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The Topp-Leone-Gompertz-Exponentiated Half Logistic-G Family of Distributions with Applications

La media familia logística exponencial de Topp-Leone-Gompertz de distribución con aplicaciones

Neo Dingalo[a], Broderick Oluyede[b], Fastel Chipepa[c]

Department of Mathematics and Statistical Sciences, Faculty of Sciences, Botswana International University of Science and Technology, Palapye, Botswana

Abstract

This paper introduces and investigates a new family of distributions called the Topp-Leone-Gompertz-exponentiated half logistic-G (TL-Gom-EHL-G) distribution. Some mathematical and statistical properties of this family of distributions are derived. To estimate and evaluate the model parameters, the maximum likelihood estimation technique is used, and the consistency of maximum likelihood estimators is examined using Monte Carlo simulation. Applications to three real data sets from different areas were used to demonstrates the usefulness and versatility of the TL-Gom-EHL-G family of distributions.

***Key words*:** Exponentiated Half Logistic Distribution; Maximum Likelihood Estimation; Gompertz Distribution; Goodness-of-fit Statistics; Simulation Study; Topp-Leone Distribution.

Resumen

Este artículo presenta e investiga una nueva familia de distribuciones denominada distribución Topp-Leone-Gompertz-exponenciada media logística-G (TL-Gom-EHL-G). Se derivan algunas propiedades matemáticas y estadísticas de esta familia de distribuciones. Para estimar y evaluar los parámetros del modelo se utiliza la técnica de estimación de máxima verosimilitud y se examina la consistencia de los estimadores de máxima verosimilitud mediante simulación de Monte Carlo. Se utilizaron aplicaciones a tres conjuntos de datos reales de diferentes áreas para demostrar la utilidad y versatilidad de la familia de distribuciones TL-Gom-EHL-G.

***Palabras clave*:** Distribución de Topp-Leone; Distribución de Gompertz; Distribución logística media exponenciada; Estimación de máxima verosimilitud; Estudio de simulación; Estadísticas de bondad de ajuste.

[a]Ph.D. E-mail: neo.dingalo@studentmail.biust.ac.bw
[b]Ph.D. E-mail: oluyedeo@biust.ac.bw
[c]Ph.D. E-mail: chipepaf@biust.ac.bw

1. Introduction

The Gompertz distribution was proposed by Gompertz (1825) and is used to study the nature of human mortality by determining the value of life contingencies. This distribution has a limitation in that it only applies to data with a monotonic hazard rate function, whereas in practice, we encounter non-monotonic data with hazard rate function that are bathtub, upside-down bathtub, and bathtub followed by upside-down bathtub. Many researchers responded to this need by extending the Gompertz distribution to produce the desired flexibility in the hazard rate function.

Extensions and generalizations of the Gompertz distribution include the generalized Gompertz distribution by El-Gohary et al. (2013) and the Gompertz-G distribution by Algarni et al. (2021). Chipepa & Oluyede (2021) developed the Marshall-Olkin-Gompertz-G family of distributions, and Oluyede, Chamunorwa, Chipepa & Alizadeh (2022) presented the Topp-Leone-Gompertz-G family of distributions. The cumulative distribution function (cdf) and probability density function (pdf) of the Gompertz-G family of distributions are given by

$$F(x;\gamma,\lambda,\underline{\psi}) = 1-\exp\left(\frac{\lambda}{\gamma}\left(1-\left[1-G(x;\underline{\psi})\right]^{-\gamma}\right)\right), \tag{1}$$

and

$$f(x;\gamma,\lambda,\underline{\psi}) = \left[1-G(x;\underline{\psi})\right]^{-\gamma-1}\exp\left(\frac{\lambda}{\gamma}\left(1-\left[1-G(x;\underline{\psi})\right]^{-\gamma}\right)\right)g(x;\underline{\psi}), \tag{2}$$

respectively, where $G(x;\underline{\psi})$ is the baseline cdf, $g(x;\underline{\psi}) = \frac{dG(x;\underline{\psi})}{dx}$, for $\gamma,\lambda > 0$ and parameter vector $\underline{\psi}$. We take $\lambda = 1$, in this paper to avoid the problem of overparameterization.

Jafari et al. (2014) developed the beta-Gompertz distribution, Roozegar et al. (2017) considered the properties and applications of McDonald-Gompertz distribution, Nzei et al. (2020) introduced Topp-Leone-Gompertz distribution, Eghwerido et al. (2021) proposed the alpha power Gompertz distribution, Lenart & Missov (2016) considered goodness-of-fit statistics for the Gompertz distribution, El-Bassiouny et al. (2017) proposed exponentiated generalized Weibull-Gompertz distribution, Khaleel et al. (2020) introduced Marshall-Olkin exponential Gompertz distribution, Benkhelifa (2017) presented the Marshall-Olkin extended generalized Gompertz distribution, Elbatal et al. (2018) proposed the modified beta Gompertz distribution, Shama et al. (2022) developed the gammaGompertz distribution, Boshi et al. (2020) proposed the generalized gammageneralized Gompertz distribution, El-Morshedy et al. (2020) proposed Kumaraswamy inverse Gompertz distribution, and De Andrade et al. (2019) introduced the exponentiated generalized extended Gompertz distribution.

Some recent generalizations of the exponentiated half logistic distribution include: exponentiated half logistic-odd Burr III-G family of distributions by Oluyede, Peter, Ndwapi & Bindele (2022), exponentiated half logistic-power generalized Weibull-G family of distributions by Oluyede et al. (2021), type II exponentiated half logistic-Topp-Leone-Marshall-Olkin-G family of distributions by Moakofi et al. (2021), exponentiated half logistic-odd Lindley-G family of distributions by Sengweni et al. (2021), exponentiated half logistic-odd Weibull-Topp-Leone-G family of distributions by Chipepa et al. (2021), and exponentiated half logistic-log-logistic Weibull distribution by Chamunorwa et al. (2021).

The motivations for developing TL-Gom-EHL-G family of distributions are as follows:

- The ability of the special case of the new family of distributions in providing better fits than other equi-parameter distributions available in the literature and the nested models;
- The TL-Gom-EHL-G family of distributions provides flexibility in data fitting, and can be applied to data sets with monotonic or non-monotonic hazard rate shapes;
- The TL-Gom-EHL-G family of distributions creates heavy-tailed distributions for modelling various real-world data sets;
- This new family of distributions makes and kurtosis more flexible compared to that of the baseline distribution.

The rest of the paper is structured as follows: Section 2 covers the hazard rate function, series expansion of the density function, quantile function, subfamilies, moments and generating, probability weighted moments, distribution of order statistics, Rényi entropy, and stochastic ordering. Section 3 contains the maximum likelihood method for estimating the unknown parameters, followed by special cases in Section 4. A Monte Carlo simulation study is presented in Section 5. Section 6 presents three applications to real-world data sets, followed by some concluding remarks in Section 7.

2. The New Family of Distributions and Some Properties

In this section, we derive the new Topp-Leone-Gompertz-exponentiated half logistic-G (TL-Gom-EHL-G) family of distributions and some of the statistical properties including sub-families, hazard rate function, series expansion of the density function, quantile function, moments and moment generating function, probability weighted moments, distribution of order statistics, Rényi entropy, and stochastic ordering.

2.1. Topp-Leone-Gompertz-Exponentiated Half Logistic-G Family of Distributions

Consider the exponentiated half logistic-G (EHL-G) family of distributions, see Seo & Kang (2015) and Topp-Leone-Gompertz-G (TL-Gom-G) family of distributions by Oluyede, Chamunorwa, Chipepa & Alizadeh (2022). The cdf of the EHL-G family of distributions is given by

$$F(x;\alpha,\underline{\psi}) = \left(\frac{G(x;\underline{\psi})}{1+\overline{G}(x;\underline{\psi})}\right)^{\alpha}, \tag{3}$$

for $\alpha > 0$, where $\overline{G}(x;\underline{\psi}) = 1 - G(x;\underline{\psi})$ is the survival function with parameter vector $\underline{\psi}$, and the cdf of TL-Gom-G family of distributions is given by

$$F(x;\gamma,b,\underline{\psi}) = \left\{1-\exp\left(\frac{2}{\gamma}\left(1-\left[1-G(x;\underline{\psi})\right]^{-\gamma}\right)\right)\right\}^{b}, \tag{4}$$

for $\gamma, b > 0$, and parameter vector $\underline{\psi}$.

The function $G(x;\underline{\psi})$ in equation (4) is replaced by equation (3) to obtain TL-Gom-EHL-G family of distributions. The cdf of the new TL-Gom-EHL-G family of distributions is given by

$$F(x;b,\gamma,\alpha,\underline{\psi}) = \left[1-\exp\left(\frac{2}{\gamma}\left(1-\left[1-\left(\frac{G(x;\underline{\psi})}{1+\overline{G}(x;\underline{\psi})}\right)^{\alpha}\right]^{-\gamma}\right)\right)\right]^{b}, \tag{5}$$

for $b, \gamma, \alpha > 0$ and parameter vector $\underline{\psi}$. The corresponding pdf is

$$\begin{aligned} f(x;b,\gamma,\alpha,\underline{\psi}) &= 4b\alpha\left[1-\exp\left(\frac{2}{\gamma}\left(1-\left[1-\left(\frac{G(x;\underline{\psi})}{1+\overline{G}(x;\underline{\psi})}\right)^{\alpha}\right]^{-\gamma}\right)\right)\right]^{b-1} \\ &\times \exp\left(\frac{2}{\gamma}\left(1-\left[1-\left(\frac{G(x;\underline{\psi})}{1+\overline{G}(x;\underline{\psi})}\right)^{\alpha}\right]^{-\gamma}\right)\right) \\ &\times \left[1-\left(\frac{G(x;\underline{\psi})}{1+\overline{G}(x;\underline{\psi})}\right)^{\alpha}\right]^{-\gamma-1}\left[1+\overline{G}(x;\underline{\psi})\right]^{-(\alpha+1)} \\ &\times \left[G(x;\underline{\psi})\right]^{\alpha-1} g(x;\underline{\psi}) \end{aligned} \tag{6}$$

for $b, \gamma, \alpha > 0$ and parameter vector $\underline{\psi}$.

2.2. Hazard Rate Function

The hazard rate function (hrf) is a very important concept in survival analysis. It is obtained by dividing the pdf by the survival function. Mathematically,

$$h(x;b,\gamma,\alpha,\underline{\psi}) = f(x;b,\gamma,\alpha,\underline{\psi})/\left(1-F(x;b,\gamma,\alpha,\underline{\psi})\right).$$

The hrf of the TL-Gom-EHL-G family of distributions is given by

$$\begin{aligned} h(x;b,\gamma,\alpha,\underline{\psi}) &= 4b\alpha\left[1-\exp\left(\frac{2}{\gamma}\left(1-\left[1-\left(\frac{G(x:\underline{\psi})}{1+\overline{G}(x;\underline{\psi})}\right)^{\alpha}\right]^{-\gamma}\right)\right)\right]^{b-1} \\ &\times \exp\left(\frac{2}{\gamma}\left(1-\left[1-\left(\frac{G(x;\underline{\psi})}{1+\overline{G}(x;\underline{\psi})}\right)^{\alpha}\right]^{-\gamma}\right)\right)\left[G(x;\underline{\psi})\right]^{\alpha-1} \\ &\times \left[1-\left(\frac{G(x;\underline{\psi})}{1+\overline{G}(x;\underline{\psi})}\right)^{\alpha}\right]^{-\gamma-1}\left[1+\overline{G}(x;\underline{\psi})\right]^{-(\alpha+1)} g(x;\underline{\psi}) \\ &\times \left(1-\left[1-\exp\left(\frac{2}{\gamma}\left(1-\left[1-\left(\frac{G(x;\underline{\psi})}{1+\overline{G}(x;\underline{\psi})}\right)^{\alpha}\right]^{-\gamma}\right)\right)\right]^{b}\right)^{-1}, \end{aligned} \tag{7}$$

for $b, \gamma, \alpha > 0$ and parameter vector $\underline{\psi}$.

2.3. Linear Representation

The pdf of the TL-Gom-EHL-G family of distributions can be expressed an infinite linear combination of exponentiated-G (Exp-G) densities, that is,

$$f(x;b,\gamma,\alpha,\underline{\psi}) = \sum_{q=0}^{\infty} a_{q+1} g_{q+1}(x;\underline{\psi}), \tag{8}$$

where $g_{q+1}(x;\underline{\psi}) = (q+1)[G(x;\underline{\psi})]^q g(x;\underline{\psi})$ is the exponentiated-G (Exp-G) pdf with the power parameter $(q+1)$ and parameter vector $\underline{\psi}$, and

$$\begin{aligned} a_{q+1} &= \sum_{l,i,j,k,m,p=0}^{\infty} \binom{b-1}{l}\binom{i}{j}\binom{\gamma(j+1)+k}{k}(-1)^{l+i+m+p+q}\frac{\left(\frac{2(l+j)}{\gamma}\right)^i}{i!} \\ &\times \binom{\alpha(k+1)+m}{m}\binom{\alpha(k+1)-1}{p}\binom{m+p}{q}\left(\frac{4b\alpha}{q+1}\right). \end{aligned} \tag{9}$$

Consequently, the mathematical and statistical properties of the TL-Gom-EHL-G family of distributions follows directly from those of the exponentiated-G (Exp-G) family of distributions. See the web-appendix for details.

2.4. Quantile Function

Let the random variable X be from the TL-Gom-EHL-G family of distributions, then the quantile function of $Q_X(u)$ can be obtained by solving the non-linear equation:

$$F(x;b,\gamma,\alpha,\underline{\psi}) = \left[1-\exp\left(\frac{2}{\gamma}\left(1-\left[1-\left(\frac{G(x;\underline{\psi})}{1+\overline{G}(x;\underline{\psi})}\right)^{\alpha}\right]^{-\gamma}\right)\right)\right]^{b} = u,$$

for $0 \leq u \leq 1$. Note that (after simplification),

$$G(x;\underline{\psi}) = 2\left[\left(1-\left[1-\frac{\gamma}{2}\log\left(1-u^{\frac{1}{b}}\right)\right]^{\frac{-1}{\gamma}}\right)^{\frac{-1}{\alpha}}+1\right]^{-1}.$$

Therefore, the quantile function of the TL-Gom-EHL-G family of distributions is given by

$$Q_X(u) = G^{-1}\left(2\left[\left(1-\left[1-\frac{\gamma}{2}\log\left(1-u^{\frac{1}{b}}\right)\right]^{\frac{-1}{\gamma}}\right)^{\frac{-1}{\alpha}}+1\right]^{-1}\right). \qquad (10)$$

Consequently, for a given baseline cdf G, equation (10) can be very useful for the generation of random numbers and simulations.

2.5. Sub-Families

In this subsection, we present sub-families of the TL-Gom-EHL-G family of distributions.

- When $\alpha = 1$, we obtain the Topp-Leone-Gompertz-Half Logistic-G (TL-Gom-HL-G) family of distributions.

- When b=1, we obtain a new family of distributions with the cdf

  $$F(x;\gamma,\alpha,\underline{\psi}) = \left[1-\exp\left(\frac{2}{\gamma}\left(1-\left[1-\left(\frac{G(x;\underline{\psi})}{1+\overline{G}(x;\underline{\psi})}\right)^{\alpha}\right]^{-\gamma}\right)\right)\right],$$

  for $\gamma, \alpha > 0$ and parameter vector $\underline{\psi}$.

- When $\gamma = 1$, we obtain a new family of distributions.

- When $b = \gamma = 1$, we obtain a new family of distributions.

- When $b = \alpha = 1$, we obtain a new family of distributions.

- When $\gamma = \alpha = 1$, we obtain a new family of distributions.

- When $b = \gamma = \alpha = 1$, we obtain a new family of distributions.

2.6. Moments and Generating Function

Moments are used to describe the characteristics of a distribution, and moment generating functions aid in the generation of moments of the statistical distributions. This helps in determining the measures of central tendency and dispersion for the new proposed distribution. The moments and moment generation function of the TL-Gom-EHL-G family of distributions are presented in this subsection. Let $Y_{q+1} \sim Exp - G(q+1, \underline{\psi})$, then the s^{th} raw moment, μ'_s of the TL-Gom-EHL-G family of distributions is given by

$$\mu'_n = E(X^s) = \int_{-\infty}^{\infty} x^s f(x) dx = \sum_{q=0}^{\infty} a_{q+1} E(Y^s_{q+1}),$$

where $E(Y^s_{q+1})$ is the s^{th} moment of Y_{q+1} and a_{q+1} is given by equation (9). The moment generating function (MGF), for $|t| < 1$, is given by:

$$M_X(t) = \sum_{q=0}^{\infty} a_{q+1} M_{q+1}(t),$$

where $M_{q+1}(t)$ is the MGF of Y_{q+1} and a_{q+1} is given by equation (9).

2.7. Probability Weighted Moments (PWMs)

In this subsection, we present the probability weighted moments (PWMs) of the TL-Gom-EHL-G family of distribution. The PWMs are the expectation of certain function of a random variable whose mean is known. The primary application of PWMs is in the estimation of parameters for a probability distribution whose inverse form cannot be expressed explicitly. For a more detailed description of PWMs, see Hosking et al. (1985). Let the pdf and cdf of the TL-Gom-EHL-G family of distributions be denoted by $f(x)$ and $F(x)$, respectively. The PWMs of a random variable X is defined by

$$\phi_{n,z} = E\left(X^n \left(F(X)\right)^z\right) = \int_{-\infty}^{\infty} x^n (F(x))^z f(x) dx.$$

We note that

$$\begin{aligned}
(F(x))^z f(x) &= 4b\alpha \left[1 - \exp\left(\frac{2}{\gamma}\left(1 - \left[1 - \left(\frac{G(x;\underline{\psi})}{1+\overline{G}(x:\underline{\psi})}\right)^{\alpha}\right]^{-\gamma}\right)\right)\right]^{b(z+1)-1} \\
&\times \exp\left(\frac{2}{\gamma}\left(1 - \left[1 - \left(\frac{G(x;\underline{\psi})}{1+\overline{G}(x;\underline{\psi})}\right)^{\alpha}\right]^{-\gamma}\right)\right) \\
&\times \left[1 - \left(\frac{G(x;\underline{\psi})}{1+\overline{G}(x;\underline{\psi})}\right)^{\alpha}\right]^{-\gamma-1} \left[1+\overline{G}(x;\underline{\psi})\right]^{-(\alpha+1)} \\
&\times \left[G(x;\underline{\psi})\right]^{\alpha-1} g(x;\underline{\psi}).
\end{aligned}$$

Now following the same steps leading to equation (8), we obtain

$$(F(x))^z f(x) = \sum_{q=0}^{\infty} C_{q+1} g_{q+1}(x;\underline{\psi}),$$

where

$$C_{q+1} = \sum_{l,i,j,k,m,p=0}^{\infty} \binom{b(z+1)-1}{l}\binom{i}{j}\binom{\gamma(j+1)+k}{k}(-1)^{l+i+m+p+q}\frac{\left(\frac{2(l+j)}{\gamma}\right)^i}{i!} \times \binom{\alpha(k+1)+m}{m}\binom{\alpha(k+1)-1}{p}\binom{m+p}{q}\left(\frac{4b\alpha}{q+1}\right).$$

Thus, the probability weighted moments of TL-Gom-EHL-G family of distributions reduces to

$$\phi_{n,z} = E\left(X^n\left(F(X)\right)^z\right) = \sum_{q=0}^{\infty} C_{q+1}\int_{-\infty}^{\infty} x^n g_{q+1}(x;\underline{\psi})dx,$$

where $g_{q+1}(x;\underline{\psi}) = (q+1)[G(x;\underline{\psi})]^q g(x;\underline{\psi})$ is the Exp-G pdf with the power parameter $(q+1)$ and parameter vector $\underline{\psi}$.

2.8. Distribution of Order Statistics

The distribution of the order statistics of the TL-Gom-EHL-G family of distributions are presented in this subsection. Order statistics are very useful in probability and statistics, and have a wide range of applications, including estimating distribution parameters and the distribution of quantiles such as the median, which are derived from the distribution of order statistics. The pdf of the r^{th} order statistic (Arnold et al., 2008) for the TL-Gom-EHL-G family of distributions can be written as

$$f_{r:n}(x) = \frac{n!f(x)}{(r-1)!(n-r)!}\sum_{w=0}^{n-r}(-1)^w\binom{n-r}{w}[F(x)]^{w+r-1}. \quad (11)$$

Using equations (5) and (6), we have

$$\begin{aligned} f(x)[F(x)]^{w+r-1} &= 4b\alpha\left[1-\exp\left(\frac{2}{\gamma}\left(1-\left[1-\left(\frac{G(x;\underline{\psi})}{1+\overline{G}(x;\underline{\psi})}\right)^{\alpha}\right]^{-\gamma}\right)\right)\right]^{b(w+r)-1} \\ &\times \exp\left(\frac{2}{\gamma}\left(1-\left[1-\left(\frac{G(x;\underline{\psi})}{1+\overline{G}(x;\underline{\psi})}\right)^{\alpha}\right]^{-\gamma}\right)\right) \\ &\times \left[1-\left(\frac{G(x;\underline{\psi})}{1+\overline{G}(x;\underline{\psi})}\right)^{\alpha}\right]^{-\gamma-1}\left[1+\overline{G}(x;\underline{\psi})\right]^{-(\alpha+1)} \\ &\times \left[G(x;\underline{\psi})\right]^{\alpha-1} g(x;\underline{\psi}). \end{aligned}$$

Now following the same steps leading to equation (8), we obtain

$$f(x)[F(x)]^{w+r-1} = \sum_{q=0}^{\infty} d_{q+1} g_{q+1}(x; \underline{\psi}), \quad (12)$$

where $g_{q+1}(x;\underline{\psi}) = (q+1)[G(x;\underline{\psi})]^q g(x;\underline{\psi})$ is the Exp-G pdf with the power parameter $(q+1)$ and parameter vector $\underline{\psi}$, and

$$\begin{aligned} d_{q+1} &= \sum_{l,i,j,k,m,p=0}^{\infty} \binom{b(w+r)-1}{l}\binom{i}{j}\binom{\gamma(j+1)+k}{k}(-1)^{l+i+m+p+q}\frac{\left(\frac{2(l+j)}{\gamma}\right)^i}{i!} \\ &\times \binom{\alpha(k+1)+m}{m}\binom{\alpha(k+1)-1}{p}\binom{m+p}{q}\left(\frac{4b\alpha}{q+1}\right). \end{aligned}$$

Thus, by substituting (12) into (11), the pdf of the r^{th} order statistic for the TL-Gom-EHL-G family of distributions can be written as

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!}\sum_{q=0}^{\infty}\sum_{w=0}^{n-r}(-1)^w\binom{n-r}{w} d_{q+1} g_{q+1}(x;\underline{\psi}). \quad (13)$$

2.9. Rényi Entropy

In this subsection, Rényi entropy for TL-Gom-EHL-G family of distributions is derived. Rényi entropy is important in information theory, ecology, and statistics as a measure of diversity. An entropy is a measure of uncertainty or variation of a random variable. Rényi entropy (Rényi, 1961) is an extension of Shannon entropy (Shannon, 1951). Rényi entropy for the TL-Gom-EHL-G family of distributions (see appendix for details) is given by

$$I_R(v) = \frac{1}{1-v}\log\left[\sum_{q=0}^{\infty}\tau_q \exp\left((1-v)I_{REG}\right)\right], \quad (14)$$

for $v > 0, v \neq 1$, where

$$I_{REG} = \frac{1}{1-v}\log\left[\int_0^{\infty}\left(\left[1+\frac{q}{v}\right]\left(G(x;\underline{\psi})\right)^{\frac{q}{v}}\left(g(x;\underline{\psi})\right)\right)^v dx\right]$$

is the Rényi entropy of Exp-G distribution with power parameter $(\frac{q}{v}+1)$ and

$$\begin{aligned} \tau_q &= \sum_{l,i,j,k,m,p=0}^{\infty}\binom{v(b-1)}{l}\binom{i}{j}(-1)^{l+j+m+q}\frac{\left(\frac{2(v+l)}{\gamma}\right)^i}{i!}\binom{\gamma(v+j)+v+k-1}{k} \\ &\times \binom{\alpha(v+k)+v+m-1}{m}\binom{m+p}{q}\binom{\alpha(v+k)-v}{p}\frac{(4b\alpha)^v}{\left[1+\frac{q}{v}\right]^v}. \end{aligned}$$

Therefore, Rényi entropy of the TL-Gom-EHL-G family of distributions can be obtained from those of the Exp-G family of distributions.

2.10. Stochastic Ordering

In this subsection, we present likelihood ratio ordering. Stochastic orderings have many applications in probability and statistics. They are useful in probability theory for deducing probability inequalities and for comparing lifetime distributions with relation to some of their characteristics.

Suppose we have two random variables W and X with distribution functions $F_W(r)$ and $F_X(r)$, respectively, and $\overline{F}_W(r) = 1 - F_W(r)$ the survival function. Note that W is stochastically smaller than X if $\overline{F}_W(r) \leq \overline{F}_X(r)$ for all r or $F_W(r) \geq F_X(r)$ for all r. This is denoted by $W <_{st} X$. Hazard rate order and likelihood ratio order are stronger and are given by $W <_{hr} X$ if $h_W(r) \geq h_X(r)$ for all r, and $W <_{lr} X$ if $\frac{f_W(r)}{f_X(r)}$ is decreasing in r (Shaked & Shanthikumar, 2007). We know that $W <_{lr} X \Rightarrow W <_{hr} X \Rightarrow W <_{st} X$.

Consider X_1 and X_2 to be independent random variables with pdfs $f_{TL-Gom-EHL-G}(x; b_1, \gamma, \alpha, \underline{\psi})$ and $f_{TL-Gom-EHL-G}(x; b_2, \gamma, \alpha, \underline{\psi})$, respectively. If $b_2 > b_1$, then the random variables X_1 and X_2 are stochastically ordered.

Note that,

$$
\begin{aligned}
f_1(x; b_1, \gamma, \alpha, \underline{\psi}) = 4b_1\alpha &\left[1 - \exp\left(\frac{2}{\gamma}\left(1 - \left[1 - \left(\frac{G(x;\underline{\psi})}{1+\overline{G}(x;\underline{\psi})}\right)^{\alpha}\right]^{-\gamma}\right)\right)\right]^{b_1-1} \\
&\times \exp\left(\frac{2}{\gamma}\left(1 - \left[1 - \left(\frac{G(x;\underline{\psi})}{1+\overline{G}(x;\underline{\psi})}\right)^{\alpha}\right]^{-\gamma}\right)\right) \\
&\times \left[1 - \left(\frac{G(x;\underline{\psi})}{1+\overline{G}(x;\underline{\psi})}\right)^{\alpha}\right]^{-\gamma-1} \left[1+\overline{G}(x;\underline{\psi})\right]^{-(\alpha+1)} \\
&\times \left[G(x;\underline{\psi})\right]^{\alpha-1} g(x;\underline{\psi}),
\end{aligned}
$$

$$
\begin{aligned}
f_2(x; b_2, \gamma, \alpha, \underline{\psi}) = 4b_2\alpha &\left[1 - \exp\left(\frac{2}{\gamma}\left(1 - \left[1 - \left(\frac{G(x;\underline{\psi})}{1+\overline{G}(x:\underline{\psi})}\right)^{\alpha}\right]^{-\gamma}\right)\right)\right]^{b_2-1} \\
&\times \exp\left(\frac{2}{\gamma}\left(1 - \left[1 - \left(\frac{G(x;\underline{\psi})}{1+\overline{G}(x;\underline{\psi})}\right)^{\alpha}\right]^{-\gamma}\right)\right) \\
&\times \left[1 - \left(\frac{G(x;\underline{\psi})}{1+\overline{G}(x;\underline{\psi})}\right)^{\alpha}\right]^{-\gamma-1} \left[1+\overline{G}(x;\underline{\psi})\right]^{-(\alpha+1)} \\
&\times \left[G(x;\underline{\psi})\right]^{\alpha-1} g(x;\underline{\psi}),
\end{aligned}
$$

and,

$$\begin{aligned}\frac{f_1(x; b_1, \gamma, \delta, \underline{\psi})}{f_2(x; b_2, \gamma, \delta, \underline{\psi})} &= \frac{f_1(x)}{f_2(x)} \\ &= \frac{b_1}{b_2}\left[1-\exp\left(\frac{2}{\gamma}\left(1-\left[1-\left(\frac{G(x;\underline{\psi})}{1+\overline{G}(x;\underline{\psi})}\right)^{\alpha}\right]^{-\gamma}\right)\right)\right]^{b_1-b_2}.\end{aligned} \tag{15}$$

If we differentiate equation (15) with respect to x, we get

$$\begin{aligned}\frac{d}{dx}\left[\frac{f_1(x)}{f_2(x)}\right] &= \frac{4\alpha b_1}{b_2}(b_1-b_2)\left[1-\exp\left(\frac{2}{\gamma}\left(1-\left[1-\left(\frac{G(x;\underline{\psi})}{1+\overline{G}(x:\underline{\psi})}\right)^{\alpha}\right]^{-\gamma}\right)\right)\right]^{b_1-b_2-1} \\ &\times \exp\left(\frac{2}{\gamma}\left(1-\left[1-\left(\frac{G(x;\underline{\psi})}{1+\overline{G}(x;\underline{\psi})}\right)^{\alpha}\right]^{-\gamma}\right)\right) \\ &\times \left[1-\left(\frac{G(x;\underline{\psi})}{1+\overline{G}(x;\underline{\psi})}\right)^{\alpha}\right]^{-\gamma-1}\left[1+\overline{G}(x;\underline{\psi})\right]^{-(\alpha+1)} \\ &\times \left[G(x;\underline{\psi})\right]^{\alpha-1} g(x;\underline{\psi}).\end{aligned}$$

Consequently, $\frac{d}{dx}\left[\frac{f_1(x)}{f_2(x)}\right] < 0$ if $b_2 > b_1$. We conclude that, $X_1 <_{lr} X_2, X_1 <_{hr} X_2$ and $X_1 <_{st} X_2$, and the random variables X_1 and X_2 are stochastically ordered.

3. Maximum Likelihood Estimation

In this section, we use the maximum likelihood estimation technique to estimate the parameters of the TL-Gom-EHL-G family of distributions. The log-likelihood function $\ell_n = \ell_n(\mathbf{\Delta})$ for the parameters from the observed values has the form

$$\begin{aligned}\ell_n(\mathbf{\Delta}) &= n\ln(4b\alpha) \\ &+ (b-1)\sum_{i=1}^{n}\log\left[1-\exp\left(\frac{2}{\gamma}\left(1-\left[1-\left(\frac{G(x_i;\underline{\psi})}{1+\overline{G}(x_i;\underline{\psi})}\right)^{\alpha}\right]^{-\gamma}\right)\right)\right] \\ &+ \sum_{i=1}^{n}\left(\frac{2}{\gamma}\left(1-\left[1-\left(\frac{G(x_i;\underline{\psi})}{1+\overline{G}(x_i;\underline{\psi})}\right)^{\alpha}\right]^{-\gamma}\right)\right) \\ &+ (\alpha-1)\sum_{i=1}^{n}\log\left[G(x_i;\underline{\psi})\right] \\ &- (\gamma+1)\sum_{i=1}^{n}\log\left[1-\left(\frac{G(x_i;\underline{\psi})}{1+\overline{G}(x_i;\underline{\psi})}\right)^{\alpha}\right] - (\alpha+1)\sum_{i=1}^{n}\log\left[1+\overline{G}(x_i;\underline{\psi})\right] \\ &+ \sum_{i=1}^{n}\log\left(g(x_i;\underline{\psi})\right).\end{aligned} \tag{16}$$

The first derivative of the log-likelihood function with respect to each component of the parameter vector $\mathbf{\Delta} = (b, \gamma, \alpha, \underline{\psi})^T$, that is, elements of the score vector $U(\mathbf{\Delta})$ are given in the appendix. These partial derivatives are not in closed form and the equations obtained from them can be solved using R, MATLAB and SAS software by use of iterative method such as the NewtonRaphson procedure for a specified baseline cdf $G(x; \underline{\psi})$.

4. Some Special Cases

In this section, some special cases of the TL-Gom-EHL-G family of distributions are presented by specifying the baseline distribution to be Weibull, Burr XII, and Lindley distributions, respectively.

4.1. Topp-Leone-Gompertz-Exponentiated Half Logistic-Weibull (TL-Gom-EHL-W) Distribution

Consider the Weibull distribution as the baseline distribution, with cdf and pdf given by $G(x; \lambda) = 1 - e^{-x^{\lambda}}$ and $g(x; \lambda) = \lambda x^{\lambda - 1} e^{-x^{\lambda}}$, respectively, for $\lambda > 0$ and $x > 0$. Then, the cdf of the TL-Gom-EHL-W distribution is given by

$$F(x; b, \gamma, \alpha, \lambda) = \left[1 - \exp\left(\frac{2}{\gamma}\left(1 - \left[1 - \left(\frac{1 - e^{-x^{\lambda}}}{1 + e^{-x^{\lambda}}}\right)^{\alpha}\right]^{-\gamma}\right)\right)\right]^{b},$$

and the corresponding pdf is

$$\begin{aligned} f(x; b, \gamma, \alpha, \lambda) &= 4b\alpha \left[1 - \exp\left(\frac{2}{\gamma}\left(1 - \left[1 - \left(\frac{1 - e^{-x^{\lambda}}}{1 + e^{-x^{\lambda}}}\right)^{\alpha}\right]^{-\gamma}\right)\right)\right]^{b-1} \\ &\times \exp\left(\frac{2}{\gamma}\left(1 - \left[1 - \left(\frac{1 - e^{-x^{\lambda}}}{1 + e^{-x^{\lambda}}}\right)^{\alpha}\right]^{-\gamma}\right)\right) \\ &\times \left[1 - \left(\frac{1 - e^{-x^{\lambda}}}{1 + e^{-x^{\lambda}}}\right)^{\alpha}\right]^{-\gamma-1} \left[1 + e^{-x^{\lambda}}\right]^{-(\alpha+1)} \\ &\times \left[1 - e^{-x^{\lambda}}\right]^{\alpha-1} \lambda x^{\lambda-1} e^{-x^{\lambda}}, \end{aligned}$$

for $b, \gamma, \alpha, \lambda > 0$. The hrf is given by

$$
\begin{aligned}
h(x; b, \gamma, \alpha, \lambda) \quad = \quad & 4b\alpha \left[1 - \exp\left(\frac{2}{\gamma}\left(1 - \left[1 - \left(\frac{1-e^{-x^{\lambda}}}{1+e^{-x^{\lambda}}}\right)^{\alpha}\right]^{-\gamma}\right)\right)\right]^{b-1} \\
\times \quad & \exp\left(\frac{2}{\gamma}\left(1 - \left[1 - \left(\frac{1-e^{-x^{\lambda}}}{1+e^{-x^{\lambda}}}\right)^{\alpha}\right]^{-\gamma}\right)\right) \\
\times \quad & \left[1 - \left(\frac{1-e^{-x^{\lambda}}}{1+e^{-x^{\lambda}}}\right)^{\alpha}\right]^{-\gamma-1} \left[1 + e^{-x^{\lambda}}\right]^{-(\alpha+1)} \\
\times \quad & \left[1 - e^{-x^{\lambda}}\right]^{\alpha-1} \lambda x^{\lambda-1} e^{-x^{\lambda}} \\
\times \quad & \left(1 - \left[1 - \exp\left(\frac{2}{\gamma}\left(1 - \left[1 - \left(\frac{1-e^{-x^{\lambda}}}{1+e^{-x^{\lambda}}}\right)^{\alpha}\right]^{-\gamma}\right)\right)\right]^{b}\right)^{-1},
\end{aligned}
$$

for $b, \gamma, \alpha, \lambda > 0$.

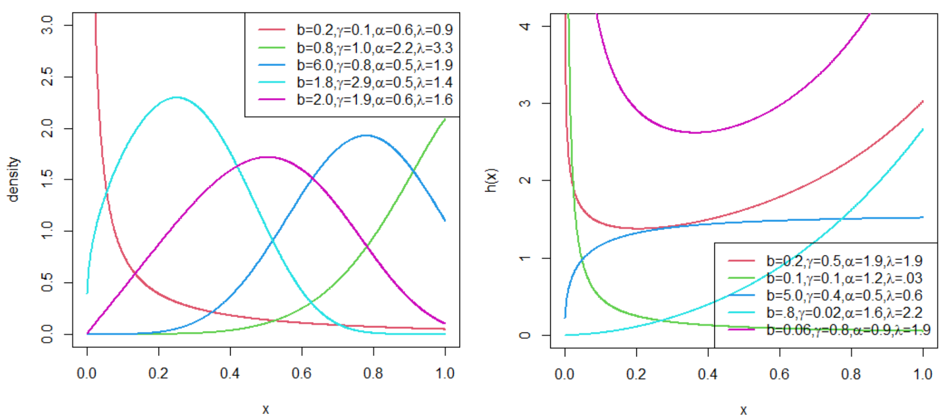

Figure 1: Plots of the pdf and hrf of the TL-Gom-EHL-W distribution

The plots of the pdf and hrf of the TL-Gom-EHL-W distribution are shown in Figure 1. The shape of the pdf can be unimodal, reverse-J, and left or right-skewed. Also, the plot of the hrf for the selected parameter values shows increasing, decreasing, and bathtub shapes. Figures 2 and 3 depicts the skewness and kurtosis plots for the TL-Gom-EHL-W distribution.

- When we fix the parameters b and α, the skewness and kurtosis of TL-Gom-EHL-W distribution increase as γ and λ increase, and

- When we fix the parameters γ and λ, the skewness and kurtosis of TL-Gom-EHL-W distribution increase as b and α increase.

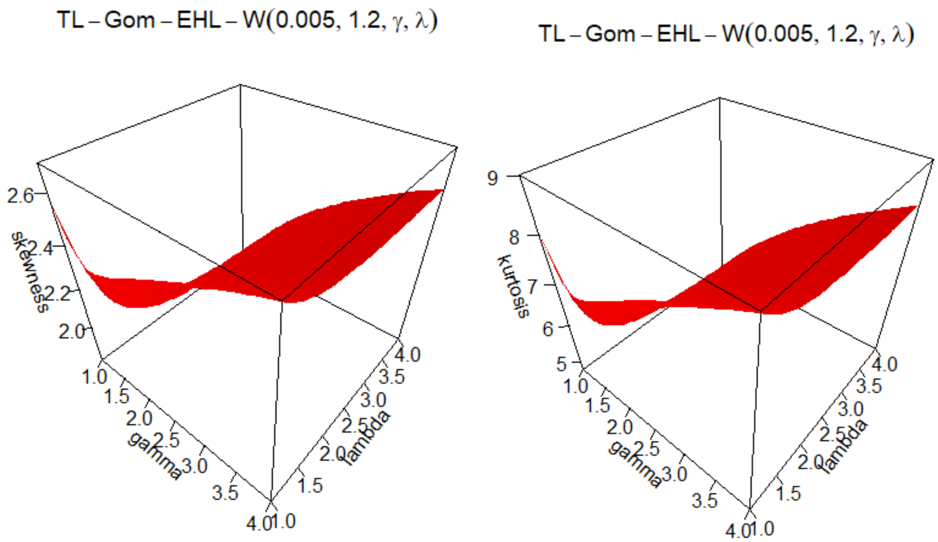

Figure 2: Plots of skewness and kurtosis of the TL-Gom-EHL-W distribution

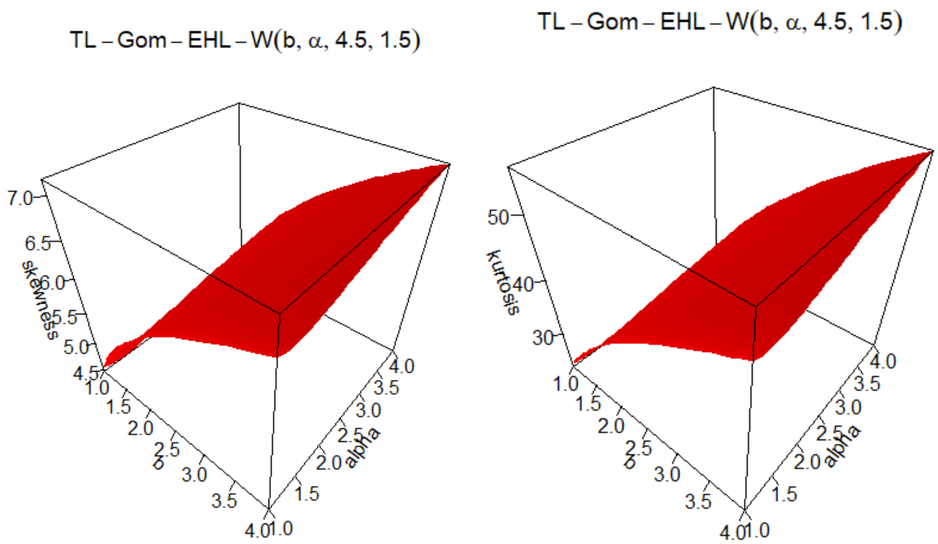

Figure 3: Plots of skewness and kurtosis of the TL-Gom-EHL-W distribution

4.2. Topp-Leone-Gompertz-Exponentiated Half Logistic-Burr XII (TL-Gom-EHL-BXII) Distribution

Consider the Burr XII distribution as the baseline distribution with the cdf and pdf given by $G(x;c,k) = 1-(1+x^c)^{-k}$ and $g(x;c) = cx^{c-1}(1+x^c)^{-k-1}$, respectively, for $c,k>0$. Then, the cdf and pdf of the TL-Gom-EHL-BXII distribution are given by

$$F(x;b,\gamma,\alpha,c,k)=\left[1-\exp\left(\frac{2}{\gamma}\left(1-\left[1-\left(\frac{1-(1+x^c)^{-k}}{1+(1+x^c)^{-k}}\right)^{\alpha}\right]^{-\gamma}\right)\right)\right]^{b},$$

and

$$\begin{aligned}
f(x;b,\gamma,\alpha,c,k) &= 4b\alpha\left[1-\exp\left(\frac{2}{\gamma}\left(1-\left[1-\left(\frac{1-(1+x^c)^{-k}}{1+(1+x^c)^{-k}}\right)^{\alpha}\right]^{-\gamma}\right)\right)\right]^{b-1}\\
&\times \exp\left(\frac{2}{\gamma}\left(1-\left[1-\left(\frac{1-(1+x^c)^{-k}}{1+(1+x^c)^{-k}}\right)^{\alpha}\right]^{-\gamma}\right)\right)\\
&\times \left[1-\left(\frac{1-(1+x^c)^{-k}}{1+(1+x^c)^{-k}}\right)^{\alpha}\right]^{-\gamma-1}\left[1+(1+x^c)^{-k}\right]^{-(\alpha+1)}\\
&\times \left[1-(1+x^c)^{-k}\right]^{\alpha-1}cx^{c-k}(1+x^c)^{-k-1},
\end{aligned}$$

for $b,\gamma,\alpha,c,k>0$.

When $c=1$, we obtain the Topp-Leone-Gompertz-exponentiated half logistic-Lomax (TL-Gom-EHL-Lomax) distribution and when $k=1$, we obtain the Topp-Leone-Gompertz-exponentiated half logistic-log-logistic (TL-Gom-EHL-LLoG) distribution.

The hrf of the TL-Gom-EHL-BXII distribution is given by

$$\begin{aligned}
h(x;b,\gamma,\alpha,c,k) &= 4b\alpha\left[1-\exp\left(\frac{2}{\gamma}\left(1-\left[1-\left(\frac{1-(1+x^c)^{-k}}{1+(1+x^c)^{-k}}\right)^{\alpha}\right]^{-\gamma}\right)\right)\right]^{b-1}\\
&\times \exp\left(\frac{2}{\gamma}\left(1-\left[1-\left(\frac{1-(1+x^c)^{-k}}{1+(1+x^c)^{-k}}\right)^{\alpha}\right]^{-\gamma}\right)\right)\\
&\times \left[1-\left(\frac{1-(1+x^c)^{-k}}{1+(1+x^c)^{-k}}\right)^{\alpha}\right]^{-\gamma-1}\left[1+(1+x^c)^{-k}\right]^{-(\alpha+1)}\\
&\times \left[1-(1+x^c)^{-k}\right]^{\alpha-1}cx^{c-1}(1+x^c)^{-k-1}\\
&\times \left(1-\left[1-\exp\left(\frac{2}{\gamma}\left(1-\left[1-\left(\frac{1-(1+x^c)^{-k}}{1+(1+x^c)^{-k-1}}\right)^{\alpha}\right]^{-\gamma}\right)\right)\right]^{b}\right)^{-1},
\end{aligned}$$

for $b,\gamma,\alpha,c,k>0$. The plots of the pdf and hrf of the TL-Gom-EHL-BXII distribution are shown in Figure 4. The pdf can be reverse-J, left-skewed, right-skewed and almost symmetric. Also, the shapes of the hrf for the TL-Gom-EHL-BXII distribution include bathtub, upside-down bathtub, bathtub followed by upside-down bathtub, increasing, and decreasing. Figures 5 and 6 depicts the skewness and kurtosis plots for the TL-Gom-EHL-BXII distribution.

- When we fix the parameters b,γ and α, the skewness and kurtosis of TL-Gom-EHL-BXII distribution decrease as c and k increase, and
- When we fix the parameters b,c and k, the skewness and kurtosis of TL-Gom-EHL-BXII distribution decrease as α and γ increase.

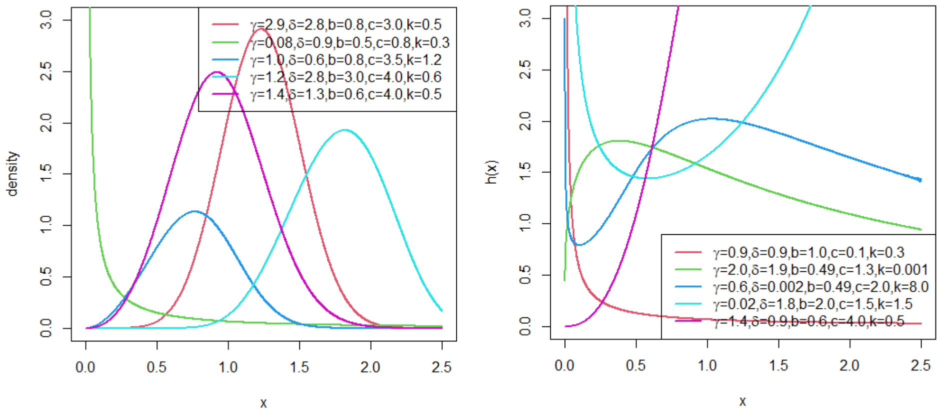

FIGURE 4: Plots of the pdf and hrf of the TL-Gom-EHL-BXII distribution

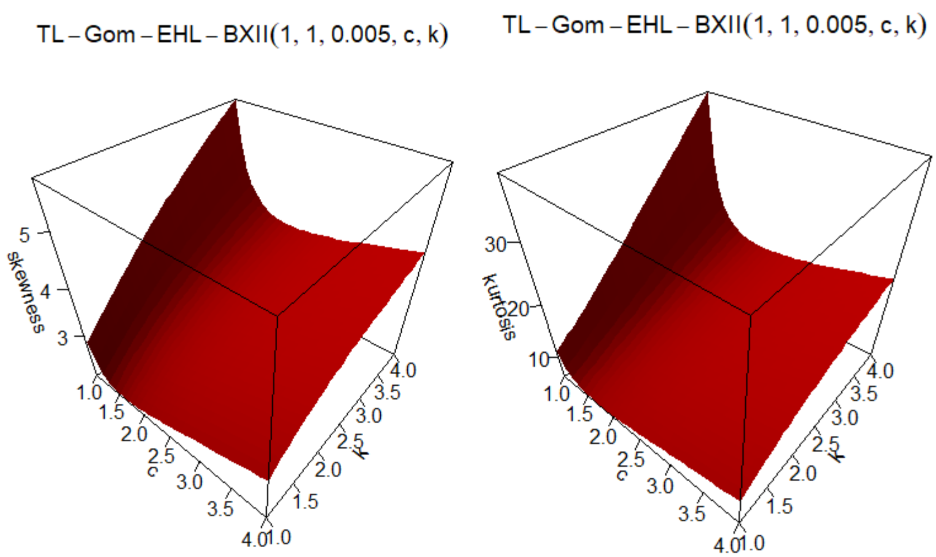

FIGURE 5: Plots of skewness and kurtosis of the TL-Gom-EHL-BXII distribution

4.3. Topp-Leone-Gompertz Exponentiated Half Logistic-Lindley (TL-Gom-EHL-L) Distribution

Taking baseline distribution to be Lindley distribution with cdf and pdf given by $G(x;\lambda) = 1 - (1 + \frac{\lambda x}{1+\lambda})e^{-\lambda x}$ and $g(x;\lambda) = \frac{\lambda^2}{(1+\lambda)}(1+x)e^{-\lambda x}$, for $\lambda > 0$, the cdf and pdf of the TL-Gom-EHL-L distribution are given by

$$F(x;b,\gamma,\alpha,\lambda) = \left[1 - \exp\left(\frac{2}{\gamma}\left(1 - \left[1 - \left(\frac{1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}{1+(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}\right)^{\alpha}\right]^{-\gamma}\right)\right)\right]^{b},$$

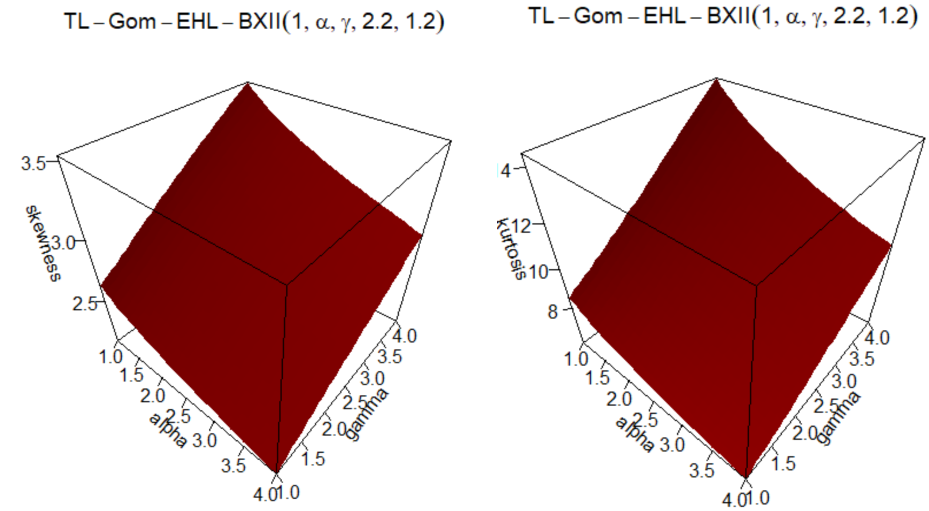

FIGURE 6: Plots of skewness and kurtosis of the TL-Gom-EHL-BXII distribution

and

$$\begin{aligned}
f(x;b,\gamma,\alpha,\lambda) = 4b\alpha & \left[1-\exp\left(\frac{2}{\gamma}\left(1-\left[1-\left(\frac{1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}{1+(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}\right)^{\alpha}\right]^{-\gamma}\right)\right)\right]^{b-1} \\
& \times \exp\left(\frac{2}{\gamma}\left(1-\left[1-\left(\frac{1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}{1+(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}\right)^{\alpha}\right]^{-\gamma}\right)\right) \\
& \times \left[1-\left(\frac{1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}{1+(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}\right)^{\alpha}\right]^{-\gamma-1} \left[1+(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}\right]^{-(\alpha+1)} \\
& \times \left[1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}\right]^{\alpha-1} \frac{\lambda^2}{(1+\lambda)}(1+x)e^{-\lambda x},
\end{aligned}$$

for $b,\gamma,\alpha,\lambda>0$. The hrf of the TL-Gom-EHL-L distribution is given by

$$\begin{aligned}
h(x;b,\gamma,\alpha,\lambda) = 4b\alpha & \left[1-\exp\left(\frac{2}{\gamma}\left(1-\left[1-\left(\frac{1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}{1+(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}\right)^{\alpha}\right]^{-\gamma}\right)\right)\right]^{b-1} \\
& \times \exp\left(\frac{2}{\gamma}\left(1-\left[1-\left(\frac{1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}{1+(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}\right)^{\alpha}\right]^{-\gamma}\right)\right) \\
& \times \left[1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}\right]^{\alpha-1} \left[1-\left(\frac{1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}{1+(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}\right)^{\alpha}\right]^{-\gamma-1} \\
& \times \left[1+(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}\right]^{-(\alpha+1)} \frac{\lambda^2}{(1+\lambda)}(1+x)e^{-\lambda x} \\
& \times \left(1-\left[1-\exp\left(\frac{2}{\gamma}\left(1-\left[1-\left(\frac{1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}{1+(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}\right)^{\alpha}\right]^{-\gamma}\right)\right)\right]^{b}\right)^{-1},
\end{aligned}$$

for $b,\gamma,\alpha,\lambda>0$.

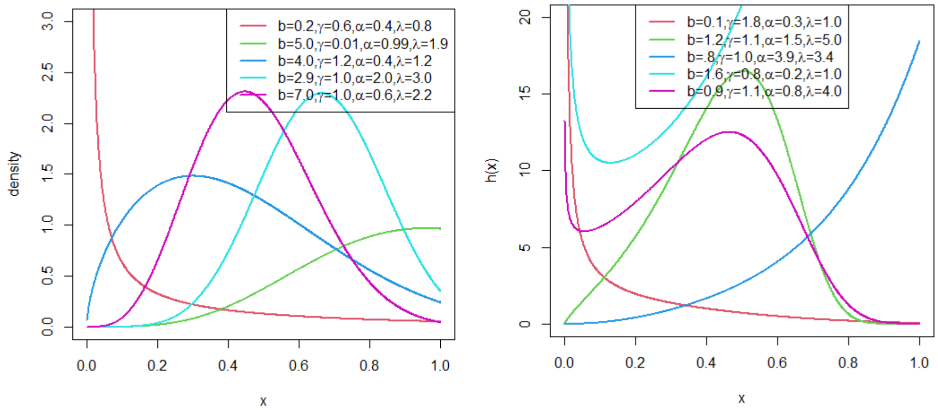

Figure 7: Plots of the pdf and hrf of the TL-Gom-EHL-L distribution

The plots of the pdf and hrf of the TL-Gom-EHL-L distribution are shown in Figure 7. The pdf can be reverse-J, left-skewed, right-skewed and almost symmetric. Also, the hrf for the TL-Gom-EHL-L distribution show bathtub, bathtub followed by upside-down bathtub, upside-down bathtub, increasing and decreasing shapes. Figures 8 and 9 depicts the skewness and kurtosis plots for the

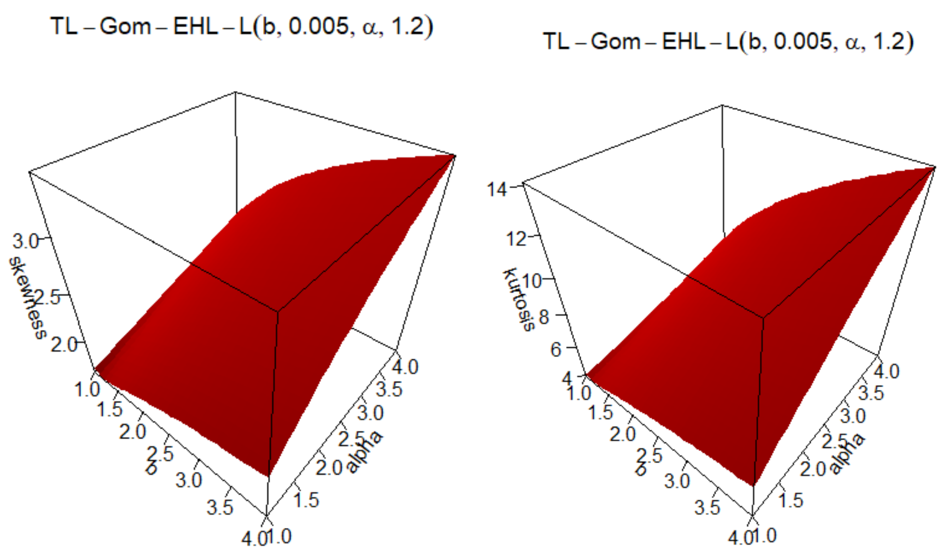

Figure 8: Plots of skewness and kurtosis of the TL-Gom-EHL-L distribution

TL-Gom-EHL-L distribution.

- When we fix the parameters γ and λ, the skewness and kurtosis of TL-Gom-EHL-L distribution increase as b and α increase, and

- When we fix the parameters γ and α, the skewness and kurtosis of TL-Gom-EHL-L distribution decrease and then increase as b and λ increase.

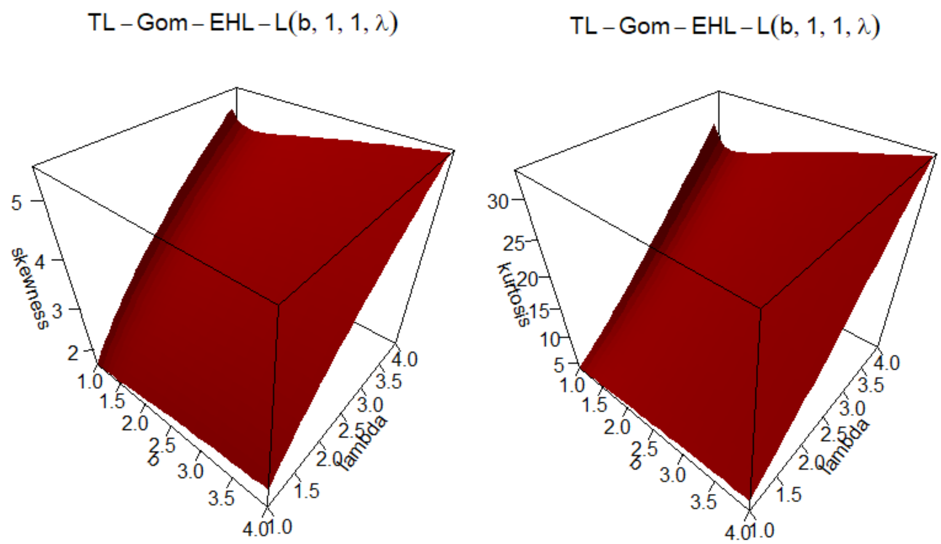

FIGURE 9: Plots of skewness and kurtosis of the TL-Gom-EHL-L distribution

5. Simulation Study

The performance of the TL-Gom-EHL-W distribution is investigated in this section by running various simulations for different sizes (n = 35, 50, 100, 200, 400, 800, 1000, and 1600) using the R software program. For the true parameter values shown in Table 1, we simulate $N = 3000$ samples. The table shows mean MLEs of the model parameters, as well as the Average bias (ABias) and root mean squared error (RMSE). The mean MLEs, RMSE, and ABias measures are used to assess the accuracy of maximum likelihood estimates. The ABias and RMSE of an estimated parameter, say $\hat{\eta}$ are given by

$$ABias(\hat{\eta}) = \frac{\sum_{i=1}^{N} \hat{\eta}_i}{N} - \eta, \quad and \quad RMSE(\hat{\eta}) = \sqrt{\frac{\sum_{i=1}^{N} (\hat{\eta}_i - \eta)^2}{N}}.$$

Tables 1 shows that as sample sizes increase, the mean estimates are consistent as the mean estimates tend to be closer to the true parameter values, while the RMSE and ABias decrease.

6. Applications

In this section, three data examples are provided to illustrate the significance and importance of the TL-Gom-EHL-G family of distributions. This is done by fitting the TL-Gom-EHL-W distribution, which is a special case of the TL-Gom-EHL-G family of distributions. The Adequacy Model package in R software (Team, 2022) was used to evaluate model performance and the `nlm` package in R software

Table 1: Monte Carlo Simulation Results

Parameter	n	(0.4,1.5,0.5,2.0)			(1.0,1.0,0.5,1.2)			(0.4,0.4,0.4,3.0)		
		Mean	RMSE	ABias	Mean	RMSE	ABias	Mean	RMSE	ABias
b	35	0.7013	0.6856	0.3013	1.1865	0.8211	0.1865	0.5490	0.7060	0.1490
	50	0.6480	0.5509	0.2480	1.2029	0.9241	0.2029	0.6463	1.3881	0.2463
	100	0.5523	0.3319	0.1523	1.1791	0.6645	0.1791	0.5076	0.3106	0.1076
	200	0.4912	0.2430	0.0911	1.1384	0.5665	0.1384	0.4645	0.1854	0.0645
	400	0.4712	0.1945	0.0712	1.1212	0.4397	0.1212	0.4614	0.1489	0.0614
	800	0.4363	0.1453	0.0363	1.0777	0.3365	0.0777	0.4391	0.0740	0.0391
	1600	0.4335	0.1239	0.0335	1.0467	0.2780	0.0467	0.4252	0.0431	0.025
α	35	2.5807	2.6112	1.0807	1.7094	1.2575	0.7094	0.6205	0.8073	0.2205
	50	2.6017	2.5605	1.1017	1.6563	1.2025	0.6563	0.6779	1.3100	0.2779
	100	2.1305	1.7630	0.6305	1.4505	0.8586	0.4505	0.5448	0.2901	0.1448
	200	1.8671	1.2301	0.3671	1.2558	0.5752	0.2558	0.5004	0.1986	0.1004
	400	1.7155	0.6813	0.2155	1.1412	0.3581	0.1412	0.4945	0.1615	0.0945
	800	1.7074	0.6344	0.2074	1.1045	0.2544	0.1045	0.4768	0.0928	0.0768
	1600	1.5272	0.4163	0.0272	1.0632	0.1649	0.0632	0.4134	0.0134	0.0134
γ	35	1.2483	1.3474	0.8483	0.7918	0.7378	0.2918	0.8094	4.3895	0.4094
	50	1.2152	1.2755	0.8152	0.7296	0.6612	0.2296	0.5330	0.3967	0.1330
	100	1.1608	1.1863	0.7608	0.6327	0.3490	0.1327	0.5269	0.3451	0.1269
	200	0.8542	0.8290	0.4542	0.5894	0.2654	0.0894	0.4791	0.1933	0.0791
	400	0.6770	0.5204	0.2770	0.5353	0.1954	0.0353	0.4543	0.1400	0.0543
	800	0.5660	0.3298	0.1660	0.5064	0.1355	0.0064	0.4283	0.0589	0.0283
	1600	0.4946	0.2248	0.0946	0.4881	0.1056	-0.0118	0.4230	0.0433	0.023
λ	35	1.2483	1.3474	0.8483	1.0314	0.5533	-0.1685	2.5847	1.3418	-0.4152
	50	1.6826	0.6984	-0.3173	1.0215	0.5189	-0.1784	2.6764	1.1238	-0.3235
	100	1.6841	0.7204	-0.3158	1.0181	0.4681	-0.1818	2.6797	0.9464	-0.3202
	200	1.7443	0.5902	-0.2556	1.0671	0.3672	-0.1328	2.6955	0.7702	-0.3044
	400	1.8027	0.5292	-0.1972	1.0834	0.3223	-0.1165	2.8581	0.6519	-0.1418
	800	1.9042	0.4507	-0.0957	1.1081	0.3105	-0.0918	2.9947	0.0378	-0.0052
	1600	1.9888	0.3846	-0.0111	1.1513	0.2702	-0.0486	3.0103	0.0103	0.0103

was used to estimate model parameters using the maximum likelihood estimation technique (Marinho et al., 2019).

These goodness-of-fit statistics are: -2 log-likelihood statistic ($-2\ln(L)$), Akaike Information Criterion ($AIC = 2p{-}2\ln(L)$), Bayesian Information Criterion ($BIC = p\ln(n) - 2\ln(L)$), $AICC{=}AIC + 2\frac{p(p+1)}{n-p-1}$, where $L = L(\hat{\Delta})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameter, Cramér-von Mises (W^*), Anderson-Darling (A^*), Kolmogorov-Smirnov (K-S) statistic, as well as its associated p-value and Sum of Squares (SS) from the probability plots were also used to assess goodness-of-fit. The Sum of Squares (SS) from the probability plots (Chambers et al., 1983) is given by

$$SS = \sum_{j=1}^{n} \left[F(x_{(j)}) - \left(\frac{j - 0.375}{n + 0.25} \right) \right]^2,$$

where $j = 1, 2 \ldots, n$ and $x_{(j)}$ are the ordered values of the observed data. The best-fitting model is the one with the smallest goodness-of-fit statistics and the highest p-value for the K-S statistic.

The total time on test (TTT) or its scaled TTT transform proposed by Aarset (1987) is used to evaluate the empirical behaviour of the hrf. For constant hazard rates, it is a straight diagonal, convex for decreasing hazard rates, and concave for increasing hazard rates. If the hazard rate is shaped like a bathtub, it is first convex and then concave. If the hazard rate is an upside-down bathtub, it is concave at first and then convex at the end.

The TL-Gom-EHL-W distribution is compared to some selected non-nested models, namely, Topp-Leone-Marshall-Olkin-Weibull (TL-MO-W) distribution by Aldahlan & Afify (2018), Topp-Leone generated Weibull (TL-GW) distribution by Aryal et al. (2017), Topp-Leone-odd Burr III-log-logistic (TL-OBIII-LLoG) distribution by Moakofi et al. (2022), Kumaraswamy-Weibull (K-W) distribution by Cordeiro et al. (2010), and odd generalised half logistic Weibull-Weibull (OGHLWW) distribution by Chipepa et al. (2020), and their pdfs are as follows:

$$f_{TL-MO-W}(x; b, \alpha, \lambda, \beta) = \frac{2b\alpha^2\beta\lambda^\beta x^{\beta-1}e^{-2(\lambda x)^\beta}}{\left(1-\bar{\alpha}e^{-(\lambda x)^\beta}\right)^3}\left(1-\left(1-\bar{\alpha}e^{-(\lambda x)^\beta}\right)^2\right)^{b-1},$$

for b, α, λ, $\beta > 0$, $\bar{\alpha} = 1 - \alpha$ and $x > 0$,

$$\begin{aligned} f_{TL-GW}(x; \alpha, \theta, \lambda, \beta) &= 2\alpha\theta\beta\lambda^\beta x^{\beta-1}\exp(-(\lambda x)^\beta)(1-\exp(-(\lambda x)^\beta))^{\theta\alpha-1} \\ &\times \left[1-(1-\exp(-(\lambda x)^\beta)\right]^\theta\left[2-(1-\exp(-(\lambda x)^\beta))^\theta\right]^{\alpha-1}, \end{aligned}$$

for α, θ, λ, $\beta > 0$ and $x > 0$,

$$\begin{aligned} f_{TL-OBIII-LLoG}(x; \alpha, \beta, b, \lambda) = 2\alpha\beta b &\left[1-\left(1-\left(1+\left(\frac{1-(1+x)^{-1}}{(1+x)^{-1}}\right)^{-\alpha}\right)^{-\beta}\right)^2\right]^{b-1} \\ &\times \left(1-\left(1+\left(\frac{1-(1+x)^{-1}}{(1+x)^{-1}}\right)^{-\alpha}\right)^{-\beta}\right)\frac{g(x;\lambda)}{((1+x^\lambda)^{-1})^2} \\ &\times \left(1+\left(\frac{1-(1+x)^{-1}}{(1+x)^{-1}}\right)^{-\alpha}\right)^{-\beta-1}\left(\frac{1-(1+x)^{-1}}{(1+x)^{-1}}\right)^{-\alpha-1}, \end{aligned}$$

for α, β, b, $\lambda > 0$ and $x > 0$,

$$f_{K-W}(x; \alpha, \theta, \lambda, \beta) = ab\beta\alpha^\beta x^{\beta-1}e^{-\alpha x^\beta}(1-e^{-\alpha x^\beta})^{a-1}(1-(1-e^{-\alpha x^\beta})^a)^{b-1},$$

for α, θ, λ, $\beta > 0$ and $x > 0$, and

$$f_{OGHLWW}(x; \alpha, \beta, \lambda, \gamma) = \frac{2\alpha\beta\lambda\gamma x^{\gamma-1}e^{-\lambda x^\gamma}\left(1-e^{-\lambda x^\gamma}\right)^{\beta-1}\exp\left[-\alpha\left(\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right)^\beta\right]}{e^{-(\beta+1)\lambda x^\gamma}\left(1+\exp\left[-\alpha\left(\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right)\right]\right)^2},$$

for α, β, λ, $\gamma > 0$.

We also perform likelihood ratio (LR) test, to compare TL-Gom-EHL-W distribution with its nested models. The nested models considered in this paper are obtained by setting some of the parameters of the TL-Gom-EHL-W distribution to unit.

6.1. Mexico COVID-19 Data

The first data set relates to the mortality rates of patients infected by the COVID-19 pandemic in Mexico, (see `https://covid19.wh`). The data points are given in the web-appendix.

The estimated variance-covariance matrix is given by

$$\begin{pmatrix} 79.7448 & 7.6906 & -1.2466 & -0.2597 \\ 7.6906 & 0.7416 & -0.1202 & -0.0250 \\ -1.2466 & -0.1202 & 0.0200 & 0.0041 \\ -0.2597 & -0.0250 & 0.0041 & 0.0008 \end{pmatrix}$$

and the approximate 95% two-sided confidence intervals for b, γ, α and λ are given by 11.3380 ± 17.5027, 14.0725 ± 1.6879, 2.3646 ± 0.2778 and 0.1000 ± 0.0581, respectively.

The maximum likelihood estimates of Mexico COVID-19 data set and standard errors in parenthesis are shown in Table 2.

Table 2: Estimates of Models for Mexico COVID-19 Data

Model	Estimates			
	b	γ	α	λ
TL-Gom-EHL-W$(b, \gamma, \alpha, \lambda)$	11.3380	14.0725	2.3646	0.1000
	(8.9299)	(0.8612)	(0.1417)	(0.0298)
TL-Gom-EHL-W$(b, \gamma, 1, \lambda)$	19.2720	1.4194×10^{-08}	-	0.9999
	(1.9575)	(0.0149)	-	(0.3729)
TL-Gom-EHL-W$(1, \gamma, \alpha, \lambda)$	-	1.7037	3.7624	0.5000
	-	(2.01339)	(0.3767)	(0.1345)
TL-Gom-EHL-W$(b, 1, \alpha, \lambda)$	523.6400	-	0.3063	0.1553
	(4.8459×10^{-08})	-	(0.0038)	(0.0098)
TL-Gom-EHL-W$(b, 1, 1, \lambda)$	0.9617	-	-	0.2745
	(0.0935)	-	-	(0.2745)
TL-MO-W$(b, \alpha, \lambda, \beta)$	3.1274	642.6500	5.2910	0.2502
	(0.0335)	(0.0035)	(0.0002)	(0.0940)
TL-GW$(\alpha, \theta, \lambda, \beta)$	0.5385	540.3600	406.9000	0.2508
	(0.1086)	(8.9958×10^{-05})	(6.0018×10^{-05})	(0.0043)
TL-OBIII-LLoG$(\alpha, \beta, b, \lambda)$	1.0002	15.7024	0.2835	1.7205
	(3.2378)	(15.0574)	(0.2901)	(5.5692)
K-W$(\alpha, \theta, \lambda, \beta)$	24.1993	431.7437	0.1847	2.8364
	(5.3787)	(0.0146)	(0.0196)	(2.5335)
OGHLWW$(\alpha, \beta, \lambda, \gamma)$	2.4172×10^{-05}	0.5020	18.4240	0.1446
	(2.4172×10^{-05})	(0.5020)	(18.4240)	(0.1446)

In Table 3 we observe that the TL-Gom-EHL-W distribution has the lowest values for the goodness-of-fit statistics for the first data set and the highest p-value for the K-S statistics when compared to the selected non-nested models.

TABLE 3: Goodness-of-Fit Statistics for Mexico COVID-19 Data

Model	Estimates								
	$-2\log$	AIC	AICC	BIC	W*	A*	K-S	p-value	SS
TL-Gom-EHL-W$(b,\gamma,\alpha,\lambda)$	376.4615	384.4615	384.8475	395.1152	0.0537	0.2970	0.0659	0.7459	0.0560
TL-Gom-EHL-W$(b,\gamma,1,\lambda)$	569.7615	575.7615	576.0001	583.7552	0.0913	0.5144	0.2814	1.0220×10^{-07}	2.6631
TL-Gom-EHL-W$(1,\gamma,\alpha,\lambda)$	384.8548	390.8548	391.0901	398.8451	0.1206	0.7904	0.0714	0.6508	0.1566
TL-Gom-EHL-W$(b,1,\alpha,\lambda)$	384.4620	390.4620	390.6973	398.4523	0.1303	0.8094	0.0847	0.4311	0.1255
TL-Gom-EHL-W$(b,1,1,\lambda)$	728.4274	732.7274	734.0841	739.2945	0.0599	0.3566	0.7846	2.2000×10^{-16}	24.7074
TL-MO-W(b,α,λ,β)	378.7113	386.7113	387.1073	397.3651	0.0739	0.4125	0.0682	0.7064	0.0709
TL-GW$(\alpha,\theta,\lambda,\beta)$	379.9907	387.9907	388.3867	398.6444	0.08558	0.5084	0.0736	0.6130	0.0783
TL-OBIII-LLoG(α,β,b,λ)	382.4029	390.4029	390.8023	401.0600	0.1105	0.6657	0.0823	0.4688	0.1059
K-W$(\alpha,\theta,\lambda,\beta)$	378.1836	386.1847	386.5828	396.8405	0.0693	0.4241	0.0743	0.6007	0.0693
OGHLWW$(\alpha,\beta,\lambda,\gamma)$	390.2730	398.2730	398.6690	408.9268	0.1862	1.2028	0.0808	0.4923	0.1510

Figure 10 shows the flexibility of the TL-Gom-EHL-W distribution in fitting heavy-tailed data, such as the Mexico COVID-19 data set.

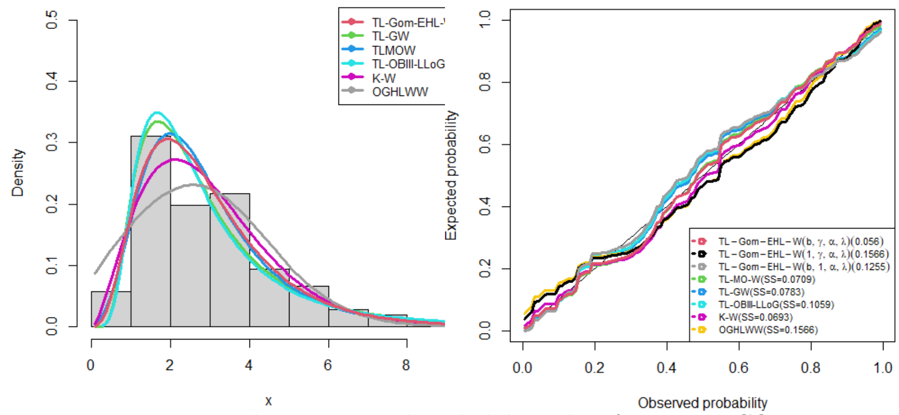

FIGURE 10: Fitted Densities and Probability Plots for Mexico COVID-19 Data

Figure 11 show the TTT plot for Mexico COVID-19 data set with increasing hazard rate function.

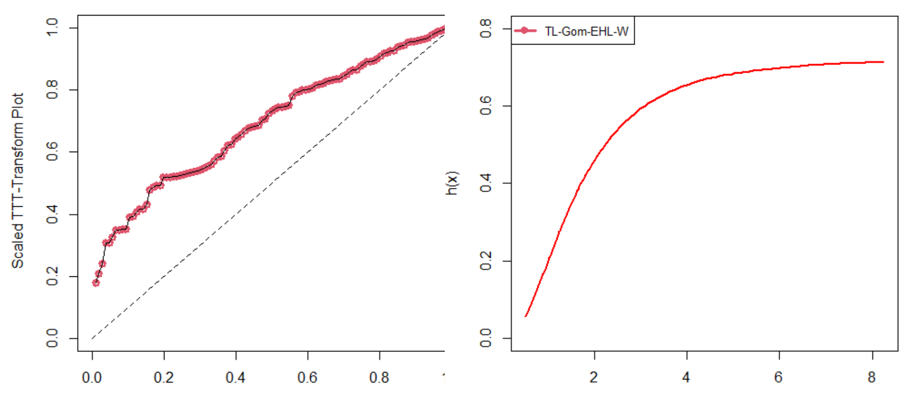

FIGURE 11: Fitted TTT and Hrf Plots for Mexico COVID-19 Data

The Kaplan-Meier (K-M) survival and Empirical Cumulative Function (ECDF) curves for Mexico COVID-19 data set are shown in Figure 12. Based on the closeness of the empirical and fitted lines in Figure 12, we conclude that TL-Gom-EHL-W adequately fits the Mexico COVID-19 data set.

We show the profile log-likelihood functions of the maximum likelihood estimates of b, γ, α, and λ in Figure 13. Figure 13 shows that the maximum likelihood estimates for the TL-Gom-EHL-W distribution exist and can be obtained uniquely.

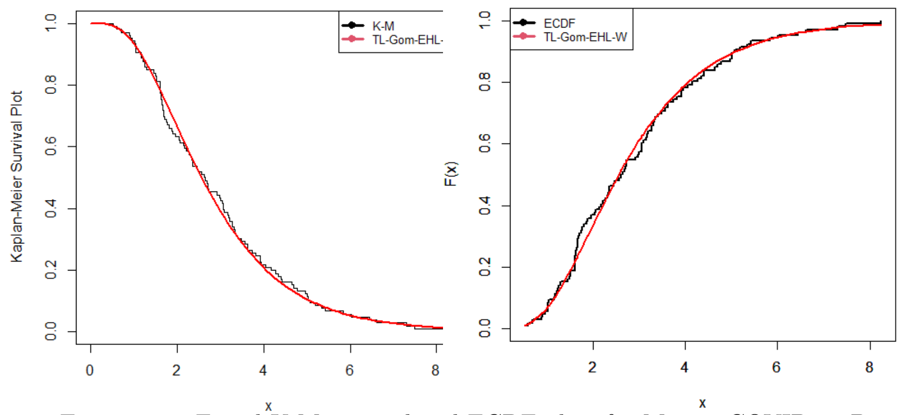

FIGURE 12: Fitted K-M survival and ECDF plots for Mexico COVID-19 Data

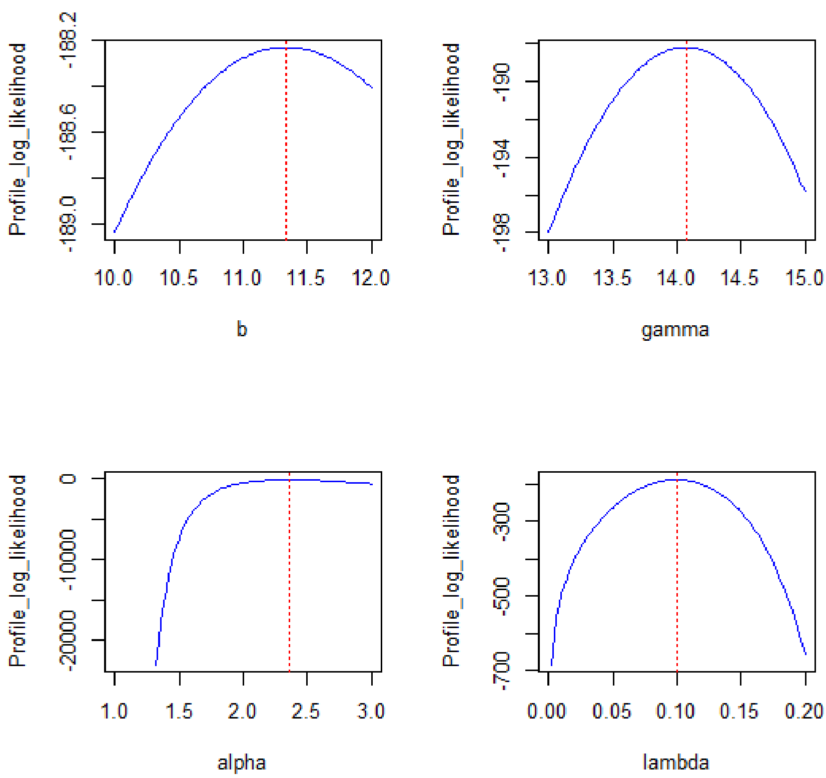

FIGURE 13: Plots of the profile log-likelihood functions of the parameters of the TL-Gom-EHL-W distribution on Mexico COVID-19 data

6.2. Earthquakes Data

The current data is derived from Castillo et al. (2005). The data represents the time in days between successive major earthquakes around the world. An earthquake is included if its magnitude was at least 7.5 on the Richter scale or if over 1000 people were killed. There were 63 recorded earthquakes in total, resulting in 62 recorded waiting times. The data observations are shown in the web-appendix.

The estimated variance-covariance matrix is given by

$$\begin{pmatrix} 0.0115 & -0.0959 & 0.0004 & 8.0256\text{x}10^{-04} \\ -0.0959 & 1.7847 & -0.0047 & -0.0124 \\ 0.0004 & -0.0047 & 1.6437\text{x}10^{-05} & 3.6630\text{x}10^{-05} \\ 8.0256\text{x}10^{-04} & -0.0124 & 3.6630\text{x}10^{-05} & 9.9867\text{x}10^{-05} \end{pmatrix}$$

and the approximate 95% two-sided confidence intervals for b, γ, α and λ are given by 0.5559 ± 0.2102, 2.1649 ± 2.6184, 20.8754 ± 0.0079 and 0.1928 ± 0.0195, respectively.

Table 4 shows the maximum likelihood estimates of the earthquake dataset and standard errors in parentheses.

Table 4: Estimates of Models for Earthquakes Data

Model	Estimates			
	b	γ	α	λ
TL-Gom-EHL-W$(b, \gamma, \alpha, \lambda)$	0.5559	2.1649	20.8754	0.1928
	(0.1072)	(1.3359)	(0.0.0040)	(0.0099)
TL-Gom-EHL-W$(b, \gamma, 1, \lambda)$	57.0048	0.6808	-	0.1158
	(23.0494)	(0.6032)	-	(0.0374)
TL-Gom-EHL-W$(1, \gamma, \alpha, \lambda)$	-	0.0372	15.7053	0.2100
	-	(0.3255)	(2.7364)	(0.0110)
TL-Gom-EHL-W$(b, 1, \alpha, \gamma)$	445.3400	-	0.6233	0.0791
	(8.3245×10^{-06})	-	(0.0274)	(0.0051)
TL-Gom-EHL-W$(b, 1, 1, \lambda)$	70.4852	-	-	0.0986
	(13.3741)	-	-	(0.0039)
TL-MO-W$(b, \alpha, \lambda, \beta)$	547.2100	0.1673	0.5013	0.2241
	(9.9610×10^{-05})	(0.1516)	(0.3835)	(0.0627)
TL-GW$(\alpha, \theta, \lambda, \beta)$	0.4075	96.44418	2.7799	0.1991
	(0.3843)	(0.1006)	(8.4110)	(0.0624)
TL-OBIII-LLoG$(\alpha, \beta, b, \lambda)$	5.5447	91.0270	0.4259	0.1212
	(0.0001)	(0.0003)	(0.0991)	(0.1212)
K-W$(\alpha, \theta, \lambda, \beta)$	5.8865	6.0009	0.3209	0.0053
	(3.1638)	(2.8003)	(0.0742)	(0.0094)
OGHLWW$(\alpha, \beta, \lambda, \gamma)$	0.1952	0.0941	0.9617	0.4938
	(0.1254)	(0.1477)	(1.0987)	(0.1451)

Table 5 shows that the TL-Gom-EHL-W distribution has the lowest values for the goodness-of-fit statistics and the highest p-value for the K-S statistics when compared to the selected non-nested models used on the earthquake data set.

TABLE 5: Goodness-of-Fit Statistics for Earthquakes Data

Model	Estimates								
	$-2\log$	AIC	AICC	BIC	W*	A*	K-S	p-value	SS
TL-Gom-EHL-W$(b,\gamma,\alpha,\lambda)$	876.1842	884.1842	884.8859	892.6927	0.0373	0.3033	0.0624	0.9691	0.0358
TL-Gom-EHL-W$(b,\gamma,1,\lambda)$	886.0019	892.0019	892.4156	898.3833	0.1973	1.1990	0.1103	0.4369	0.2034
TL-Gom-EHL-W$(1,\gamma,\alpha,\lambda)$	883.7150	889.7150	890.1287	896.0964	0.1616	0.9962	0.0957	0.6201	0.1583
TL-Gom-EHL-W$(b,1,\alpha,\lambda)$	889.5642	895.5641	895.9779	901.9455	0.2474	1.4840	0.1187	0.3465	0.2381
TL-Gom-EHL-W$(b,1,1,\lambda$	886.2214	890.2214	890.4248	894.4757	0.2005	1.2172	0.1111	0.4275	0.2065
TL-MO-W(b,α,λ,β)	892.1244	900.1243	900.8261	908.6329	0.2814	1.6725	0.1292	0.2512	0.2860
TL-GW$(\alpha,\theta,\lambda,\beta)$	884.0287	892.0287	892.7305	900.5373	0.1681	1.0306	0.0974	0.5977	0.1607
TL-OBIII-LLoG(α,β,b,λ)	894.5799	902.5799	903.2816	911.0884	0.3075	1.8198	0.1281	0.2601	20.5047
K-W$(\alpha,\theta,\lambda,\beta)$	879.4400	887.4400	888.1417	895.9485	0.0909	0.5883	0.0871	0.7343	0.0889
OGHLWW$(\alpha,\beta,\lambda,\gamma)$	900.9073	908.9073	909.6091	917.4159	0.1923	1.4680	0.1410	0.1694	20.6226

The fitted densities and probability plots are provided in Figure 14. Figure 14 shows the adaptability of the TL-Gom-EHL-W distribution using the earthquake data set.

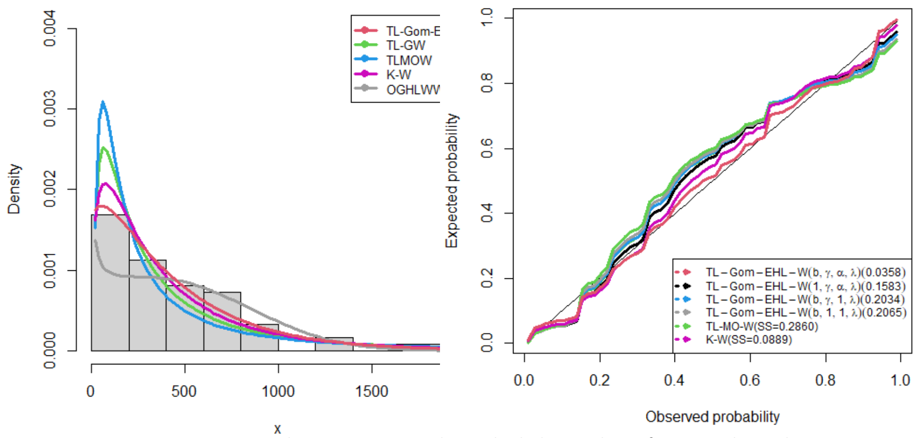

FIGURE 14: Fitted Densities and Probability Plots for Earthquake Data

The TTT plot for earthquake data with an upside-down bathtub followed by bathtub hazard rate function is shown in Figure 15.

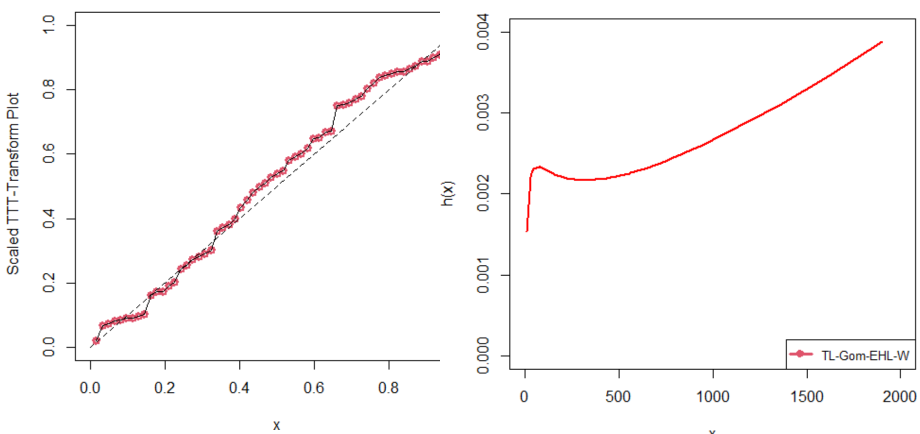

FIGURE 15: Fitted TTT and Hrf Plots for Earthquake Data

The K-M survival and ECDF curves for the earthquake data set are shown in Figure 16. Based on the closeness of the empirical and fitted lines in Figure 16, we conclude that TL-Gom-EHL-W distribution adequately fits earthquake data set.

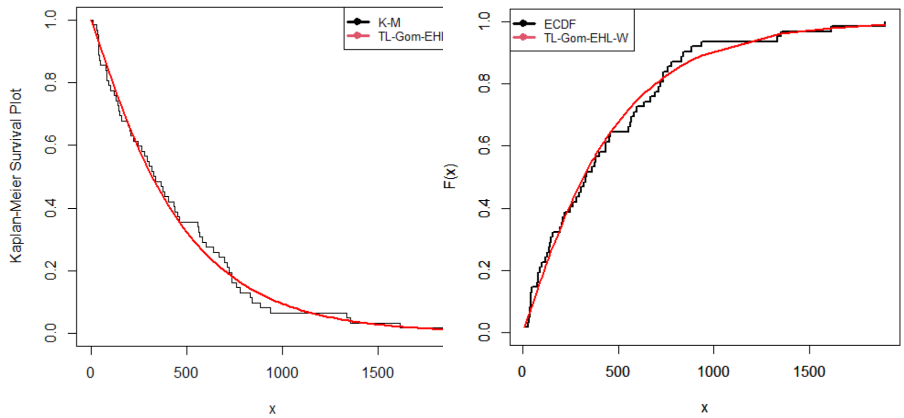

FIGURE 16: Fitted K-M survival and ECDF plots for Earthquake Data

The profile log-likelihood functions of the maximum likelihood estimates of b, γ, α, and λ are shown in Figures 17 and 18. Figures 17 and 18 shows that the maximum likelihood estimates b, γ, α, and λ exist and can be obtained uniquely.

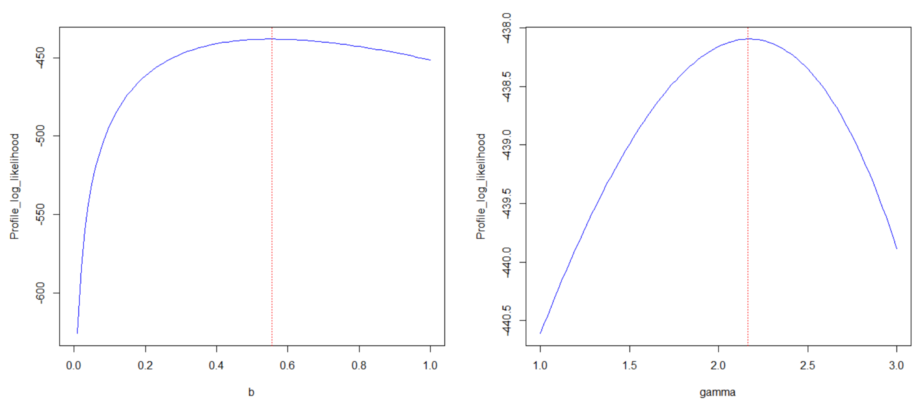

FIGURE 17: Plots of the profile log-likelihood functions of the parameters of the TL-Gom-EHL-W distribution on Earthquake data

6.3. Unemployment Insurance Data

This unemployment insurance data set presents monthly unemployment insurance metrics from July 2008 to April 2013 from the Department of Labour, Licensing and Regulation. It consists of 58 observations and 21 variables. It is available at: `https://catalog.data.gov/dataset/unemployment-insurance-data-july-2008-to-april-2013`.

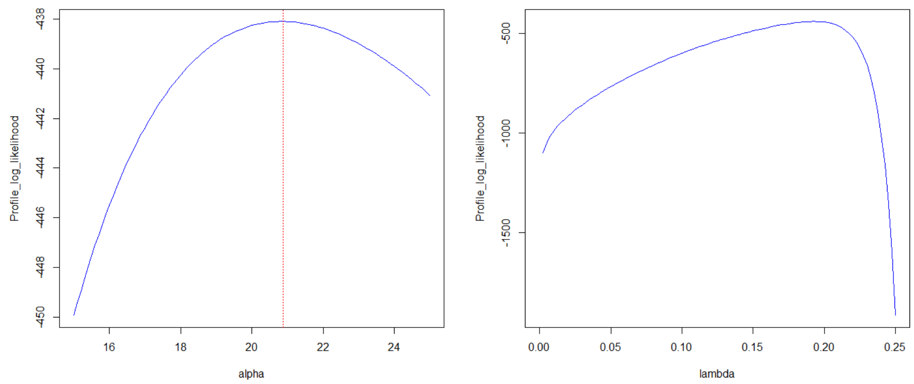

Figure 18: Plots of the profile log-likelihood functions of the parameters of the TL-Gom-EHL-W distribution on Earthquake data

The estimated variance-covariance matrix is given by

$$\begin{pmatrix} 0.7618 & 5.7560 & -8.2286 & -0.0785 \\ 5.7560 & 136.8862 & -101.6347 & -1.2337 \\ -8.2286 & -101.6347 & 108.1100 & 1.1386 \\ -0.0785 & -1.2337 & 1.1386 & 0.0126 \end{pmatrix}$$

and the approximate 95% two-sided confidence intervals for b, γ, α and λ are given by 1.3399 ± 1.7107, 8.9737 ± 22.9316, 11.2968 ± 20.3792 and 0.2410 ± 0.2204, respectively.

The maximum likelihood estimates of the unemployment insurance data set are shown in Table 6, along with standard errors in parentheses.

In comparison to the selected non-nested distributions, the TL-Gom-EHL-W distribution fits the unemployment insurance data well because it has the smallest values of the goodness-of-fit statistics (AIC, AICC, BIC, W, A, K-S, SS) and the highest p-value for the K-S statistics.

The fitted pdf and probability plots for the unemployment insurance data set are shown in Figure 19. These fitted plots shows that the TL-Gom-EHL-W distribution fits the data better than the non-nested models.

The TTT plot for the unemployment insurance data set shown in Figure 20 shows an increasing hazard rate function. Figure 21 shows the Kaplan-Meier (K-M) and Empirical Cumulative Distribution Function (ECDF) curves for the unemployment insurance data set. According to the closeness of the observed and fitted lines in Figure 21, TL-Gom-EHL-W distribution fit unemployment insurance data well. In Figures 22 and 23 we demonstrate the profile log-likelihood functions of the maximum likelihood estimates of b, γ, α, and λ. From Figures 22 and 23, we observe that the maximum likelihood estimates for TL-Gom-EHL-W distribution are obtained uniquely.

TABLE 6: Estimates of Models for Unemployment Insurance Data

	Estimates			
Model	b	γ	α	λ
TL-Gom-EHL-W$(b,\gamma,\alpha,\lambda)$	1.3399	8.9737	11.2968	0.2410
	(0.8728)	(11.6998)	(10.3976)	(0.1124)
TL-Gom-EHL-W$(b,\gamma,1,\lambda)$	131.8559	0.0161	-	0.3136
	(34.3302)	(0.1162)	-	(0.0366)
TL-Gom-EHL-W$(1,\gamma,\alpha,\lambda)$	-	4.3309×10^{-09}	1.1217	0.0531
	-	(0.0113)	(0.1083)	(0.1348)
TL-Gom-EHL-W$(b,1,\alpha,\gamma)$	3.3520×10^{03}	-	0.5255	0.1286
	(7.1862×10^{-07})	-	(0.0249)	(0.0081)
TL-Gom-EHL-W$(b,1,1,\lambda)$	194.4000	-	-	0.1707
	(3.4068×10^{-07})	-	-	(0.0032)
TL-MO-W(b,α,λ,β)	79.0799	0.6110	0.2868	0.5113
	(0.0024)	(0.4512)	(0.2271)	(0.1292)
TL-GW$(\alpha,\theta,\lambda,\beta)$	0.8024	706.8900	47.7450	0.2395
	(0.2281)	(0.0002)	(0.0015)	(0.0048)
TL-OBIII-LLoG(α,β,b,λ)	0.0524	286.8700	0.6295	26.2820
	(0.0024)	(0.0002)	(0.1829)	(5.1868×10^{-06})
K-W$(\alpha,\theta,\lambda,\beta)$	592.0100	3.5560	0.1884	273.3100
	(0.0024)	(1.0337)	(0.0031)	(0.0022)
OGHLWW$(\alpha,\beta,\lambda,\gamma)$	0.0938	0.0979	0.5196	0.9953
	(0.0874)	(0.2313)	(1.0656)	(0.3226)

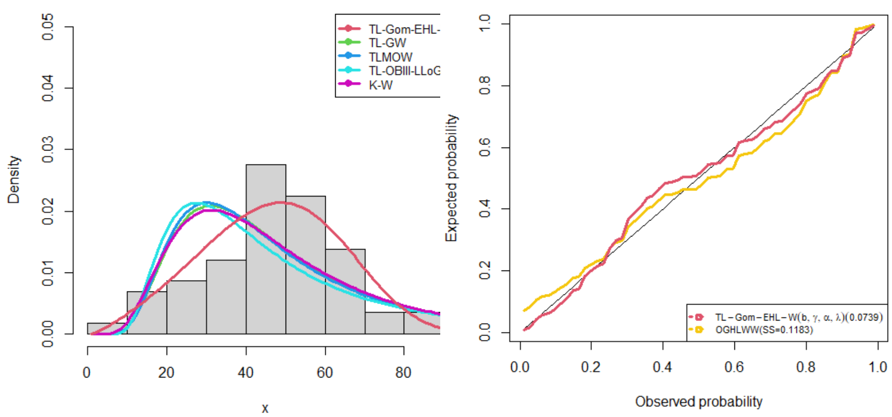

FIGURE 19: Fitted Densities and Probability Plots for Unemployent Insurance Data

6.4. Likelihood Ratio Test

This section contains the likelihood ratio test results for comparing the TL-Gom-EHL-W distribution and nested models.

The performance of the nested models and the TL-Gom-EHL-W distribution differ significantly as shown by the results in Table 8. This demonstrates the importance of the additional parameters added to the baseline distribution in improving model flexibility.

TABLE 7: Goodness-of-Fit Statistics for Unemployment Insurance Data

Model	Estimates								
	$-2\log$	AIC	AICC	BIC	W^*	A^*	K-S	p-value	SS
TL-Gom-EHL-W$(b,\gamma,\alpha,\lambda)$	495.5564	503.5564	504.3111	511.7982	0.0724	0.3950	0.0869	0.7730	0.0739
TL-Gom-EHL-W$(b,\gamma,1,\lambda)$	525.5429	531.5429	531.9873	537.7242	0.5597	3.0044	0.1923	0.0274	0.5895
TL-Gom-EHL-W$(1,\gamma,\alpha,\lambda)$	793.8212	799.8229	800.2673	806.0042	0.3649	1.9323	0.8145	2.2000×10^{-16}	13.3008
TL-Gom-EHL-W$(b,1,\alpha,\lambda)$	527.3638	533.3637	533.8081	539.5450	0.5868	3.1534	0.1940	0.0253	0.5768
TL-Gom-EHL-W$(b,1,1,\lambda$	520.9290	524.9290	525.1471	529.0498	0.4998	2.6760	0.1945	0.0283	1.4001
TL-MO-W(b,α,λ,β)	522.7145	530.7145	531.4692	538.9563	0.5238	2.8080	0.1931	0.0263	0.5320
TL-GW$(\alpha,\theta,\lambda,\beta)$	520.9682	528.9684	529.7231	537.2101	0.4958	2.6555	0.1814	0.0438	19.2905
TL-OBIII-LLoG(α,β,b,λ)	531.9278	539.9278	540.9825	548.1696	0.6415	3.4544	0.1981	0.0210	0.5966
K-W$(\alpha,\theta,\lambda,\beta)$	517.7306	525.7306	526.4852	533.9733	0.4339	2.3143	0.1806	0.0453	0.5145
OGHLWW$(\alpha,\beta,\lambda,\gamma)$	505.3387	513.3387	514.0934	521.5804	0.1286	0.9147	0.0955	0.6645	0.1183

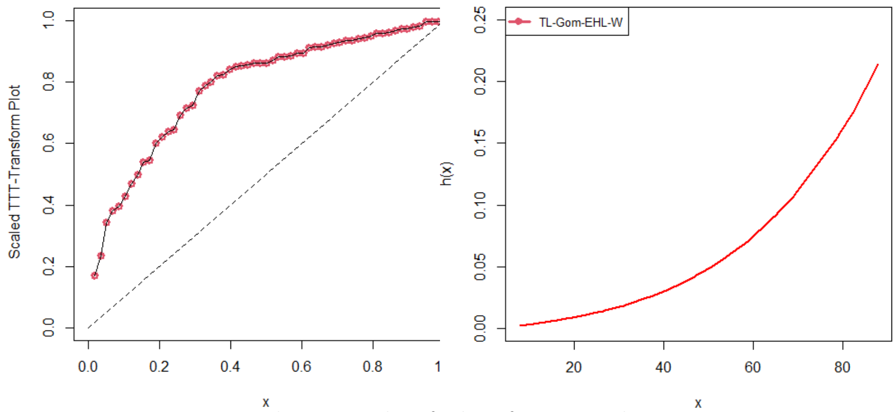

FIGURE 20: Fitted TTT and Hrf Plots for Unemployment Insurance Data

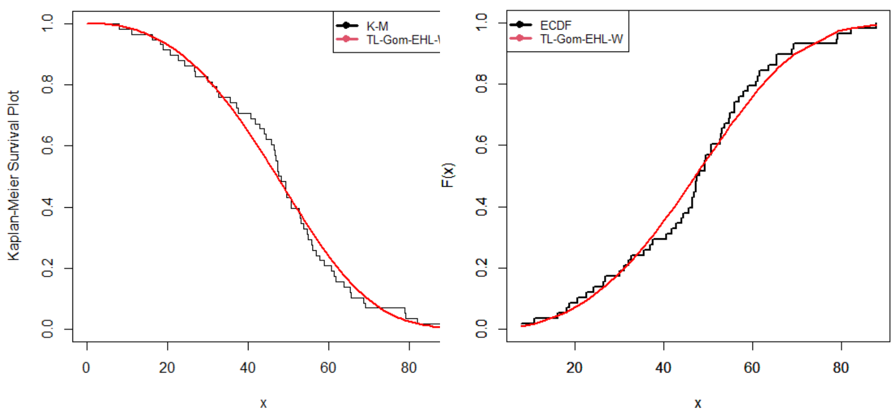

FIGURE 21: Fitted K-M survival and ECDF plots for Unemployment Insurance Data

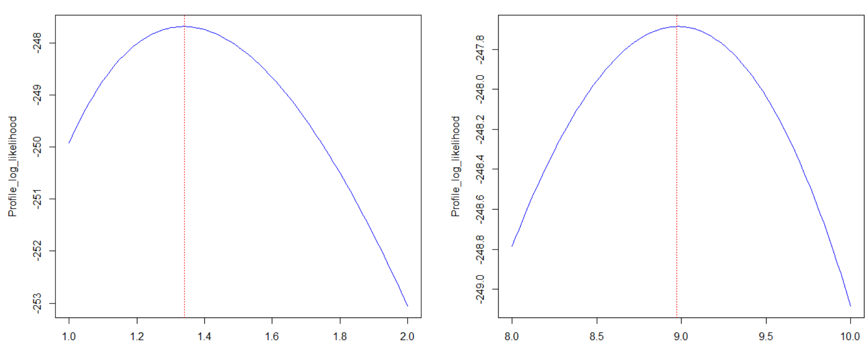

FIGURE 22: Plots of the profile log-likelihood functions of the parameters of the TL-Gom-EHL-W distribution on Unemployment Insurance data

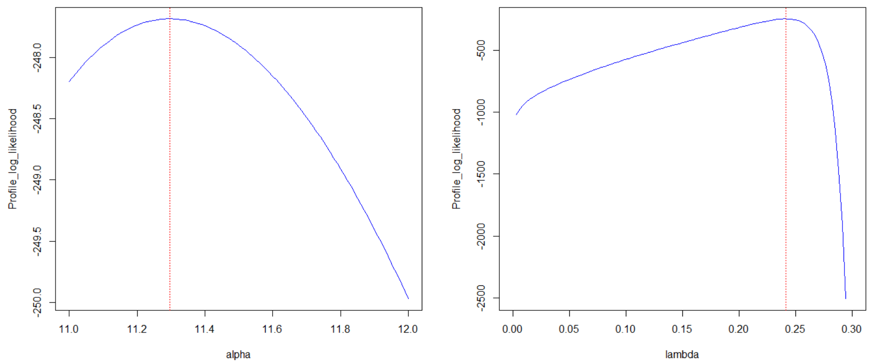

FIGURE 23: Plots of the profile log-likelihood functions of the parameters of the TL-Gom-EHL-W distribution on Unemployment Insurance data

TABLE 8: Likelihood Ratio Test Results

		Mexico COVID-19 Data	Earthquake Data	Unemployment Insurance Data
Model	df	χ^2(p-value)	χ^2(p-value)	χ^2(p-value)
TL-Gom-EHL-W(b, γ, 1, λ)	1	193.3000(<0.00001)	9.8177(0.001728)	29.9865(<0.00001)
TL-Gom-EHL-W(1, γ, α, λ)	1	8.3933(0.003766)	7.5308(0.006065)	298.2648(<0.00001)
TL-Gom-EHL-W(b, 1, α, λ)	1	8.0005(0.004676)	13.3800(0.000254)	31.8074(<0.00001)
TL-Gom-EHL-W(b, 1, 1, λ)	2	351.9559(<0.00001)	10.0372(0.001534)	25.3696(<0.00001)

7. Conclusions

We have proposed the TL-Gom-EHL-G family of distributions. Statistical properties of the new family of distributions including hazard rate function, quantile function, moments and moment generating functions, probability weighted moments, distribution of order statistics, Rényi entropy, and stochastic ordering are presented. The maximum likelihood estimation technique was used to estimate the model parameters. The TL-Gom-EHL-W, TL-Gom-EHL-BXII, and TL-Gom-EHL-L distributions were discussed as the special cases of the TL-Gom-EHL-G family of distributions. The performance of the TL-Gom-EHL-W distribution was investigated using various simulations with different sample sizes. Finally, the TL-Gom-EHL-W distribution was fitted to three real data sets to demonstrate the usefulness and effectiveness of the new family of distributions. The TL-Gom-EHL-W distribution outperformed the nested and non-nested models used for comparison in this paper.

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Appendix

The appendix materials can be found at the following url. `https://drive.google.com/file/d/1H7NyC-cTGOPMPkfnhwSpixxkbVQyNr-j/view?usp=sharing`