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An Improved Class of Ratio and Product Estimators Based on ORRT Models

Una clase mejorado de estimadores de razones y productos basados en modelos ORRT

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Abstract

In this study, we introduced a problem for the estimation of the population mean of two sensitive variables using ORRT models. A class of estimators has been developed for the estimation of ratio and product estimators. Up to the first degree of approximation, properties of the proposed estimators including bias and mean square error have been studied. To show the efficacy of the proposed class of estimators, we compare it with the conventional estimators and the PRE of the proposed class of estimators is obtained with respect to the usual ratio and product estimator. The simulation study justified that our suggested estimator is more efficient than the existing estimator in terms of having higher PRE.

***Key words*:** Auxiliary variable; Bias; Mean square error (MSE); Non-response; Optional RRT; Sensitive study variables.

Resumen

En este estudio, presentamos un problema para la estimación de la media poblacional de dos variables sensibles utilizando modelos ORRT. Se ha desarrollado una clase de estimadores para la estimación de razón y producto. Hasta el primer grado de aproximación, se han estudiado las propiedades de los estimadores propuestos, incluido el sesgo y el error cuadrático medio. Para mostrar la eficacia de la clase de estimadores propuesta, la comparamos

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con los estimadores convencionales, y se obtiene el PRE de la clase de estimadores propuesta con respecto al estimador habitual de razón y producto. El estudio de simulación que nuestro estimador sugerido es más eficiente que el estimador existente en términos de tener un PRE más alto.

***Palabras clave*:** Error cuadrático medio (MSE); Falta de respuesta; Sesgo; TSR opcional; Variables sensibles del estudio, Variable auxiliar.

1. Introduction

There is a possibility that survey sampling will yield incorrect information if the variable of interest is sensitive. Such surveys often lead to respondents refusing to be honest when questions regarding drug addiction, illegal behavior, risky driving, etc. As a result of having to share such personal information, respondents may provide false information unintentionally or refuse to participate in the survey directly. Direct questioning could not provide comprehensive and trustworthy information, which leads to bias. Therefore, it is a significant concern in such investigations. The randomized response technique (RRT), first described by Warner (1965), minimizes potential bias and is used to gather reliable data while protecting respondent's privacy. Further, Greenberg et al. (1971) expand the Randomized Response model (RRM) to estimate the mean of a sensitive variable. In addition, Eichhorn & Hayre (1983), Singh & Mathur (2005), Gjestvang & Singh (2006), Gupta et al. (2010), Diana & Perri (2011), Chaudhuri & Pal (2015), Bouza et al. (2018), Zhang et al. (2018), Murtaza et al. (2020), Waseem et al. (2021), Kumar et al. (2021), Qureshi et al. (2022), and others have also done research in this area.

Surveys on sensitive topics may also result in non-response if respondents decline to provide the information or show no interest. Initially, Hansen & Hurwitz (1946) put forth the notion of gathering a sub-sample of non-respondents and then individually interviewing this sub-sample to acquire information. A response rate of 40–50% is considered acceptable, but in practice, it is substantially lower, according to Fan & Yan (2010) and Miller & Dillman (2011). Furthermore, Foradori (1961), Cochran (1977), Khare & Srivastava (1997), Singh & Kumar (2009), Singh & Kumar (2011), and Kumar & Bhougal (2011) investigated the methodology and proposed many forms of estimators for population parameters based on Hansen & Hurwitz (1946).

A key goal in every statistical estimating technique is to get more precise estimators of parameters of interest. It is also generally known that adding additional information to an estimation procedure leads to better estimates. The ratio and product estimation approach develops better estimators by using auxiliary information that is linearly related to the variable under investigation and is used to estimate the population means. Cochran (1940) and Murthy (1964) estimators of ratio and product are notable examples, respectively. The ratio and product estimator of two parameters is crucial in practice. For example, measuring the proportions of people under different modes of livelihood and total crop production (i.e., a product of cultivated area and yield rate) in a region is equally significant as estimating the total population and total cultivated area (Singh (1965)). In addition, Koyuncu & Kadilar (2009), Haq & Shabbir (2013), Yadav & Kadilar (2013), etc., developed more ratio and product estimators. Sousa et al. (2010) established the ratio estimator, which uses non-sensitive auxiliary data and the ASR model to estimate the mean of a sensitive study variable. The estimation of population means of a sensitive variable based on an ORR model, Kalucha et al. (2015) developed a ratio estimator. By applying ORRT models, Kumar & Kour (2021), Kumar & Kour (2022), Kumar, Kour & Zhang (2023), Choudhary et al. (2023), Kumar, Kour, Gupta & Joorel (2023) developed the mean estimator for estimating the population mean of sensitive variable.

Reviewing the previous studies and motivated by the work of Singh & Karpe (2009), we developed a class of estimators for estimating the ratio and product estimators of two population mean of sensitive variables using ORRT models. Further, the rest of the paper is arranged in such a manner that section 2 describes the optional randomized response technique. Some existing estimators are discussed in 3. The proposed estimator along with its properties are discussed in section 4. In section 5, we compare the proposed estimator with the other conventional estimators and the efficiency conditions are developed. Section 6 provides a simulation study to check the effectiveness of the proposed estimator. At Last, the conclusion is discussed in section 7.

2. Optional Randomized Response Technique

Let us consider a finite population $U_i = U_1, U_2, \ldots, U_N$ of size N and a simple random sample (without replacement) of size n is taken from N. Assume that Y_1 and Y_2 be two sensitive study variables and X be an auxiliary variable. Let S_1 and S_2 be two scrambling variables with mean $(\bar{S}_1, \bar{S}_2)$ and variances $(S^2_{S_1}, S^2_{S_2})$, respectively. Also, let w be the probability that the respondent will find the question is sensitive. In the ORRT version, the respondent may answer in the two ways given in (1) depending on whether the respondent considers the question sensitive or not. So, the general scrambling model for sensitive study variables $Y_i; i = 1, 2$ is given as

$$Z_i = \begin{cases} Y_i & \text{with probability (1-w)} \\ S_1 Y_i + S_2 & \text{with probability w,} \end{cases} \tag{1}$$

The mean and variance of Z_i; $i = 1, 2$ is given by

$E(Z_i) = E(Y_i)(1-w) + E(S_1 Y_i + S_2)w = E(Y_i)$

and $Var(Z_i) = E(Z_i^2) - E^2(Z_i) = S^2_{y_i} + S^2_{S_2} w + S^2_{S_1}(S^2_{y_i} + \bar{Y}_i^2)w.$

We write the randomized linear model as $Z_i = (S_1 Y_i + S_2)J + Y_i(1-J)$ and the expectation and variance of the randomized mechanism are $E_R(Z_i) = (\bar{S}_1 w + 1 - w)Y_i + \bar{S}_2 w$ and $Var_R(Z_i) = (Y_i^2 S^2_{S_1} + S^2_{S_2})w$.

where, S_1 and S_2 follows a normal distribution with mean $(1, 0)$ and variances $(S^2_{S_1}, S^2_{S_2})$, i.e., $S_1 \sim N(1, S^2_{S_1})$ and $S_2 \sim N(0, S^2_{S_2})$, $J \sim$ Bernoulli(w) with $E(J) = w, Var(J) = w(1-w)$ and $E(J^2) = Var(J) + E^2(J) = w$.

2.1. Modified Hansen and Hurwitz Technique

Hansen & Hurwitz (1946) suggests a sub-sampling procedure in which they assume that from n only n_1 units provide response on the first call and remaining $n_2 = n - n_1$ units do not respond. Then, a sub-sample of size $n_s = \frac{n_2}{k}$; k is inverse sampling ratio and $(k > 1)$ is taken from the n_2 non-responding units. If the survey is sensitive in the second phase then they give respondents the opportunity to scramble their responses by using ORRT. In this case, using Hansen and Hurwitz's technique by assuming that the respondent group provides direct responses in the first phase, and then the ORRT model i.e. (1) is used to provide responses from a sample of non-respondents in the second phase.

Let $\hat{y}_{ij}$ denotes a transformation of the randomised response on the j^{th} block, the expectation of which is real response y_{ij} and is given as

$$\hat{y}_{ij} = \frac{z_i - \bar{S}_2}{\bar{S}_1 w + 1 - w}$$

with $E(\hat{y}_{ij}) = y_{ij}$ and $Var(\hat{y}_{ij}) = \frac{(y_{ij}^2 S_{S_1}^2 + S_{S_2}^2)w}{(\bar{S}_1 w + 1 - w)^2} = \kappa_{ij}$; $i = 1, 2$. Hansen & Hurwitz (1946) estimator in the presence of non-response by using ORRT is given as

$$\hat{\bar{y}}_i = w_1 \bar{y}_{i1} + w_2 \hat{\bar{y}}_{i2}$$

where $\hat{\bar{y}}_{i2} = \sum_{i=1}^{n_s} (\frac{\hat{y}_{ik}}{n_s})$.

It is easy to verify that

$$E(\hat{\bar{y}}_i) = Y_i$$

The variance of $\hat{\bar{y}}_i$ is

$$Var(\hat{\bar{y}}_i) = \bar{Y}_i \Big[\gamma C_{y_i}^2 + \gamma^* C_{y_{i(2)}}^2 + K_i \Big] \quad (2)$$

where,

$\gamma = \frac{1}{n} - \frac{1}{N}$, $\gamma^* = \frac{W_2(k-1)}{n}$, $W_2 = \frac{N_2}{N}$, $K_i = \frac{W_2 k}{n} \left\{ \frac{\{(C_{y_{i(2)}}^2 + \bar{y}_{i(2)}^2) S_{S_1}^2 + S_{S_2}^2\} w}{(\bar{S}_1 w + 1 - w)^2} \right\}$, $i = 1, 2$,

$C_{y_i}^2 = \frac{S_{y_i}^2}{\bar{Y}_i}$; $i = 1, 2$; is the coefficient of Variation for the respondent class, and

$C_{y_{i(2)}}^2 = \frac{S_{y_{i(2)}}^2}{\bar{Y}_{i(2)}}$; $i = 1, 2$; is the coefficient of Variation for the non-respondent class.

3. Conventional Estimators

For the estimation of the population ratio $\hat{R}$ and product $\hat{P}$, it is assumed that the population mean $\bar{X}$ of the auxiliary variable x is known. The usual ratio and product estimators using ORRT models are defined as

$$\hat{R} = \frac{\bar{y}_1}{\bar{y}_2} \quad and \quad \hat{P} = \bar{y}_1 \bar{y}_2$$

The MSE of the above estimators are given as

$$MSE(\hat{R}) = R^2\big[\gamma\{C_{y1}^2 + C_{y2}^2 - 2\rho_{y1y2}C_{y1}C_{y2}\} + \gamma^*\{C_{y1(2)}^2 + C_{y2(2)}^2 - 2\rho_{y1y2(2)}C_{y1(2)}C_{y2(2)}\} + (K_1 + K_2)\big] \tag{3}$$

$$MSE(\hat{P}) = P^2\big[\gamma\{C_{y1}^2 + C_{y2}^2 + 2\rho_{y1y2}C_{y1}C_{y2}\} + \gamma^*\{C_{y1(2)}^2 + C_{y2(2)}^2 + 2\rho_{y1y2(2)}C_{y1(2)}C_{y2(2)}\} + (K_1 + K_2)\big] \tag{4}$$

where
$K_1 = \frac{W_2 k}{n}\left\{\frac{\{(C_{y_{1(2)}}^2 + \bar{y}_{1(2)}^2)S_{S_1}^2 + S_{S_2}^2\}w}{(\bar{S}_1 w + 1 - w)^2}\right\}$ and $K_2 = \frac{W_2 k}{n}\left\{\frac{\{(C_{y_{2(2)}}^2 + \bar{y}_{2(2)}^2)S_{S_1}^2 + S_{S_2}^2\}w}{(\bar{S}_1 w + 1 - w)^2}\right\}$.

Also, we considered Singh & Karpe (2009) estimators using ORRT models and is defined as

$$t_{1R} = \hat{R}\left(\frac{\bar{X}}{\bar{x}}\right)$$

$$t_{2R} = \hat{R}\left(\frac{\bar{x}}{\bar{X}}\right)$$

for estimating ratio R,

$$t_{1P} = \hat{P}\left(\frac{\bar{X}}{\bar{x}}\right)$$

$$t_{2P} = \hat{P}\left(\frac{\bar{x}}{\bar{X}}\right)$$

for estimating product P,
The mean square error (MSE) of the above defining estimators are as

$$MSE(t_{1R}) = R^2\big[\gamma\big(C_{y1}^2 + C_{y2}^2 + C_x^2 - 2\rho_{y1y2}C_{y1}C_{y2} + 2\rho_{y2x}C_{y2}C_x - 2\rho_{y1x}C_{y1}C_x\big) + \gamma^*\big(C_{y1(2)}^2 + C_{y2(2)}^2 - 2\rho_{y1y2(2)}C_{y1(2)}C_{y2(2)}\big) + \big(K_1 + K_2\big)\big] \tag{5}$$

$$MSE(t_{2R}) = R^2\big[\gamma\big(C_{y1}^2 + C_{y2}^2 + C_x^2 - 2\rho_{y1y2}C_{y1}C_{y2} - 2\rho_{y2x}C_{y2}C_x + 2\rho_{y1x}C_{y1}C_x\big) + \gamma^*\big(C_{y1(2)}^2 + C_{y2(2)}^2 - 2\rho_{y1y2(2)}C_{y1(2)}C_{y2(2)}\big) + \big(K_1 + K_2\big)\big] \tag{6}$$

$$MSE(t_{1P}) = P^2\big[\gamma\big(C_{y1}^2 + C_{y2}^2 + C_x^2 + 2\rho_{y1y2}C_{y1}C_{y2} - 2\rho_{y2x}C_{y2}C_x - 2\rho_{y1x}C_{y1}C_x\big) + \gamma^*\big(C_{y1(2)}^2 + C_{y2(2)}^2 + 2\rho_{y1y2(2)}C_{y1(2)}C_{y2(2)}\big) + \big(K_1 + K_2\big)\big] \tag{7}$$

$$MSE(t_{2P}) = P^2\big[\gamma\big(C_{y1}^2 + C_{y2}^2 + C_x^2 + 2\rho_{y1y2}C_{y1}C_{y2} + 2\rho_{y2x}C_{y2}C_x + 2\rho_{y1x}C_{y1}C_x\big) + \gamma^*\big(C_{y1(2)}^2 + C_{y2(2)}^2 + 2\rho_{y1y2(2)}C_{y1(2)}C_{y2(2)}\big) + \big(K_1 + K_2\big)\big] \tag{8}$$

Some other existing estimator by combining the estimators (t_{1R} with $\hat{R}$), (t_{2R} with $\hat{R}$), (t_{1P} with $\hat{P}$), (t_{1P} with $\hat{P}$) are then defined as

$$t_\theta = \theta t_{1R} + (1-\theta)\hat{R}$$

$$t_\phi = \phi t_{2R} + (1-\phi)\hat{R}$$

for estimating ratio R,

$$t_\eta = \eta t_{1P} + (1-\eta)\hat{P}$$

$$t_\delta = \delta t_{2P} + (1-\delta)\hat{P}$$

for estimating product P, where, θ, ϕ, η and δ are appropriately chosen characterizing scalars.

The defining estimators above have the following mean square errors (MSE) as

$$MSE(t_\theta) = R^2\big[\gamma\big(C_{y1}^2 + C_{y2}^2 + \theta^2C_x^2 - 2\rho_{y1y2}C_{y1}C_{y2} + 2\theta\rho_{y2x}C_{y2}C_x - 2\theta\rho_{y1x}C_{y1}C_x\big) + \gamma^*\big(C_{y1(2)}^2 + C_{y2(2)}^2 - 2\rho_{y1y2(2)}C_{y1(2)}C_{y2(2)}\big) + \big(K_1 + K_2\big)\big] \tag{9}$$

$$MSE(t_\phi) = R^2\big[\gamma\big(C_{y1}^2 + C_{y2}^2 + \phi^2C_x^2 - 2\rho_{y1y2}C_{y1}C_{y2} - 2\phi\rho_{y2x}C_{y2}C_x + 2\phi\rho_{y1x}C_{y1}C_x\big) + \gamma^*\big(C_{y1(2)}^2 + C_{y2(2)}^2 - 2\rho_{y1y2(2)}C_{y1(2)}C_{y2(2)}\big) + \big(K_1 + K_2\big)\big] \tag{10}$$

$$MSE(t_\eta) = P^2\big[\gamma\big(C_{y1}^2 + C_{y2}^2 + \eta^2C_x^2 + 2\rho_{y1y2}C_{y1}C_{y2} - 2\eta\rho_{y2x}C_{y2}C_x - 2\eta\rho_{y1x}C_{y1}C_x\big) + \gamma^*\big(C_{y1(2)}^2 + C_{y2(2)}^2 + 2\rho_{y1y2(2)}C_{y1(2)}C_{y2(2)}\big) + \big(K_1 + K_2\big)\big] \tag{11}$$

$$MSE(t_\delta) = P^2\big[\gamma\big(C_{y1}^2 + C_{y2}^2 + \delta^2C_x^2 + 2\rho_{y1y2}C_{y1}C_{y2} + 2\delta\rho_{y2x}C_{y2}C_x + 2\delta\rho_{y1x}C_{y1}C_x\big) + \gamma^*\big(C_{y1(2)}^2 + C_{y2(2)}^2 + 2\rho_{y1y2(2)}C_{y1(2)}C_{y2(2)}\big) + \big(K_1 + K_2\big)\big] \tag{12}$$

To find the minimum MSE of t_θ, t_ϕ, t_η and t_δ, we differentiate it w.r.t. θ, ϕ, η, and δ, respectively and equating to zero, then we get the optimum values as

$$\theta = \frac{\rho_{y1x}C_{y1} - \rho_{y2x}C_{y2}}{C_x} = \theta_{opt} \tag{13}$$

$$\phi = -\frac{\rho_{y1x}C_{y1} - \rho_{y2x}C_{y2}}{C_x} = \phi_{opt} \tag{14}$$

$$\eta = \frac{\rho_{y1x}C_{y1} + \rho_{y2x}C_{y2}}{C_x} = \eta_{opt} \tag{15}$$

$$\delta = -\frac{\rho_{y1x}C_{y1} + \rho_{y2x}C_{y2}}{C_x} = \delta_{opt} \tag{16}$$

After substituting the optimum values from (13-16) in (9-12), respectively, then we get the minimum MSE's as

$$\begin{aligned} min.MSE(t_\theta) &= R^2\big[\gamma\big(C_{y1}^2 + C_{y2}^2 - \rho_{y1x}^2 C_{y1}^2 - \rho_{y2x}^2 C_{y2}^2 - 2\rho_{y1y2}C_{y1}C_{y2} \\ &+ 2\rho_{y1x}\rho_{y2x}C_{y1}C_{y2}\big) + \gamma^*\big(C_{y1(2)}^2 + C_{y2(2)}^2 - 2\rho_{y1y2(2)}C_{y1(2)}C_{y2(2)}\big) \\ &+ \big(K_1 + K_2\big)\big] \\ &= min.MSE(t_\phi) \end{aligned} \tag{17}$$

Similarly,

$$\begin{aligned} min.MSE(t_\eta) &= P^2\big[\gamma\big(C_{y1}^2 + C_{y2}^2 - \rho_{y1x}^2 C_{y1}^2 - \rho_{y2x}^2 C_{y2}^2 + 2\rho_{y1y2}C_{y1}C_{y2} \\ &- 2\rho_{y1x}\rho_{y2x}C_{y1}C_{y2}\big) + \gamma^*\big(C_{y1(2)}^2 + C_{y2(2)}^2 + 2\rho_{y1y2(2)}C_{y1(2)}C_{y2(2)}\big) \\ &+ \big(K_1 + K_2\big)\big] \\ &= min.MSE(t_\delta) \end{aligned} \tag{18}$$

4. Proposed Estimator

Following the general class of estimators given by Srivastava & Jhajj (1981), we introduced a novel class of estimators for estimating the ratio and product estimators of two population means of sensitive study variables using ORRT models when the variance of the auxiliary variable is known at the estimation stage and are defined as follows

$$t_{mR} = \hat{R}H_R(u, v), \tag{19}$$

for ratio R,

$$t_{mP} = \hat{P}H_P(u, v), \tag{20}$$

for product P, where, $R = \frac{\bar{Y}_1}{\bar{Y}_2}$, $P = \bar{Y}_1\bar{Y}_2$, $\hat{R} = \frac{\bar{y_1}}{\bar{y_2}}$, $\hat{P} = \bar{y_1}\bar{y_2}$, $u = \frac{\bar{x}}{\bar{X}}$, $v = \frac{s_x^2}{S_X^2}$ and $H_R(u, v)$ and $H_P(u, v)$is a function of u and v such that,

(a) The point (u, v) assumes the value in a closed convex subset R_2 of two-dimensional real space containing the point $(1, 1)$;

(b) The function $H_R(u, v)$ and $H_P(u, v)$ is continuous and bounded in R_2;

(c) $H_R(1, 1) = H_P(1, 1) = 1$;

(d) The first and second order partial derivative of $H_R(u, v)$ and $H_P(u, v)$ exist, continuous and bounded in R_2.

To determine the bias and MSE of the proposed estimator up to first order approximation, use the following transformation as

$$\hat{e}^*_{01} = \frac{\bar{y_1}}{\bar{Y_1}} - 1, \hat{e}^*_{02} = \frac{\bar{y_2}}{\bar{Y_2}} - 1, e_1 = \frac{\bar{x}}{\bar{X}} - 1 \text{ and } e_2 = \frac{s_x^2}{S_X^2} - 1$$

such that

$$E(\hat{e}^*_{01}) = E(\hat{e}^*_{02}) = E(e_1) = E(e_2) = 0;$$
$$E(\hat{e}^{*2}_{01}) = \gamma C^2_{y1} + \gamma^* C^2_{y1(2)} + K_1, E(\hat{e}^{*2}_{02}) = \gamma C^2_{y2} + \gamma^* C^2_{y2(2)} + K_2;$$
$$E(e_1^2) = \gamma C_x^2, E(e_2^2) = \gamma(\lambda_{004} - 1), E(e_1 e_2) = \gamma C_x \lambda_{003};$$
$$E(\hat{e}^*_{01}\hat{e}^*_{02}) = \gamma \rho_{y1y2} C_{y1} C_{y2} + \gamma^* \rho_{y1y2(2)} C_{y1(2)} C_{y2(2)};$$
$$E(\hat{e}^*_{01} e_1) = \gamma \rho_{y1x} C_{y1} C_x, E(\hat{e}^*_{02} e_1) = \gamma \rho_{y2x} C_{y2} C_x;$$
$$E(\hat{e}^*_{01} e_2) = \gamma C_{y1} \lambda_{102}, E(\hat{e}^*_{02} e_2) = \gamma C_{y2} \lambda_{012};$$

where, $\lambda_{rsq} = \frac{\mu_{rsq}}{\mu_{200}^{r/2}\mu_{020}^{s/2}\mu_{002}^{q/2}}$ and $\mu_{rsq} = \frac{1}{N-1}\sum_{i=1}^{N}(Y_{1i} - \bar{Y})^r (Y_{2i} - \bar{Y})^s (X_i - \bar{X})^q$; ($r, s$ and q) are non-negative integer and μ_{200}, μ_{020} and μ_{002} are the second order moment, and λ_{rsq} is the moment ratio.

4.1. Bias and MSE of Proposed Estimator

The class of estimators t_{mR} and t_{mP} are expressed in terms of expected values and by expanding $H_R(u, v)$ and $H_P(u, v)$ about the point $(1, 1)$ in a second order Taylor's series, we obtain. For ratio R,

$$\begin{aligned} t_{mR} &= \frac{\bar{y_1}}{\bar{y_2}} H_R\big[1 + (u-1), 1 + (v-1)\big] \\ &\approx \frac{\bar{y_1}}{\bar{y_2}}\Big[H_R(1,1) + (u-1)H_{1R} + (v-1)H_{2R} + (u-1)^2 H_{3R} \\ &+ (v-1)^2 H_{4R} + (u-1)(v-1)H_{5R}\Big] \end{aligned}$$

where,

$$H_{1R} = \frac{\partial H_R}{\partial u}\Big|_{(1,1)}, H_{2R} = \frac{\partial H_R}{\partial v}\Big|_{(1,1)}, H_{3R} = \frac{1}{2}\frac{\partial^2 H_R}{\partial u^2}\Big|_{(1,1)}, H_{4R} = \frac{1}{2}\frac{\partial^2 H_R}{\partial v^2}\Big|_{(1,1)},$$

and $H_{5R} = \frac{1}{2}\frac{\partial^2 H_R}{\partial u \partial v}\Big|_{(1,1)}$.

Similarly for product P,

$$\begin{aligned} t_{mP} &= \bar{y_1}\bar{y_2} H_P\big[1 + (u-1), 1 + (v-1)\big] \\ &\approx \bar{y_1}\bar{y_2}\Big[H_P(1,1) + (u-1)H_{1P} + (v-1)H_{2P} + (u-1)^2 H_{3P} \\ &+ (v-1)^2 H_{4P} + (u-1)(v-1)H_{5P}\Big] \end{aligned}$$

where, $H_{1P} = \frac{\partial H_P}{\partial u}\Big|_{(1,1)}$, $H_{2P} = \frac{\partial H_P}{\partial v}\Big|_{(1,1)}$, $H_{3P} = \frac{1}{2}\frac{\partial^2 H_P}{\partial u^2}\Big|_{(1,1)}$, $H_{4P} = \frac{1}{2}\frac{\partial^2 H_P}{\partial v^2}\Big|_{(1,1)}$, and $H_{5P} = \frac{1}{2}\frac{\partial^2 H_P}{\partial u \partial v}\Big|_{(1,1)}$.

Then, the expression of t_{mR} and t_{mP} in term of $e's$, we have

$$t_{mR} = \frac{\bar{Y}_1(1+\hat{e}^*_{01})}{\bar{Y}_2(1+\hat{e}^*_{02})}\Big[1 + e_1H_{1R} + e_2H_{2R} + e_1^2H_{3R} + e_2^2H_{4R} + e_1e_2H_{5R}\Big] \qquad (21)$$

$$t_{mP} = \bar{Y}_1(1+\hat{e}^*_{01})\bar{Y}_2(1+\hat{e}^*_{02})\Big[1 + e_1H_{1P} + e_2H_{2P} + e_1^2H_{3P} + e_2^2H_{4P} + e_1e_2H_{5P}\Big] \qquad (22)$$

Taking expectation on both side of equation (21) and (22), the bias of t_{mR} and t_{mP} are obtained as

$$\begin{aligned} B(t_{mR}) = R\Big[&\gamma\big\{H_{3R}C_x^2 + H_{4R}(\lambda_{004}-1) + H_{5R}C_x\lambda_{003} + H_{1R}C_x(\rho_{y1x}C_{y1} \\ &- \rho_{y2x}C_{y2}) + H_{2R}(C_{y1}\lambda_{102} - C_{y2}\lambda_{012}) - \rho_{y1y2}C_{y1}C_{y2} + C_{y2}^2\big\} \\ &+ \gamma^*\big\{C^2_{y2(2)} - \rho_{y1y2(2)}C_{y1(2)}C_{y2(2)}\big\} + K_2\Big] \end{aligned} \qquad (23)$$

$$\begin{aligned} B(t_{mP}) = P\Big[&\gamma\big\{H_{3P}C_x^2 + H_{4P}(\lambda_{004}-1) + H_{5P}C_x\lambda_{003} + H_{1P}C_x(\rho_{y1x}C_{y1} \\ &+ \rho_{y2x}C_{y2}) + H_{2P}(C_{y1}\lambda_{102} + C_{y2}\lambda_{012}) + \rho_{y1y2}C_{y1}C_{y2}\big\} \\ &+ \gamma^*\big\{\rho_{y1y2(2)}C_{y1(2)}C_{y2(2)}\big\}\Big] \end{aligned} \qquad (24)$$

The MSE of the proposed estimators is obtained by squaring and taking expectations on both sides of equations (21) and (22) as

$$\begin{aligned} MSE(t_{mR}) = R^2\Big[&\gamma\big\{H_{1R}^2C_x^2 + H_{2R}^2(\lambda_{004}-1) + C_{y1}^2 + C_{y2}^2 + 2H_{1R}H_{2R}C_x\lambda_{003} \\ &+2H_{1R}C_x(\rho_{y1x}C_{y1} - \rho_{y2x}C_{y2}) + 2H_{2R}(C_{y1}\lambda_{102} - C_{y2}\lambda_{012}) - 2\rho_{y1y2} \\ &C_{y1}C_{y2}\big\} + \gamma^*\big\{C^2_{y1(2)} + C^2_{y2(2)} - 2\rho_{y1y2(2)}C_{y1(2)}C_{y2(2)}\big\} + (K_1+K_2)\Big] \end{aligned} \qquad (25)$$

$$\begin{aligned} MSE(t_{mP}) = P^2\Big[&\gamma\big\{H_{1P}^2C_x^2 + H_{2P}^2(\lambda_{004}-1) + C_{y1}^2 + C_{y2}^2 + 2H_{1P}H_{2P}C_x\lambda_{003} \\ &+2H_{1P}C_x(\rho_{y1x}C_{y1} + \rho_{y2x}C_{y2}) + 2H_{2P}(C_{y1}\lambda_{102} + C_{y2}\lambda_{012}) + 2\rho_{y1y2} \\ &C_{y1}C_{y2}\big\} + \gamma^*\big\{C^2_{y1(2)} + C^2_{y2(2)} + 2\rho_{y1y2(2)}C_{y1(2)}C_{y2(2)}\big\} + (K_1+K_2)\Big] \end{aligned} \qquad (26)$$

To obtain the minimum MSE of t_{mR} and t_{mP}, differentiate it with respect to H_{1R}, H_{2R} and H_{1P}, H_{2P}, respectively, we get

$$H_{1R} = \frac{C_{y1}\delta_0 - C_{y2}\delta_1}{C_x\delta_3} = H_{1R(opt)}$$

$$H_{2R} = \frac{\lambda_{003}(\rho_{y1x} - \rho_{y2x}) - C_{y1}\lambda_{102} + C_{y2}\lambda_{012}}{\delta_3} = H_{2R(opt)}$$

$$H_{1P} = \frac{C_{y1}\delta_0 + C_{y2}\delta_1}{C_x\delta_3} = H_{1P(opt)}$$

$$H_{2P} = \frac{\lambda_{003}(\rho_{y1x} + \rho_{y2x}) - C_{y1}\lambda_{102} - C_{y2}\lambda_{012}}{\delta_3} = H_{2P(opt)}$$

where,

$\delta_0 = [\lambda_{102}\lambda_{003} - \rho_{y1x}(\lambda_{004} - 1)]$, $\delta_1 = [\lambda_{012}\lambda_{003} - \rho_{y2x}(\lambda_{004} - 1)]$, $\delta_3 = [\lambda_{004} - 1 - \lambda_{003}^2]$

After substituting the optimum value of H_{1R}, H_{2R} and H_{1P}, H_{2P} in (25) and (26), respectively, then the minimum MSE of proposed estimators are as

$$\begin{aligned} min.MSE(t_{mR}) = R^2\Big[&\gamma\big\{H^2_{1R(opt)}C^2_x + H^2_{2R(opt)}(\lambda_{004} - 1) + C^2_{y1} + C^2_{y2} \\ &+ 2H_{1R(opt)}H_{2R(opt)}C_x\lambda_{003} + 2H_{1R(opt)}C_x(\rho_{y1x}C_{y1} - \rho_{y2x}C_{y2}) \\ &+ 2H_{2R(opt)}(C_{y1}\lambda_{102} - C_{y2}\lambda_{012}) - 2\rho_{y1y2}C_{y1}C_{y2}\big\} + \gamma^*\big\{C^2_{y1(2)} \\ &+ C^2_{y2(2)} - 2\rho_{y1y2(2)}C_{y1(2)}C_{y2(2)}\big\} + (K_1 + K_2)\Big] \end{aligned} \tag{27}$$

$$\begin{aligned} min.MSE(t_{mP}) = P^2\Big[&\gamma\big\{H^2_{1P(opt)}C^2_x + H^2_{2P(opt)}(\lambda_{004} - 1) + C^2_{y1} + C^2_{y2} \\ &+ 2H_{1P(opt)}H_{2P(opt)}C_x\lambda_{003} + 2H_{1P(opt)}C_x(\rho_{y1x}C_{y1} + \rho_{y2x}C_{y2}) \\ &+ 2H_{2P(opt)}(C_{y1}\lambda_{102} + C_{y2}\lambda_{012}) + 2\rho_{y1y2}C_{y1}C_{y2}\big\} + \gamma^*\big\{C^2_{y1(2)} \\ &+ C^2_{y2(2)} + 2\rho_{y1y2(2)}C_{y1(2)}C_{y2(2)}\big\} + (K_1 + K_2)\Big] \end{aligned} \tag{28}$$

5. Efficiency Comparisons

In this section, we find the conditions by comparing the MSE of proposed class of estimator with the MSE of usual ratio and product estimator and other considered existing estimators.
For Ratio R,

(i) $min.MSE(t_{mR}) < MSE(\hat{R})$, iff

$$\begin{aligned} &\Big[\gamma\big\{H^2_{1R(opt)}C^2_x + H^2_{2R(opt)}(\lambda_{004} - 1) + 2H_{1R(opt)}H_{2R(opt)}C_x\lambda_{003} \\ &+2H_{1R(opt)}C_x(\rho_{y1x}C_{y1} - \rho_{y2x}C_{y2}) + 2H_{2R(opt)}(C_{y1}\lambda_{102} - C_{y2}\lambda_{012})\big\}\Big] < 0 \end{aligned}$$

(ii) $min.MSE(t_{mR}) < MSE(t_{1R})$, iff

$$\begin{aligned} &\Big[\gamma\big\{H^2_{1R(opt)}C^2_x + H^2_{2R(opt)}(\lambda_{004} - 1) + 2H_{1R(opt)}H_{2R(opt)}C_x\lambda_{003} \\ &+2H_{1R(opt)}C_x(\rho_{y1x}C_{y1} - \rho_{y2x}C_{y2}) + 2H_{2R(opt)}(C_{y1}\lambda_{102} - C_{y2}\lambda_{012}) \\ &-C^2_x - 2\rho_{y2x}C_{y2}C_x + 2\rho_{y1x}C_{y1}C_x\big\}\Big] < 0 \end{aligned}$$

(iii) $min.MSE(t_{mR}) < MSE(t_{2R})$, iff

$$\Big[\gamma\big\{H^2_{1R(opt)}C^2_x + H^2_{2R(opt)}(\lambda_{004}-1) + 2H_{1R(opt)}H_{2R(opt)}C_x\lambda_{003} + 2H_{1R(opt)}C_x(\rho_{y1x}C_{y1} - \rho_{y2x}C_{y2}) + 2H_{2R(opt)}(C_{y1}\lambda_{102} - C_{y2}\lambda_{012}) - C^2_x + 2\rho_{y2x}C_{y2}C_x - 2\rho_{y1x}C_{y1}C_x\big\}\Big] < 0$$

(iv) $min.MSE(t_{mR}) < min.MSE(t_{\theta}) = min.MSE(t_{\phi})$, iff

$$\Big[\gamma\big\{H^2_{1R(opt)}C^2_x + H^2_{2R(opt)}(\lambda_{004}-1) + 2H_{1R(opt)}H_{2R(opt)}C_x\lambda_{003} + 2H_{1R(opt)}C_x(\rho_{y1x}C_{y1} - \rho_{y2x}C_{y2}) + 2H_{2R(opt)}(C_{y1}\lambda_{102} - C_{y2}\lambda_{012}) + \rho^2_{y1x}C^2_{y1} + \rho^2_{y2x}C^2_{y2} + 2\rho_{y1x}\rho_{y2x}C_{y1}C_{y2}\big\}\Big] < 0$$

For product P,

(v) $min.MSE(t_{mP}) < MSE(\hat{P})$, iff

$$\Big[\gamma\big\{H^2_{1P(opt)}C^2_x + H^2_{2P(opt)}(\lambda_{004}-1) + 2H_{1P(opt)}H_{2P(opt)}C_x\lambda_{003} + 2H_{1P(opt)}C_x(\rho_{y1x}C_{y1} + \rho_{y2x}C_{y2}) + 2H_{2P(opt)}(C_{y1}\lambda_{102} + C_{y2}\lambda_{012})\big\}\Big] < 0$$

(vi) $min.MSE(t_{mP}) < MSE(t_{1P})$, iff

$$\Big[\gamma\big\{H^2_{1P(opt)}C^2_x + H^2_{2P(opt)}(\lambda_{004}-1) + 2H_{1P(opt)}H_{2P(opt)}C_x\lambda_{003} + 2H_{1P(opt)}C_x(\rho_{y1x}C_{y1} + \rho_{y2x}C_{y2}) + 2H_{2P(opt)}(C_{y1}\lambda_{102} + C_{y2}\lambda_{012}) - C^2_x + 2\rho_{y2x}C_{y2}C_x + 2\rho_{y1x}C_{y1}C_x\big\}\Big] < 0$$

(vii) $min.MSE(t_{mP}) < MSE(t_{2P})$, iff

$$\Big[\gamma\big\{H^2_{1P(opt)}C^2_x + H^2_{2P(opt)}(\lambda_{004}-1) + 2H_{1P(opt)}H_{2P(opt)}C_x\lambda_{003} + 2H_{1P(opt)}C_x(\rho_{y1x}C_{y1} + \rho_{y2x}C_{y2}) + 2H_{2P(opt)}(C_{y1}\lambda_{102} + C_{y2}\lambda_{012}) - C^2_x - 2\rho_{y2x}C_{y2}C_x - 2\rho_{y1x}C_{y1}C_x\big\}\Big] < 0$$

(viii) $min.MSE(t_{mP}) < min.MSE(t_{\eta}) = min.MSE(t_{\delta})$, iff

$$\Big[\gamma\big\{H^2_{1P(opt)}C^2_x + H^2_{2P(opt)}(\lambda_{004}-1) + 2H_{1P(opt)}H_{2P(opt)}C_x\lambda_{003} + 2H_{1P(opt)}C_x(\rho_{y1x}C_{y1} + \rho_{y2x}C_{y2}) + 2H_{2P(opt)}(C_{y1}\lambda_{102} + C_{y2}\lambda_{012}) + \rho^2_{y1x}C^2_{y1} + \rho^2_{y2x}C^2_{y2} + 2\rho_{y1x}\rho_{y2x}C_{y1}C_{y2}\big\}\Big] < 0$$

If the above conditions from (i-viii) holds true, then proposed class of ratio and product estimator is more efficient than other considered conventional estimators.

6. Simulation Study

This section of the paper discusses how well the proposed class of estimators performs as compared to the usual ratio and product estimators when the study variables are sensitive by nature. Now, we perform a simulation study using R Software to verify the performance of the proposed class of estimator over the considered existing estimator. For this we generated an artificial population of size $N = 150$ and draw a sample of size $n = 50$ using SRSWOR with 53.3% response rate. From the sample only $10(n_1)$ provide a response to the survey question and $40(n_2)$ of them do not respond. Then, we take another sample $(n_s = \frac{n_2}{k})$ from the non-respondent group by using $k = 2, 3, 4$. Here, we take two different model for ratio and product estimators with scrambling variable $S_1 \sim N(1, 10)$ and $S_2 \sim N(0, 10)$.

For ratio estimator, $X = rnorm(N, 15, 1)$, $Y_1 = X + rnorm(N, 15, 1)$ and $Y_2 = X + rnorm(N, 15, 1)$.

For product estimator, $X = rnorm(N, 10, 1)$, $Y_1 = X + rnorm(N, 5, 1)$ and $Y_2 = X + rnorm(N, 5, 1)$.

Next, we calculate PRE's of the proposed and existing estimator against usual ratio and product estimators, respectively, for different values of w.

Thus, the percent relative efficiency of the conventional and proposed class of estimators(t_{mR} and t_{mP}) with respect to the usual ratio $\hat{R}$ and product $\hat{P}$ estimators, respectively, are as

$$PRE(E_R) = \left[\frac{MSE(\hat{R})}{MSE(E_R)}\right] \times 100 \tag{29}$$

where, $E_R = t_{mR}, t_{1R}, t_{2R}, t_\theta$ and t_ϕ

$$PRE(E_P) = \left[\frac{MSE(\hat{P})}{MSE(E_P)}\right] \times 100 \tag{30}$$

where, $E_P = t_{mP}, t_{1P}, t_{2P}, t_\eta$ and t_δ.

The findings are shown in Table 1 to 6, respectively.

Following points are noted from Table 1 to 3 as

- Table 1 to 3 represent the performance of proposed class of ratio estimator against the considered conventional ratio estimators.
- It is envisaged from Table 1 to 3, that our proposed estimator is more efficient then all other considered conventional estimator.
- Also, from Table 1 to 3, with the increasing value of w, the PRE of proposed estimator (i.e., t_{mR}) are decreases.
- The PRE of t_{1R} first decreases then increases, but the behaviour of t_{2R} opposite to t_{1R} i.e., first increases then decreases, and the PRE of t_θ & t_ϕ first decreases then increases and then again decreases with the increasing value of w.

- The performance of t_{2R} is very less than all the considered estimator and t_θ & t_ϕ are equally efficient.

Table 1: PRE of the proposed and existing ratio estimator with respect to usual ratio estimator for $k = 2$

Estimator	$w = 0.2$	$w = 0.4$	$w = 0.6$	$w = 0.8$	$w = 1$
			$n = 40$		
t_{mR}	134.6828	190.5363	170.8708	118.5198	100.0117
t_{1R}	110.8748	99.2541	100.2286	102.3050	98.5560
t_{2R}	99.9981	99.9970	99.9957	99.9986	100.0000
t_θ	100.0000	100.0000	100.0001	100.0000	100.0000
t_ϕ	100.0000	100.0000	100.0001	100.0000	100.0000
			$n = 45$		
t_{mR}	350.9810	272.9284	243.8051	211.0215	100.0141
t_{1R}	99.9855	99.9913	99.9909	99.9913	100.0000
t_{2R}	99.9898	99.9920	99.9946	99.9952	100.0000
t_θ	99.1488	100.7468	99.7698	101.2729	99.7828
t_ϕ	99.1488	100.7468	99.7698	101.2729	99.7828
			$n = 50$		
t_{mR}	154.2558	136.0052	131.3350	130.1940	100.8953
t_{1R}	104.9325	103.1830	99.0614	98.6202	100.2205
t_{2R}	99.9956	99.9966	99.9994	100.0002	99.9999
t_θ	100.0003	100.0003	100.0002	100.0005	100.0000
t_ϕ	100.0003	100.0003	100.0002	100.0005	100.0000

Table 2: PRE of the proposed and existing ratio estimator with respect to usual ratio estimator for $k = 3$

Estimator	$w = 0.2$	$w = 0.4$	$w = 0.6$	$w = 0.8$	$w = 1$
			$n = 40$		
t_{mR}	117.4478	124.1317	120.0394	110.5994	100.0059
t_{1R}	112.7617	107.2395	103.7318	102.84270	98.5450
t_{2R}	99.9989	99.9984	99.9988	99.9992	100.0000
t_θ	100.0000	100.0000	100.0000	100.0000	100.0000
t_ϕ	100.0000	100.0000	100.0000	100.0000	100.0000
			$n = 45$		
t_{mR}	267.6842	170.0753	162.7086	152.3073	100.0070
t_{1R}	99.9907	99.9943	99.9941	99.9944	100.0000
t_{2R}	99.9934	99.9948	99.9965	99.9969	100.0000
t_θ	98.9049	100.9735	99.6993	101.6682	99.7845
t_ϕ	98.9049	100.9735	99.6993	101.6682	99.7845
			$n = 50$		
t_{mR}	128.5935	120.4318	118.2384	117.6661	100.5929
t_{1R}	106.3194	104.1169	98.7916	98.2230	100.2932
t_{2R}	99.9972	99.9978	99.9996	100.0001	99.9999
t_θ	100.0002	100.0002	100.0001	100.0003	100.0000
t_ϕ	100.0002	100.0002	100.0001	100.0005	100.0000

Table 3: PRE of the proposed and existing ratio estimator with respect to usual ratio estimator for $k = 4$

Estimator	$n = 40$				
	$w = 0.2$	$w = 0.4$	$w = 0.6$	$w = 0.8$	$w = 1$
t_{mR}	111.6557	113.1468	107.8658	107.4243	100.0039
t_{1R}	113.5450	108.7787	104.5756	103.0823	98.5414
t_{2R}	99.9984	99.9990	99.9995	99.9994	100.0000
t_θ	100.0000	100.0000	100.0000	100.0000	100.0000
t_ϕ	100.0000	100.0000	100.0000	100.0000	100.0000
	$n = 45$				
t_{mR}	185.9612	143.9406	140.0967	134.2134	100.0047
t_{1R}	99.9931	99.9958	99.9956	99.9958	100.0000
t_{2R}	99.9952	99.9961	99.9974	99.9977	100.0000
t_θ	99.7892	101.0831	99.6651	101.8609	99.7851
t_ϕ	99.7892	101.0831	99.6651	101.8609	99.7851
	$n = 50$				
t_{mR}	119.4119	114.2638	112.8625	112.4856	100.4432
t_{1R}	106.9729	104.5631	98.6633	98.0340	100.3294
t_{2R}	99.9980	99.9984	99.9997	100.0001	100.0000
t_θ	100.0002	100.0001	100.0001	100.0003	100.0000
t_ϕ	100.0002	100.0001	100.0001	100.0003	100.0000

Following points are noted from Table 4 to 6 as

- Table 4 to 6 shows the results of the suggested class of product estimators in comparison to the existing product estimators under consideration.
- It is envisaged from Table 4 to 6, that the performance of our proposed estimator is outstanding then all other considered conventional estimator.
- Also, from Table 4 to 6, the PRE of proposed estimator (i.e., t_{mP}) are decreases, but the PRE of t_{2P} are increases with the increasing value of w.
- The PRE of t_{1P} first increases then decreases and then again increases and decreases, and the PRE of t_θ & t_ϕ first increases then decreases with the increasing value of w.
- The performance of t_θ & t_ϕ are equally effective but less than all the considered estimators.

Table 4: PRE of the proposed and existing product estimator with respect to usual product estimator for $k = 2$

Estimator	$w = 0.2$	$w = 0.4$	$w = 0.6$	$w = 0.8$	$w = 1$
			$n = 40$		
t_{mP}	160.2515	139.6268	129.2720	128.5368	100.0123
t_{1P}	99.9944	99.9954	99.9968	99.9961	98.5414
t_{2P}	99.9971	99.9985	99.9981	99.9992	100.0000
t_θ	90.1909	93.2487	96.4017	97.7465	101.4652
t_ϕ	90.1909	93.2487	96.4017	97.7465	101.4652
			$n = 45$		
t_{mP}	118.4956	114.0844	111.1203	110.9805	100.0041
t_{1P}	99.9977	99.9980	99.9986	99.9983	100.0000
t_{2P}	99.9988	99.9994	99.9992	99.9996	100.0000
t_θ	88.0705	91.3670	95.2845	97.0097	101.4802
t_ϕ	88.0705	91.3670	95.2845	97.0097	101.4802
			$n = 50$		
t_{mP}	193.2220	155.6005	140.7693	138.0035	101.2111
t_{1P}	99.9940	99.9957	99.9943	99.9936	99.9998
t_{2P}	99.9940	99.9955	99.9977	99.9987	99.9999
t_η	95.2994	96.9150	100.9446	100.3958	99.7799
t_δ	95.2994	96.9150	100.9446	100.3958	99.7799

Table 5: PRE of the proposed and existing product estimator with respect to usual product estimator for $k = 3$

Estimator	$w = 0.2$	$w = 0.4$	$w = 0.6$	$w = 0.8$	$w = 1$
			$n = 40$		
t_{mP}	128.3030	116.1176	107.8658	115.8588	100.0062
t_{1P}	99.9967	99.9980	104.5756	99.9976	100.0000
t_{2P}	99.9983	99.9988	99.9995	99.9995	100.0000
t_θ	88.6822	95.6241	100.0000	97.2357	101.4764
t_ϕ	88.6822	95.6241	100.0000	97.2357	101.4764
			$n = 45$		
t_{mP}	118.4956	114.0844	111.1203	110.9805	100.0041
t_{1P}	99.9977	99.9980	99.9986	99.9983	100.0000
t_{2P}	99.9988	99.9994	99.9992	99.9996	100.0000
t_θ	88.0705	91.3670	95.2845	97.0097	101.4802
t_ϕ	88.0705	91.3670	95.2845	97.0097	101.4802
			$n = 50$		
t_{mP}	144.7289	130.0515	122.9612	121.5892	100.8017
t_{1P}	99.9962	99.9972	99.9963	99.9959	99.9999
t_{2P}	99.9962	99.9971	99.9985	99.9991	99.9999
t_η	94.0563	96.0458	101.2213	101.8070	99.7076
t_δ	94.0563	96.0458	101.2213	101.8070	99.7076

TABLE 6: PRE of the proposed and existing product estimator with respect to usual product estimator for $k = 4$

Estimator	$n = 40$				
	$w = 0.2$	$w = 0.4$	$w = 0.6$	$w = 0.8$	$w = 1$
t_{mP}	118.4956	114.0844	111.1203	110.9805	100.0041
t_{1P}	99.9977	99.9980	99.9986	99.9983	100.0000
t_{2P}	99.9988	99.9994	99.9992	99.9996	100.0000
t_{θ}	88.0705	91.3670	95.2845	97.0097	101.4802
t_{ϕ}	88.0705	91.3670	95.2845	97.0097	101.4802
			$n = 45$		
t_{mP}	185.9612	114.0844	111.1203	110.9805	100.0041
t_{1P}	99.9825	99.9942	99.9965	99.9956	100.0000
t_{2P}	99.9855	99.9994	99.9992	99.9996	100.0000
t_{θ}	99.9898	91.3670	95.2845	97.0097	101.4802
t_{ϕ}	88.0705	91.3670	95.2845	97.0097	101.4802
			$n = 50$		
t_{mP}	129.4232	120.5901	115.9808	115.0772	100.5992
t_{1P}	99.9972	99.9979	99.9973	99.9969	99.9999
t_{2P}	99.9972	99.9978	99.9989	99.9994	99.9999
t_{η}	93.4818	95.6360	101.3534	102.0038	99.6717
t_{δ}	93.4818	95.6360	101.3534	102.0038	99.6717

6.1. Numerical Illustration Using Real Dataset

The dataset is based on Census 2011 literacy rates in India. The data is of $N = 35$ Indian states and union territories and sample of size $n = 15$ is drawn from the population. The literacy rate is spread across the major parameters-Overall, Rural and Urban. Let y_1, y_2 and x denotes the number of literates (people) in 2001, 2011, and the total literacy rate (2011), respectively.

We have taken into account $\bar{S}_1 = 1; \bar{S}_2 = 0$ and $S^2_{s_1} = 0.5; S^2_{s_2} = 1$ in order to minimize the effect of scrambling on the real data.

From Table 7, it is clear that the performance of our proposed ratio and product estimator i.e. t_{mR} and t_{mP} is outstanding better then all other considered existing estimator(s). Also, from Table 7, the PRE of proposed ratio estimator first decreases for $w = 0.2$ to 0.4 then increases and again decreases when $w = 0.8$. But the PRE of proposed product estimator first increases for $w = 0.2$ to 0.4 then decreases when $w = 0.6$ and again increases when $w = 0.8$. So, for real dataset our proposed ratio and product estimator performs well as compared to other considered estimators.

Table 7: PRE of the proposed and existing ratio and product estimator with respect to usual ratio and product estimator for $k = 3$

Estimator	$w = 0.2$	$w = 0.4$	$w = 0.6$	$w = 0.8$
t_{mR}	365.8602	311.8663	709.0322	439.4191
t_{1R}	113.1646	131.7994	80.3672	98.1132
t_{2R}	99.9995	99.9995	99.9995	99.9996
t_{θ}	100.0001	100.0000	95.2845	100.0000
t_{ϕ}	100.0001	100.0000	95.2845	100.0000
t_{mP}	322.1020	399.3888	238.4184	245.4303
t_{1P}	99.9991	99.9989	99.9993	99.9992
t_{2P}	100.0004	100.0005	100.0003	100.0003
t_{θ}	88.3670	75.8732	124.4294	101.9233
t_{ϕ}	88.3670	75.8732	124.4294	101.9233

7. Conclusion

This article focuses on the ORRT model in the presence of non-response only on study variables and also addresses a class of estimators for estimating the population ratio and product estimator of two sensitive study variable(s). Additionally, the suggested class of estimator's characteristics (i.e., bias and MSE) have been examined. Singh & Karpe (2009) estimators are taken into consideration to evaluate the adequacy of the proposed class of estimators. To verify the theoretical findings, a simulation study using R software is conducted for both real and artificial population. The simulation results clearly demonstrate that our proposed class of estimators is better than other widely utilized conventional estimators. As a result, we recommend that our proposed ratio and product class of estimator performs well when the respondent finds the survey is sensitive.

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Appendix

All necessary derivation for Bias and MSE

For ratio R estimator,

$$\begin{aligned} t_{mR} &= \frac{\bar{y_1}}{\bar{y_2}} H_R\big[1+(u-1), 1+(v-1)\big] \\ &\approx \frac{\bar{y_1}}{\bar{y_2}} \Big[H_R(1,1) + (u-1)H_{1R} + (v-1)H_{2R} + (u-1)^2 H_{3R} \\ &+ (v-1)^2 H_{4R} + (u-1)(v-1)H_{5R}\big] \\ &= \frac{\bar{y_1}}{\bar{y_2}} \Big[1 + \Big(\frac{\bar{x}}{\bar{X}} - 1\Big) H_{1R} + \Big(\frac{s_x^2}{S_X^2} - 1\Big) H_{2R} + \Big(\frac{\bar{x}}{\bar{X}} - 1\Big)^2 H_{3R} \\ &+ \Big(\frac{s_x^2}{S_X^2} - 1\Big)^2 H_{4R} + \Big(\frac{\bar{x}}{\bar{X}} - 1\Big)\Big(\frac{s_x^2}{S_X^2} - 1\Big) H_{5R}\Big] \end{aligned}$$

Using the expected terms, we get

$$\begin{aligned} t_{mR} &= \frac{\bar{Y_1}(1+\hat{e}^*_{01})}{\bar{Y_2}(1+\hat{e}^*_{02})} \Big[1 + e_1 H_{1R} + e_2 H_{2R} + e_1^2 H_{3R} + e_2^2 H_{4R} + e_1 e_2 H_{5R}\Big] \\ &= \frac{\bar{Y_1}}{\bar{Y_2}} (1+\hat{e}^*_{01})(1+\hat{e}^*_{02})^{-1} \Big[1 + e_1 H_{1R} + e_2 H_{2R} + e_1^2 H_{3R} + e_2^2 H_{4R} + e_1 e_2 H_{5R}\Big] \\ &= \frac{\bar{Y_1}}{\bar{Y_2}} (1 + \hat{e}^*_{01} - \hat{e}^*_{02} - \hat{e}^*_{01}\hat{e}^*_{02} + \hat{e}^{*2}_{02}) \Big[1 + e_1 H_{1R} + e_2 H_{2R} + e_1^2 H_{3R} + e_2^2 H_{4R} \\ &+ e_1 e_2 H_{5R}\Big] \end{aligned}$$

After multiplication, we get

$$\begin{aligned} t_{mR} = \frac{\bar{Y_1}}{\bar{Y_2}} \big[& 1 + e_1 H_{1R} + e_2 H_{2R} + e_1^2 H_{3R} + e_2^2 H_{4R} + e_1 e_2 H_{5R} + \hat{e}^*_{01} + \hat{e}^*_{01} e_1 H_{1R} \\ & + \hat{e}^*_{01} e_2 H_{2R} - \hat{e}^*_{02} - \hat{e}^*_{02} e_1 H_{1R} - \hat{e}^*_{02} e_2 H_{2R} - \hat{e}^*_{01}\hat{e}^*_{02} + \hat{e}^{*2}_{02}\big] \end{aligned}$$

$$\begin{aligned} \Big[t_{mR} - \frac{\bar{Y_1}}{\bar{Y_2}}\Big] = \frac{\bar{Y_1}}{\bar{Y_2}} \big[& e_1 H_{1R} + e_2 H_{2R} + e_1^2 H_{3R} + e_2^2 H_{4R} + e_1 e_2 H_{5R} + \hat{e}^*_{01} + \hat{e}^*_{01} e_1 H_{1R} \\ & + \hat{e}^*_{01} e_2 H_{2R} - \hat{e}^*_{02} - \hat{e}^*_{02} e_1 H_{1R} - \hat{e}^*_{02} e_2 H_{2R} - \hat{e}^*_{01}\hat{e}^*_{02} + \hat{e}^{*2}_{02}\big] \end{aligned} \tag{A1}$$

Taking expectation on both side of equation (A1), then we get

$$\begin{aligned} E\Big[t_{mR} - \frac{\bar{Y_1}}{\bar{Y_2}}\Big] = E\Big[\frac{\bar{Y_1}}{\bar{Y_2}} \big(& e_1^2 H_{3R} + e_2^2 H_{4R} + e_1 e_2 H_{5R} + \hat{e}^*_{01} e_1 H_{1R} + \hat{e}^*_{01} e_2 H_{2R} \\ & - \hat{e}^*_{02} e_1 H_{1R} - \hat{e}^*_{02} e_2 H_{2R} - \hat{e}^*_{01}\hat{e}^*_{02} + \hat{e}^{*2}_{02}\big)\Big] \end{aligned}$$

After putting the expected value, we get the Bias of t_{mR} that is defined in equation (23).

Again taking expection and squaring on both side of equation (A1), then we get

$$E\Big[t_{mR}-\frac{\bar{Y}_1}{\bar{Y}_2}\Big]^2 = E\Bigg[\frac{\bar{Y}_1}{\bar{Y}_2}\big(e_1H_{1R}+e_2H_{2R}+e_1^2H_{3R}+e_2^2H_{4R}+e_1e_2H_{5R}+\hat{e}_{01}^* + \hat{e}_{01}^*e_1H_{1R}+\hat{e}_{01}^*e_2H_{2R}-\hat{e}_{02}^*-\hat{e}_{02}^*e_1H_{1R}-\hat{e}_{02}^*e_2H_{2R} - \hat{e}_{01}^*\hat{e}_{02}^*+\hat{e}_{02}^{*2}\big)\Bigg]^2$$

All higher order greater than degree 2 are ignored, then

$$\begin{aligned}E\Big[t_{mR}-\frac{\bar{Y}_1}{\bar{Y}_2}\Big]^2 &= E\Bigg[\frac{\bar{Y}_1}{\bar{Y}_2}\big(e_1H_{1R}+e_2H_{2R}+\hat{e}_{01}^*-\hat{e}_{02}^*\big)\Bigg]^2\\ &= \Big(\frac{\bar{Y}_1}{\bar{Y}_2}\Big)^2E\Bigg[e_1^2H_{1R}^2+e_2^2H_{2R}^2+\hat{e}_{01}^{*2}+\hat{e}_{02}^{*2}+2e_1e_2H_{1R}H_{2R}\\ &+2e_1\hat{e}_{01}^*H_{1R}-2e_1\hat{e}_{02}^*H_{1R}+2e_2\hat{e}_{01}^*H_{2R}-2e_2\hat{e}_{02}^*H_{2R}-2\hat{e}_{01}^*\hat{e}_{02}^*\Bigg]\end{aligned}$$

After putting the expected values, we get the MSE of t_{mR} that is defined in equation (25).
Similarly, for product P estimator,

$$\begin{aligned}t_{mP} &= \bar{y}_1\bar{y}_2H_P\big[1+(u-1),1+(v-1)\big]\\ &\approx \bar{y}_1\bar{y}_2\Big[H_P(1,1)+(u-1)H_{1P}+(v-1)H_{2P}+(u-1)^2H_{3P}\\ &+(v-1)^2H_{4P}+(u-1)(v-1)H_{5P}\Big]\\ &= \bar{y}_1\bar{y}_2\Big[1+\Big(\frac{\bar{x}}{\bar{X}}-1\Big)H_{1P}+\Big(\frac{s_x^2}{S_X^2}-1\Big)H_{2P}+\Big(\frac{\bar{x}}{\bar{X}}-1\Big)^2H_{3P}\\ &+\Big(\frac{s_x^2}{S_X^2}-1\Big)^2H_{4P}+\Big(\frac{\bar{x}}{\bar{X}}-1\Big)\Big(\frac{s_x^2}{S_X^2}-1\Big)H_{5P}\Big]\end{aligned}$$

Using the expected terms, we get

$$\begin{aligned}t_{mP} &= \bar{Y}_1(1+\hat{e}_{01}^*)\bar{Y}_2(1+\hat{e}_{02}^*)\Big[1+e_1H_{1P}+e_2H_{2P}+e_1^2H_{3P}+e_2^2H_{4P}+e_1e_2H_{5P}\Big]\\ &= \bar{Y}_1\bar{Y}_2(1+\hat{e}_{01}^*)(1+\hat{e}_{02}^*)\Big[1+e_1H_{1P}+e_2H_{2P}+e_1^2H_{3P}+e_2^2H_{4P}+e_1e_2H_{5P}\Big]\\ &= \bar{Y}_1\bar{Y}_2(1+\hat{e}_{01}^*+\hat{e}_{02}^*+\hat{e}_{01}^*\hat{e}_{02}^*)\Big[1+e_1H_{1P}+e_2H_{2P}+e_1^2H_{3P}+e_2^2H_{4P}\\ &+e_1e_2H_{5P}\Big]\end{aligned}$$

After multiplication, we get

$$\begin{aligned}t_{mP} = \bar{Y}_1\bar{Y}_2\big[&1+e_1H_{1P}+e_2H_{2P}+e_1^2H_{3P}+e_2^2H_{4P}+e_1e_2H_{5P}+\hat{e}_{01}^*+\hat{e}_{01}^*e_1H_{1P}\\ &+\hat{e}_{01}^*e_2H_{2P}+\hat{e}_{02}^*+\hat{e}_{02}^*e_1H_{1P}+\hat{e}_{02}^*e_2H_{2P}+\hat{e}_{01}^*\hat{e}_{02}^*\big]\end{aligned}$$

$$\left[t_{mP}-\bar{Y}_1\bar{Y}_2\right]=\bar{Y}_1\bar{Y}_2\big[e_1H_{1P}+e_2H_{2P}+e_1^2H_{3P}+e_2^2H_{4P}+e_1e_2H_{5P}+\hat{e}^*_{01}+\hat{e}^*_{01}e_1H_{1P}+\hat{e}^*_{01}e_2H_{2P}+\hat{e}^*_{02}+\hat{e}^*_{02}e_1H_{1P}+\hat{e}^*_{02}e_2H_{2P}+\hat{e}^*_{01}\hat{e}^*_{02}\big] \tag{A2}$$

Taking expectation on both side of equation (A2), then we get

$$E\left[t_{mP}-\bar{Y}_1\bar{Y}_2\right]=E\left[\bar{Y}_1\bar{Y}_2\big(e_1^2H_{3P}+e_2^2H_{4P}+e_1e_2H_{5P}+\hat{e}^*_{01}e_1H_{1P}+\hat{e}^*_{01}e_2H_{2P}+\hat{e}^*_{02}e_1H_{1P}+\hat{e}^*_{02}e_2H_{2P}+\hat{e}^*_{01}\hat{e}^*_{02}\big)\right]$$

After putting the expected value, we get the Bias of t_{mP} that is defined in equation (24).

Again taking expection and squaring on both side of equation (A2), then we get

$$E\left[t_{mP}-\bar{Y}_1\bar{Y}_2\right]^2=E\left[\bar{Y}_1\bar{Y}_2\big(e_1H_{1P}+e_2H_{2P}+e_1^2H_{3P}+e_2^2H_{4P}+e_1e_2H_{5P}+\hat{e}^*_{01}+\hat{e}^*_{01}e_1H_{1P}+\hat{e}^*_{01}e_2H_{2P}+\hat{e}^*_{02}+\hat{e}^*_{02}e_1H_{1P}+\hat{e}^*_{02}e_2H_{2P}+\hat{e}^*_{01}\hat{e}^*_{02}\big)\right]^2$$

All higher order greater than degree 2 are ignored, then

$$\begin{aligned}E\left[t_{mP}-\bar{Y}_1\bar{Y}_2\right]^2&=E\left[\bar{Y}_1\bar{Y}_2\big(e_1H_{1P}+e_2H_{2P}+\hat{e}^*_{01}-\hat{e}^*_{02}\big)\right]^2\\&=\left(\bar{Y}_1\bar{Y}_2\right)^2E\Big[e_1^2H_{1P}^2+e_2^2H_{2P}^2+\hat{e}^{*2}_{01}+\hat{e}^{*2}_{02}+2e_1e_2H_{1P}H_{2P}\\&\quad+2e_1\hat{e}^*_{01}H_{1P}+2e_1\hat{e}^*_{02}H_{1P}+2e_2\hat{e}^*_{01}H_{2P}+2e_2\hat{e}^*_{02}H_{2P}+2\hat{e}^*_{01}\hat{e}^*_{02}\Big]\end{aligned}$$

After putting the expected values, we get the MSE of t_{mP} that is defined in equation (26).