Revista Colombiana de Estadística - Applied Statistics
July 2024, volume 47, issue 2, pp. 211–235
http://doi.org/10.15446/rce.v47n2.108197

Double Sampling Plan for OPPE Model Using Combined Mean

Plan de muestreo doble para el modelo OPPE utilizando media combinada

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Abstract

This work provides Double Sampling (DS) inspection plans considering the average lifetime as a quality characteristic which follows one parameter polynomial exponential (OPPE) family of distributions. Exponential, Lindley, Akash, Aradhana, Sujatha, length-biased Lindley, etc., are a few particular cases of the OPPE family. The quality of a lot is computed in this technique, by the lot average (μ) for first sample and, for the second sample we have taken combined mean to measure the lot quality. Also, we have estimated the optimum value of parameters of the proposed plan by non-linear optimization approaches considering acceptable quality level and rejectable quality level. A comparison part of the study is given, with respect to the sample size, between the Double Sampling (DS) plan and the Single Sampling (SS) plan for the variable. To describe the proposed work, we have also taken one example.

***Key words*:** Consumer's risk; Lindley distribution; Operating characteristics curve; Producer's risk; Single sampling inspection plan.

Resumen

Este trabajo proporciona planes de inspección de doble muestreo (DS) considerando la vida útil promedio como una característica de calidad que sigue una familia de distribuciones polinomiales exponenciales (OPPE). Exponencial, Lindley, Akash, Aradhana, Sujatha, Lindley con sesgo de longitud, etc., son algunos casos particulares de la familia OPPE. La calidad de un lote se calcula en esta técnica, mediante el promedio del lote (μ) para la primera

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muestra y, para la segunda muestra, hemos tomado la media combinada para medir la calidad del lote. Además, hemos estimado el valor óptimo de los parámetros del plan propuesto mediante enfoques de optimización no lineal considerando el nivel de calidad aceptable y el nivel de calidad rechazable. Se da una parte comparativa del estudio, con respecto al tamaño de la muestra, entre el plan de Doble Muestreo (DS) y el plan de Muestreo Único (SS) para la variable. Para describir el trabajo propuesto, también hemos tomado un ejemplo.

***Palabras clave*:** Curva de características de operación; Distribución Lindley; Plan de inspección por muestreo único; Riesgo del Consumidor; Riesgo del productor.

1. Introduction

The best achievement of an industry is to supply good products to the consumers according to their satisfaction level. Therefore, the main key of success for a company is to improve product quality. In this way, quality testing become so important to every industrial house. Different statistical tools and techniques are used for this purpose. One of the most efficient and useful lot monitoring techniques is the acceptance sampling plan, which is very popular in the industry. Basically, two acceptance sampling approaches are very conventional to the industrialist. One is the attribute-type sampling approach, and the other is the variable-type sampling approach. A crucial advantage of the variable type sampling approach is that taking equal protection, it holds a smaller sample size than the attribute type sampling approach. A variable-type sampling approach is beneficial in minimizing the monitoring cost in the situation of destructive testing.

Various industry has different opinions on choosing the proper approach to monitoring a lot, like Single Sampling (SS), Cascade Sampling (CS), Two stage sampling (TSS), Double sampling (DS), etc. Before the 1990s, the DS plan for attribute and variable characteristics was studied. The Bayesian approach to the DS plan was also discussed.

Based on the Acceptable Quality Level (AQL) - Limiting Quality Level (LQL) concept, Soundararajan & Arumainayagam (1990) introduced a generalised procedure for double sampling plan for attribute through example. Govindaraju & Subramani (1992) gave some tables and procedures for finding the double sampling plan, conditional double sampling plan, link sampling plan, ChSP-4 and ChSP-4A chain sampling plans involving minimum sum of producer's and consumer's risks for specified Acceptable Quality Level and Limiting Quality Level. Baillie (1992) described double sampling plan for variable when the process standard deviation was unknown. Arumainayagam & Soundararajan (1995) quick switching double sampling plan and compared with existing plan. Vijayaraghavan & Soundararajan (1998) designed skip-lot sampling plan with double-sampling plan as the reference plan by number of non-conforming units. Balamurali & Kalyanasundaram (1999) discussed conditional double sampling plan by various combinations of entry parameters and compared with single sampling plan. Feldmann &

Krumbholz (2002) introduced ASN-minimax double sampling plans for variables where they used one-sided lower specification limit. Considering Six Sigma Quality Levels, Radhakrishnan & Sivakumaran (2009) constructed of double sampling plans by counting the number of non-conformities. Aslam et al. (2010) designed a double sampling plan for a general life distribution considering a time-truncated situation. Jamkhaneh & Gildeh (2012) constructed acceptance double sampling plan using Fuzzy Poison Distribution. Sampath & Deepa (2012) proposed optimal double sampling plan through some different way that is called genetic algorithm. Balamurali et al. (2012) explained another Bayesian double sampling plan under Gamma-Poison Distribution by considering the number of non-conformities. Using double sampling plan as the reference plan, Balamurali & Subramani (2012) constructed the skip-lot sampling plan of type SkSP-2 for optimal design. Aslam et al. (2012) optimised the designing of an SkSP-V skip-lot sampling plan using double-sampling plan as the reference plan. Vangjeli (2012) proposed the ASN-minimax double sampling plans for variables using two-sided specification limits when the standard deviation is known. Nezhad et al. (2015) explained economic optimal double sampling design with zero acceptance numbers. Nezhad & Seifi (2017) designed the optimal double-sampling plan based on process capability index. Suresh & Usha (2016) constructed of Bayesian double sampling plan using minimum angle method. Balamurali et al. (2018) described optimal designing of an SkSP-R double sampling plan. Butt et al. (2019) proposed a double sampling plan for selecting a better supplier comparing two suppliers with linear profiles. Balamurali et al. (2020) introduced a mixed double sampling plan based on process capability index. Arizono et al. (2020) proposed a stage-independent double sampling plan for variables considering acceptance quality loss limit inspection scheme. Murugeswari et al. (2021) explained the optimal format of a skip-lot re-inspection plan with a double sampling plan as a reference. Khired et al. (2021) introduced the refined double sampling scheme with measures and application. Saranya et al. (2022) proposed the design of double sampling inspection plans for life tests under time censoring based on Pareto type IV distribution.

But, no author considers the Lindley distribution [see, Lindley (1958), Ghitany et al. (2008)] as a lifetime quality model in the context of a variable type double sampling approach. And, hardly any author incorporated combined mean as a lifetime quality characteristic when drawing 2^{nd} sample. Lindley distribution has recently gained momentum as it has more flexibility with respect to mean remaining life, failure rate, etc. In the acceptance sampling plan work, Tripathi et al. (2020) and Biswas & Maiti (2022) applied Lindley distribution as a lifetime model. In this paper, we have tried to establish a double sampling inspection plan based on the combined mean for Lindley distribution, and we expect that it would be more efficient than the single sampling inspection plan recommended by Mukherjee & Maiti (2014).

We have also incorporated the OPPE distributed characteristic in the double sampling plan. The sampling plan is executed based on the mean only. We do not consider specification limits or other criteria. Since this differs from previous approaches, we compare the proposed plan with the SS plan developed by Mukherjee & Maiti (2014).

We have emphasized different sections of the work in different ways. Section 2 explains the technique of the DS approach. There is a short description of estimation of plan parameters for OPPE model in Section 3. The parameters of the plan that we estimated by considering both exponential and Lindley distribution. A comparison of the proposed DS plan with the SS plan is presented in Section 4. Section 5 describes the applied part of our work through some real datasets. Finally, a concluding remarks is pivoted in Section 6.

2. Double Sampling Inspection Plan

Let, X be the lifetime quality measure under any distributional assumption and the quality is measured by mean (μ). Now, the DS plan for variable is as follows.

Let us assume that the lifetime of the units follows a specified probability distribution with unknown mean μ. We consider that larger the value of μ better is the quality. Suppose, μ_0 and μ_1 denote the AQL and LQL based on the mean respectively.

Step 1: Choose the values of (μ_0, μ_1) at producer's risk (α) and consumer's risk (β).

Step 2: Take a random sample of size n_1, say $(X_{11}, X_{12}, \ldots, X_{1n_1})$, from the lot. Suppose, X_{1i} be the lifetime of i^{th} unit, and compute the sample mean $\bar{X}_1 = \frac{1}{n_1}\sum_{i=1}^{n_1} X_{1i}$.

Step 3: Accept the lot if $\bar{X}_1 \geq c_1$, reject the lot if $\bar{X}_1 < c_2$; and, if $c_2 \leq \bar{X}_1 < c_1$ then, follow Step-4.

Step 4: Draw the 2^{nd} random sample of size n_2, say $(X_{21}, X_{22}, \ldots, X_{2n_2})$, from the lot and compute the sample mean $\bar{X}_2 = \frac{1}{n_2}\sum_{i=1}^{n_2} X_{2i}$. Also, take $n = n_1 + n_2$ and, compute the combined mean, $\bar{\bar{X}} = \frac{n_1\bar{X}_1 + n_2\bar{X}_2}{n_1+n_2}$. Then, accept the lot if $\bar{\bar{X}} \geq c_2$, reject the lot if $\bar{\bar{X}} < c_2$.

Therefore, the Operating characteristic (OC) function of the proposed DS plan is

$$\begin{aligned}
L(\mu) &= \text{Probability of accepting a lot with quality } \mu \\
&= P\left(\bar{X}_1 \geq c_1|\mu\right) + P\left(c_2 \leq \bar{X}_1 < c_1|\mu\right) P\left(\bar{\bar{X}} \geq c_2|\mu\right) \\
&= P\left(\sum_{i=1}^{n_1} X_{1i} \geq n_1 c_1|\mu\right) + \left[P\left(\sum_{i=1}^{n_1} X_{1i} \geq n_1 c_2|\mu\right)\right. \\
&\quad \left. - P\left(\sum_{i=1}^{n_1} X_{1i} > n_1 c_1|\mu\right)\right] P\left(\sum_{i=1}^{n_1} X_{1i} + \sum_{i=1}^{n_2} X_{2i} \geq (n_1+n_2)c_2|\mu\right) \\
&= L_1(\mu) + [L_2(\mu) - L_1(\mu)]L_p(\mu),
\end{aligned} \tag{1}$$

where, $L_1(\mu) = P\left(\sum_{i=1}^{n_1} X_{1i} \geq n_1 c_1 | \mu\right)$, $L_2(\mu) = P\left(\sum_{i=1}^{n_1} X_{1i} > n_1 c_2 | \mu\right)$,
and, $L_p(\mu) = P\left(\sum_{i=1}^{n_1} X_{1i} + \sum_{i=1}^{n_2} X_{2i} \geq (n_1+n_2) c_2 | \mu\right)$.

A submitted lot may be accepted or rejected in an acceptance sampling plan based on sampled items from the lot(s). The probability of rejecting a good lot by the customer under any plan is called the producer's risk and is denoted by P_p, while the probability of accepting a bad lot by the customer is called the consumer's risk, which is denoted by P_c. The producer's risk and the consumer's risk for the proposed plan are

$$
\begin{aligned}
P_p(\mu_0) &= 1 - L(\mu_0) \\
&= 1 - L_1(\mu_0) - [L_2(\mu_0) - L_1(\mu_0)] L_p(\mu_0) \\
&= 1 - P\left(\sum_{i=1}^{n_1} X_{1i} \geq n_1 c_1 | \mu_0\right) - \left[P\left(\sum_{i=1}^{n_1} X_{1i} \geq n_1 c_2 | \mu_0\right)\right. \\
&\quad \left. - P\left(\sum_{i=1}^{n_1} X_{1i} > n_1 c_1 | \mu_0\right)\right] P\left(\sum_{i=1}^{n_1} X_{1i} + \sum_{i=1}^{n_2} X_{2i} \geq (n_1+n_2) c_2 | \mu_0\right)
\end{aligned} \tag{2}
$$

and,

$$
\begin{aligned}
P_c(\mu_1) &= L(\mu_1) \\
&= L_1(\mu_1) + [L_2(\mu_1) - L_1(\mu_1)] L_p(\mu_1) \\
&= P\left(\sum_{i=1}^{n_1} X_{1i} \geq n_1 c_1 | \mu_1\right) + \left[P\left(\sum_{i=1}^{n_1} X_{1i} \geq n_1 c_2 | \mu_1\right)\right. \\
&\quad \left. - P\left(\sum_{i=1}^{n_1} X_{1i} > n_1 c_1 | \mu_1\right)\right] P\left(\sum_{i=1}^{n_1} X_{1i} + \sum_{i=1}^{n_2} X_{2i} \geq (n_1+n_2) c_2 | \mu_1\right)
\end{aligned} \tag{3}
$$

where, μ_0 and μ_1 are AQL and LQL respectively.

3. Estimation of the Plan Parameters

The consumer often establishes a sampling plan for a continuous supply of raw materials regarding an AQL. AQL represents the minimum level of quality for the supplier's process that the consumer would consider acceptable at a process average. Note that AQL is a property of the supplier's manufacturing process, not a property of the sampling plan. The consumer will often design the sampling procedure so that the OC curve gives a high probability of acceptance at AQL. The consumer will also be interested in the other end of the OC curve, i.e., in the protection obtained for an individual lot of poor quality. In such a situation, the consumer may establish an LQL. It is the minimum level of quality that the consumer is ready to accept in an individual lot. It is to be noted that LQL is also not a characteristic of the sampling plan but is a level of lot quality specified by the consumer. It is possible to design a sampling plan that gives the specified probability of acceptance at LQL. Subsequently, the plan parameters are determined to minimize the sample size n such that α is satisfied at AQL(μ_0) and β is met at LQL(μ_1). Thus, consider the following non-linear optimization problem.

Minimize n, subject to

$$P_p(\mu_0) = 1 - L(\mu_0) \geq (1-\alpha), \tag{4}$$

and

$$P_c(\mu_1) = L(\mu_1) \leq \beta. \tag{5}$$

To determine n_i and c_i for $i = 1,\ 2$; first we fix a value of n_i and then choose that c_i for which (4) is satisfied. In this way, one can have several combinations of (n_1, n_2, c_1, c_2) for which (4) is true. Hence, that pair of (n_1, n_2, c_1, c_2) is chosen for which (5) is to be satisfied and will be very nearer to β, if not exactly equal. This procedure is to be followed in all the considered models under discussion.

Suppose X is the quality characteristic that follows any one of the distributions mentioned earlier with scale parameter θ. The OC function, producer's risk, and consumer's risk of the plan for the considered models are given in the sequel.

3.1. One Parameter Polynomial Exponential Family of Distribution

Bouchahed & Zeghdoudi (2018) have proposed the distribution known as one parameter polynomial exponential (OPPE) family of distribution. The pdf of a random variable X from OPPE distribution can be written as

$$f(x) = h(\theta)p(x)e^{-\theta x}; x, \theta > 0,$$

where, $h(\theta) = \frac{1}{\sum_{k=0}^{r} a_k \frac{\Gamma(k+1)}{\theta^{k+1}}}$ and, $p(x) = \sum_{k=0}^{r} a_k x^k$, with known non-negative constants, a_k's and known non-negative integer, r.
The distribution can also be written as

$$\begin{aligned} f(x) &= h(\theta) \sum_{k=0}^{r} a_k x^k e^{-\theta x} \\ &= \frac{\sum_{k=0}^{r} a_k \frac{\Gamma(k+1)}{\theta^{k+1}} f_{GA}(x; k+1, \theta)}{\sum_{k=0}^{r} a_k \frac{\Gamma(k+1)}{\theta^{k+1}}}, \end{aligned} \tag{6}$$

where, $f_{GA}(x; k+1, \theta)$ is the pdf of gamma distribution with shape parameter $(k+1)$ and scale parameter θ.

The OC function is given by (detailed derivation is in Appendix)

$$\begin{aligned}
L(\theta) &= L_1(\theta) + [L_2(\theta) - L_1(\theta)]L_p(\theta) \\
&= 1 - \{h(\theta)\}^{n_1} \sum_{q_0}\sum_{q_1}\cdots\sum_{q_r} \frac{n_1!}{q_0!q_1!\cdots q_r!} \prod_{k=0}^{r} (a_k\Gamma(k+1))^{q_k}\, \theta^{-\sum\limits_{k=0}^{r}(k+1)q_k} \\
&\quad \times\Gamma\left(\theta n_1 c_1, \sum_{k=0}^{r}(k+1)q_k\right) + \Bigg[\{h(\theta)\}^{n_1} \sum_{q_0}\sum_{q_1}\cdots\sum_{q_r} \frac{n_1!}{q_0!q_1!\cdots q_r!} \\
&\quad \times \prod_{k=0}^{r} (a_k\Gamma(k+1))^{q_k}\, \theta^{-\sum\limits_{k=0}^{r}(k+1)q_k} \Gamma\left(\theta n_1 c_1, \sum_{k=0}^{r}(k+1)q_k\right) - \{h(\theta)\}^{n_1} \\
&\quad \times \sum_{q_0}\sum_{q_1}\cdots\sum_{q_r} \frac{n_1!}{q_0!q_1!\cdots q_r!} \prod_{k=0}^{r} (a_k\Gamma(k+1))^{q_k}\, \theta^{-\sum\limits_{k=0}^{r}(k+1)q_k} \\
&\quad \times\Gamma\left(\theta n_1 c_2, \sum_{k=0}^{r}(k+1)q_k\right)\Bigg] \Bigg[1 - \{h(\theta)\}^{(n_1+n_2)} \sum_{q_0}\sum_{q_1}\cdots\sum_{q_r} \frac{(n_1+n_2)!}{q_0!q_1!\cdots q_r!} \\
&\quad \times \prod_{k=0}^{r} (a_k\Gamma(k+1))^{q_k}\, \theta^{-\sum\limits_{k=0}^{r}(k+1)q_k} \Gamma\left(\theta(n_1+n_2)c_2, \sum_{k=0}^{r}(k+1)q_k\right)\Bigg]. \qquad (7)
\end{aligned}$$

The parameters of the plan are determined to minimize the sample size $n = n_1 + n_2$ by the following non-linear optimization problem.

$$P_p(\theta_0) \geq 1 - \alpha \text{ and, } P_c(\theta_1) \leq \beta.$$

ie. Minimize $n = n_1 + n_2$, subject to

$$\begin{aligned}
&1 - \{h(\theta_0)\}^{n_1} \sum_{q_0}\sum_{q_1}\cdots\sum_{q_r} \frac{n_1!}{q_0!q_1!\cdots q_r!} \prod_{k=0}^{r} (a_k\Gamma(k+1))^{q_k}\, \theta_0^{-\sum\limits_{k=0}^{r}(k+1)q_k} \\
&\times\Gamma\left(\theta_0 n_1 c_1, \sum_{k=0}^{r}(k+1)q_k\right) + \Bigg[\{h(\theta_0)\}^{n_1} \sum_{q_0}\sum_{q_1}\cdots\sum_{q_r} \frac{n_1!}{q_0!q_1!\cdots q_r!} \prod_{k=0}^{r} (a_k\Gamma(k+1))^{q_k} \\
&\times\theta_0^{-\sum\limits_{k=0}^{r}(k+1)q_k} \Gamma\left(\theta_0 n_1 c_1, \sum_{k=0}^{r}(k+1)q_k\right) - \{h(\theta_0)\}^{n_1} \sum_{q_0}\sum_{q_1}\cdots\sum_{q_r} \frac{n_1!}{q_0!q_1!\cdots q_r!} \\
&\times \prod_{k=0}^{r} (a_k\Gamma(k+1))^{q_k}\, \theta_0^{-\sum\limits_{k=0}^{r}(k+1)q_k} \Gamma\left(\theta_0 n_1 c_2, \sum_{k=0}^{r}(k+1)q_k\right)\Bigg] \Big[1 - \{h(\theta_0)\}^{(n_1+n_2)} \\
&\times \sum_{q_0}\sum_{q_1}\cdots\sum_{q_r} \frac{(n_1+n_2)!}{q_0!q_1!\cdots q_r!} \prod_{k=0}^{r} (a_k\Gamma(k+1))^{q_k}\, \theta_0^{-\sum\limits_{k=0}^{r}(k+1)q_k} \\
&\times\Gamma\left(\theta_0(n_1+n_2)c_2, \sum_{k=0}^{r}(k+1)q_k\right)\Bigg] \geq 1 - \alpha,
\end{aligned}$$

and,

$$1-\{h(\theta_1)\}^{n_1}\sum_{q_0}\sum_{q_1}\cdots\sum_{q_r}\frac{n_1!}{q_0!q_1!\cdots q_r!}\prod_{k=0}^{r}(a_k\Gamma(k+1))^{q_k}\,\theta_1^{-\sum\limits_{k=0}^{r}(k+1)q_k}$$
$$\times\Gamma\left(\theta_1 n_1 c_1,\sum_{k=0}^{r}(k+1)q_k\right)+\left[\{h(\theta_1)\}^{n_1}\sum_{q_0}\sum_{q_1}\cdots\sum_{q_r}\frac{n_1!}{q_0!q_1!\cdots q_r!}\prod_{k=0}^{r}(a_k\Gamma(k+1))^{q_k}\right.$$
$$\times\theta_1^{-\sum\limits_{k=0}^{r}(k+1)q_k}\Gamma\left(\theta_1 n_1 c_1,\sum_{k=0}^{r}(k+1)q_k\right)-\{h(\theta_1)\}^{n_1}\sum_{q_0}\sum_{q_1}\cdots\sum_{q_r}\frac{n_1!}{q_0!q_1!\cdots q_r!}$$
$$\left.\times\prod_{k=0}^{r}(a_k\Gamma(k+1))^{q_k}\,\theta_1^{-\sum\limits_{k=0}^{r}(k+1)q_k}\Gamma\left(\theta_1 n_1 c_2,\sum_{k=0}^{r}(k+1)q_k\right)\right]\Big[1-\{h(\theta_1)\}^{(n_1+n_2)}$$
$$\times\sum_{q_0}\sum_{q_1}\cdots\sum_{q_r}\frac{(n_1+n_2)!}{q_0!q_1!\cdots q_r!}\prod_{k=0}^{r}(a_k\Gamma(k+1))^{q_k}\,\theta_1^{-\sum\limits_{k=0}^{r}(k+1)q_k}$$
$$\times\Gamma\left(\theta_1(n_1+n_2)c_2,\sum_{k=0}^{r}(k+1)q_k\right)\Big]\leq\beta.$$

We will go ahead with two particular choices.

- For $r = 0$, $a_0 = 1$, OPPE reduces to the exponential distribution with parameter θ.
- For $r = 1$, $a_0 = 1$, $a_1 = 1$, OPPE reduces to the Lindley distribution with parameter θ.

3.1.1. Exponential Distribution

Let X be a quality characteristic following an exponential distribution with parameter θ. The pdf of the quality characteristic is given as

$$f(x)=\theta e^{-\theta x}; x>0,\theta>0, \tag{8}$$

where θ is the scale parameter. The population mean of the exponential distribution is $\mu=\frac{1}{\theta}$. It is to be noted that if a 1^{st} random sample $(X_{11}, X_{12}, \ldots, X_{1n_1})$ of size n_1 is drawn from the exponential distribution, then the distribution of $U_1=\sum\limits_{i=1}^{n_1}X_{1i}$ is a gamma distribution with parameters n_1, the shape and θ, the scale. Similarly, if a 2^{nd} random sample $(X_{21}, X_{22}, \ldots, X_{2n_2})$ of size n_2 is drawn from the exponential distribution, then the distribution of $U_2=\sum\limits_{i=1}^{n_2}X_{2i}$ is also a gamma distribution with parameters n_2, the shape and θ, the scale. If we take $U=U_1+U_2=\sum\limits_{i=1}^{n_1}X_{1i}+\sum\limits_{i=1}^{n_2}X_{2i}$, then u follows a gamma distribution with parameters $n=n_1+n_2$, the shape and θ, the scale. For a given value of μ, one can find the value of θ, and hence the OC function can be treated as a function of θ.

Again,

$$\begin{aligned} L_1(\theta) &= P\left(\sum_{i=1}^{n_1} X_{1i} \geq n_1c_1|\theta\right) \\ &= 1-\Gamma(\theta n_1c_1, n_1), \\ L_2(\theta) &= P\left(\sum_{i=1}^{n_1} X_{1i} \geq n_1c_2|\theta\right) \\ &= 1-\Gamma(\theta n_1c_2, n_1), \\ L_p(\theta) &= P\left(\sum_{i=1}^{n_1} X_{1i} + \sum_{i=1}^{n_2} X_{2i} \geq (n_1+n_2)c_2|\theta\right) \\ &= 1-\Gamma\left[\theta(n_1+n_2)c_2, (n_1+n_2)\right]. \end{aligned}$$

Therefore, the OC function for the exponentially distributed quality characteristic of the proposed sampling inspection plan is given by

$$\begin{aligned} L(\theta) &= L_1(\theta) + [L_2(\theta) - L_1(\theta)]L_p(\theta) \\ &= 1-\Gamma(\theta n_1c_1, n_1) + [1-\Gamma(\theta n_1c_2, n_1) - 1 + \Gamma(\theta n_1c_1, n_1)] \times \\ &\quad \{1-\Gamma\left[\theta(n_1+n_2)c_2, (n_1+n_2)\right]\} \\ &= 1-\Gamma(\theta n_1c_1, n_1) + [\Gamma(\theta n_1c_1, n_1) - \Gamma(\theta n_1c_2, n_1)] \times \\ &\quad \{1-\Gamma\left[\theta(n_1+n_2)c_2, (n_1+n_2)\right]\}, \end{aligned} \tag{9}$$

where, $\Gamma(\nu, n) = \frac{1}{\Gamma(n)} \int\limits_0^{\nu} e^{-t}t^{n-1}dt$.

Now, the producer's risk and the consumer's risk are given as :

$$\begin{aligned} P_p(\theta_0) &= 1 - L(\theta_0) \\ &= 1-1+\Gamma(\theta_0n_1c_1, n_1) - [\Gamma(\theta_0n_1c_1, n_1) - \Gamma(\theta_0n_1c_2, n_1)] \times \\ &\quad \{1-\Gamma\left[\theta_0(n_1+n_2)c_2, (n_1+n_2)\right]\} \\ &= \Gamma(\theta_0n_1c_1, n_1) - [\Gamma(\theta_0n_1c_1, n_1) - \Gamma(\theta_0n_1c_2, n_1)] \times \\ &\quad \{1-\Gamma\left[\theta_0(n_1+n_2)c_2, (n_1+n_2)\right]\}, \end{aligned}$$

and,

$$\begin{aligned} P_c(\theta_1) &= L(\theta_1) \\ &= 1-\Gamma(\theta_1n_1c_1, n_1) + [\Gamma(\theta_1n_1c_1, n_1) - \Gamma(\theta_1n_1c_2, n_1)] \times \\ &\quad \{1-\Gamma\left[\theta_1(n_1+n_2)c_2, (n_1+n_2)\right]\}. \end{aligned}$$

The plan's parameters are determined to minimize the total sample size $n = n_1 + n_2$ by the following non-linear optimization problem.

Minimize n, subject to

$$\begin{aligned} 1-\Gamma(\theta_0n_1c_1, n_1) + [\Gamma(\theta_0n_1c_1, n_1) - \Gamma(\theta_0n_1c_2, n_1)] \times \\ \{1-\Gamma\left[\theta_0(n_1+n_2)c_2, (n_1+n_2)\right]\} \geq (1-\alpha) \end{aligned} \tag{10}$$

and,

$$1-\Gamma(\theta_1 n_1 c_1, n_1)+[\Gamma(\theta_1 n_1 c_1, n_1)-\Gamma(\theta_1 n_1 c_2, n_1)]\times \\ \{1-\Gamma\left[\theta_1(n_1+n_2)c_2,(n_1+n_2)\right]\}\le\beta. \tag{11}$$

3.1.2. Lindley Distribution

Let X be a quality characteristic following a Lindley distribution with parameter θ. The pdf of the quality characteristic is given as

$$f(x)=\frac{\theta^2(1+x)e^{-\theta x}}{\theta+1}; x>0, \theta>0, \tag{12}$$

where θ is the scale parameter of the distribution. The population mean of the Lindley distribution is $\mu=\frac{(\theta+2)}{\theta(\theta+1)}$ (see, Lindley, 1958; Ghitany et al., 2008).

If $(X_{11}, X_{12}, \ldots, X_{1n_1})$ be a 1^{st} random sample of size n_1 from the Lindley with parameter θ, then the pdf of $U_1=\sum\limits_{i=1}^{n_1} X_{1i}$ is

$$g(u_1; n_1, \theta)=\sum_{k=0}^{n_1} P_{(k,n_1)}(\theta).f_{(GA)}(u_1; 2n_1-k, \theta),$$

where,

$$P_{(k,n_1)}(\theta)=\binom{n_1}{k}\frac{\theta^k}{(1+\theta)^{n_1}},$$

and

$$f_{(GA)}(u_1; 2n_1-k, \theta)=\frac{\theta^{(2n_1-k)}}{\Gamma(2n_1-k)}{u_1}^{(2n_1-k)-1}e^{-\theta u_1}.$$

Hence, U_1 have the probability density function of gamma distribution with shape and scale parameters $(2n_1-k)$ and θ, respectively (see, Al-Mutairi et al., 2013).

Again, $(X_{21}, X_{22}, \ldots, X_{2n_2})$ be a 2^{nd} random sample of size n_2 from the Lindley with parameter θ, then the pdf of $U_2=\sum\limits_{i=1}^{n_2} X_{2i}$ is

$$g(u_2; n_2, \theta)=\sum_{k=0}^{n_2} P_{(k,n_2)}(\theta).f_{(GA)}(u_2; 2n_2-k, \theta),$$

where,

$$P_{(k,n_2)}(\theta)=\binom{n_2}{k}\frac{\theta^k}{(1+\theta)^{n_2}}$$

and,

$$f_{(GA)}(u_2; 2n_2-k, \theta)=\frac{\theta^{(2n_2-k)}}{\Gamma(2n_2-k)}{u_2}^{(2n_2-k)-1}e^{-\theta u_2}.$$

So, U_2 has the pdf of gamma distribution with shape and scale parameters $(2n_2-k)$ and θ, respectively.

If we take $n = n_1 + n_2$ and $U = U_1 + U_2 = \sum_{i=1}^{n_1} X_{1i} + \sum_{i=1}^{n_2} X_{2i}$, then U have quite similar distribution as following.

$$g(u; n, \theta) = \sum_{k=0}^{n} P_{(k,n)}(\theta).f_{(GA)}(u; 2n-k, \theta),$$

where,

$$P_{(k,n)}(\theta) = \binom{n}{k} \frac{\theta^k}{(1+\theta)^n}$$

and

$$f_{(GA)}(u; 2n-k, \theta) = \frac{\theta^{(2n-k)}}{\Gamma(2n-k)} u^{(2n-k)-1} e^{-\theta u}.$$

Here, U has the pdf of gamma distribution with shape and scale parameters $(2n-k)$ and θ, respectively.

Again,

$$\begin{aligned} L_1(\theta) &= P\left(\sum_{i=1}^{n_1} X_{1i} \geq n_1 c_1 | \theta\right) \\ &= 1 - \int_0^{n_1 c_1} g(u_1; n_1, \theta) du_1 \\ &= 1 - \sum_{k=0}^{n_1} \binom{n_1}{k} \frac{\theta^k}{(1+\theta)^{n_1}} \Gamma(\theta n_1 c_1, 2n_1 - k), \\ L_2(\theta) &= P\left(\sum_{i=1}^{n_1} X_{1i} \geq n_1 c_2 | \theta\right) \\ &= 1 - \int_0^{n_1 c_2} g(u_1; n_1, \theta) du_1 \\ &= 1 - \sum_{k=0}^{n_1} \binom{n_1}{k} \frac{\theta^k}{(1+\theta)^{n_1}} \Gamma(\theta n_1 c_2, 2n_1 - k), \end{aligned}$$

$$\begin{aligned} L_p(\theta) &= P\left(\sum_{i=1}^{n_1} X_{1i} + \sum_{i=1}^{n_2} X_{2i} \geq (n_1 + n_2) c_2 | \theta\right) \\ &= 1 - \int_0^{n c_2} g(u; n, \theta) du \\ &= 1 - \sum_{k=0}^{n} \binom{n}{k} \frac{\theta^k}{(1+\theta)^n} \Gamma(\theta n c_2, 2n - k). \end{aligned}$$

Therefore, the OC function for the Lindley distributed quality characteristic of the proposed DS plan is given by

$$\begin{aligned}
L(\theta) &= L_1(\theta) + [L_2(\theta) - L_1(\theta)]L_p(\theta) \\
&= 1 - \sum_{k=0}^{n_1} \binom{n_1}{k} \frac{\theta^k}{(1+\theta)^{n_1}} \Gamma(\theta n_1 c_1, 2n_1 - k) + \Bigg[1 - \sum_{k=0}^{n_1} \binom{n_1}{k} \frac{\theta^k}{(1+\theta)^{n_1}} \Gamma(\theta n_1 c_2, 2n_1 - k) \\
&\quad -1 + \sum_{k=0}^{n_1} \binom{n_1}{k} \frac{\theta^k}{(1+\theta)^{n_1}} \Gamma(\theta n_1 c_1, 2n_1 - k)\Bigg] \left\{1 - \sum_{k=0}^{n} \binom{n}{k} \frac{\theta^k}{(1+\theta)^{n}} \Gamma(\theta n c_2, 2n - k)\right\} \\
&= 1 - \sum_{k=0}^{n_1} \binom{n_1}{k} \frac{\theta^k}{(1+\theta)^{n_1}} \Gamma(\theta n_1 c_1, 2n_1 - k) + \Bigg[\sum_{k=0}^{n_1} \binom{n_1}{k} \frac{\theta^k}{(1+\theta)^{n_1}} \Gamma(\theta n_1 c_1, 2n_1 - k) \\
&\quad - \sum_{k=0}^{n_1} \binom{n_1}{k} \frac{\theta^k}{(1+\theta)^{n_1}} \Gamma(\theta n_1 c_2, 2n_1 - k)\Bigg] \left\{1 - \sum_{k=0}^{n} \binom{n}{k} \frac{\theta^k}{(1+\theta)^{n}} \Gamma(\theta n c_2, 2n - k)\right\},
\end{aligned} \tag{13}$$

where, $\Gamma(\nu, n) = \frac{1}{\Gamma(n)} \int\limits_0^{\nu} e^{-t} t^{n-1} dt$.

Now, the producer's risk and the consumer's risk are given by

$$\begin{aligned}
P_p(\theta_0) &= 1 - L(\theta_0) \\
&= \sum_{k=0}^{n_1} \binom{n_1}{k} \frac{\theta_0^k}{(1+\theta_0)^{n_1}} \Gamma(\theta_0 n_1 c_1, 2n_1 - k) - \Bigg[\sum_{k=0}^{n_1} \binom{n_1}{k} \frac{\theta_0^k}{(1+\theta_0)^{n_1}} \Gamma(\theta_0 n_1 c_1, 2n_1 - k) \\
&\quad - \sum_{k=0}^{n_1} \binom{n_1}{k} \frac{\theta_0^k}{(1+\theta_0)^{n_1}} \Gamma(\theta_0 n_1 c_2, 2n_1 - k)\Bigg] \left\{1 - \sum_{k=0}^{n} \binom{n}{k} \frac{\theta_0^k}{(1+\theta_0)^{n}} \Gamma(\theta_0 n c_2, 2n - k)\right\},
\end{aligned} \tag{14}$$

and,

$$\begin{aligned}
P_c(\theta_1) &= L(\theta_1) \\
&= 1 - \sum_{k=0}^{n_1} \binom{n_1}{k} \frac{\theta_1^k}{(1+\theta_1)^{n_1}} \Gamma(\theta_1 n_1 c_1, 2n_1 - k) + \Bigg[\sum_{k=0}^{n_1} \binom{n_1}{k} \frac{\theta_1^k}{(1+\theta_1)^{n_1}} \Gamma(\theta_1 n_1 c_1, 2n_1 - k) \\
&\quad - \sum_{k=0}^{n_1} \binom{n_1}{k} \frac{\theta_1^k}{(1+\theta_1)^{n_1}} \Gamma(\theta_1 n_1 c_2, 2n_1 - k)\Bigg] \left\{1 - \sum_{k=0}^{n} \binom{n}{k} \frac{\theta_1^k}{(1+\theta_1)^{n}} \Gamma(\theta_1 n c_2, 2n - k)\right\}.
\end{aligned} \tag{15}$$

The plan's parameters are determined to minimize the sample size $n = n_1 + n_2$ by the following non-linear optimization problem.

Minimize n, subject to

$$\begin{aligned}
&1 - \sum_{k=0}^{n_1} \binom{n_1}{k} \frac{\theta_0^k}{(1+\theta_0)^{n_1}} \Gamma(\theta_0 n_1 c_1, 2n_1 - k) + \Bigg[\sum_{k=0}^{n_1} \binom{n_1}{k} \frac{\theta_0^k}{(1+\theta_0)^{n_1}} \Gamma(\theta_0 n_1 c_1, 2n_1 - k) - \sum_{k=0}^{n_1} \binom{n_1}{k} \\
&\times \frac{\theta_0^k}{(1+\theta_0)^{n_1}} \Gamma(\theta_0 n_1 c_2, 2n_1 - k)\Bigg] \left\{1 - \sum_{k=0}^{n} \binom{n}{k} \frac{\theta_0^k}{(1+\theta_0)^{n}} \Gamma(\theta_0 n c_2, 2n - k)\right\} \geq (1 - \alpha),
\end{aligned} \tag{16}$$

and,

$$1-\sum_{k=0}^{n_1}\binom{n_1}{k}\frac{\theta_1^k}{(1+\theta_1)^{n_1}}\Gamma(\theta_1 n_1 c_1, 2n_1-k)+\left[\sum_{k=0}^{n_1}\binom{n_1}{k}\frac{\theta_1^k}{(1+\theta_1)^{n_1}}\Gamma(\theta_1 n_1 c_1, 2n_1-k)\right.$$
$$\left.-\sum_{k=0}^{n_1}\binom{n_1}{k}\frac{\theta_1^k}{(1+\theta_1)^{n_1}}\Gamma(\theta_1 n_1 c_2, 2n_1-k)\right]\left\{1-\sum_{k=0}^{n}\binom{n}{k}\frac{\theta_1^k}{(1+\theta_1)^{n}}\Gamma(\theta_1 n c_2, 2n-k)\right\}\leq \beta. \tag{17}$$

TABLE 1: Values of (n_1, n_2, c_1, c_2) for DS Plan and (n, c) for SS Plan in Lindley and Exponential Distribution respectively, for given, $\alpha = 0.01$, $\beta = 0.01$

		Lindley Distribution								Exponential Distribution							
μ		θ		DS Plan				SS Plan		θ		DS Plan				SS Plan	
μ_0	μ_1	θ_0	θ_1	n	(n_1, n_2)	c_1	c_2	n	c	θ_0	θ_1	n	(n_1, n_2)	c_1	c_2	n	c
70	30	0.0282	0.0646	17	(8,9)	30.70	21.20	17	44.76	0.0143	0.0333	28	(14,14)	30.07	18.07	31	44.10
70	35	0.0282	0.0556	23	(11,12)	35.84	26.34	24	48.33	0.0143	0.0286	43	(22,21)	36.97	28.47	46	48.25
70	40	0.0282	0.0488	35	(17,18)	41.94	34.44	37	52.19	0.0143	0.0250	65	(33,32)	42.30	34.80	70	52.02
60	40	0.0328	0.0488	65	(32,33)	41.98	37.28	69	48.59	0.0167	0.0250	125	(63,62)	42.24	37.04	133	48.56
60	35	0.0328	0.0556	38	(19,19)	37.11	31.61	40	45.28	0.0167	0.0286	71	(36,35)	37.16	30.96	76	45.16
60	30	0.0328	0.0646	23	(11,12)	30.64	22.14	24	41.42	0.0167	0.0333	43	(22,21)	31.70	24.70	46	41.35

We have presented plan parameter values (n_1, n_2, c_1, c_2) for DS Plan and (n, c) for single sampling plan with respect to some specified values of θ_0, θ_1, α and β in the Tables 1-5. We have used R software for calculation purposes; the R codes may be available from the corresponding author upon request.

To ensure quality, minimize risk from the producer's side of rejecting a good lot or from the consumer's side of accepting a bad one. It is natural to inspect a large sample, which is reflected in the tables. For example, in Table 3, for $\mu_0 = 60$ and $\mu_1 = 40$, the values of (n_1, n_2, c_1, c_2) for DS Plan is (45, 44, 43.01, 37.41) for exponential distribution and (23, 23, 43.60, 38.60) for Lindley distribution respectively, whereas, the corresponding (n, c) values of exponential and Lindley distribution for Single Sampling Plan (SSP) are (95, 50.25) and (50, 50.34).

So, a proper sampling plan and model selection for the quality characteristic is essential in saving cost and time in the industry. It is evident from Tables 1-5 that, on the one hand, the optimal sample size (total) in the DS plan is generally less than that of the SS plan. On the other hand, if the quality characteristic data fits the Lindley distribution better than the exponential distribution, then the sample needed for making a decision based on the DS plan is less. Hence, the use of DS plan is more economical.

TABLE 2: Values of (n_1, n_2, c_1, c_2) for DS Plan and (n, c) for SS Plan in Lindley and Exponential Distribution respectively, for given, $\alpha = 0.02$, $\beta = 0.01$

		Lindley Distribution								Exponential Distribution							
μ		θ		DS Plan				SS Plan		θ		DS Plan				SS Plan	
μ_0	μ_1	θ_0	θ_1	n	(n_1, n_2)	c_1	c_2	n	c	θ_0	θ_1	n	(n_1, n_2)	c_1	c_2	n	c
70	30	0.0282	0.0646	15	(7,8)	31.29	21.79	15	45.96	0.0143	0.0333	26	(13,13)	31.77	21.77	27	45.18
70	35	0.0282	0.0556	20	(10,10)	37.09	28.09	21	49.33	0.0143	0.0286	38	(19,19)	37.52	28.52	40	49.19
70	40	0.0282	0.0488	30	(15,15)	42.70	34.70	32	52.96	0.0143	0.0250	58	(29,29)	43.06	35.26	62	52.97
60	40	0.0328	0.0488	57	(28,29)	42.51	37.81	61	49.22	0.0167	0.0250	110	(55,55)	42.73	37.23	117	49.17
60	35	0.0328	0.0556	33	(16,17)	37.27	31.27	35	45.99	0.0167	0.0286	62	(31,31)	37.59	31.09	66	45.82
60	30	0.0328	0.0646	20	(10,10)	31.71	23.71	21	42.28	0.0167	0.0333	38	(19,19)	32.17	24.67	40	42.16

Table 3: Values of (n_1, n_2, c_1, c_2) for DS Plan and (n, c) for SS Plan in Lindley and Exponential Distribution respectively, for given, $\alpha = 0.05$, $\beta = 0.01$

		Lindley Distribution										Exponential Distribution							
μ		θ		DS Plan				SS Plan		θ		DS Plan				SS Plan	
μ_0	μ_1	θ_0	θ_1	n	(n_1, n_2)	c_1	c_2	n	c	θ_0	θ_1	n	(n_1, n_2)	c_1	c_2	n	c
70	30	0.0282	0.0646	12	(6,6)	33.52	24.52	12	47.94	0.0143	0.0333	21	(11,10)	33.88	23.88	22	47.39
70	35	0.0282	0.0556	17	(8,9)	38.20	29.20	17	51.20	0.0143	0.0286	31	(16,15)	39.49	30.49	33	51.23
70	40	0.0282	0.0488	25	(12,13)	43.83	35.83	26	54.59	0.0143	0.0250	46	(23,23)	44.12	35.62	50	54.55
60	40	0.0328	0.0488	46	(23,23)	43.60	38.60	50	50.34	0.0167	0.0250	89	(45,44)	43.76	37.86	95	50.25
60	35	0.0328	0.0556	27	(13,14)	38.38	31.88	28	47.26	0.0167	0.0286	50	(25,25)	38.65	31.45	54	47.23
60	30	0.0328	0.0646	17	(8,9)	32.64	24.63	17	43.89	0.0167	0.0333	31	(16,15)	33.82	25.82	33	43.91

Table 4: Values of (n_1, n_2, c_1, c_2) for DS Plan and (n, c) for SS Plan in Lindley and Exponential Distribution respectively, for given, $\alpha = 0.02$, $\beta = 0.02$

		Lindley Distribution										Exponential Distribution							
μ		θ		DS Plan				SS Plan		θ		DS Plan				SS Plan	
μ_0	μ_1	θ_0	θ_1	n	(n_1, n_2)	c_1	c_2	n	c	θ_0	θ_1	n	(n_1, n_2)	c_1	c_2	n	c
70	30	0.0282	0.0646	13	(7,6)	31.23	20.13	13	44.39	0.0143	0.0333	24	(12,12)	30.44	19.44	25	44.33
70	35	0.0282	0.0556	19	(9,10)	35.40	24.40	19	48.37	0.0143	0.0286	34	(17,17)	35.85	24.85	36	48.17
70	40	0.0282	0.0488	28	(14,14)	41.82	33.82	29	52.16	0.0143	0.0250	51	(26,25)	41.67	32.67	55	52.01
60	40	0.0328	0.0488	50	(25,25)	41.49	35.49	54	48.57	0.0167	0.0250	97	(49,48)	41.77	35.57	104	48.54
60	35	0.0328	0.0556	29	(14,15)	35.69	25.69	31	45.18	0.0167	0.0286	55	(28,27)	36.51	29.01	59	45.07
60	30	0.0328	0.0646	19	(9,10)	30.26	20.26	19	41.46	0.0167	0.0333	34	(17,17)	30.72	20.72	36	41.29

Table 5: Values of (n_1, n_2, c_1, c_2) for DS Plan and (n, c) for SS Plan in Lindley and Exponential Distribution respectively, for given, $\alpha = 0.05$, $\beta = 0.02$

		Lindley Distribution										Exponential Distribution							
μ		θ		DS Plan				SS Plan		θ		DS Plan				SS Plan	
μ_0	μ_1	θ_0	θ_1	n	(n_1, n_2)	c_1	c_2	n	c	θ_0	θ_1	n	(n_1, n_2)	c_1	c_2	n	c
70	30	0.0282	0.0646	10	(5,5)	29.82	16.82	10	46.04	0.0143	0.0333	19	(10,9)	32.21	20.71	19	45.84
70	35	0.0282	0.0556	15	(7,8)	35.89	24.39	15	50.08	0.0143	0.0286	28	(14,14)	37.53	27.53	29	50.08
70	40	0.0282	0.0488	22	(11,11)	42.69	34.19	23	53.67	0.0143	0.0250	41	(21,20)	42.96	33.46	44	53.59
60	40	0.0328	0.0488	40	(20,20)	42.38	36.38	43	49.61	0.0167	0.0250	79	(40,39)	42.82	36.42	83	49.59
60	35	0.0328	0.0556	22	(11,11)	36.43	27.93	25	46.55	0.0167	0.0286	44	(22,22)	37.31	29.31	47	46.37
60	30	0.0328	0.0646	15	(7,8)	30.69	20.69	15	42.93	0.0167	0.0333	28	(14,14)	32.14	23.14	29	42.92

4. Comparative Study

The proposed DS plan is based on the sample means for the first sample and combined mean for the second sample, which is different from other DS plans (based on the number of non-conforming, percentage of non-conforming, variance, specification limits, minimum angle method, cost function, ASN-minimax, process capability index, etc.) exist in literature. Therefore, there is little scope to compare the current DS plan with other DS plans. However, we have compared the plan with the corresponding SS Plan. A comparative study is also made for the Lindley as well as the exponentially distributed quality characteristic.

The features of the proposed DS plan are studied under different conditions of the plan parameters. Tables 1−5 display the plan parameters ($n = n_1 + n_2$, c_1, c_2) for DS plan and (n, c) for the Lindley as well as the exponentially distributed quality characteristic with various producer's risk, α and consumer's risk, β. By studying the nature of the distribution of the quality characteristic through model selection methods and with the help of the Tables, practitioners can determine the size of a sample required for inspection and the associated rules for lot sentencing.

It is observed from the Tables that
(i) the required sample size, n, decreases as α and/or β risk increases. For example,

a sample of size $n = 38$ is required for $(\alpha,\ \beta) = (0.01,\ 0.01)$, $n = 33$ for $(\alpha,\ \beta) = (0.02,\ 0.01)$, $n = 29$ for $(\alpha,\ \beta) = (0.02,\ 0.02)$ under the same quality conditions $(\mu_0,\ \mu_1) = (60,\ 35)$. This indicates that relatively smaller sample size is needed as long as the producer and/or consumer is ready to suffer larger risks for making incorrect decisions.
(ii) The required sample size, n, decreases as the difference between μ_0 and μ_1 increases. For example, a sample of size $n = 15$ is required for $(\mu_0,\ \mu_1) = (60,\ 30)$ and a sample of size $n = 40$ is required for $(\mu_0,\ \mu_1) = (60,\ 40)$ under the same risk conditions, i.e., $(\alpha,\ \beta) = (0.05,\ 0.02)$. This indicates that comparatively, smaller sample size is required because it becomes easier to make correct decisions if the difference between two specified quality levels increases.
(iii) The sample sizes for the Lindley distributed quality characteristic is smaller than that for the exponentially distributed one.
(iv) The sample sizes for DS plan is smaller than that for SS Plan.

5. Examples and Data Analysis

The data is from Lawless (2003), that specifies the number of cycles to failure for twenty-five 100 cm specimens of yarn, tested at a particular strain level, shown in the following.

Table 6: The number of cycles to failure data

15	20	38	42	61
76	86	98	121	146
149	157	175	176	180
180	198	220	224	251
264	282	321	325	653

From Table-7, we find that the OPPE distribution for $r = 1, a_0 = 0.8$, $a_1 = 0.1$ best fits the data as the value of AIC is the least. The histogram and fitted distributions have been shown in Figure 1.

Table 7: Comparison between exponential and OPPE distribution with $r = 1, a_0 = 0.8$, $a_1 = 0.1$ for Data Set

Distribution	Estimate of θ	Negative Log-likelihood	AIC
Exponential	0.005605705	154.5895	311.1790
OPPE distribution with $r = 1, a_0 = 0.8$, $a_1 = 0.1$	0.01076875	152.4577	306.9154

Table 8 indicates the estimation of plan parameters for the Data Set, using exponential and OPPE distribution for $r = 1, a_0 = 0.8$, $a_1 = 0.1$ if we consider the producer's value, $\mu_0 = 305$ and producer's risk, $\alpha = 0.02$, and for consumer's value, $\mu_1 = 125$, and consumer's risk, $\beta = 0.01$. We summarize the analysis and decision for this data set in Table 10. The decision is the same for the single sampling plan and for the double sampling plan (on the basis of 1st sample) in the case of exponential distribution and OPPE distribution for $r = 1, a_0 = 0.8$,

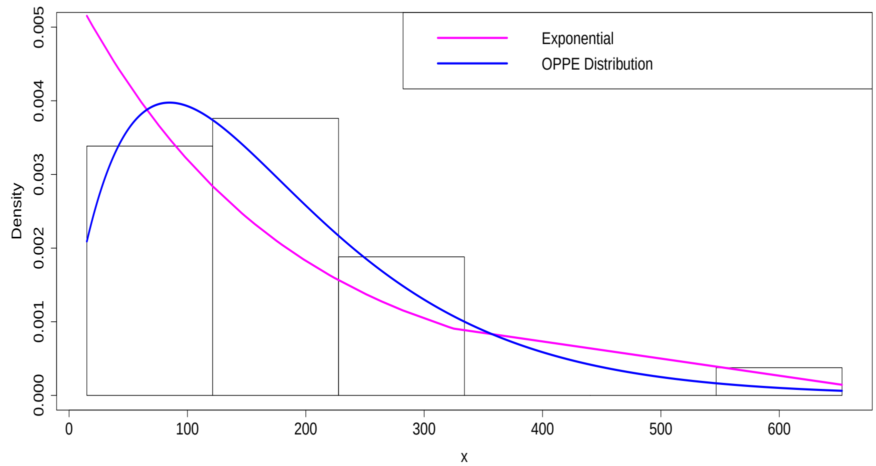

FIGURE 1: Histogram and fitted distributions for the Data Set.

and $a_1 = 0.1$. Here, the sample size is minimum(n, n_1), i.e., we need to draw just $n_1 = 6$, sample observations for the double sampling inspection plan using OPPE distribution for $r = 1, a_0 = 0.8,\ a_1 = 0.1$ (that is a better fit in this data). OC curves for Exponential distribution and OPPE distribution for $r = 1, a_0 = 0.8$, $a_1 = 0.1$ have been presented respectively in Figure 2, and it is observed that the OC curve under the DS Plan for OPPE distribution for $r = 1, a_0 = 0.8,\ a_1 = 0.1$ assumption is steeper than the others. This fact also justifies more protection to the customers.

TABLE 8: Plan Parameter Estimation for the Data Set, using exponential and OPPE distribution for $r = 1, a_0 = 0.8,\ a_1 = 0.1$, for given $\alpha = 0.02,\ \beta = 0.01$, where, $\mu_0 = 305,\ \mu_1 = 125$

Distribution	$\mu = 178.32$		θ		DS plan				SS plan	
	μ_0	μ_1	θ_0	θ_1	n	(n_1, n_2)	c_1	c_2	n	c
Exponential	305	125	0.0033	0.0080	21	(11,10)	126.30	78.30	25	193.15
OPPE with $r = 1$, $a_0 = 0.8,\ a_1 = 0.1$			0.0065	0.0159	12	(6,6)	125.74	75.74	13	195.79

TABLE 9: Plan Parameter Estimation for the Data Set, using exponential distribution and OPPE distribution for $r = 1, a_0 = 0.8,\ a_1 = 0.1$, for given $\alpha = 0.05,\ \beta = 0.02$, where, $\mu_0 = 240,\ \mu_1 = 120$

Distribution	$\mu = 178.32$		θ		DS plan				SS plan	
	μ_0	μ_1	θ_0	θ_1	n	(n_1, n_2)	c_1	c_2	n	c
Exponential	240	120	0.0042	0.0083	23	(12,11)	120.00	70.00	29	171.69
OPPE with $r = 1$, $a_0 = 0.8,\ a_1 = 0.1$			0.0083	0.0165	14	(7,7)	123.53	43.53	15	172.46

Table 9 indicates the estimation of plan parameters for the Data Set, using exponential distribution and OPPE distribution for $r = 1, a_0 = 0.8,\ a_1 = 0.1$ if we consider the producer's value, $\mu_0 = 240$ and producer's risk, $\alpha = 0.05$, and for consumer's value, $\mu_1 = 120$, and consumer's risk, $\beta = 0.02$. We summarize the analysis and decision for this data set in Table 11. We can't take decision if we perform single sampling plan using exponential distribution as there is not enough sample to conclude. And, for single sampling plan using OPPE distribution for $r = 1, a_0 = 0.8,\ a_1 = 0.1$, our decision is to reject the lot. On the other hand, we need to take 2nd sample under double sampling plan using both exponential distribution and OPPE distribution for $r = 1$, $a_0 = 0.8$, $a_1 = 0.1$ and finally accept the lot on the basis of combined sample. In this situation, double sampling plan using OPPE distribution for $r = 1$, $a_0 = 0.8$, $a_1 = 0.1$ also, provides more benefits as the sample size, $n = 14$ have drawn, which is, again, least.

Table 10: Sampling Inspection plans for Data Set, at level $\alpha = 0.02,\ \beta = 0.01$

Distribution	Sampling Plan	(n, n_1, n_2)	c_1	c_2	n	c	mean/combined mean	Decision
Exponential	Single Sampling Inspection Plan	—	—	—	25	193.15	$\bar{X} = 178.32$	Reject the lot
	Double Sampling Inspection Plan (1st Sample)	(21,11,10)	126.30	78.30	—	—	$\bar{X}_1 = 77.45$	Reject the lot on the basis of 1st sample
OPPE with $r = 1$, $a_0 = 0.8$, $a_1 = 0.1$	Single Sampling Inspection Plan	—	—	—	13	195.79	$\bar{X} = 91.08$	Reject the lot
	Double Sampling Inspection Plan (1st Sample)	(12,6,6)	125.74	75.74	—	—	$\bar{X}_1 = 42.00$	Reject the lot on the basis of 1st sample

Table 11: Sampling Inspection plans for Data Set, at level $\alpha = 0.05,\ \beta = 0.02$

Distribution	Sampling Plan	(n, n_1, n_2)	c_1	c_2	n	c	sample mean / combined mean	Decision
Exponential	Single Sampling Inspection Plan	—	—	—	29	171.69	——	Not enough sample observations to take decision
	Double Sampling Inspection Plan (1st Sample)	(23,12,11)	120.00	70.00	—	—	$\bar{X}_1 = 84.08$	Need to draw 2^{nd} sample of size $n_2 = 11$
	Double Sampling Inspection Plan (Combined Sample)	(23,12,11)	120.00	70.00	—	—	$\bar{\bar{X}} = 151.30$	Accept the lot and on the basis of combined sample
OPPE with $r = 1$, $a_0 = 0.8$, $a_1 = 0.1$	Single Sampling Inspection Plan	—	—	—	15	172.46	$\bar{X} = 102.67$	Reject the lot
	Double Sampling Inspection Plan (1st Sample)	(14,7,7)	123.53	43.53	—	—	$\bar{X}_1 = 48.29$	Need to draw 2^{nd} sample of size $n_2 = 7$
	Double Sampling Inspection Plan (Combined Sample)	(14,7,7)	123.53	43.53	—	—	$\bar{\bar{X}} = 97.14$	Accept the lot and on the basis of combined sample

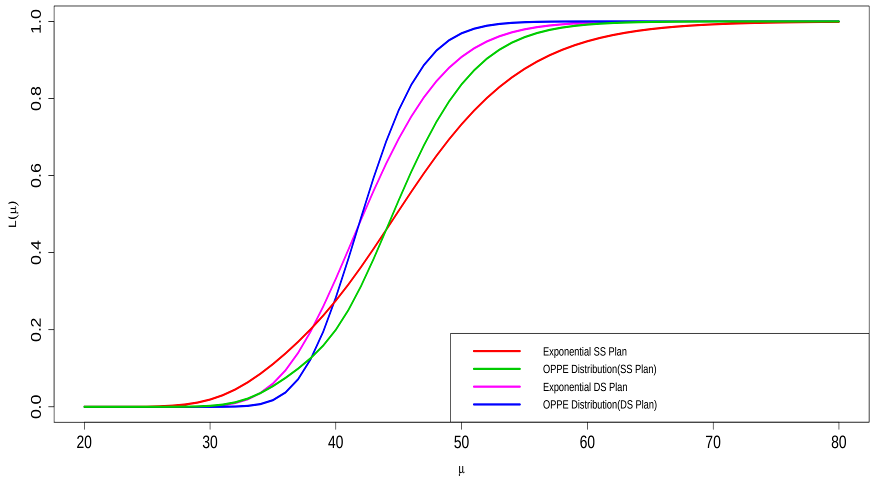

FIGURE 2: OC curve for the single sampling plan and double sampling plan for both exponential distribution and OPPE distribution for $r = 1, a_0 = 0.8$, $a_1 = 0.1$ for the Data Set.

6. Concluding Remarks

In this article, we have discussed the DS sampling inspection plans for the variable for the OPPE family of distributions. The plan parameters are estimated based on a two-point OC curve approach, the AQL and the LQL. In a comparison study, we have observed that the DS sampling inspection plans for the OPPE family of distributions works more efficiently in terms of sample size rather than an SS inspection plan. In other words, the plan for the double sampling required a smaller sample size to draw a verdict about acceptance or rejection of the lot compared to an SS inspection plan since it has more flexibility in fitting the data. The DS plan is recommended using the OPPE family of distributions. Tables of plan parameters have been obtained through computational techniques at various values of $[AQL(\mu_0), \alpha]$, $[LQL(\mu_1), \beta]$ for ready reference for practitioners. The proposed DS plan has more benefits of achieving the goal of cost and time saving of the organization's quality monitoring process. Also, the sampling plan has been explained through a real-life example. We expect the industrial engineers and statisticians to implement the DS sampling approach to OPPE distributed lifetime data effectively.

Acknowledgements

We are grateful to the Editor-in-Chief and anonymous referees for making some constructive suggestions and comments on an earlier version of the manuscript, which resulted in this much-improved version.

[Received: March 2023 — Accepted: November 2023]

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Appendix A.

We discuss the derivation of the sampling distributions of the sum of the first sample and that of the sum of the combined sample (both the first and second samples).

If $X_1, X_2, \ldots, X_n$ are i.i.d. random variables from the Oppe distribution with parameter θ, then the derivation of the distribution of $Z = \sum_{i=1}^{n} X_i$ is as follows.

Let $X \sim Oppe(\theta)$, then the moment generating function of X will be

$$\begin{aligned}
M_x(t) &= E(e^{tx}) \\
&= \int_0^\infty e^{tx} f(x)dx \\
&= \int_0^\infty e^{tx} h(\theta) \sum_{k=0}^{r} a_k x^k e^{-\theta x} dx \\
&= h(\theta) \sum_{k=0}^{r} a_k \int_0^\infty e^{-(\theta-t)x} x^{(k+1)-1} dx \\
&= h(\theta) \sum_{k=0}^{r} a_k \frac{\Gamma(k+1)}{(\theta-t)^{k+1}},
\end{aligned}$$

and, the moment generating function of $Z = \sum_{i=1}^{n} X_i$ will be

$$\begin{aligned}
M_z(t) &= E(e^{tz}) \\
&= E\left(e^{t\sum_{i=1}^{n} X_i}\right) \\
&= \left\{E(e^{tx})\right\}^n \\
&= \{h(\theta)\}^n \left\{\sum_{k=0}^{r} a_k \frac{\Gamma(k+1)}{(\theta-t)^{k+1}}\right\}^n \\
&= \{h(\theta)\}^n \left\{\sum_{k=0}^{r} a_k \frac{\Gamma(k+1)}{\theta^{k+1}} \left(1-\frac{t}{\theta}\right)^{-(k+1)}\right\}^n \\
&= \{h(\theta)\}^n \sum_{q_0}\sum_{q_1}\cdots\sum_{q_r} \frac{n!}{q_0! q_1! \cdots q_r!} \prod_{k=0}^{r} \left(a_k \Gamma(k+1)\right)^{q_k} \times \\
&\quad \theta^{-\sum_{k=0}^{r}(k+1)q_k} \left(1-\frac{t}{\theta}\right)^{-\sum_{k=0}^{r}(k+1)q_k}.
\end{aligned}$$

Hence, the distribution of $Z = \sum_{i=1}^{n} X_i$ is given by

$$
\begin{aligned}
f(z) = \{h(\theta)\}^n \sum_{q_0}\sum_{q_1}\cdots\sum_{q_r} \frac{n!}{q_0!q_1!\cdots q_r!} \prod_{k=0}^{r} (a_k\Gamma(k+1))^{q_k} \times \\
\theta^{-\sum\limits_{k=0}^{r}(k+1)q_k} f_{GA}\left(z, \sum_{k=0}^{r}(k+1)q_k, \theta\right) \\
= \{h(\theta)\}^n \sum_{q_0}\sum_{q_1}\cdots\sum_{q_r} \frac{n!}{q_0!q_1!\cdots q_r!} \prod_{k=0}^{r} (a_k\Gamma(k+1))^{q_k} \times \\
\theta^{-\sum\limits_{k=0}^{r}(k+1)q_k} \frac{\theta^{\sum\limits_{k=0}^{r}(k+1)q_k}}{\Gamma\left(\sum\limits_{k=0}^{r}(k+1)q_k\right)} z^{\sum\limits_{k=0}^{r}(k+1)q_k - 1} e^{-\theta z} \\
= \{h(\theta)\}^n \sum_{q_0}\sum_{q_1}\cdots\sum_{q_r} \frac{n!}{q_0!q_1!\cdots q_r!} \prod_{k=0}^{r} (a_k\Gamma(k+1))^{q_k} \times \\
\frac{z^{\sum\limits_{k=0}^{r}(k+1)q_k - 1} e^{-\theta z}}{\Gamma\left(\sum\limits_{k=0}^{r}(k+1)q_k\right)}.
\end{aligned} \tag{A1}
$$

If $(X_{11}, X_{12}, \ldots, X_{1n_1})$ be a 1^{st} random sample of size n_1 from the Oppe distribution with parameter θ, then the pdf of $U_1 = \sum_{i=1}^{n_1} X_{1i}$ is

$$
\begin{aligned}
g(u_1; n_1, \theta) = \{h(\theta)\}^{n_1} \sum_{q_0}\sum_{q_1}\cdots\sum_{q_r} \frac{n_1!}{q_0!q_1!\cdots q_r!} \prod_{k=0}^{r} (a_k\Gamma(k+1))^{q_k} \times \\
\theta^{-\sum\limits_{k=0}^{r}(k+1)q_k} f_{GA}\left(u_1, \sum_{k=0}^{r}(k+1)q_k, \theta\right) \\
= \{h(\theta)\}^{n_1} \sum_{q_0}\sum_{q_1}\cdots\sum_{q_r} \frac{n_1!}{q_0!q_1!\cdots q_r!} \prod_{k=0}^{r} (a_k\Gamma(k+1))^{q_k} \times \\
\theta^{-\sum\limits_{k=0}^{r}(k+1)q_k} \frac{\theta^{\sum\limits_{k=0}^{r}(k+1)q_k}}{\Gamma\left(\sum\limits_{k=0}^{r}(k+1)q_k\right)} u_1^{\sum\limits_{k=0}^{r}(k+1)q_k - 1} e^{-\theta u_1} \\
= \{h(\theta)\}^{n_1} \sum_{q_0}\sum_{q_1}\cdots\sum_{q_r} \frac{n_1!}{q_0!q_1!\cdots q_r!} \prod_{k=0}^{r} (a_k\Gamma(k+1))^{q_k} \times \\
\frac{u_1^{\sum\limits_{k=0}^{r}(k+1)q_k - 1} e^{-\theta u_1}}{\Gamma\left(\sum\limits_{k=0}^{r}(k+1)q_k\right)}.
\end{aligned}
$$

Next,

$$
\begin{aligned}
L_1(\theta) &= P\left(\sum_{i=1}^{n_1} X_{1i} \geq n_1 c_1 | \theta\right) \\
&= P\left(U_1 \geq n_1 c_1 | \theta\right) \\
&= 1 - \{h(\theta)\}^{n_1} \sum_{q_0}\sum_{q_1}\cdots\sum_{q_r} \frac{n_1!}{q_0! q_1! \cdots q_r!} \prod_{k=0}^{r} \left(a_k \Gamma(k+1)\right)^{q_k} \theta^{-\sum\limits_{k=0}^{r}(k+1)q_k} \\
&\quad \times \int_0^{n_1 c_1} \frac{\theta^{\sum\limits_{k=0}^{r}(k+1)q_k}}{\Gamma\left(\sum\limits_{k=0}^{r}(k+1)q_k\right)} u_1^{\sum\limits_{k=0}^{r}(k+1)q_k - 1} e^{-\theta u_1} du_1 \\
&= 1 - \{h(\theta)\}^{n_1} \sum_{q_0}\sum_{q_1}\cdots\sum_{q_r} \frac{n_1!}{q_0! q_1! \cdots q_r!} \prod_{k=0}^{r} \left(a_k \Gamma(k+1)\right)^{q_k} \theta^{-\sum\limits_{k=0}^{r}(k+1)q_k} \\
&\quad \times \Gamma\left(\theta n_1 c_1, \sum_{k=0}^{r}(k+1)q_k\right).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
L_2(\theta) &= P\left(\sum_{i=1}^{n_1} X_{1i} > n_1 c_2 | \theta\right) \\
&= P\left(U_1 \geq n_1 c_2 | \theta\right) \\
&= 1 - \{h(\theta)\}^{n_1} \sum_{q_0}\sum_{q_1}\cdots\sum_{q_r} \frac{n_1!}{q_0! q_1! \cdots q_r!} \prod_{k=0}^{r} \left(a_k \Gamma(k+1)\right)^{q_k} \theta^{-\sum\limits_{k=0}^{r}(k+1)q_k} \\
&\quad \times \Gamma\left(\theta n_1 c_2, \sum_{k=0}^{r}(k+1)q_k\right).
\end{aligned}
$$

And, if $(X_{21}, X_{22}, \ldots, X_{2n_2})$ be a 2^{nd} random sample of size n_2 [say, $n = n_1 + n_2$] from the Oppe distribution, also with parameter θ, then the pdf of $U = U_1 + U_2 = \sum\limits_{i=1}^{n_1} X_{1i} + \sum\limits_{i=1}^{n_2} X_{2i}$ is given by

$$g(u;n,\theta) = \{h(\theta)\}^n \sum_{q_0}\sum_{q_1}\cdots\sum_{q_r} \frac{n!}{q_0!q_1!\cdots q_r!} \prod_{k=0}^{r} \left(a_k\Gamma(k+1)\right)^{q_k} \times$$
$$\theta^{-\sum\limits_{k=0}^{r}(k+1)q_k} f_{GA}\left(u, \sum_{k=0}^{r}(k+1)q_k, \theta\right)$$
$$= \{h(\theta)\}^n \sum_{q_0}\sum_{q_1}\cdots\sum_{q_r} \frac{n!}{q_0!q_1!\cdots q_r!} \prod_{k=0}^{r} \left(a_k\Gamma(k+1)\right)^{q_k} \times$$
$$\theta^{-\sum\limits_{k=0}^{r}(k+1)q_k} \frac{\theta^{\sum\limits_{k=0}^{r}(k+1)q_k}}{\Gamma\left(\sum\limits_{k=0}^{r}(k+1)q_k\right)} u^{\sum\limits_{k=0}^{r}(k+1)q_k-1} e^{-\theta u}$$
$$= \{h(\theta)\}^n \sum_{q_0}\sum_{q_1}\cdots\sum_{q_r} \frac{n!}{q_0!q_1!\cdots q_r!} \prod_{k=0}^{r} \left(a_k\Gamma(k+1)\right)^{q_k} \times$$
$$\frac{u^{\sum\limits_{k=0}^{r}(k+1)q_k-1} e^{-\theta u}}{\Gamma\left(\sum\limits_{k=0}^{r}(k+1)q_k\right)}.$$

Therefore,

$$L_p(\theta) = P\left(\sum_{i=1}^{n_1} X_{1i} + \sum_{i=1}^{n_2} X_{2i} \geq (n_1+n_2)c_2|\theta\right)$$
$$= P\left(U \geq (n_1+n_2)c_2|\theta\right)$$
$$= 1 - \{h(\theta)\}^{(n_1+n_2)} \sum_{q_0}\sum_{q_1}\cdots\sum_{q_r} \frac{(n_1+n_2)!}{q_0!q_1!\cdots q_r!} \prod_{k=0}^{r} \left(a_k\Gamma(k+1)\right)^{q_k} \theta^{-\sum\limits_{k=0}^{r}(k+1)q_k}$$
$$\times\Gamma\left(\theta(n_1+n_2)c_2, \sum_{k=0}^{r}(k+1)q_k\right). \quad \text{(A2)}$$