M survival, TTT-transform and estimated hrf plots for remission times data set. The cdf line for TIIEHL-MO-W distribution is very close to the empirical cdf while the survival function for
TIIEHL-MO-W distribution is also closer to the K-M curve which shows that indeed our model is the best in explaining remission times data. TTT-transform plot indicates a uni-modal hazard rate function for the remission times data.

7.2. Time to Failure (in Hours) of 59 Test Conductors of 400 Micrometer Length


The estimated variance-covariance matrix of TIIEHL-MO-W distribution for time to failure of test conductors data is given by

\[
\begin{bmatrix}
9.3767 \times 10^{-02} & 6.2245 \times 10^{-06} & -1.5292 \times 10^{-02} \\
6.2245 \times 10^{-06} & 4.1335 \times 10^{-10} & -1.0210 \times 10^{-06} \\
-1.5292 \times 10^{-02} & -1.0210 \times 10^{-06} & 2.7401 \times 10^{-03}
\end{bmatrix}
\]
Table 7: Estimates of models for time to failure of 59 test conductors

<table>
<thead>
<tr>
<th>Model</th>
<th>Estimates</th>
<th>Statistics</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$a$</td>
<td>$\delta$</td>
</tr>
<tr>
<td>THEHL-MO-W</td>
<td>$7.0507 \times 10^{-01}$</td>
<td>$4.0874 \times 10^{03}$</td>
</tr>
<tr>
<td></td>
<td>$3.0613 \times 10^{-01}$</td>
<td>$2.0317 \times 10^{-05}$</td>
</tr>
<tr>
<td>THEHLW</td>
<td>$0.7276$</td>
<td>$0.0664$</td>
</tr>
<tr>
<td></td>
<td>$(0.4275)$</td>
<td>$(0.0245)$</td>
</tr>
<tr>
<td>MOIW</td>
<td>$1.4673 \times 10^{+03}$</td>
<td>$7.5117 \times 10^{+00}$</td>
</tr>
<tr>
<td></td>
<td>$(1.0752 \times 10^{-05})$</td>
<td>$(8.1356 \times 10^{-01})$</td>
</tr>
<tr>
<td>MOEFr</td>
<td>$0.0004$</td>
<td>$68.3500$</td>
</tr>
<tr>
<td></td>
<td>$(0.0003)$</td>
<td>$(0.0001)$</td>
</tr>
<tr>
<td>MOEGE</td>
<td>$0.0005$</td>
<td>$1.0359$</td>
</tr>
<tr>
<td></td>
<td>$(0.0004)$</td>
<td>$(0.0002)$</td>
</tr>
<tr>
<td>NMOW</td>
<td>$4.6541 \times 10^{+00}$</td>
<td>$8.0650 \times 10^{-05}$</td>
</tr>
<tr>
<td></td>
<td>$(1.7883 \times 10^{-09})$</td>
<td>$(0.0981 \times 10^{-05})$</td>
</tr>
<tr>
<td>MOTITLW</td>
<td>$296.3100$</td>
<td>$0.5294$</td>
</tr>
<tr>
<td></td>
<td>$(0.0004)$</td>
<td>$(0.1892)$</td>
</tr>
<tr>
<td>WE</td>
<td>$5.4349 \times 10^{+02}$</td>
<td>$4.2393 \times 10^{+00}$</td>
</tr>
<tr>
<td></td>
<td>$(5.5751 \times 10^{-05})$</td>
<td>$(3.6954 \times 10^{-01})$</td>
</tr>
</tbody>
</table>
Table 7, shows that the values of the goodness-of-fit statistics: AIC, AICC and BIC are the smallest for the TIEHLMO-W distribution as compared to the models listed in the table. The value of SS from the probability plots and the values of the goodness-of-fit statistics: $A^*$, $W^*$ and K-S are smallest for the TIEHLMO-W distribution, and the K-S p-value is closer to 1, hence the TIEHLMO-W distribution is a better fit for failure times of 59 test conductors data.

Figures 12 and 13 above gives the estimated cdf, K-M survival plot, TTT-transform and estimated hrf plots for time to failure of test conductors data set. The cdf line for TIEHLMO-W distribution is close to the empirical cdf, while the survival function for TIEHLMO-W distribution is also closer to the K-M curve which
shows that our model is better in explaining time to failure of test conductors data. TTT-transform plot is concave indicating an increasing hrf for time to failure of test conductors data.

7.3. COVID-19 Data in Canada


The estimated variance-covariance matrix of TIIEHL-MO-W distribution for Europe COVID-19 data is given by

\[
\begin{bmatrix}
0.0491 & 1.0651 & -0.0383 \\
1.0651 & 2966.7550 & 1.8663 \\
-0.0383 & 1.8663 & 0.0355
\end{bmatrix}.
\]
Table 8: Estimates of models for Canada COVID-19 data

<table>
<thead>
<tr>
<th>Model</th>
<th>Estimates</th>
<th>Statistics</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>-2 logL</td>
</tr>
<tr>
<td>THEHL-MO-W</td>
<td>0.4071</td>
<td>115.0155</td>
</tr>
<tr>
<td></td>
<td>(0.2215)</td>
<td>(54.4679)</td>
</tr>
<tr>
<td>TIEHLW</td>
<td>25.7059</td>
<td>0.7364</td>
</tr>
<tr>
<td></td>
<td>(6.0515)</td>
<td>(0.3688)</td>
</tr>
<tr>
<td>MOIW</td>
<td>2.6234x10^-05</td>
<td>0.5908</td>
</tr>
<tr>
<td></td>
<td>(4.0461x10^-05)</td>
<td>(8.3759x10^-05)</td>
</tr>
<tr>
<td>MOEFr</td>
<td>0.0001</td>
<td>7.5237</td>
</tr>
<tr>
<td></td>
<td>(0.0001)</td>
<td>(0.0001)</td>
</tr>
<tr>
<td>MOEGE</td>
<td>43.7075</td>
<td>12.3057</td>
</tr>
<tr>
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<td>(75.0656)</td>
<td>(4.9945)</td>
</tr>
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<td>NMOW</td>
<td>1.7131</td>
<td>0.4040</td>
</tr>
<tr>
<td></td>
<td>(0.3487)</td>
<td>(0.2864)</td>
</tr>
<tr>
<td>MOTITILW</td>
<td>17.9001</td>
<td>0.4802</td>
</tr>
<tr>
<td></td>
<td>(27.3796)</td>
<td>(0.6804)</td>
</tr>
<tr>
<td>WE</td>
<td>24830.0000</td>
<td>3.2218</td>
</tr>
<tr>
<td></td>
<td>(4.6583x10^-07)</td>
<td>(0.3415)</td>
</tr>
</tbody>
</table>
The values of the goodness-of-fit statistics: AIC, AICC and BIC are the least for TIIEHL-MO-W distribution in Table 8 as compared to the other models listed. The value of SS from the probability plots and the values of the goodness-of-fit statistics: $A^*$, $W^*$ and K-S are smallest for TIIEHL-MO-W distribution, and the K-S p-value is largest for the new distribution. Therefore, the TIIEHL-MO-W distribution is a good fit for Canada COVID-19 data.

The cdf line for TIIEHL-MO-W distribution is superimposed on the empirical cdf, while the survival function for TIIEHL-MO-W distribution is also closer to the K-M curve which shows that our model is the better in explaining Canada COVID-19 data. TTT-transform plot indicates an increasing lrf for Canada COVID-19 data.

Figure 14: Fitted densities and probability plots for Canada COVID-19 data

Figure 15: Estimated cdf and K-M survival plots for Canada COVID-19 data
8. Concluding Remarks

A new family of distributions referred to as the type II Exponentiated Half-Logistic-Marshall-Olkin-G distribution was introduced. Some statistical properties of the new family of distributions were presented. Three methods of estimation namely, maximum likelihood, ordinary and weighted least squares were used to estimate model parameters. The method of maximum likelihood was implemented in full. We also studied special cases of the new family of distributions including the Type II Exponentiated Half Logistic-Marshall-Olkin-Weibull, the Type II Exponentiated Half Logistic-Marshall-Olkin-Burr XII and the Type II Exponentiated Half Logistic-Marshall-Olkin-Burr III distributions. The new family of distributions can model data with heavy tails with different levels of skewness and kurtosis. The usefulness of the new family of distributions was examined by means of applications to three real data sets.

In future research, we will also like to consider other estimation techniques including Bayesian method. Bayesian techniques are of interest because they are easily dealt with for almost all parametric techniques and are increasingly used in global health modeling. This method can also be used for estimating the parameters of the proposed family of distributions.

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References


**Appendix**

https://drive.google.com/file/d/10EFbhtqNfXTIs5iSnhk_Wq90nxuezFpX/view?usp=sharing