An Improved Class of Ratio and Product Estimators Based on ORRT Models

Una clase mejorado de estimadores de razones y productos basados en modelos ORRT

MONICA CHOUDHARY\textsuperscript{1,a}, SUNIL KUMAR\textsuperscript{2,b}, SANAM PREET KOUR\textsuperscript{2,c}, TANIA VERMA\textsuperscript{2,d}

\textsuperscript{1}ICMR Advanced Centre for Evidence Based Child Health, PGIMER, Chandigarh, India
\textsuperscript{2}University of Jammu, Jammu, India

Abstract

In this study, we introduced a problem for the estimation of the population mean of two sensitive variables using ORRT models. A class of estimators has been developed for the estimation of ratio and product estimators. Up to the first degree of approximation, properties of the proposed estimators including bias and mean square error have been studied. To show the efficacy of the proposed class of estimators, we compare it with the conventional estimators and the PRE of the proposed class of estimators is obtained with respect to the usual ratio and product estimator. The simulation study justified that our suggested estimator is more efficient than the existing estimator in terms of having higher PRE.

Key words: Auxiliary variable; Bias; Mean square error (MSE); Non-response; Optional RRT; Sensitive study variables.

Resumen

En este estudio, presentamos un problema para la estimación de la media poblacional de dos variables sensibles utilizando modelos ORRT. Se ha desarrollado una clase de estimadores para la estimación de razón y producto. Hasta el primer grado de aproximación, se han estudiado las propiedades de los estimadores propuestos, incluido el sesgo y el error cuadrático medio. Para mostrar la eficacia de la clase de estimadores propuesta, la comparamos...
Monica Choudhary, Sunil Kumar, Sanam Preet Kour & Tania Verma

con los estimadores convencionales, y se obtiene el PRE de la clase de estimadores propuesta con respecto al estimador habitual de razón y producto. El estudio de simulación que nuestro estimator sugerido es más eficiente que el estimator existente en términos de tener un PRE más alto.

Palabras clave: Error cuadrático medio (MSE); Falta de respuesta; Sesgo; TSR opcional; Variables sensibles del estudio, Variable auxiliar.

1. Introduction

There is a possibility that survey sampling will yield incorrect information if the variable of interest is sensitive. Such surveys often lead to respondents refusing to be honest when questions regarding drug addiction, illegal behavior, risky driving, etc. As a result of having to share such personal information, respondents may provide false information unintentionally or refuse to participate in the survey directly. Direct questioning could not provide comprehensive and trustworthy information, which leads to bias. Therefore, it is a significant concern in such investigations. The randomized response technique (RRT), first described by Warner (1965), minimizes potential bias and is used to gather reliable data while protecting respondent’s privacy. Further, Greenberg et al. (1971) expand the Randomized Response model (RRM) to estimate the mean of a sensitive variable. In addition, Eichhorn & Hayre (1983), Singh & Mathur (2005), Gjestvang & Singh (2006), Gupta et al. (2010), Diana & Perri (2011), Chaudhuri & Pal (2015), Bouza et al. (2018), Zhang et al. (2018), Murtaza et al. (2020), Waseem et al. (2021), Kumar et al. (2021), Qureshi et al. (2022), and others have also done research in this area.

Surveys on sensitive topics may also result in non-response if respondents decline to provide the information or show no interest. Initially, Hansen & Hurwitz (1946) put forth the notion of gathering a sub-sample of non-respondents and then individually interviewing this sub-sample to acquire information. A response rate of 40-50% is considered acceptable, but in practice, it is substantially lower, according to Fan & Yan (2010) and Miller & Dillman (2011). Furthermore, Foradori (1961), Cochran (1977), Khare & Srivastava (1997), Singh & Kumar (2009), Singh & Kumar (2011), and Kumar & Bhongal (2011) investigated the methodology and proposed many forms of estimators for population parameters based on Hansen & Hurwitz (1946).

A key goal in every statistical estimating technique is to get more precise estimators of parameters of interest. It is also generally known that adding additional information to an estimation procedure leads to better estimates. The ratio and product estimation approach develops better estimators by using auxiliary information that is linearly related to the variable under investigation and is used to estimate the population means. Cochran (1940) and Murthy (1964) estimators of ratio and product are notable examples, respectively. The ratio and product estimator of two parameters is crucial in practice. For example, measuring the proportions of people under different modes of livelihood and total crop production (i.e., a product of cultivated area and yield rate) in a region is equally significant.
as estimating the total population and total cultivated area (Singh (1965)). In addition, Koyuncu & Kadilar (2009), Haq & Shabbir (2013), Yadav & Kadilar (2013), etc., developed more ratio and product estimators. Sousa et al. (2010) established the ratio estimator, which uses non-sensitive auxiliary data and the ASR model to estimate the mean of a sensitive study variable. The estimation of population means of a sensitive variable based on an ORR model, Kalucha et al. (2015) developed a ratio estimator. By applying ORRT models, Kumar & Kour (2021), Kumar & Kour (2022), Kumar, Kour & Zhang (2023), Choudhary et al. (2023), Kumar, Kour, Gupta & Joorel (2023) developed the mean estimator for estimating the population mean of sensitive variable.

Reviewing the previous studies and motivated by the work of Singh & Karpe (2009), we developed a class of estimators for estimating the ratio and product estimators of two population mean of sensitive variables using ORRT models. Further, the rest of the paper is arranged in such a manner that section 2 describes the optimal randomized response technique. Some existing estimators are discussed in 3. The proposed estimator along with its properties are discussed in section 4. In section 5, we compare the proposed estimator with the other conventional estimators and the efficiency conditions are developed. Section 6 provides a simulation study to check the effectiveness of the proposed estimator. At Last, the conclusion is discussed in section 7.

2. Optional Randomized Response Technique

Let us consider a finite population \( U_i = U_1, U_2, \ldots, U_N \) of size \( N \) and a simple random sample (without replacement) of size \( n \) is taken from \( N \). Assume that \( Y_1 \) and \( Y_2 \) be two sensitive study variables and \( X \) be an auxiliary variable. Let \( S_1 \) and \( S_2 \) be two scrambling variables with mean \((\bar{S}_1, \bar{S}_2)\) and variances \((S^2_{1Y}, S^2_{2Y})\), respectively. Also, let \( w \) be the probability that the respondent will find the question is sensitive. In the ORRT version, the respondent may answer in the two ways given in (1) depending on whether the respondent considers the question sensitive or not. So, the general scrambling model for sensitive study variables \( Y_i; i = 1, 2 \) is given as

\[
Z_i = \begin{cases} 
Y_i & \text{with probability } (1-w) \\
S_1Y_i + S_2 & \text{with probability } w,
\end{cases}
\]

(1)

The mean and variance of \( Z_i; i = 1, 2 \) is given by

\[
E(Z_i) = E(Y_i)(1-w) + E(S_1Y_i + S_2)w = E(Y_i)
\]

and \( \text{Var}(Z_i) = E(Z^2_i) - E^2(Z_i) = S^2_{1Y_i} + S^2_{2Y_i} w + S^2_{1Y} + Y^2_i w. \)

We write the randomized linear model as \( Z_i = (S_1Y_i + S_2)J + Y_i(1 - J) \) and the expectation and variance of the randomized mechanism are \( E_R(Z_i) = (\bar{S}_1 w + 1 - w)Y_i + S_2 w \) and \( \text{Var}_R(Z_i) = (Y^2_i S^2_{3Y} + S^2_{2Y}) w. \)

where, \( S_1 \) and \( S_2 \) follows a normal distribution with mean \((1, 0)\) and variances \((S^2_{1Y}, S^2_{2Y})\), i.e., \( S_1 \sim N(1, S^2_{3Y}) \) and \( S_2 \sim N(0, S^2_{2Y}) \), \( J \sim \text{Bernoulli}(w) \) with \( E(J) = w, \text{Var}(J) = w(1-w) \) and \( E(J^2) = \text{Var}(J) + E^2(J) = w. \)
2.1. Modified Hansen and Hurwitz Technique

Hansen & Hurwitz (1946) suggests a sub-sampling procedure in which they assume that from \( n \) only \( n_1 \) units provide response on the first call and remaining \( n_2 = n - n_1 \) units do not respond. Then, a sub-sample of size \( n_s = \frac{n_2}{k} \); \( k \) is inverse sampling ratio and \((k > 1)\) is taken from the \( n_2 \) non-responding units. If the survey is sensitive in the second phase then they give respondents the opportunity to scramble their responses by using ORRT. In this case, using Hansen and Hurwitz’s technique by assuming that the respondent group provides direct responses in the first phase, and then the ORRT model i.e. (1) is used to provide responses from a sample of non-respondents in the second phase.

Let \( \hat{y}_{ij} \) denotes a transformation of the randomised response on the \( j^{th} \) block, the expectation of which is real response \( y_{ij} \) and is given as

\[
\hat{y}_{ij} = \frac{z_i - \bar{S}_2}{S_1 w + 1 - w}
\]

with \( E(\hat{y}_{ij}) = y_{ij} \) and \( \text{Var}(\hat{y}_{ij}) = \frac{(\bar{y}_{i1}^2 S_1^2 + \bar{y}_{i2}^2) w}{(S_1 w + 1 - w)^2} = \kappa_{ij}; i = 1, 2 \). Hansen & Hurwitz (1946) estimator in the presence of non-response by using ORRT is given as

\[
\hat{\bar{y}}_i = w_1 \bar{y}_{i1} + w_2 \bar{y}_{i2}
\]

where \( \bar{y}_{i2} = \sum_{i=1}^{n_s} \left( \frac{\hat{y}_{ik}}{n_s} \right) \).

It is easy to verify that

\[
E(\hat{\bar{y}}_i) = Y_i
\]

The variance of \( \hat{\bar{y}}_i \) is

\[
\text{Var}(\hat{\bar{y}}_i) = \frac{\bar{Y}_i}{\gamma C^2_{y_i} + \gamma^* C^2_{y_{i(2)}} + K_i}
\]

where,

\[
\gamma = \frac{1}{n} - \frac{1}{N}, \quad \gamma^* = \frac{W_2(k-1)}{n}, \quad W_2 = \frac{N_2}{N}, \quad K_i = \frac{W_2 k}{n} \left\{ \frac{\left( (C^2_{y_{i(2)}} + \bar{y}_{i(2)}^2) S_1^2 + S_2^2 \right) w}{(S_1 w + 1 - w)^2} \right\}, i = 1, 2,
\]

\[
C^2_{y_i} = \frac{S^2_{y_i}}{Y_i}; i = 1, 2; \text{is the coefficient of Variation for the respondent class, and}
\]

\[
C^2_{y_{i(2)}} = \frac{S^2_{y_{i(2)}}}{Y_{i(2)}}; i = 1, 2; \text{is the coefficient of Variation for the non-respondent class.}
\]

3. Conventional Estimators

For the estimation of the population ratio \( \hat{R} \) and product \( \hat{P} \), it is assumed that the population mean \( \bar{X} \) of the auxiliary variable \( x \) is known. The usual ratio and product estimators using ORRT models are defined as

\[
\hat{R} = \frac{\bar{y}_1}{\bar{y}_2} \quad \text{and} \quad \hat{P} = \bar{y}_1 \bar{y}_2
\]
The MSE of the above estimators are given as

\[
MSE(\hat{R}) = R^2 \left[ \gamma \left\{ C_{y_1}^2 + C_{y_2}^2 - 2\rho_{y_1y_2}C_{y_1}C_{y_2} \right\} + \gamma^* \left\{ C_{y_1(2)}^2 + C_{y_2(2)}^2 - 2\rho_{y_1y_2(2)}C_{y_1(2)}C_{y_2(2)} \right\} + (K_1 + K_2) \right] \tag{3}
\]

\[
MSE(\hat{P}) = P^2 \left[ \gamma \left\{ C_{y_1}^2 + C_{y_2}^2 + 2\rho_{y_1y_2}C_{y_1}C_{y_2} \right\} + \gamma^* \left\{ C_{y_1(2)}^2 + C_{y_2(2)}^2 + 2\rho_{y_1y_2(2)}C_{y_1(2)}C_{y_2(2)} \right\} + (K_1 + K_2) \right] \tag{4}
\]

where

\[
K_1 = \frac{w_n k}{n} \left\{ \frac{\{(C_{y_1(2)} + \bar{y}_{y_2})S_{y_1}^2 + S_{y_2}^2\}}{(S_{y_1} + w - 1)^2} \right\} \quad \text{and} \quad K_2 = \frac{w_n k}{n} \left\{ \frac{\{(C_{y_2(2)} + \bar{y}_{y_2})S_{y_1}^2 + S_{y_2}^2\}}{(S_{y_1} + w - 1)^2} \right\}.
\]

Also, we considered Singh & Karpe (2009) estimators using ORRT models and is defined as

\[
t_{1R} = \hat{R} \left( \frac{\bar{x}}{x} \right)
\]

\[
t_{2R} = \hat{R} \left( \frac{\bar{x}}{X} \right)
\]

for estimating ratio \( R \),

\[
t_{1P} = \hat{P} \left( \frac{\bar{x}}{x} \right)
\]

\[
t_{2P} = \hat{P} \left( \frac{\bar{x}}{X} \right)
\]

for estimating product \( P \).

The mean square error (MSE) of the above defining estimators are as

\[
MSE(t_{1R}) = \frac{1}{n^2} \left[ \gamma \left\{ C_{y_1}^2 + C_{y_2}^2 + C_x^2 - 2\rho_{y_1y_2}C_{y_1}C_{y_2} + 2\rho_{y_2}C_{y_2}C_x \right\} + \gamma^* \left\{ C_{y_1(2)}^2 + C_{y_2(2)}^2 - 2\rho_{y_1y_2(2)}C_{y_1(2)}C_{y_2(2)} \right\} + (K_1 + K_2) \right] \tag{5}
\]

\[
MSE(t_{2R}) = \frac{1}{n^2} \left[ \gamma \left\{ C_{y_1}^2 + C_{y_2}^2 + C_x^2 - 2\rho_{y_1y_2}C_{y_1}C_{y_2} - 2\rho_{y_2}C_{y_2}C_x \right\} + \gamma^* \left\{ C_{y_1(2)}^2 + C_{y_2(2)}^2 + 2\rho_{y_1y_2(2)}C_{y_1(2)}C_{y_2(2)} \right\} + (K_1 + K_2) \right] \tag{6}
\]

\[
MSE(t_{1P}) = \frac{1}{n^2} \left[ \gamma \left\{ C_{y_1}^2 + C_{y_2}^2 + C_x^2 + 2\rho_{y_1y_2}C_{y_1}C_{y_2} - 2\rho_{y_2}C_{y_2}C_x \right\} + \gamma^* \left\{ C_{y_1(2)}^2 + C_{y_2(2)}^2 - 2\rho_{y_1y_2(2)}C_{y_1(2)}C_{y_2(2)} \right\} + (K_1 + K_2) \right] \tag{7}
\]
Some other existing estimator by combining the estimators \((t_{1R} \text{ with } \hat{R}), \ (t_{2R} \text{ with } \hat{R}), \ (t_{1P} \text{ with } \hat{P}), \ (t_{1P} \text{ with } \hat{P})\) are then defined as

\[
 t_\theta = \theta t_{1R} + (1 - \theta)\hat{R}
\]

\[
 t_\phi = \phi t_{2R} + (1 - \phi)\hat{R}
\]

for estimating ratio \(R\),

\[
 t_\eta = \eta t_{1P} + (1 - \eta)\hat{P}
\]

\[
 t_\delta = \delta t_{2P} + (1 - \delta)\hat{P}
\]

for estimating product \(P\), where, \(\theta, \phi, \eta\) and \(\delta\) are appropriately chosen characterizing scalars.

The defining estimators above have the following mean square errors (MSE) as

\[
 MSE(t_\theta) = \frac{t_{1R}^2}{\theta^2 C_1^2 + \phi^2 C_2^2 - 2\phi \rho_{xy2} C_{y1} C_{y2} + 2\phi \rho_{x2} C_{y2} C_x} + (K_1 + K_2)
\]

\[
 MSE(t_\phi) = \frac{t_{2R}^2}{\phi^2 C_1^2 + \delta^2 C_2^2 - 2\delta \rho_{xy2} C_{y1} C_{y2} - 2\phi \rho_{x2} C_{y2} C_x} + (K_1 + K_2)
\]

\[
 MSE(t_\eta) = \frac{t_{1P}^2}{\eta^2 C_1^2 + \gamma^2 C_2^2 + 2\gamma \rho_{xy2} C_{y1} C_{y2} - 2\eta \rho_{x2} C_{y2} C_x} + (K_1 + K_2)
\]

\[
 MSE(t_\delta) = \frac{t_{2P}^2}{\gamma^2 C_1^2 + \delta^2 C_2^2 + 2\delta \rho_{xy2} C_{y1} C_{y2} + 2\delta \rho_{x2} C_{y2} C_x} + (K_1 + K_2)
\]

To find the minimum MSE of \(t_\theta, t_\phi, t_\eta\) and \(t_\delta\), we differentiate it w.r.t. \(\theta, \phi, \eta, \) and \(\delta\), respectively and equating to zero, then we get the optimum values as

\[
 \theta = \frac{\rho_{y1x} C_{y1} - \rho_{x2} C_{y2}}{C_x} = \theta_{opt}
\]

\[
 \phi = -\frac{\rho_{y1x} C_{y1} - \rho_{x2} C_{y2}}{C_x} = \phi_{opt}
\]
\[ \eta = \frac{\rho y_1x + \rho y_2x}{C_x} = \eta_{opt} \]  
(15)

\[ \delta = \frac{-\rho y_1x + \rho y_2x}{C_x} = \delta_{opt} \]  
(16)

After substituting the optimum values from (13-16) in (9-12), respectively, then we get the minimum MSE’s as

\[
min.MSE(t_\theta) = R^2 \left[ \gamma \left( C^2_{y_1} + C^2_{y_2} - \rho^2_{y_1x} C^2_{y_1} - \rho^2_{y_2x} C^2_{y_2} - 2 \rho y_{1y_2} C_{y_1} C_{y_2} \right) 
+ 2 \rho y_{1x} \rho y_{2x} C_{y_1} C_{y_2} \right] 
+ \gamma^* \left( C^2_{y_1(2)} + C^2_{y_2(2)} - 2 \rho y_{1y_2(2)} C_{y_1(2)} C_{y_2(2)} \right) 
+ (K_1 + K_2) \right] 
= min.MSE(t_\phi) \]  
(17)

Similarly,

\[
min.MSE(t_\eta) = P^2 \left[ \gamma \left( C^2_{y_1} + C^2_{y_2} - \rho^2_{y_1x} C^2_{y_1} - \rho^2_{y_2x} C^2_{y_2} + 2 \rho y_{1y_2} C_{y_1} C_{y_2} \right) 
- 2 \rho y_{1x} \rho y_{2x} C_{y_1} C_{y_2} \right] 
+ \gamma^* \left( C^2_{y_1(2)} + C^2_{y_2(2)} + 2 \rho y_{1y_2(2)} C_{y_1(2)} C_{y_2(2)} \right) 
+ (K_1 + K_2) \right] 
= min.MSE(t_\delta) \]  
(18)

4. Proposed Estimator

Following the general class of estimators given by Srivastava & Jhajj (1981), we introduced a novel class of estimators for estimating the ratio and product estimators of two population means of sensitive study variables using ORRT models when the variance of the auxiliary variable is known at the estimation stage and are defined as follows

\[ t_{mR} = \hat{R}H_R(u, v), \]  
(19)

for ratio R,

\[ t_{mP} = \hat{P}H_P(u, v), \]  
(20)

for product P, where, \( R = \frac{\bar{Y}_1}{\bar{Y}_2}, \) \( P = \bar{Y}_1\bar{Y}_2, \) \( \hat{R} = \frac{\bar{y}_1}{\bar{y}_2}, \) \( \hat{P} = \bar{y}_1\bar{y}_2, \) \( u = \frac{x}{\bar{x}}, \) \( v = \frac{x}{\bar{x}} \) and \( H_R(u, v) \) and \( H_P(u, v) \) is a function of \( u \) and \( v \) such that,

(a) The point \((u, v)\) assumes the value in a closed convex subset \( R_2 \) of two-dimensional real space containing the point \((1, 1)\);

(b) The function \( H_R(u, v) \) and \( H_P(u, v) \) is continuous and bounded in \( R_2 \);

(c) \( H_R(1, 1) = H_P(1, 1) = 1 \);

(d) The first and second order partial derivative of \( H_R(u, v) \) and \( H_P(u, v) \) exist, continuous and bounded in \( R_2 \).
To determine the bias and MSE of the proposed estimator up to first order approximation, use the following transformation as

\[
\hat{e}_{01}^* = \frac{\tilde{y}_1}{Y_1} - 1, \hat{e}_{02}^* = \frac{\tilde{y}_2}{Y_2} - 1, e_1 = \frac{\bar{x}}{X} - 1 \text{ and } e_2 = \frac{s_x^2}{S_X^2} - 1
\]

such that

\[
E(\hat{e}_{01}^*) = E(\hat{e}_{02}^*) = E(e_1) = E(e_2) = 0;
\]

\[
E(\hat{e}_{01}^{*2}) = \gamma C_{y1}^2 + \gamma^* C_{y1(1)}^2 + K_1, E(\hat{e}_{02}^{*2}) = \gamma C_{y2}^2 + \gamma^* C_{y2(2)}^2 + K_2;
\]

\[
E(e_1^2) = \gamma C_{x}^2, E(e_2^2) = \gamma (\lambda_{004} - 1), E(e_1 e_2) = \gamma C_x \lambda_{003};
\]

\[
E(\hat{e}_{01}^* \hat{e}_{02}^*) = \gamma \rho_{y1y2} C_{y1} C_{y2} + \gamma^* \rho_{y1y2(2)} C_{y1(2)} C_{y2(2)};
\]

\[
E(\hat{e}_{01} e_1) = \gamma \rho_{y1x} C_{y1} C_x, E(\hat{e}_{02} e_1) = \gamma \rho_{y2x} C_{y2} C_x;
\]

\[
E(\hat{e}_{01} e_2) = \gamma C_{y1} \lambda_{002}, E(\hat{e}_{02} e_2) = \gamma C_{y2} \lambda_{012};
\]

where, \(\lambda_{rsq} = \frac{\mu_{rsq}}{\mu_{200} \mu_{020} \mu_{002}}\) and \(\mu_{rsq} = \frac{1}{N} \sum_{i=1}^{N} (Y_{1i} - \bar{Y})^r (Y_{2i} - \bar{Y})^s (X_i - \bar{X})^q\); \(r, s \text{ and } q\) are non-negative integer and \(\mu_{200}, \mu_{020} \text{ and } \mu_{002}\) are the second order moment, and \(\lambda_{rsq}\) is the moment ratio.

### 4.1. Bias and MSE of Proposed Estimator

The class of estimators \(t_{mR}\) and \(t_{mP}\) are expressed in terms of expected values and by expanding \(H_R(u, v)\) and \(H_P(u, v)\) about the point \((1, 1)\) in a second order Taylor’s series, we obtain. For ratio \(R\),

\[
t_{mR} = \frac{\tilde{y}_1}{\tilde{y}_2} H_R [1 + (u - 1), 1 + (v - 1)]
\]

\[
\approx \frac{\tilde{y}_1}{\tilde{y}_2} [H_R(1, 1) + (u - 1)H_{1R} + (v - 1)H_{2R} + (u - 1)^2 H_{3R}\]

\[
+ (v - 1)^2 H_{4R} + (u - 1)(v - 1)H_{5R}]
\]

where,

\[
H_{1R} = \frac{\partial H_R}{\partial u} \bigg|_{(1,1)}, H_{2R} = \frac{\partial H_R}{\partial v} \bigg|_{(1,1)}, H_{3R} = \frac{1}{2} \frac{\partial^2 H_R}{\partial u^2} \bigg|_{(1,1)}, H_{4R} = \frac{1}{2} \frac{\partial^2 H_R}{\partial v^2} \bigg|_{(1,1)},
\]

and \(H_{5R} = \frac{1}{2} \frac{\partial^2 H_R}{\partial u \partial v} \bigg|_{(1,1)}\).

Similarly for product \(P\),

\[
t_{mP} = \frac{\tilde{y}_1 \tilde{y}_2}{\partial H_P} \bigg|_{(1,1)}, H_{2P} = \frac{\partial H_P}{\partial v} \bigg|_{(1,1)}, H_{3P} = \frac{1}{2} \frac{\partial^2 H_P}{\partial u^2} \bigg|_{(1,1)},
\]

\[
H_{4P} = \frac{1}{2} \frac{\partial^2 H_P}{\partial u \partial v} \bigg|_{(1,1)}, \text{ and } H_{5P} = \frac{1}{2} \frac{\partial^2 H_P}{\partial v^2} \bigg|_{(1,1)}.
\]
Then, the expression of $t_{mR}$ and $t_{mP}$ in term of $e'$ s, we have

$$t_{mR} = \frac{\hat{Y}_1(1 + \hat{e}_{01})}{\hat{Y}_2(1 + \hat{e}_{02})} \left[ 1 + e_1 H_{1R} + e_2 H_{2R} + e_1^2 H_{3R} + e_2^2 H_{4R} + e_1 e_2 H_{5R} \right]$$

(21)

$$t_{mP} = \frac{\hat{Y}_1(1 + \hat{e}_{01})}{\hat{Y}_2(1 + \hat{e}_{02})} \left[ 1 + e_1 H_{1P} + e_2 H_{2P} + e_1^2 H_{3P} + e_2^2 H_{4P} + e_1 e_2 H_{5P} \right]$$

(22)

Taking expectation on both sides of equation (21) and (22), the bias of $t_{mR}$ and $t_{mP}$ are obtained as

$$B(t_{mR}) = R \left[ \gamma \left\{ H_{3R}C_x^2 + H_{4R}(\lambda_{004} - 1) + H_{5R}C_x \lambda_{003} + H_{1R}C_x (\rho y_{1x} C_{y1} - \rho y_{2x} C_{y2}) + H_{2R}(C_{y1} \lambda_{102} - C_{y2} \lambda_{012}) - \rho y_{1y2} C_{y1} C_{y2} + C_{y2}^2 \right\} + \gamma^* \left\{ C_{y1(2)}^2 - \rho y_{1y2(2)} C_{y1(2)} C_{y2(2)} \right\} + K_2 \right]$$

(23)

$$B(t_{mP}) = P \left[ \gamma \left\{ H_{3P}C_x^2 + H_{4P}(\lambda_{004} - 1) + H_{5P}C_x \lambda_{003} + H_{1P}C_x (\rho y_{1x} C_{y1} + \rho y_{2x} C_{y2}) + H_{2P}(C_{y1} \lambda_{102} + C_{y2} \lambda_{012}) + \rho y_{1y2} C_{y1} C_{y2} \right\} + \gamma^* \left\{ C_{y1(2)}^2 + C_{y2(2)}^2 - 2 \rho y_{1y2(2)} C_{y1(2)} C_{y2(2)} \right\} + (K_1 + K_2) \right]$$

(24)

The MSE of the proposed estimators is obtained by squaring and taking expectations on both sides of equations (21) and (22) as

$$MSE(t_{mR}) = R^2 \left[ \gamma \left\{ H_{1R}^2 C_x^2 + H_{2R}^2 (\lambda_{004} - 1) + C_{y1}^2 + C_{y2}^2 + 2 H_{1R} H_{2R} C_x \lambda_{003} + 2 H_{1R} C_x (\rho y_{1x} C_{y1} - \rho y_{2x} C_{y2}) + 2 H_{2R} (C_{y1} \lambda_{102} - C_{y2} \lambda_{012}) - 2 \rho y_{1y2} C_{y1} C_{y2} \right\} + \gamma^* \left\{ C_{y1(2)}^2 + C_{y2(2)}^2 - 2 \rho y_{1y2(2)} C_{y1(2)} C_{y2(2)} \right\} + (K_1 + K_2) \right]$$

(25)

$$MSE(t_{mP}) = P^2 \left[ \gamma \left\{ H_{1P}^2 C_x^2 + H_{2P}^2 (\lambda_{004} - 1) + C_{y1}^2 + C_{y2}^2 + 2 H_{1P} H_{2P} C_x \lambda_{003} + 2 H_{1P} C_x (\rho y_{1x} C_{y1} + \rho y_{2x} C_{y2}) + 2 H_{2P} (C_{y1} \lambda_{102} + C_{y2} \lambda_{012}) + 2 \rho y_{1y2} C_{y1} C_{y2} \right\} + \gamma^* \left\{ C_{y1(2)}^2 + C_{y2(2)}^2 + 2 \rho y_{1y2(2)} C_{y1(2)} C_{y2(2)} \right\} + (K_1 + K_2) \right]$$

(26)

To obtain the minimum MSE of $t_{mR}$ and $t_{mP}$, differentiate it with respect to $H_{1R}$, $H_{2R}$ and $H_{1P}$, $H_{2P}$, respectively, we get

$$H_{1R} = \frac{C_{y1} \delta_0 - C_{y2} \delta_1}{C_x \delta_3} = H_{1R(\text{opt})}$$

$$H_{2R} = \frac{\lambda_{003} (\rho y_{1x} - \rho y_{2x}) - C_{y1} \lambda_{102} + C_{y2} \lambda_{012}}{\delta_3} = H_{2R(\text{opt})}$$

$$H_{1P} = \frac{C_{y1} \delta_0 + C_{y2} \delta_1}{C_x \delta_3} = H_{1P(\text{opt})}$$

Revista Colombiana de Estadística - Theoretical Statistics 47 (2024) 1-23
For Ratio $R$, consider existing estimators. A class of estimator with the MSE of usual ratio and product estimator and other estimators is:

$$H_{2P} = \frac{\lambda_{003} (\rho_{y1x} + \rho_{y2x}) - C_y \lambda_{102} - C_y \lambda_{012}}{\delta_3} = H_{2P(opt)}$$

where,

$$\delta_0 = [\lambda_{102} \lambda_{003} - \rho_{y1x} (\lambda_{004} - 1)], \delta_1 = [\lambda_{012} \lambda_{003} - \rho_{y2x} (\lambda_{004} - 1)], \delta_3 = [\lambda_{004} - 1 - \lambda_{003}^2]$$

After substituting the optimum value of $H_{1R}, H_{2R}$ and $H_{1P}, H_{2P}$ in (25) and (26), respectively, then the minimum MSE of proposed estimators are as

$$\min.MSE(t_{mR}) = R^2 \left[ \gamma \left\{ H_{1R(opt)} C_x^2 + H_{2R(opt)} (\lambda_{004} - 1) + C_y^2 + C_{y2}^2 \right. \right.$$  
$$+ 2H_{1R(opt)} H_{2R(opt)} C_x \lambda_{003} + 2H_{1R(opt)} C_x (\rho_{y1x} C_y - \rho_{y2x} C_{y2}) \right.$$  
$$+ 2H_{2R(opt)} (C_y \lambda_{102} - C_y \lambda_{012}) - 2\rho_{y1y2} C_y C_{y2} \right.$$  
$$+ C_{y2}^2 - 2\rho_{y1y2} (C_y C_{y2} + (K_1 + K_2) \right]$$  

(27)

$$\min.MSE(t_{mP}) = P^2 \left[ \gamma \left\{ H_{1P(opt)} C_x^2 + H_{2P(opt)} (\lambda_{004} - 1) + C_y^2 + C_{y2}^2 \right. \right.$$  
$$+ 2H_{1P(opt)} H_{2P(opt)} C_x \lambda_{003} + 2H_{1P(opt)} C_x (\rho_{y1x} C_y + \rho_{y2x} C_{y2}) \right.$$  
$$+ 2H_{2P(opt)} (C_y \lambda_{102} + C_y \lambda_{012}) + 2\rho_{y1y2} C_y C_{y2} \right.$$  
$$+ C_{y2}^2 + 2\rho_{y1y2} (C_y C_{y2} + (K_1 + K_2) \right]$$  

(28)

5. Efficiency Comparisons

In this section, we find the conditions by comparing the MSE of proposed class of estimator with the MSE of usual ratio and product estimator and other considered existing estimators.

For Ratio $R$,

(i) $\min.MSE(t_{mR}) < MSE(\hat{R})$, iff

$$\left[ \gamma \left\{ H_{1R(opt)}^2 C_x^2 \right. \right.$$
$$+ 2H_{1R(opt)} C_x (\rho_{y1x} C_y - \rho_{y2x} C_{y2}) + 2H_{2R(opt)} (C_y \lambda_{102} - C_y \lambda_{012}) \right.$$  

$$< 0$$

(ii) $\min.MSE(t_{mR}) < MSE(t_{1R})$, iff

$$\left[ \gamma \left\{ H_{1R(opt)}^2 C_x^2 \right. \right.$$
$$+ 2H_{1R(opt)} C_x (\rho_{y1x} C_y - \rho_{y2x} C_{y2}) + 2H_{2R(opt)} (C_y \lambda_{102} - C_y \lambda_{012}) \right.$$  
$$- C_x^2 + 2\rho_{y2x} C_y C_x + 2\rho_{y1x} C_y C_x \right]$$  

$$< 0$$
(iii) \( \text{min.MSE}(t_{mR}) < \text{MSE}(t_{2R}) \), if

\[
\gamma \{ H_{1R}^2 C_x^2 + H_{2R}^2 (\lambda_{004} - 1) + 2H_{1R}(\lambda_{004})H_{2R}(\lambda_{004})C_x \lambda_{003} \\
+ 2H_{1R}(\lambda_{004})C_x (\rho_{y1x} C_y - \rho_{y2x} C_y) + 2H_{2R}(\lambda_{004})C_y (\lambda_{102} - C_{y2} \lambda_{012}) \\
- C_x^2 + 2\rho_{y2x} C_y C_x - 2\rho_{y1x} C_y C_x \} < 0
\]

(iv) \( \text{min.MSE}(t_{mR}) < \text{min.MSE}(t_\theta) = \text{min.MSE}(t_\phi) \), if

\[
\gamma \{ H_{1P}^2 C_x^2 + H_{2P}^2 (\lambda_{004} - 1) + 2H_{1P}(\lambda_{004})H_{2P}(\lambda_{004})C_x \lambda_{003} \\
+ 2H_{1P}(\lambda_{004})C_x (\rho_{y1x} C_y - \rho_{y2x} C_y) + 2H_{2P}(\lambda_{004})C_y (\lambda_{102} - C_{y2} \lambda_{012}) \\
+ \rho_{y1x}^2 C_y^2 + \rho_{y2x}^2 C_y^2 + 2\rho_{y1x} \rho_{y2x} C_y C_x \} < 0
\]

For product \( P \),

(v) \( \text{min.MSE}(t_{mP}) < \text{MSE}(\hat{P}) \), if

\[
\gamma \{ H_{1P}^2 C_x^2 + H_{2P}^2 (\lambda_{004} - 1) + 2H_{1P}(\lambda_{004})H_{2P}(\lambda_{004})C_x \lambda_{003} \\
+ 2H_{1P}(\lambda_{004})C_x (\rho_{y1x} C_y - \rho_{y2x} C_y) + 2H_{2P}(\lambda_{004})C_y (\lambda_{102} - C_{y2} \lambda_{012}) \} < 0
\]

(vi) \( \text{min.MSE}(t_{mP}) < \text{MSE}(t_1 P) \), if

\[
\gamma \{ H_{1P}^2 C_x^2 + H_{2P}^2 (\lambda_{004} - 1) + 2H_{1P}(\lambda_{004})H_{2P}(\lambda_{004})C_x \lambda_{003} \\
+ 2H_{1P}(\lambda_{004})C_x (\rho_{y1x} C_y - \rho_{y2x} C_y) + 2H_{2P}(\lambda_{004})C_y (\lambda_{102} - C_{y2} \lambda_{012}) \\
- C_x^2 + 2\rho_{y2x} C_y C_x + 2\rho_{y1x} C_y C_x \} < 0
\]

(vii) \( \text{min.MSE}(t_{mP}) < \text{MSE}(t_2 P) \), if

\[
\gamma \{ H_{1P}^2 C_x^2 + H_{2P}^2 (\lambda_{004} - 1) + 2H_{1P}(\lambda_{004})H_{2P}(\lambda_{004})C_x \lambda_{003} \\
+ 2H_{1P}(\lambda_{004})C_x (\rho_{y1x} C_y - \rho_{y2x} C_y) + 2H_{2P}(\lambda_{004})C_y (\lambda_{102} - C_{y2} \lambda_{012}) \\
- C_x^2 - 2\rho_{y2x} C_y C_x - 2\rho_{y1x} C_y C_x \} < 0
\]

(viii) \( \text{min.MSE}(t_{mP}) < \text{min.MSE}(t_\eta) = \text{min.MSE}(t_\zeta) \), if

\[
\gamma \{ H_{1P}^2 C_x^2 + H_{2P}^2 (\lambda_{004} - 1) + 2H_{1P}(\lambda_{004})H_{2P}(\lambda_{004})C_x \lambda_{003} \\
+ 2H_{1P}(\lambda_{004})C_x (\rho_{y1x} C_y - \rho_{y2x} C_y) + 2H_{2P}(\lambda_{004})C_y (\lambda_{102} - C_{y2} \lambda_{012}) \\
+ \rho_{y1x}^2 C_y^2 + \rho_{y2x}^2 C_y^2 + 2\rho_{y1x} \rho_{y2x} C_y C_x \} < 0
\]

If the above conditions from (i-viii) holds true, then proposed class of ratio and product estimator is more efficient than other considered conventional estimators.
6. Simulation Study

This section of the paper discusses how well the proposed class of estimators performs as compared to the usual ratio and product estimators when the study variables are sensitive by nature. Now, we perform a simulation study using R Software to verify the performance of the proposed class of estimator over the considered existing estimator. For this we generated an artificial population of size $N = 150$ and draw a sample of size $n = 50$ using SRSWOR with $53.3\%$ response rate. From the sample only $10(n_1)$ provide a response to the survey question and $40(n_2)$ of them do not respond. Then, we take another sample ($n_3 = \frac{2n_1}{3}$) from the non-respondent group by using $k = 2, 3, 4$. Here, we take two different model for ratio and product estimators with scrambling variable $S_1 \sim N(1, 10)$ and $S_2 \sim N(0, 10)$.

For ratio estimator, $X = rnorm(N, 15, 1)$, $Y_1 = X + rnorm(N, 15, 1)$ and $Y_2 = X + rnorm(N, 15, 1)$.

For product estimator, $X = rnorm(N, 10, 1)$, $Y_1 = X + rnorm(N, 5, 1)$ and $Y_2 = X + rnorm(N, 5, 1)$.

Next, we calculate PRE’s of the proposed and existing estimator against usual ratio and product estimators, respectively, for different values of $w$.

Thus, the percent relative efficiency of the conventional and proposed class of estimators ($t_{mR}$ and $t_{mP}$) with respect to the usual ratio $\hat{R}$ and product $\hat{P}$ estimators, respectively, are as

$$PRE(E_R) = \left[ \frac{MSE(\hat{R})}{MSE(E_R)} \right] \times 100 \quad (29)$$

where, $E_R = t_{mR}, t_{1R}, t_{2R}, t_\theta$ and $t_\phi$

$$PRE(E_P) = \left[ \frac{MSE(\hat{P})}{MSE(E_P)} \right] \times 100 \quad (30)$$

where, $E_P = t_{mP}, t_{1P}, t_{2P}, t_\eta$ and $t_\delta$.

The findings are shown in Table 1 to 6, respectively. Following points are noted from Table 1 to 3 as

- Table 1 to 3 represent the performance of proposed class of ratio estimator against the considered conventional ratio estimators.
- It is envisaged from Table 1 to 3, that our proposed estimator is more efficient then all other considered conventional estimator.
- Also, from Table 1 to 3, with the increasing value of $w$, the PRE of proposed estimator (i.e., $t_{mR}$) are decreases.
- The PRE of $t_{1R}$ first decreases then increases, but the behaviour of $t_{2R}$ opposite to $t_{1R}$ i.e., first increases then decreases, and the PRE of $t_\theta$ & $t_\phi$ first decreases then increases and then again decreases with the increasing value of $w$. 
The performance of $t_{2R}$ is very less than all the considered estimator and $t_\theta$ & $t_\phi$ are equally efficient.

### Table 1: PRE of the proposed and existing ratio estimator with respect to usual ratio estimator for $k = 2$

<table>
<thead>
<tr>
<th>Estimator</th>
<th>$n = 40$</th>
<th>$n = 45$</th>
<th>$n = 50$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_{mR}$</td>
<td>134.6828</td>
<td>350.9810</td>
<td>154.2558</td>
</tr>
<tr>
<td>$t_1R$</td>
<td>110.8748</td>
<td>99.9855</td>
<td>104.9325</td>
</tr>
<tr>
<td>$t_2R$</td>
<td>99.9981</td>
<td>99.9991</td>
<td>99.9996</td>
</tr>
<tr>
<td>$t_\theta$</td>
<td>100.0000</td>
<td>100.0000</td>
<td>100.0000</td>
</tr>
<tr>
<td>$t_\phi$</td>
<td>100.0000</td>
<td>100.0000</td>
<td>100.0000</td>
</tr>
</tbody>
</table>

### Table 2: PRE of the proposed and existing ratio estimator with respect to usual ratio estimator for $k = 3$

<table>
<thead>
<tr>
<th>Estimator</th>
<th>$n = 40$</th>
<th>$n = 45$</th>
<th>$n = 50$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_{mR}$</td>
<td>117.4478</td>
<td>154.2558</td>
<td>128.5935</td>
</tr>
<tr>
<td>$t_1R$</td>
<td>112.7617</td>
<td>104.9325</td>
<td>106.3194</td>
</tr>
<tr>
<td>$t_2R$</td>
<td>99.9989</td>
<td>99.9996</td>
<td>99.9994</td>
</tr>
<tr>
<td>$t_\theta$</td>
<td>100.0000</td>
<td>100.0000</td>
<td>100.0000</td>
</tr>
<tr>
<td>$t_\phi$</td>
<td>100.0000</td>
<td>100.0000</td>
<td>100.0000</td>
</tr>
</tbody>
</table>

Revista Colombiana de Estadística - Theoretical Statistics 47 (2024) 1-23
Table 3: PRE of the proposed and existing ratio estimator with respect to usual ratio estimator for \( k = 4 \)

<table>
<thead>
<tr>
<th>Estimator</th>
<th>( n = 40 )</th>
<th>( n = 45 )</th>
<th>( n = 50 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( w = 0.2 )</td>
<td>( w = 0.4 )</td>
<td>( w = 0.6 )</td>
</tr>
<tr>
<td>( t_mR )</td>
<td>111.6557</td>
<td>113.1468</td>
<td>107.8658</td>
</tr>
<tr>
<td>( t_\theta )</td>
<td>100.0000</td>
<td>100.0000</td>
<td>100.0000</td>
</tr>
<tr>
<td>( t_\phi )</td>
<td>100.0000</td>
<td>100.0000</td>
<td>100.0000</td>
</tr>
</tbody>
</table>

Following points are noted from Table 4 to 6 as

- Table 4 to 6 shows the results of the suggested class of product estimators in comparison to the existing product estimators under consideration.
- It is envisaged from Table 4 to 6, that the performance of our proposed estimator is outstanding then all other considered conventional estimator.
- Also, from Table 4 to 6, the PRE of proposed estimator (i.e., \( t_mP \)) are decreases, but the PRE of \( t_2R \) are increases with the increasing value of \( w \).
- The PRE of \( t_1R \) first increases then decreases and then again increases and decreases, and the PRE of \( t_\theta \) & \( t_\phi \) first increases then decreases with the increasing value of \( w \).
- The performance of \( t_\theta \) & \( t_\phi \) are equally effective but less than all the considered estimators.
Table 4: PRE of the proposed and existing product estimator with respect to usual product estimator for $k = 2$

<table>
<thead>
<tr>
<th>Estimator</th>
<th>$w = 0.2$</th>
<th>$w = 0.4$</th>
<th>$w = 0.6$</th>
<th>$w = 0.8$</th>
<th>$w = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_{mP}$</td>
<td>100.2515</td>
<td>139.6268</td>
<td>128.5368</td>
<td>100.0123</td>
<td></td>
</tr>
<tr>
<td>$t_{1P}$</td>
<td>99.9944</td>
<td>99.9054</td>
<td>99.9968</td>
<td>99.9961</td>
<td>98.5114</td>
</tr>
<tr>
<td>$t_{2P}$</td>
<td>99.9971</td>
<td>99.9985</td>
<td>99.9981</td>
<td>99.9992</td>
<td>100.0000</td>
</tr>
<tr>
<td>$t_0$</td>
<td>90.1909</td>
<td>93.2487</td>
<td>96.4017</td>
<td>97.7465</td>
<td>101.4652</td>
</tr>
<tr>
<td>$t_8$</td>
<td>90.1909</td>
<td>93.2487</td>
<td>96.4017</td>
<td>97.7465</td>
<td>101.4652</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>n = 45</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_{mP}$</td>
</tr>
<tr>
<td>$t_{1P}$</td>
</tr>
<tr>
<td>$t_{2P}$</td>
</tr>
<tr>
<td>$t_0$</td>
</tr>
<tr>
<td>$t_8$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>n = 50</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_{mP}$</td>
</tr>
<tr>
<td>$t_{1P}$</td>
</tr>
<tr>
<td>$t_{2P}$</td>
</tr>
<tr>
<td>$t_0$</td>
</tr>
<tr>
<td>$t_8$</td>
</tr>
</tbody>
</table>

Table 5: PRE of the proposed and existing product estimator with respect to usual product estimator for $k = 3$

<table>
<thead>
<tr>
<th>Estimator</th>
<th>$w = 0.2$</th>
<th>$w = 0.4$</th>
<th>$w = 0.6$</th>
<th>$w = 0.8$</th>
<th>$w = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_{mP}$</td>
<td>128.3030</td>
<td>116.1176</td>
<td>107.8656</td>
<td>115.8588</td>
<td>100.0062</td>
</tr>
<tr>
<td>$t_{1P}$</td>
<td>99.9967</td>
<td>99.9980</td>
<td>104.5756</td>
<td>99.9976</td>
<td>100.0000</td>
</tr>
<tr>
<td>$t_{2P}$</td>
<td>99.9983</td>
<td>99.9988</td>
<td>99.9995</td>
<td>99.9995</td>
<td>100.0000</td>
</tr>
<tr>
<td>$t_0$</td>
<td>88.6822</td>
<td>95.6241</td>
<td>100.0000</td>
<td>97.2357</td>
<td>101.4764</td>
</tr>
<tr>
<td>$t_8$</td>
<td>88.6822</td>
<td>95.6241</td>
<td>100.0000</td>
<td>97.2357</td>
<td>101.4764</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>n = 45</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_{mP}$</td>
</tr>
<tr>
<td>$t_{1P}$</td>
</tr>
<tr>
<td>$t_{2P}$</td>
</tr>
<tr>
<td>$t_0$</td>
</tr>
<tr>
<td>$t_8$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>n = 50</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_{mP}$</td>
</tr>
<tr>
<td>$t_{1P}$</td>
</tr>
<tr>
<td>$t_0$</td>
</tr>
<tr>
<td>$t_8$</td>
</tr>
</tbody>
</table>
Table 6: PRE of the proposed and existing product estimator with respect to usual product estimator for $k = 4$

<table>
<thead>
<tr>
<th>Estimator</th>
<th>$n = 40$</th>
<th>$n = 45$</th>
<th>$n = 50$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$w = 0.2$</td>
<td>$w = 0.4$</td>
<td>$w = 0.6$</td>
</tr>
<tr>
<td>$t_{mP}$</td>
<td>118.4956</td>
<td>114.0844</td>
<td>111.1203</td>
</tr>
<tr>
<td>$t_{1P}$</td>
<td>99.9977</td>
<td>99.9980</td>
<td>99.9986</td>
</tr>
<tr>
<td>$t_{2P}$</td>
<td>99.9988</td>
<td>99.9994</td>
<td>99.9992</td>
</tr>
<tr>
<td>$t_\theta$</td>
<td>88.0705</td>
<td>91.3670</td>
<td>95.2845</td>
</tr>
<tr>
<td>$t_\phi$</td>
<td>88.0705</td>
<td>91.3670</td>
<td>95.2845</td>
</tr>
</tbody>
</table>

6.1. Numerical Illustration Using Real Dataset

The dataset is based on Census 2011 literacy rates in India. The data is of $N = 35$ Indian states and union territories and sample of size $n = 15$ is drawn from the population. The literacy rate is spread across the major parameters—Overall, Rural and Urban. Let $y_1$, $y_2$ and $x$ denotes the number of literates (people) in 2001, 2011, and the total literacy rate (2011), respectively.

We have taken into account $\bar{S}_1 = 1$; $\bar{S}_2 = 0$ and $S_{x1}^2 = 0.5$; $S_{x2}^2 = 1$ in order to minimize the effect of scrambling on the real data.

From Table 7, it is clear that the performance of our proposed ratio and product estimator i.e. $t_{mR}$ and $t_{mP}$ is outstanding better than all other considered existing estimator(s). Also, from Table 7, the PRE of proposed ratio estimator first decreases for $w = 0.2$ to $0.4$ then increases and again decreases when $w = 0.8$. But the PRE of proposed product estimator first increases for $w = 0.2$ to $0.4$ then decreases when $w = 0.6$ and again increases when $w = 0.8$. So, for real dataset our proposed ratio and product estimator performs well as compared to other considered estimators.
Table 7: PRE of the proposed and existing ratio and product estimator with respect to usual ratio and product estimator for $k = 3$

<table>
<thead>
<tr>
<th>Estimator</th>
<th>$w = 0.2$</th>
<th>$w = 0.4$</th>
<th>$w = 0.6$</th>
<th>$w = 0.8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_mR$</td>
<td>365.8602</td>
<td>311.9663</td>
<td>709.0322</td>
<td>439.4191</td>
</tr>
<tr>
<td>$t_1R$</td>
<td>113.1646</td>
<td>131.7994</td>
<td>80.3672</td>
<td>98.1132</td>
</tr>
<tr>
<td>$t_2R$</td>
<td>99.9995</td>
<td>99.9995</td>
<td>99.9995</td>
<td>99.9996</td>
</tr>
<tr>
<td>$t_θ$</td>
<td>100.0001</td>
<td>100.0000</td>
<td>95.2845</td>
<td>100.0000</td>
</tr>
<tr>
<td>$t_φ$</td>
<td>100.0001</td>
<td>100.0000</td>
<td>95.2845</td>
<td>100.0000</td>
</tr>
<tr>
<td>$t_mP$</td>
<td>322.1020</td>
<td>399.3888</td>
<td>238.4184</td>
<td>245.4303</td>
</tr>
<tr>
<td>$t_1P$</td>
<td>99.9991</td>
<td>99.9989</td>
<td>99.9993</td>
<td>99.9992</td>
</tr>
<tr>
<td>$t_2P$</td>
<td>100.0004</td>
<td>100.0005</td>
<td>100.0003</td>
<td>100.0003</td>
</tr>
<tr>
<td>$t_θ$</td>
<td>88.3670</td>
<td>75.8732</td>
<td>124.4294</td>
<td>101.9233</td>
</tr>
<tr>
<td>$t_φ$</td>
<td>88.3670</td>
<td>75.8732</td>
<td>124.4294</td>
<td>101.9233</td>
</tr>
</tbody>
</table>

7. Conclusion

This article focuses on the ORRT model in the presence of non-response only on study variables and also addresses a class of estimators for estimating the population ratio and product estimator of two sensitive study variable(s). Additionally, the suggested class of estimator’s characteristics (i.e., bias and MSE) have been examined. Singh & Karpe (2009) estimators are taken into consideration to evaluate the adequacy of the proposed class of estimators. To verify the theoretical findings, a simulation study using R software is conducted for both real and artificial population. The simulation results clearly demonstrate that our proposed class of estimators is better than other widely utilized conventional estimators. As a result, we recommend that our proposed ratio and product class of estimator performs well when the respondent finds the survey is sensitive.

Acknowledgement

We are very thankful to the learned referee’s for their valuable comments and suggestions that improved the quality of the paper.

[Received: January 2023 — Accepted: September 2023]

References


Appendix

All necessary derivation for Bias and MSE

For ratio $R$ estimator,

$$t_{mR} = \frac{\bar{Y}_1}{\bar{Y}_2} H_R [1 + (u - 1), 1 + (v - 1)]$$

$$= \frac{\bar{Y}_1}{\bar{Y}_2} \left[ H_R (1, 1) + (u - 1) H_1 R + (v - 1) H_2 R + (u - 1)^2 H_3 R + (v - 1) (v - 1) H_5 R \right]$$

$$= \frac{\bar{Y}_1}{\bar{Y}_2} \left[ 1 + \left( \frac{\bar{x}}{X} - 1 \right) H_1 R + \left( \frac{s^2}{S^2 X} - 1 \right) H_2 R + \left( \frac{\bar{x}}{X} - 1 \right)^2 H_3 R + \left( \frac{s^2}{S^2 X} - 1 \right) H_5 R \right]$$

Using the expected terms, we get

$$t_{mR} = \frac{\bar{Y}_1}{\bar{Y}_2} (1 + \hat{e}_{01}^* + \hat{e}_{02}^*) \left[ 1 + e_1 H_1 R + e_2 H_2 R + e_1^2 H_3 R + e_2^2 H_4 R + e_1 e_2 H_5 R \right]$$

$$= \frac{\bar{Y}_1}{\bar{Y}_2} (1 + \hat{e}_{01}^* + \hat{e}_{02}^*)^{-1} \left[ 1 + e_1 H_1 R + e_2 H_2 R + e_1^2 H_3 R + e_2^2 H_4 R + e_1 e_2 H_5 R \right]$$

$$= \frac{\bar{Y}_1}{\bar{Y}_2} (1 + \hat{e}_{01}^* - \hat{e}_{02}^* - \hat{e}_{01}^* \hat{e}_{02}^* + \hat{e}_{02}^2) \left[ 1 + e_1 H_1 R + e_2 H_2 R + e_1^2 H_3 R + e_2^2 H_4 R \right.$$

$$+ e_1 e_2 H_5 R \right]$$

After multiplication, we get

$$t_{mR} = \frac{\bar{Y}_1}{\bar{Y}_2} \left[ 1 + e_1 H_1 R + e_2 H_2 R + e_1^2 H_3 R + e_2^2 H_4 R + e_1 e_2 H_5 R + \hat{e}_{01}^* e_1 H_1 R \right.$$  

$$+ \hat{e}_{02}^* e_2 H_2 R - \hat{e}_{01}^* \hat{e}_{02}^* e_1 H_1 R - \hat{e}_{02}^* e_2 e_1 H_1 R - \hat{e}_{01}^* \hat{e}_{02}^* e_2 H_2 R - \hat{e}_{01}^* \hat{e}_{02}^* + \hat{e}_{02}^2 \right]$$

$$\left[ t_{mR} - \frac{\bar{Y}_1}{\bar{Y}_2} \right] = \frac{\bar{Y}_1}{\bar{Y}_2} \left[ e_1 H_1 R + e_2 H_2 R + e_1^2 H_3 R + e_2^2 H_4 R + e_1 e_2 H_5 R + \hat{e}_{01}^* e_1 H_1 R \right.$$  

$$+ \hat{e}_{02}^* e_2 H_2 R - \hat{e}_{01}^* \hat{e}_{02}^* e_1 H_1 R - \hat{e}_{02}^* e_2 e_1 H_1 R - \hat{e}_{01}^* \hat{e}_{02}^* \right]$$

Taking expectation on both side of equation (A1), then we get

$$E \left[ t_{mR} - \frac{\bar{Y}_1}{\bar{Y}_2} \right] = E \left[ \frac{\bar{Y}_1}{\bar{Y}_2} \left( e_1^2 H_3 R + e_2^2 H_4 R + e_1 e_2 H_5 R + \hat{e}_{01}^* e_1 H_1 R + \hat{e}_{01}^* e_2 H_2 R \right.$$  

$$\left. - \hat{e}_{02}^* e_1 H_1 R - \hat{e}_{02}^* e_2 H_2 R - \hat{e}_{01}^* \hat{e}_{02}^* + \hat{e}_{02}^2 \right]$$

After putting the expected value, we get the Bias of $t_{mR}$ that is defined in equation (23).
Again taking expectation and squaring on both side of equation (A1), then we get

\[
E \left[ t_{mR} - \frac{\bar{Y}_1}{\bar{Y}_2} \right]^2 = E \left[ \frac{\bar{Y}_1}{\bar{Y}_2} \left( e_1 H_{1R} + e_2 H_{2R} + \hat{e}_{01}^* + \hat{e}_{02}^* - \hat{e}_{01}^* - \hat{e}_{02}^* \right)^2 \right]
\]

After putting the expected values, we get the MSE of \( t_{mR} \) that is defined in equation (25).

Similarly, for product \( P \) estimator,

\[
t_{mP} = \tilde{y}_1 \tilde{y}_2 H_P \left[ 1 + (u - 1), 1 + (v - 1) \right]
\]

\[
\approx \tilde{y}_1 \tilde{y}_2 \left[ H_P(1, 1) + (u - 1)H_{1P} + (v - 1)H_{2P} + (u - 1)^2 H_{3P} 
+ (v - 1)^2 H_{4P} + (u - 1)(v - 1)H_{5P} \right]
\]

\[
= \tilde{y}_1 \tilde{y}_2 \left[ 1 + \left( \frac{\bar{x}}{X} - 1 \right)H_{1P} + \left( \frac{S^2_x}{S^2_Y} - 1 \right)H_{2P} + \left( \frac{\bar{x}}{X} - 1 \right)^2 H_{3P} 
+ \left( \frac{S^2_x}{S^2_Y} - 1 \right)^2 H_{4P} + \left( \frac{\bar{x}}{X} - 1 \right) \left( \frac{S^2_x}{S^2_Y} - 1 \right)H_{5P} \right]
\]

Using the expected terms, we get

\[
t_{mP} = \bar{Y}_1 \left( 1 + \hat{e}_{01}^* \right) \bar{Y}_2 \left( 1 + \hat{e}_{02}^* \right) \left[ 1 + e_1 H_{1P} + e_2 H_{2P} + e_3 H_{3P} + e_4 H_{4P} + e_5 H_{5P} \right]
\]

\[
= \bar{Y}_1 \bar{Y}_2 (1 + \hat{e}_{01}^*) (1 + \hat{e}_{02}^*) \left[ 1 + e_1 H_{1P} + e_2 H_{2P} + e_3 H_{3P} + e_4 H_{4P} + e_5 H_{5P} \right]
\]

\[
= \bar{Y}_1 \bar{Y}_2 \left( 1 + \hat{e}_{01}^* + \hat{e}_{02}^* + \hat{e}_{01}^* \hat{e}_{02}^* \right) \left[ 1 + e_1 H_{1P} + e_2 H_{2P} + e_3 H_{3P} + e_4 H_{4P} + e_5 H_{5P} \right]
\]

\[
+ e_1 e_2 H_{5P}
\]

After multiplication, we get

\[
t_{mP} = \bar{Y}_1 \bar{Y}_2 \left[ 1 + e_1 H_{1P} + e_2 H_{2P} + e_3 H_{3P} + e_4 H_{4P} + e_5 H_{5P} + \hat{e}_{01} + \hat{e}_{01} e_1 H_{1P} 
+ \hat{e}_{02} e_2 H_{2P} + \hat{e}_{02} e_1 H_{1P} + \hat{e}_{02} e_2 H_{2P} + \hat{e}_{01} \hat{e}_{02} \right]
\]
\[ t_{mP} - \bar{Y}_1 \bar{Y}_2 = \bar{Y}_1 \bar{Y}_2 (e_1 H_1 P + e_2 H_2 P + \hat{e}_1^* H_3 P + \hat{e}_2^* H_4 P + e_1 e_2 H_5 P + \hat{e}_{01}^* + \hat{e}_{02}^* e_1 H_1 P + \hat{e}_{02}^* e_2 H_2 P + \hat{e}_{01}^* \hat{e}_{02}^*) \] (A2)

Taking expectation on both side of equation (A2), then we get

\[ E \left[ t_{mP} - \bar{Y}_1 \bar{Y}_2 \right] = E \left[ \bar{Y}_1 \bar{Y}_2 (e_1 H_1 P + e_2 H_2 P + \hat{e}_1^* e_1 H_1 P + \hat{e}_1^* e_2 H_2 P + \hat{e}_{02}^* e_1 H_1 P + \hat{e}_{02}^* e_2 H_2 P + \hat{e}_{01}^* \hat{e}_{02}^*) \right] \]

After putting the expected value, we get the Bias of \( t_{mP} \) that is defined in equation (24).

Again taking expectation and squaring on both side of equation (A2), then we get

\[ E \left[ t_{mP} - \bar{Y}_1 \bar{Y}_2 \right]^2 = E \left[ \bar{Y}_1 \bar{Y}_2 (e_1 H_1 P + e_2 H_2 P + \hat{e}_1^* e_1 H_1 P + \hat{e}_1^* e_2 H_2 P + \hat{e}_{02}^* e_1 H_1 P + \hat{e}_{02}^* e_2 H_2 P + \hat{e}_{01}^* \hat{e}_{02}^*) \right]^2 \]

All higher order greater than degree 2 are ignored, then

\[ E \left[ t_{mP} - \bar{Y}_1 \bar{Y}_2 \right]^2 = E \left[ \bar{Y}_1 \bar{Y}_2 \left( e_1 H_1 P + e_2 H_2 P + \hat{e}_{01}^* - \hat{e}_{02}^* \right) \right]^2 \]

\[ = (\bar{Y}_1 \bar{Y}_2)^2 E \left[ e_1^2 H_1^2 P + e_2^2 H_2^2 P + \hat{e}_{01}^{*2} + \hat{e}_{02}^{*2} + 2 e_1 e_2 H_1 P H_2 P + 2 e_1 \hat{e}_{01}^* H_1 P + 2 e_2 \hat{e}_{02}^* H_2 P + 2 \hat{e}_{01}^* \hat{e}_{02}^* \right] \]

After putting the expected values, we get the MSE of \( t_{mP} \) that is defined in equation (26).