

## Some Improved Combined Estimators of Population Mean in Stratified Ranked Set Sampling

Algunos estimadores combinados mejorados de la media de la población en el muestreo de conjuntos clasificados estratificados

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### Abstract

This paper presents improved population mean estimators using auxiliary variables in Stratified Ranked Set Sampling. We have derived the expressions for bias and mean square errors up to the first order of approximation and shown that the proposed estimators under optimum conditions are more efficient than other estimators taken in this paper. In an attempt to verify the efficiencies of proposed estimators, theoretical results are supported by numerical illustrations and simulation study for which we have considered two populations.

**Key words:** Auxiliary variable; Bias; Mean square error; Ranked set sampling; Study variable.

### Resumen

Este artículo presenta estimadores mejorados de la media de la población utilizando variables auxiliares en el muestreo de conjuntos ordenados estratificados. Hemos derivado las expresiones para el sesgo y los errores cuadráticos medios hasta el primer orden de aproximación y hemos demostrado que los estimadores propuestos en condiciones óptimas son más eficientes que otros estimadores tomados en este artículo. En un intento por verificar las eficiencias de los estimadores propuestos, los resultados teóricos están respaldados por ilustraciones numéricas y estudios de simulación para los cuales hemos considerado dos poblaciones.

**Palabras clave:** Error cuadrático medio; Inclinación; Muestreo de conjuntos clasificados; Variable de estudio, Variable auxiliar.

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## 1. Introduction

In theory of sampling it is evident that suitable use of auxiliary information improves the efficiency of the estimator. These auxiliary information may be used either at the design phase or the estimation phase or at both phases. [Cochran \(1977\)](#) was the first to introduce a ratio estimator of Population Mean using auxiliary information. [Shabbir & Gupta \(2007\)](#), [Koyuncu & Kadilar \(2009\)](#) and [Chaudhary et al. \(2009\)](#) have considered the problem of estimating population mean taking into consideration information on auxiliary variable.

When population is heterogeneous stratified random sampling (SSRS) is used for better accuracy. Several authors like [Kadilar & Cingi \(2003\)](#), [Shabbir & Gupta \(2006\)](#) and [Haq & Shabbir \(2013\)](#) have proposed estimators in stratified random sampling using information on a single auxiliary variable. [Singh & Kumar \(2012\)](#) have proposed improved estimators of population mean using two auxiliary variables in stratified random sampling. Recently, [Muneer et al. \(2017\)](#) have proposed family of chain exponential estimators in SSRS.

Ranked set sampling (RSS) is an improved sampling method over Simple Random Set Sampling (SRS). [McIntyre \(1952\)](#) was the first to explain RSS for estimating the population means. [Takahasi & Wakimoto \(1968\)](#) gave the necessary mathematical theory of RSS. [Samawi & Muttlak \(1996\)](#) suggested ratio estimators of population mean in RSS and showed that the RSS estimators gave improved results over their SRS counterparts. [Ganeslingam & Ganesh \(2006\)](#) compared RSS with SRS for estimation of the unknown mean of study variable and the ratio of study variable to auxiliary variable. He concluded that RSS gives a better estimate for both the mean and the ratio. [Singh et al. \(2014\)](#) suggested a general procedure for estimating the population mean using RSS. [Al-Omari & Bouza \(2014\)](#) and [Bouza et al. \(2018\)](#) provided a review of RSS, its modification, and its application. Stratified ranked set sampling (SRSS) was first introduced by [Samawi & Muttlak \(1996\)](#) for estimation of population mean. [Samawi & Siam \(2003\)](#) have proposed the combined and the separate ratio estimators in SRSS.

Following is how the rest of the article is organized. The SRSS sampling approach is presented in section 2. Section 3 examines a review of existing works. In section 4 and 5, the proposed estimators and their characteristics are discussed. Numerical illustrations are provided in section 6. In section 7, using a simulated populations, the effectiveness of proposed estimators is compared to that of existing estimators. Section 8 draws the conclusion.

## 2. Sampling Methodology

In ranked set sampling (RSS), we rank randomly selected units from the population merely by observation or prior experience after which only a few of these sampled units are measured. In RSS,  $k$  independent random sets each of size  $k$  are selected from the population and each unit in the set is being selected with equal probability (SRS). The members of each random set are ranked with respect to the characteristic of the auxiliary variable. Then the smallest unit is selected from the

first ordered set and the second smallest unit is selected from the second ordered set. By this way, this procedure is continued until the largest rank is chosen from the  $k^{th}$  set. This cycle may be repeated  $r$  times, so  $r(k = n)$  units have been measured during this process.

SRSS takes the following steps.

- Step 1: Select  $k_h^2$  bivariate sample units randomly from the  $h^{th}$  stratum of the population.
- Step 2: Arrange these selected units randomly into  $k_h$  sets, each of size  $k_h$ .
- Step 3: The procedure of ranked set sampling (RSS) is then applied, on each of the sets to obtain the  $k_h$  ranked set sample unit. Here ranking is done with respect to the auxiliary variable  $X_h$ .
- Step 4: Repeat the above steps  $r$  times for each stratum to get the desired sample of size  $n_h = k_h r$ .

Consider a finite population  $U = (U_1, U_2, \dots, U_N)$  based on  $N$  identifiable units with a study variable  $Y$  and auxiliary variables  $X$  associated with each unit  $U_i$ ,  $i = 1, 2, \dots, N$  of the population. Let the population be divided into  $L$  disjoint strata with stratum  $h$  based on  $N_h$ ,  $h = 1, 2, \dots, L$  units.

Let  $(Y_{h[1]j}, X_{h(1)j}), (Y_{h[2]j}, X_{h(2)j}), \dots, (Y_{h[k_h]j}, X_{h(k_h)j})$  be the stratified ranked set sample for  $j^{th}$ ,  $j = 1, 2, \dots, r$  cycle in  $h^{th}$  stratum.

$$\text{Let } \bar{y}_{SRSS} = \sum_{h=1}^L W_h \bar{y}_{h[rss]} \text{ and } \bar{x}_{SRSS} = \sum_{h=1}^L W_h \bar{x}_{h(rss)}$$

respectively be the stratified ranked set sample means corresponding to the population means  $\bar{Y} = \sum_{h=1}^L W_h \bar{Y}_h$  and  $\bar{X} = \sum_{h=1}^L W_h \bar{X}_h$  of variables  $Y$  and  $X$ , where  $W_h = \frac{N_h}{N}$  is the weight in stratum  $h$ .

Let  $\bar{y}_{h[rss]} = \sum_{i=1}^{k_h} \sum_{j=1}^r \frac{Y_{h[i]j}}{k_h r}$  and  $\bar{x}_{h(rss)} = \sum_{i=1}^{k_h} \sum_{j=1}^r \frac{X_{h[i]j}}{k_h r}$  be the stratified ranked set sample means corresponding to the population means  $\bar{Y}_h = \sum_{j=1}^{N_h} \frac{Y_{h[i]j}}{N_h}$  and  $\bar{X}_h = \sum_{j=1}^{N_h} \frac{X_{h(i)j}}{N_h}$  of variables  $Y$  and  $X$  in stratum  $h$ .

Let  $s_{yh}^2 = \frac{1}{n_h - 1} \sum_{h=1}^L (Y_{h[i]} - \bar{y}_{h[rss]})^2$ ,  $s_{xh}^2 = \frac{1}{n_h - 1} \sum_{h=1}^L (X_{h(i)} - \bar{x}_{h(rss)})^2$  and  $s_{xyh} = \frac{1}{n_h - 1} \sum_{h=1}^L (Y_{h[i]} - \bar{y}_{h[rss]})(X_{h(i)} - \bar{x}_{h(rss)})$ , respectively be the sample variances and covariances corresponding to the population variances and covariances  $S_{yh}^2 = \frac{1}{N_h - 1} \sum_{h=1}^L (Y_{h[i]} - \bar{Y}_h)^2$ ,  $S_{xh}^2 = \frac{1}{N_h - 1} \sum_{h=1}^L (X_{h(i)} - \bar{X}_h)^2$  and  $S_{xyh} = \frac{1}{N_h - 1} \sum_{h=1}^L (Y_{h[i]} - \bar{Y}_h)(X_{h(i)} - \bar{X}_h)$  in the stratum  $h$ .

Let  $C_{yh}$  and  $C_{xh}$  respectively be the population coefficient of variation of variables  $Y$  and  $X$ .

### 3. Existing Estimators

The conventional combined estimator of the population mean  $\bar{Y}$  under SRSS is given by

$$t^c = \bar{y}_{SRSS} \tag{1}$$

The variance of the estimator  $t^c$  is given by

$$Var(t^c) = \bar{Y}^2 V_{20} \quad (2)$$

The classical combined ratio estimator of the population mean  $\bar{Y}$  given by Samawi & Siam (2003) under SRSS is defined as

$$t_r^c = \bar{y}_{SRSS} \frac{\bar{X}}{\bar{x}_{SRSS}} \quad (3)$$

The Mean Squared Error (MSE) of the estimator  $t_r^c$  is given by

$$MSE(t_r^c) = \bar{Y}^2 [V_{20} + V_{02} - 2V_{11}] \quad (4)$$

The classical combined regression estimator of the population mean  $\bar{Y}$  under SRSS is given as

$$t_{lr}^c = \bar{y}_{SRSS} + \beta(\bar{X} - \bar{x}_{SRSS}) \quad (5)$$

The Mean Squared Error (MSE) of the estimator  $t_{lr}^c$  is given by

$$MSE(t_{lr}^c) = \bar{Y}^2 V_{20} + \beta^2 \bar{X}^2 V_{02} - 2\beta \bar{Y} \bar{X} V_{11} \quad (6)$$

where  $\beta$  is the regression coefficient of  $Y$  on  $X$ .

## 4. Proposed Estimators

Motivated by Bhushan et al. (2020), we suggest some estimators of the population mean  $\bar{Y}$  using SRSS as

$$t_{p1}^c = \bar{y}_{SRSS} \exp\left(\alpha_1 \left(\frac{\bar{x}_{SRSS}}{\bar{X}} - 1\right)\right) \quad (7)$$

$$t_{p2}^c = \bar{y}_{SRSS} \exp\left(\alpha_2 \log \frac{\bar{x}_{SRSS}}{\bar{X}}\right) \quad (8)$$

where  $\alpha_1$  and  $\alpha_2$  are constants such that MSE of the estimators is minimum.

**Proposition 1.** *The Bias, MSE and min MSE of the proposed estimators are*

$$Bias(t_{p1}^c) = \bar{Y} \left( \frac{\alpha_1^2}{2} V_{02} + \alpha_1 V_{11} \right)$$

$$Bias(t_{p2}^c) = \bar{Y} \left( \frac{(\alpha_2^2 - \alpha_2)}{2} V_{02} + \alpha_2 V_{11} \right)$$

$$MSE(t_{p1}^c) = \bar{Y}^2 (V_{20} + \alpha_1^2 V_{02} + 2\alpha_1 V_{11})$$

$$MSE(t_{p2}^c) = \bar{Y}^2 (V_{20} + \alpha_2^2 V_{02} + 2\alpha_2 V_{11})$$

$$MinMSE(t_{p1}^c) = \bar{Y}^2 \left( V_{20} - \frac{V_{11}^2}{V_{02}} \right)$$

$$MinMSE(t_{p2}^c) = \bar{Y}^2 \left( V_{20} - \frac{V_{11}^2}{V_{02}} \right)$$

**Proof.** Outline of the derivations are given in Appendix. □

### 5. Some Other Proposed Estimators

We propose modified estimators of population mean by  $\bar{Y}$  under SRSS as

$$t_{p3}^c = [(1 + w_1)\bar{y}_{SRSS} + w_2(\bar{X} - \bar{x}_{SRSS})] \frac{\bar{X}}{\bar{x}_{SRSS}} \tag{9}$$

$$t_{p4}^c = [(1 + w_3)\bar{y}_{SRSS} + w_4(\bar{X} - \bar{x}_{SRSS})] \exp\left(\frac{\bar{X} - \bar{x}_{SRSS}}{\bar{X} + \bar{x}_{SRSS}}\right) \tag{10}$$

$$t_{p5}^c = w_5\bar{y}_{SRSS} + w_6 \exp\left(\frac{\bar{X} - \bar{x}_{SRSS}}{\bar{X} + \bar{x}_{SRSS}}\right) \left(1 + \log\frac{\bar{x}_{SRSS}}{\bar{X}}\right) \tag{11}$$

$$t_{p6}^c = w_7\bar{y}_{SRSS} + w_8 \left(\frac{\bar{X}}{\bar{x}_{SRSS}}\right) \exp\left(\frac{\bar{X} - \bar{x}_{SRSS}}{\bar{X} + \bar{x}_{SRSS}}\right) \tag{12}$$

**Proposition 2.** *The Bias, MSE and min MSE of the proposed estimators are*

$$bias(t_{p3}^c) = \bar{Y}w_1 + \bar{Y}(V_{02} + w_1V_{02} + w_2\delta V_{02} - V_{11} - w_1V_{11})$$

$$bias(t_{p4}^c) = \bar{Y}w_3 + \bar{Y}\left(\frac{3}{8}V_{02} + \frac{3}{8}w_3V_{02} + \frac{1}{2}w_4\delta V_{02} - \frac{1}{2}V_{11} - \frac{1}{2}w_3V_{11}\right)$$

$$Bias(t_{p5}^c) = (w_5 - 1)\bar{Y} + w_6\left(1 - \frac{5}{8}V_{02}\right)$$

$$Bias(t_{p6}^c) = (w_7 - 1)\bar{Y} + w_8\left(1 + \frac{15}{8}V_{02}\right)$$

$$MSE(t_{p3}^c) = \bar{Y}^2(A_1 + w_1^2B_1 + w_2^2C_1 + 2w_1D_1 - 2w_2E_1 - 2w_1w_2F_1)$$

$$MSE(t_{p4}^c) = \bar{Y}^2(A_2 + w_3^2B_2 + w_4^2C_2 + 2w_3D_2 - 2w_4E_2 - 2w_3w_4F_2)$$

$$minMSE(t_{p3}^c) = \bar{Y}^2\left(A_1 + \frac{C_1D_1^2 + B_1E_1^2 - 2D_1E_1F_1}{F_1^2 - B_1C_1}\right)$$

$$minMSE(t_{p4}^c) = \bar{Y}^2\left(A_2 + \frac{C_2D_2^2 + B_2E_2^2 - 2D_2E_2F_2}{F_2^2 - B_2C_2}\right)$$

**Proof.** Outline of the derivations are given in Appendix. □

**Case 1: Sum of Weights is Unity** ( $w_5 + w_6 = 1$  &  $w_7 + w_8 = 1$ ).

**Proposition 3.** *The MSE and min MSE of the proposed estimators are The MSE of the estimator  $t_{p5}^c$  is given by*

$$MSE(t_{p5}^c) = \bar{Y}^2(V_{20} + w_6^2V_{02} - 2w_6V_{11})$$

$$MSE(t_{p6}^c) = \bar{Y}^2(V_{20} + w_8^2V_{02} - 2w_8V_{11})$$

$$MinMSE(t_{p5}^c) = \bar{Y}^2\left(V_{20} - \frac{V_{11}^2}{V_{02}}\right)$$

$$MinMSE(t_{p6}^c) = \bar{Y}^2\left(V_{20} - \frac{V_{11}^2}{V_{02}}\right)$$

**Proof.** Outline of the derivations are given in Appendix.  $\square$

**Case 2: Sum of Weights is Flexible** ( $w_5 + w_6 \neq 1$  &  $w_7 + w_8 \neq 1$ )

**Proposition 4.** *The MSE and min MSE of the proposed estimators are*

$$MSE(t_{p5}^c) = C_3 + w_5^2 A_3 + w_6^2 B_3 - 2w_5 C_3 - 2w_6 D_3 + 2w_5 w_6 E_3$$

$$MSE(t_{p6}^c) = C_4 + w_7^2 A_4 + w_8^2 B_4 - 2w_7 C_4 - 2w_8 D_4 + 2w_7 w_8 E_4$$

$$\min MSE(t_{p5}^c) = C_3 + \frac{B_3 C_3^2 + A_3 D_3^2 - 2C_3 D_3 E_3}{E_3^2 - A_3 B_3}$$

$$\min MSE(t_{p6}^c) = C_4 + \frac{B_4 C_4^2 + A_4 D_4^2 - 2C_4 D_4 E_4}{E_4^2 - A_4 B_4}$$

**Proof.** Outline of the derivations are given in Appendix.  $\square$

## 6. Numerical Illustrations

In this section, we compare the performance of the proposed estimators with the other estimators considered in this paper. For comparison, we have taken a real data of area under tobacco production and production of tobacco of specified countries during 1998 given by Singh (2003) (Appendix) given in Table 1, where  $y$  is production (study variable) in metric tons and  $x$  is area (auxiliary variable) in hectares. For the above population, the parameters are given as below: For total,  $N = 67$ ,  $\bar{Y} = 72247.6$ ,  $\bar{X} = 26438$ .

TABLE 1: Data

| Stratum 1               | Stratum 2                | Stratum 3                 |
|-------------------------|--------------------------|---------------------------|
| $N_1 = 20$              | $N_2 = 30$               | $N_3 = 17$                |
| $n_1 = 12$              | $n_2 = 18$               | $n_3 = 9$                 |
| $W_1 = 0.29851$         | $W_2 = 0.44776$          | $W_3 = 0.25373$           |
| $\bar{X}_1 = 6801.25$   | $\bar{X}_2 = 11025.3$    | $\bar{X}_3 = 82464.1$     |
| $\bar{Y}_1 = 17511.7$   | $\bar{Y}_2 = 18937.4$    | $\bar{Y}_3 = 377960.5$    |
| $S_{x1}^2 = 175539558$  | $S_{x2}^2 = 595679198.4$ | $S_{x3}^2 = 20255478994$  |
| $S_{y1}^2 = 1366895911$ | $S_{y2}^2 = 2421559069$  | $S_{y3}^2 = 687956456787$ |
| $S_{y1x1} = 489224338$  | $S_{y2x2} = 1174423304$  | $S_{y3x3} = 46735680920$  |
| $C_{x1} = 1.94804$      | $C_{x2} = 2.21368$       | $C_{x3} = 1.72586$        |
| $R_1 = 2.57477$         | $R_2 = 1.71763$          | $R_3 = 4.58333$           |

Case 1 (when initial sample is taken by SRS): From this population we took ranked set samples of sizes  $k_1 = 4$ ,  $k_2 = 6$  and  $k_3 = 3$  from the stratum 1<sup>st</sup>, 2<sup>nd</sup> and 3<sup>rd</sup> respectively. Further each ranked set sample from each stratum were repeated with number of cycles  $r = 3$ .

Case 2 (when initial sample is taken by PPS): From this population  $k$  independent random sets (initial sample) each of size  $k$  are selected from the population

TABLE 2: The MSE and PRE of the estimators

| Estimators | MSE        | PRE     |
|------------|------------|---------|
| $t^c$      | 566408434  | 100     |
| $t_r^c$    | 201822821  | 280.64  |
| $t_{lr}^c$ | 201169647  | 281.55  |
| $t_{p1}^c$ | 201169647  | 281.55  |
| $t_{p2}^c$ | 201169647  | 281.55  |
| $t_{p3}^c$ | 193114626  | 293.30  |
| $t_{p4}^c$ | 155280723  | 364.76  |
| $t_{p5}^c$ | 55059116.5 | 1028.72 |
| $t_{p6}^c$ | 41886045.6 | 1352.26 |

by using probability proportion to size (pps) sampling. We took ranked set samples of size  $k = 3$  from each stratum. Further each ranked set sample from each stratum were repeated with number of cycles  $r_1 = 3$ ,  $r_2 = 2$  and  $r_3 = 2$  from the stratum  $1^{st}$ ,  $2^{nd}$  and  $3^{rd}$ , respectively.

TABLE 3: The MSE and PRE of the estimators under PPS sampling

| Estimators | MSE         | PRE     |
|------------|-------------|---------|
| $t^c$      | 98340720.29 | 100     |
| $t_r^c$    | 14809479.67 | 664.03  |
| $t_{lr}^c$ | 14474608.66 | 679.40  |
| $t_{p1}^c$ | 14474608.66 | 679.40  |
| $t_{p2}^c$ | 14474608.66 | 679.40  |
| $t_{p3}^c$ | 14192040.24 | 692.92  |
| $t_{p4}^c$ | 6775003.38  | 1451.52 |
| $t_{p5}^c$ | 3539691.30  | 2778.22 |
| $t_{p6}^c$ | 2914206.65  | 3374.52 |

The formula for Percent Relative Efficiency (PRE) is  $PRE(estimators) = \frac{MSE(t^c)}{MSE(estimator)} \times 100$ .

From Table 2, it is observed that

- The estimators  $t_{p1}^c$  and  $t_{p2}^c$  are almost equally efficient estimators as combined linear regression estimators under SRSS as these estimators show the MSE almost equal to the MSE of the combined linear regression estimator ( $t_{lr}^c$ ). These two estimators  $t_{p1}^c$  and  $t_{p2}^c$  are more efficient estimators than that the other competitive estimators.
- $t_{p3}^c, t_{p4}^c, t_{p5}^c$  and  $t_{p6}^c$  are more efficient than other estimators used in this paper. It is observed that  $t_{p3}^c, t_{p4}^c, t_{p5}^c$  and  $t_{p6}^c$  are more efficient than convention, ratio estimator and linear regression estimator under SRSS.
- From Table 2, we can conclude that the proposed estimators perform better than existing estimators as our proposed estimators have greater PRE.
- From Table 3, we can conclude that the proposed estimators when initial sample is taken by PPS show same trend as usual proposed estimators. The

estimators  $t_{p1}^c$  and  $t_{p2}^c$  are almost equally efficient estimators as combined linear regression estimators under SRSS.  $t_{p3}^c$ ,  $t_{p4}^c$ ,  $t_{p5}^c$  and  $t_{p6}^c$  are more efficient than other estimators used in this paper. From Table 3, we can conclude that the proposed estimators perform better than existing estimators as our proposed estimators have greater PRE.

## 7. Simulation Study

To generalize the results of the numerical study, we have conducted simulation study over two hypothetically generated normal populations. The simulation procedure is explained in the following points:

- We generated bivariate random observations of size  $N = 600$  units from a bivariate normal distribution with parameters  $\mu_y = 20$ ,  $\sigma_y = 15$ , and  $\mu_x = 15$ ,  $\sigma_x = 10$  and passably chosen values of  $\rho = 0.3, 0.5, 0.6, 0.7, 0.8, 0.9$ .
- Similarly, generate the population-2 with the parameters  $\mu_y = 120$ ,  $\sigma_y = 25$ , and  $\mu_x = 100$ ,  $\sigma_x = 20$ .
- The population generated above is divided into 3 equal strata and a stratified ranked set sample of size 12 units with number of cycles 4 and set size 3 is drawn from each stratum.
- Compute the required statistics.
- Iterate the above steps 10000 times to calculate the MSE and PRE of various combined estimators using the following expression.

$$MSE(T) = \frac{1}{10000} \sum_{i=1}^{10000} (T_i - \bar{Y})^2 \quad (13)$$

$$PRE = \frac{Var(t^c)}{MSE(T)} \times 100 \quad (14)$$

The MSE and PRE of the combined estimators are calculated using (13) and (14) and the results are reported for various values of correlation coefficients in Table 4.

Table 4: The MSE and PRE of the estimators

| $r_{yx}$ | Estimators | Population 1 |            | Population 2 |            |
|----------|------------|--------------|------------|--------------|------------|
|          |            | MSE          | PRE        | MSE          | PRE        |
| 0.9      | $t^c$      | 0.009399     | 100        | 0.009977     | 100        |
|          | $t_r^c$    | 0.008983     | 104.632314 | 0.007649     | 130.441398 |
|          | $t_{r'}^c$ | 0.008140     | 115.470738 | 0.007368     | 135.409878 |
|          | $t_{p1}^c$ | 0.008165     | 115.112126 | 0.007377     | 135.246341 |
|          | $t_{p2}^c$ | 0.008143     | 115.426215 | 0.007323     | 136.245249 |

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Table 4. Continued from previous page

| $r_{yx}$ | Estimators | Population 1 |            | Population 2 |            |
|----------|------------|--------------|------------|--------------|------------|
|          |            |              |            |              |            |
|          | $t_{p3}^c$ | 0.005397     | 174.155787 | 0.004591     | 217.296845 |
|          | $t_{p4}^c$ | 0.004126     | 227.787861 | 0.003749     | 266.095249 |
|          | $t_{p5}^c$ | 0.001845     | 509.395864 | 0.003496     | 285.359492 |
|          | $t_{p6}^c$ | 0.001612     | 582.794752 | 0.002739     | 364.269207 |
| 0.8      | $t^c$      | 0.008071     | 100        | 0.007391     | 100        |
|          | $t_r^c$    | 0.008009     | 100.781241 | 0.006988     | 105.760711 |
|          | $t_{lr}^c$ | 0.003936     | 205.057579 | 0.006895     | 107.191263 |
|          | $t_{p1}^c$ | 0.003954     | 204.104535 | 0.006832     | 108.179959 |
|          | $t_{p2}^c$ | 0.003923     | 205.726039 | 0.006898     | 107.139678 |
|          | $t_{p3}^c$ | 0.002592     | 311.383468 | 0.005186     | 142.501132 |
|          | $t_{p4}^c$ | 0.001537     | 524.918546 | 0.003622     | 204.034130 |
|          | $t_{p5}^c$ | 0.001390     | 580.591840 | 0.002845     | 259.729142 |
|          | $t_{p6}^c$ | 0.001186     | 680.007413 | 0.002617     | 282.358244 |
| 0.7      | $t^c$      | 0.007355     | 100        | 0.009191     | 100        |
|          | $t_r^c$    | 0.005736     | 128.219695 | 0.008556     | 107.426159 |
|          | $t_{lr}^c$ | 0.004616     | 159.320334 | 0.004802     | 191.378071 |
|          | $t_{p1}^c$ | 0.004687     | 156.904577 | 0.004823     | 190.561527 |
|          | $t_{p2}^c$ | 0.004643     | 158.398764 | 0.004855     | 189.293929 |
|          | $t_{p3}^c$ | 0.003559     | 206.628985 | 0.002697     | 340.760512 |
|          | $t_{p4}^c$ | 0.003389     | 217.030976 | 0.002335     | 393.501322 |
|          | $t_{p5}^c$ | 0.002699     | 272.486505 | 0.001865     | 492.606328 |
|          | $t_{p6}^c$ | 0.001856     | 396.239252 | 0.001135     | 809.643520 |
| 0.6      | $t^c$      | 0.009910     | 100        | 0.018945     | 100        |
|          | $t_r^c$    | 0.008116     | 122.109429 | 0.014026     | 135.069295 |
|          | $t_{lr}^c$ | 0.008083     | 122.603248 | 0.012800     | 148.004437 |
|          | $t_{p1}^c$ | 0.008044     | 123.195799 | 0.012770     | 148.354436 |
|          | $t_{p2}^c$ | 0.008078     | 122.674271 | 0.012438     | 152.320271 |
|          | $t_{p3}^c$ | 0.005600     | 176.957222 | 0.010820     | 175.097502 |
|          | $t_{p4}^c$ | 0.005533     | 179.089460 | 0.005823     | 325.314051 |
|          | $t_{p5}^c$ | 0.004191     | 236.468491 | 0.005641     | 335.814228 |
|          | $t_{p6}^c$ | 0.004054     | 244.411288 | 0.003085     | 614.006261 |
| 0.5      | $t^c$      | 0.011061     | 100        | 0.032002     | 100        |
|          | $t_r^c$    | 0.014576     | 75.886529  | 0.032978     | 97.039310  |
|          | $t_{lr}^c$ | 0.009580     | 115.454465 | 0.025516     | 125.420912 |
|          | $t_{p1}^c$ | 0.009572     | 115.560486 | 0.025929     | 125.848529 |
|          | $t_{p2}^c$ | 0.009578     | 115.488096 | 0.025836     | 123.864115 |
|          | $t_{p3}^c$ | 0.008217     | 134.614214 | 0.020097     | 159.234941 |
|          | $t_{p4}^c$ | 0.008118     | 136.249085 | 0.010095     | 292.123303 |
|          | $t_{p5}^c$ | 0.005430     | 203.696951 | 0.010087     | 317.254054 |
|          | $t_{p6}^c$ | 0.004944     | 223.693975 | 0.008502     | 376.398745 |
| 0.3      | $t^c$      | 0.019186     | 100        | 0.036245     | 100        |
|          | $t_r^c$    | 0.030482     | 62.941238  | 0.055141     | 65.731736  |
|          | $t_{lr}^c$ | 0.018669     | 102.764342 | 0.035146     | 103.126333 |
|          | $t_{p1}^c$ | 0.018532     | 103.524003 | 0.035112     | 103.227368 |
|          | $t_{p2}^c$ | 0.018933     | 101.335755 | 0.035409     | 102.361822 |
|          | $t_{p3}^c$ | 0.018019     | 106.471769 | 0.033719     | 107.489961 |
|          | $t_{p4}^c$ | 0.017812     | 107.712086 | 0.032991     | 109.863842 |
|          | $t_{p5}^c$ | 0.015881     | 120.810271 | 0.029482     | 122.937858 |
|          | $t_{p6}^c$ | 0.014371     | 133.496614 | 0.027864     | 130.076404 |

Table 4 also shows that our proposed estimators perform better than the existing estimators. The MSE of the estimators decreases when the correlation and sample size increases for the population 1 and 2.

## 8. Conclusions

In this article we have proposed estimators for the population mean in stratified Ranked set sampling using the information of auxiliary variables. The expressions for Bias and MSE of the suggested estimators have been derived up to the first order of approximation. Numerical illustrations and simulation study for comparing the efficiency of the proposed estimators with other estimators have been used. The results have been shown in the Tables 2, 3 and 4. The Tables 2 and 3 show that the proposed estimators turn out to be more efficient as compared to the other estimators for both cases when initial samples are taken by using SRS and PPS respectively.

The proposed estimators are found to be rather improved in terms of lesser MSE and greater PRE as compared to the existing estimators in both real and simulated data sets. It is also observed from the simulation that the MSE of the proposed estimators decreases as the values of the correlation coefficient increase whereas the PRE of the suggested estimators increases as the values of the correlation coefficients increase. Based on our numerical illustrations and simulation study, we can conclude that our proposed estimators can be preferred over the other estimators taken in this paper in several real situations.

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## Appendix

This section consider the proof of the Theorems of Section 4 and 5. To derive the MSE of the proposed estimators, the following notations will be used throughout the paper.

$$\begin{aligned}\bar{y}_{srss} &= \bar{Y}(1 + \epsilon_0) \\ \bar{x}_{srss} &= \bar{X}(1 + \epsilon_1) \\ V_{rs} &= \sum_{h=1}^L W_h^{r+s} \frac{E[(\bar{y}_{srss} - \bar{Y})^r (\bar{x}_{srss} - \bar{X})^s]}{\bar{Y}^r \bar{X}^s}\end{aligned}$$

such that  $E(\epsilon_0) = E(\epsilon_1) = 0$

$$\begin{aligned}E(\epsilon_0^2) &= \sum_{h=1}^L W_h^2 (\eta_h C_{yh}^2 - D_{yh[i]}^2) = V_{20} \\ E(\epsilon_1^2) &= \sum_{h=1}^L W_h^2 (\eta_h C_{xh}^2 - D_{xh[i]}^2) = V_{02} \\ E(\epsilon_0 \epsilon_1) &= \sum_{h=1}^L W_h^2 (\eta_h C_{xyh} - D_{xyh[i]}) = V_{11}\end{aligned}$$

where  $\eta_h = \frac{1}{k_h r}$ ,  $C_{xh} = \frac{S_{xh}}{\bar{X}}$ ,  $C_{yh} = \frac{S_{yh}}{\bar{Y}}$ ,  $D_{xh[i]}^2 = \frac{1}{k_h^2 r \bar{X}^2} \sum_{i=1}^{k_h} (\bar{X}_{h(i)} - \bar{X}_h)^2$ ,  $D_{yh[i]}^2 = \frac{1}{k_h^2 r \bar{Y}^2} \sum_{i=1}^{k_h} (\bar{Y}_{h[i]} - \bar{Y}_h)^2$  and  $D_{xyh[i]} = \frac{1}{k_h^2 r \bar{Y} \bar{X}} \sum_{i=1}^{k_h} (\bar{Y}_{h[i]} - \bar{Y}_h)(\bar{X}_{h(i)} - \bar{X}_h)$  where  $\bar{Y}_{h[i]}$  and  $\bar{X}_{h(i)}$  are the means of the  $i^{th}$  is ranked set and are given by

$$\bar{Y}_{h[i]} = \frac{1}{r} \sum_{j=1}^r Y_{h[i]j}, \bar{X}_{h(i)} = \frac{1}{r} \sum_{j=1}^r X_{h(i)j}$$

Now, consider the estimator

$$t_{p1}^c = \bar{y}_{SRSS} \exp \left( \alpha_1 \left( \frac{\bar{x}_{SRSS}}{\bar{X}} - 1 \right) \right)$$

Using the above notations we have

$$t_{p1}^c = \bar{Y}(1 + \epsilon_0) \exp \left( \alpha_1 \left( \frac{\bar{X}(1 + \epsilon_1)}{\bar{X}} - 1 \right) \right)$$

The bias of the estimator  $t_{p1}^c$  is given by

$$Bias(t_{p1}^c) = \bar{Y} \left( \frac{\alpha_1^2}{2} V_{02} + \alpha_1 V_{11} \right)$$

The MSE of the estimator  $t_{p1}^c$  is given by

$$MSE(t_{p1}^c) = \bar{Y}^2 (V_{20} + \alpha_1^2 V_{02} + 2\alpha_1 V_{11}) \quad (A1)$$

To find out the minimum MSE for  $t_{p1}^c$ , we partially differentiate equation (A1) w.r.t.  $\alpha_1$  and equating to zero we get

$$\alpha_1^* = -\frac{V_{11}}{V_{02}}$$

Putting the optimum value of  $\alpha_1$  in the equation (A1), we get a minimum MSE of  $t_{p1}^c$  as

$$MinMSE(t_{p1}^c) = \bar{Y}^2 \left( V_{20} - \frac{V_{11}^2}{V_{02}} \right)$$

Similarly, we can obtain the optimum values of constants and minimum MSEs of other proposed estimators which are given as

$$t_{p2}^c = \bar{Y}(1 + \epsilon_0) \exp \left( \alpha_2 \log \frac{\bar{X}(1 + \epsilon_1)}{\bar{X}} \right)$$

The bias of the estimator  $t_{p2}^c$  is given by

$$Bias(t_{p2}^c) = \bar{Y} \left( \frac{(\alpha_2^2 - \alpha_2)}{2} V_{02} + \alpha_2 V_{11} \right)$$

The MSE of the estimator  $t_{p2}^c$  is given by

$$MSE(t_{p2}^c) = \bar{Y}^2 (V_{20} + \alpha_2^2 V_{02} + 2\alpha_2 V_{11}) \tag{A2}$$

To find out the minimum MSE for  $t_{p2}^c$ , we partially differentiate equation (A2) w.r.t.  $\alpha_2$  and equating to zero we get

$$\alpha_2^* = -\frac{V_{11}}{V_{02}}$$

Putting the optimum value of  $\alpha_2$  in the equation (A2), we get a minimum MSE of  $t_{p2}^c$  as

$$MinMSE(t_{p2}^c) = \bar{Y}^2 \left( V_{20} - \frac{V_{11}^2}{V_{02}} \right)$$

$$t_{p3}^c = [(1 + w_1)\bar{Y}(1 + \epsilon_0) + w_2\epsilon_1](1 - \epsilon_1 + \epsilon_1^2)$$

$$t_{p3}^c - \bar{Y} = \bar{Y}[(\epsilon_0 + W_1 + \epsilon_0 w_1 - \epsilon_1 - \epsilon_1 w_1 - \epsilon_0 \epsilon_1 - \epsilon_0 \epsilon_1 w_1 + \epsilon_1^2 + w_1 \epsilon_1^2) - w_2 \delta(\epsilon_1 - \epsilon_1^2)]$$

The bias of the estimator  $t_{p3}^c$  is given by

$$Bias(t_{p3}^c) = \bar{Y} w_1 + \bar{Y}(V_{02} + w_1 V_{02} + w_2 \delta V_{02} - V_{11} - w_1 V_{11})$$

The MSE of the estimator  $t_{p3}^c$  is given by

$$MSE(t_{p3}^c) = \bar{Y}^2 (V_{20} + V_{02} - 2V_{11} + w_1^2(1 + V_{20} + 3V_{02} - 4V_{11}) + w_2^2 \delta^2 V_{02} + 2w_1(V_{20} + 2V_{02} - 3V_{11}) - 2w_2 \delta(V_{11} - V_{02}) - 2w_1 w_2 \delta(V_{11} - 2V_{02}))$$

$$MSE(t_{p3}^c) = \bar{Y}^2(A_1 + w_1^2 B_1 + w_2^2 C_1 + 2w_1 D_1 - 2w_2 E_1 - 2w_1 w_2 F_1) \quad (\text{A3})$$

where

$$\begin{aligned} A_1 &= V_{20} + V_{02} - 2V_{11} \\ B_1 &= 1 + V_{20} + 3V_{02} - 4V_{11} \\ C_1 &= \delta^2 V_{02}, \delta = \frac{\bar{X}}{\bar{Y}} \\ D_1 &= V_{20} + 2V_{02} - 3V_{11} \\ E_1 &= \delta(V_{02} - V_{11}) \\ F_1 &= \delta(V_{02} - 2V_{11}) \end{aligned}$$

To find out the minimum MSE for  $t_{p3}^c$ , we partially differentiate equation (A3) w.r.t.  $w_1$  and  $w_2$  and equating to zero we get

$$\begin{aligned} w_1^* &= \frac{C_1 D_1 - E_1 F_1}{F_1^2 - B_1 C_1} \\ w_2^* &= \frac{D_1 F_1 - B_1 C_1}{F_1^2 - B_1 C_1} \end{aligned}$$

Putting the optimum values of  $w_1$  and  $w_2$  in the equation (A3), we get a minimum MSE of  $t_{p3}^c$  as

$$MinMSE(t_{p3}^c) = \bar{Y}^2 \left( A_1 + \frac{C_1 D_1^2 + B_1 E_1^2 - 2D_1 E_1 F_1}{F_1^2 - B_1 C_1} \right)$$

$$t_{p4}^c = [(1 + w_3)\bar{Y}(1 + \epsilon_0) + w_4 \epsilon_1] \left( 1 - \frac{3}{2}\epsilon_1 + \frac{15}{8}\epsilon_1^2 \right)$$

$$\begin{aligned} t_{p4}^c - \bar{Y} &= \bar{Y}[(\epsilon_0 + W_3 + \epsilon_0 w_3 - \frac{1}{2}\epsilon_1 - \frac{1}{2}\epsilon_1 w_3 - \frac{1}{2}\epsilon_0 \epsilon_1 - \frac{1}{2}\epsilon_0 \epsilon_1 w_3 + \frac{3}{8}\epsilon_1^2 \\ &\quad + \frac{3}{8}w_3 \epsilon_1^2) - w_4 \delta(\epsilon_1 - \epsilon_1^2)] \end{aligned}$$

The bias of the estimator  $t_{p4}^c$  is given by

$$bias(t_{p4}^c) = \bar{Y}w_3 + \bar{Y} \left( \frac{3}{8}V_{02} + \frac{3}{8}w_3 V_{02} + \frac{1}{2}w_4 \delta V_{02} - \frac{1}{2}V_{11} - \frac{1}{2}w_3 V_{11} \right)$$

The MSE of the estimator  $t_{p4}^c$  is given by

$$\begin{aligned} MSE(t_{p4}^c) &= \bar{Y}^2 \left( V_{20} + \frac{1}{4}V_{02} - V_{11} + w_3^2(1 + V_{20} + V_{02} - 2V_{11}) + w_4^2 \delta^2 V_{02} + 2w_3(V_{20} \right. \\ &\quad \left. + \frac{5}{4}V_{02} - \frac{3}{2}V_{11}) + 2w_4 \delta(V_{11} - \frac{1}{2}V_{02}) - 2w_3 w_4 \delta(V_{11} - V_{02}) \right) \end{aligned}$$

$$MSE(t_{p4}^c) = \bar{Y}^2 (A_2 + w_3^2 B_2 + w_4^2 C_2 + 2w_3 D_2 - 2w_4 E_2 - 2w_3 w_4 F_2) \quad (A4)$$

where

$$\begin{aligned} A_2 &= V_{20} + \frac{1}{4}V_{02} - V_{11} \\ B_2 &= 1 + V_{20} + V_{02} - 2V_{11} \\ C_2 &= \delta^2 V_{02}, \delta = \frac{\bar{X}}{\bar{Y}} \\ D_2 &= V_{20} + \frac{5}{4}V_{02} - \frac{3}{2}V_{11} \\ E_2 &= \delta \left( V_{02} - \frac{1}{2}V_{11} \right) \\ F_2 &= \delta(V_{02} - V_{11}) \end{aligned}$$

To find out the minimum MSE for  $t_{p4}^c$ , we partially differentiate equation w.r.t. (A4)  $w_3$  and  $w_4$  and equating to zero we get

$$w_3^* = \frac{C_2 D_2 - E_2 F_2}{F_2^2 - B_2 C_2}$$

$$w_4^* = \frac{D_2 F_2 - B_2 C_2}{F_2^2 - B_2 C_2}$$

Putting the optimum values of  $w_3$  and  $w_4$  in the equation (A4), we get a minimum MSE of  $t_{p4}^c$  as

$$\min MSE(t_{p4}^c) = \bar{Y}^2 \left( A_2 + \frac{C_2 D_2^2 + B_2 E_2^2 - 2D_2 E_2 F_2}{F_2^2 - B_2 C_2} \right)$$

$$t_{p5}^c = w_5 \bar{Y} (1 + \epsilon_0) + w_6 \exp\left(\frac{-\epsilon_1}{2 + \epsilon_1}\right) (1 + \log(1 + \epsilon_1))$$

$$t_{p5}^c - \bar{Y} = (w_5 - 1)\bar{Y} + w_5 \bar{Y} \epsilon_0 + w_6 \left( 1 + \frac{\epsilon_1}{2} - \frac{5}{8} \epsilon_1^2 \right)$$

$$Bias(t_{p5}^c) = (w_5 - 1)\bar{Y} + w_6 \left( 1 - \frac{5}{8} V_{02} \right)$$

$$t_{p6}^c = w_7 \bar{Y} (1 + \epsilon_0) + w_8 \exp\left(\frac{-\epsilon_1}{2 + \epsilon_1}\right) (1 + \epsilon_1)^{-1}$$

$$t_{p6}^c - \bar{Y} = (w_7 - 1)\bar{Y} + w_7 \bar{Y} \epsilon_0 + w_8 \left( 1 - \frac{3}{2} \epsilon_1 - \frac{15}{8} \epsilon_1^2 \right)$$

$$Bias(t_{p6}^c) = (w_7 - 1)\bar{Y} + w_8 \left( 1 + \frac{15}{8} V_{02} \right)$$

**Case 1: Sum of Weights is Unity** ( $w_5 + w_6 = 1$  &  $w_7 + w_8 = 1$ ). The MSE of the estimator  $t_{p5^c}$  is given by

$$MSE(t_{p5^c}^c) = \bar{Y}^2(V_{20} + w_6^2V_{02} - 2w_6V_{11}) \quad (\text{A5})$$

To find out the minimum MSE for  $t_{p5^c}^c$ , we partially differentiate equation (A5) w.r.t.  $w_6$ , and equating to zero we get

$$w_6^* = \frac{V_{11}}{V_{02}}$$

Putting the optimum value of  $w_6$  in the equation (A5), we get a minimum MSE of  $t_{p5^c}^c$  as

$$MinMSE(t_{p5^c}^c) = \bar{Y}^2 \left( V_{20} - \frac{V_{11}^2}{V_{02}} \right)$$

The MSE of the estimator  $t_{p6}^c$  is given by

$$MSE(t_{p6}^c) = \bar{Y}^2(V_{20} + w_8^2V_{02} - 2w_8V_{11}) \quad (\text{A6})$$

To find out the minimum MSE for  $t_{p6}^c$ , we partially differentiate equation (A6) w.r.t.  $w_8$ , and equating to zero we get

$$w_8^* = \frac{V_{11}}{V_{02}}$$

Putting the optimum value of  $w_8$  in the equation (A6), we get a minimum MSE of  $t_{p6}^c$  as

$$MinMSE(t_{p6}^c) = \bar{Y}^2 \left( V_{20} - \frac{V_{11}^2}{V_{02}} \right)$$

**Case 2: Sum of Weights is Flexible** ( $w_5 + w_6 \neq 1$  &  $w_7 + w_8 \neq 1$ )

$$t_{p5}^c - \bar{Y} = (W_5 - 1)\bar{Y} + w_5\bar{Y}\epsilon_0 + w_6 \left( 1 + \frac{\epsilon_1}{2} - \frac{5}{8}\epsilon_1^2 \right)$$

Squaring on both sides we get

$$\begin{aligned} (t_{p5}^c - \bar{Y})^2 &= \bar{Y}^2 + \bar{Y}^2 w_5^2(1 + \epsilon_0^2) + w_6^2(1 - \epsilon_1^2) - 2w_5\bar{Y}^2 - 2w_6\bar{Y} \left( 1 - \frac{5}{8}\epsilon_1^2 \right) \\ &\quad + 2w_5w_6 \left( 1 - \frac{5}{8}\epsilon_1^2 + \frac{1}{2}\epsilon_0\epsilon_1 \right) \end{aligned}$$

Taking expectations on both sides we get

$$\begin{aligned} MSE(t_{p5}^c) &= \bar{Y}^2 + \bar{Y}^2 w_5^2(1 + V_{20}) + w_6^2(1 - V_{02}) - 2w_5\bar{Y}^2 \\ &\quad - 2w_6\bar{Y} \left( 1 - \frac{5}{8}V_{02} \right) + 2w_5w_6 \left( 1 - \frac{5}{8}V_{02} + \frac{1}{2}V_{11} \right) \end{aligned}$$



$$MSE(t_{p5}^c) = C_3 + w_5^2 A_3 + w_6^2 B_3 - 2w_5 C_3 - 2w_6 D_3 + 2w_5 w_6 E_3 \quad (A7)$$

where

$$\begin{aligned} A_3 &= \bar{Y}^2 (1 + V_{20}) \\ B_3 &= 1 - V_{02} \\ C_3 &= \bar{Y}^2 \\ D_3 &= \bar{Y} \left( 1 - \frac{5}{8} V_{02} \right) \\ E_3 &= \bar{Y} \left( 1 - \frac{5}{8} V_{02} + \frac{1}{2} V_{11} \right) \end{aligned}$$

To find out the minimum MSE for the estimator  $t_{p5}^c$ , we partially differentiate equation (21) w.r.t.  $w_5$  and  $w_6$  and equating to zero we get

$$\begin{aligned} w_5^* &= \frac{B_3 C_3 - D_3 E_3}{A_3 B_3 - E_3^2} \\ w_6^* &= \frac{A_3 D_3 - C_3 E_3}{A_3 B_3 - E_3^2} \end{aligned}$$

Putting the optimum values of  $w_5$  and  $w_6$  in the equation (21), we get a minimum MSE of  $t_{p5}^c$  as

$$MinMSE(t_{p5}^c) = C_3 + \frac{B_3 C_3^2 + A_3 D_3^2 - 2C_3 D_3 E_3}{E_3^2 - A_3 B_3}$$

$$t_{p6}^c - \bar{Y} = (W_7 - 1)\bar{Y} + w_7 \bar{Y} \epsilon_0 + w_8 \left( 1 - \frac{3}{2} \epsilon_1 + \frac{15}{8} \epsilon_1^2 \right)$$

Squaring on both sides we get

$$\begin{aligned} (t_{p6}^c - \bar{Y})^2 &= \bar{Y}^2 + \bar{Y}^2 w_7^2 (1 + \epsilon_0^2) + w_8^2 (1 + 6\epsilon_1^2) - 2w_7 \bar{Y}^2 - 2w_8 \bar{Y} \left( 1 - \frac{15}{8} \epsilon_1^2 \right) \\ &\quad + 2w_7 w_8 \left( 1 + \frac{15}{8} \epsilon_1^2 - \frac{3}{2} \epsilon_0 \epsilon_1 \right) \end{aligned}$$

Taking expectations on both sides we get

$$\begin{aligned} MSE(t_{p6}^c) &= \bar{Y}^2 + \bar{Y}^2 w_7^2 (1 + V_{20}) + w_8^2 (1 + 6V_{02}) - 2w_7 \bar{Y}^2 - 2w_8 \bar{Y} \left( 1 + \frac{15}{8} V_{02} \right) \\ &\quad + 2w_7 w_8 \left( 1 + \frac{15}{8} V_{02} - \frac{3}{2} V_{11} \right) \end{aligned}$$

$$MSE(t_{p6}^c) = C_4 + w_7^2 A_4 + w_8^2 B_4 - 2w_7 C_4 - 2w_8 D_4 + 2w_7 w_8 E_4 \quad (A8)$$

where

$$A_4 = \bar{Y}^2 (1 + V_{20})$$

$$\begin{aligned}
 B_4 &= 1 + 6V_{02} \\
 C_4 &= \bar{Y}^2 \\
 D_4 &= \bar{Y} \left( 1 + \frac{15}{8}V_{02} \right) \\
 E_4 &= \bar{Y} \left( 1 + \frac{15}{8}V_{02} - \frac{3}{2}V_{11} \right)
 \end{aligned}$$

To find out the minimum MSE for the estimator  $t_{p6}^c$ , we partially differentiate equation (22) w.r.t.  $w_7$  and  $w_8$  and equating to zero we get

$$\begin{aligned}
 w_7^* &= \frac{B_4C_4 - D_4E_4}{A_4B_4 - E_4^2} \\
 w_8^* &= \frac{A_4D_4 - C_4E_4}{A_4B_4 - E_4^2}
 \end{aligned}$$

Putting the optimum values of  $w_7$  and  $w_8$  in the equation (22), we get a minimum MSE of  $t_{p6}^c$  as

$$\min MSE(t_{p6}^c) = C_4 + \frac{B_4C_4^2 + A_4D_4^2 - 2C_4D_4E_4}{E_4^2 - A_4B_4}$$