Revista Colombiana de Estadística - Applied Statistics
July 2024, volume 47, issue 2, pp. 301–327
http://doi.org/10.15446/rce.v47n2.110213

New Unconditional and Quantile Regression Model Erf-Weibull: An Alternative to Gamma, Gumbel and Exponentiated Exponential Distributions

Nuevo modelo incondicional y de regresión cuantiles Erf-Weibull: una alternativa a las distribuciones gamma, Gumbel y exponencial exponenciada

Caroline Cogo Carneosso[1,a], Thiago A. N. De Andrade[1,b], Cleber Bisognin[1,c]

[1]Statistics Department, Federal University of Santa Maria, Santa Maria, Brazil

Abstract

In this paper, we present a stochastic model that uses the Gaussian error function to change the likelihood of the Weibull distribution without changing the complexity of its parametric space. Several mathematical properties are derived for the proposed model, which has numerical examples to illustrate its usability in practice. The failure rate function of the resulting model presents non-monotonous shapes, such as the shape of a bathtub, which represents a gain concerning the base distribution. Two parameter estimation methods are presented and evaluated numerically. In addition to the unconditional model, a regression model for the quantiles of the distribution is derived. Both absolute and regression models have applications to actual data and simulation studies, corroborating their use in practical situations.

***Key words*:** Applied statistic; Gaussian error function; Regression models; Weibull distribution.

Resumen

En este artículo, presentamos un modelo estocástico que utiliza la función de error de Gauss para cambiar la probabilidad de la distribución de Weibull sin cambiar la complejidad de su espacio paramétrico. Se derivan varias propiedades matemáticas para el modelo propuesto, que tiene ejemplos numéricos para ilustrar su usabilidad en la práctica. La función de tasa de falla del modelo resultante presenta formas no monótonas, como la forma

[a]Bachelor's degree in Statistics. E-mail: caroline.cogo@acad.ufsm.br
[b]Ph.D. E-mail: thiagoan.andrade@gmail.com
[c]Ph.D. E-mail: cleber.bisognin@ufsm.br

de una bañera, lo que representa una ganancia con respecto a la distribución base. Se presentan y evalúan numéricamente dos métodos de estimación de parámetros. Además del modelo incondicional, se deriva un modelo de regresión para los cuantiles de la distribución. Tanto los modelos absolutos como los de regresión tienen aplicaciones a datos reales y estudios de simulación, corroborando su uso en situaciones prácticas.

***Palabras clave*:** Distribución Weibull; Estadística aplicada; Función de error gaussiano; Modelo de regresión.

1. Introduction

From both theoretical and applied perspectives, the Weibull distribution has received significant attention in the specialized academic literature, being one of the most used statistical models. For theorists, the Weilbull model is attractive because it has desirable characteristics such as a closed-form cumulative distribution function, a simple expression for the density function, risk and density functions that accommodates non-monotone shapes, among others.

From an applied point of view, the Weibull distribution has proved to be quite flexible to adjust to actual positive data in different contexts and originating from the most varied areas of knowledge since it became official in the fifties with the work of Waloddi Weibull until the present day (Rinne, 2008). To have an idea of the importance of the Weibull distribution, a seminal book by Rinne (2008) cites more than a hundred academic works that consider applications of this model in areas such as materials science, engineering, physics, chemistry, meteorology, hydrology, medicine, psychology, pharmacy, economics, and business administration, among others.

The line of research that deals with the proposition of new probability distributions has been widely considered the Weibull model as the basis for generating generalized Weibull families. In particular, works dealing with the design of new probabilistic models through the so-called distribution generators have been very fruitful. Generators like *Exponentiated* by Lehmann (1953), *Marshall-Olkin* by Marshall & Olkin (1997), *Beta* by Eugene et al. (2002), *Gamma* by Zografos & Balakrishnan (2009), Ristić & Balakrishnan (2012) and Nadarajah et al. (2015), *Kumaraswamy* by Cordeiro & de Castro (2011), *McDonald* by Alexander et al. (2012) and *Generalized E xponentiated* by Cordeiro et al. (2013), have been coupled to the Weibull model to give rise to new distributions.

Although the previously mentioned works have demonstrated success in investigating properties and in the search for new practical applications in real data, we observe a significant limitation: the use of the mentioned distribution generators introduces additional complexity in the parameter space of the derived models. This complexification can be seen as a critical weakness, as it complicates the interpretation and applicability of the models. However, this research is breaking new knowledge and considering a new distribution generator that does not attribute adding new parameters to modify the likelihood and give rise to a new model based on the bi-parametric Weibull distribution.

This work considers the Erf-G generator (Fernández & De Andrade, 2020) applied to the Weibull distribution to propose a new two-parameter stochastic model. In addition to proving to be helpful in applied situations and having well-defined properties for the standard model, this research has the following clear motivations:

- The new probabilistic model has only the two parameters of the base distribution, maintaining its characteristics and interpretations;
- All the main formulas associated with the new model are simple and manageable and have numerical examples that illustrate their use in practice;
- We inserted a regression structure to model the quantiles of the new distribution through a convenient reparametrization;
- We considered two estimation methods for the unconditional model;
- The failure rate function presents non-monotonous shapes, including the bathtub shape, representing a gain in the base model;
- Several simulation studies are presented for both unconditional and regression models and show good performance in the considered scenarios;
- Adjustment to actual data shows that the proposed model performs better than other well-established bi-parametric models.

The work is organized as follows, in addition to this introduction. In Section 2, the new Erf-Weibull (Erf-W) model is presented in detail. We derived the density, cumulative, quantile, and risk functions. In addition, mathematical properties such as moments, moment-generating function, reliability, and entropies, among others, are maintained. Section 3 deals with regression model for the quantiles of new distribution. Section 4 discusses parameter evaluation methods and simulation studies. Applications to actual data are customized in Section 5, while Section 6 is dedicated to final considerations.

2. The New Model

The error function (also referred to as the Gauss error function) was first introduced by the German mathematician and statistician Carl Friedrich Gauss in the early 19th century. This function is given by

$$\mathrm{Erf}(z) = \frac{1}{\sqrt{\pi}} \int_{-z}^{z} \exp(-t^2)\,\mathrm{d}t = \frac{2}{\sqrt{\pi}} \int_{0}^{z} \exp(-t^2)\,\mathrm{d}t, \quad z \in \mathbb{R}. \tag{1}$$

In recent work, Fernández & De Andrade (2020) proposed replacing z in the Equation (1) by $G(x)/[1-G(x)]$, for $x \in \mathcal{X} \subseteq \mathbb{R}$ and $G(x)$, to give rise to a new family of distributions, with cumulative defined by

$$F(x) = \text{Erf}\left[\frac{G(x)}{1 - G(x)}\right]. \tag{2}$$

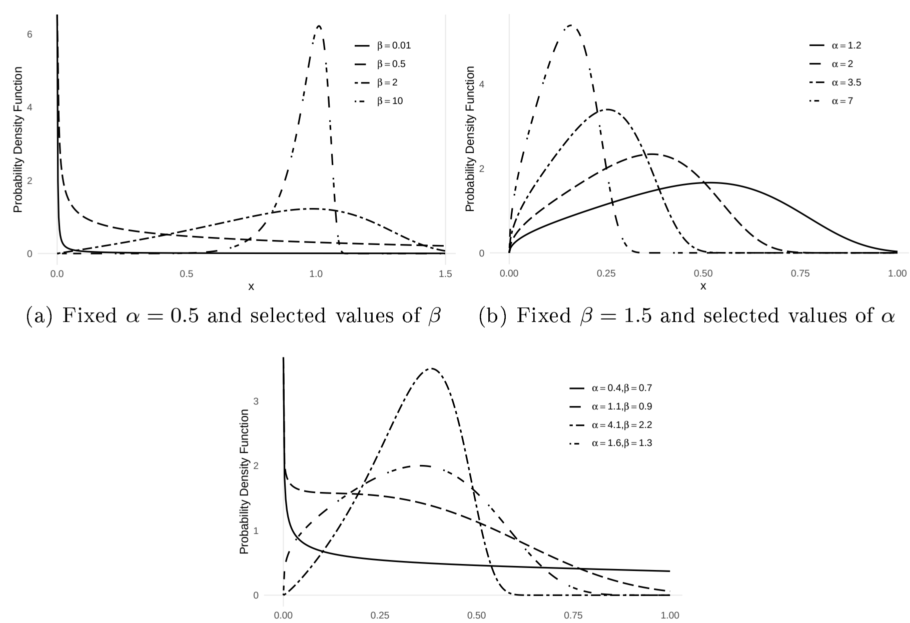
In this paper, we advocate the replacement of $G(x)$ in Equation (2) by the cdf of the Weibull model to give rise to a new and efficient probability distribution. Indeed, considering $G(x) = 1 - \exp\left(-\alpha x^{\beta}\right)$ in Equation (2), with $x \geqslant 0$, $\alpha > 0$ and $\beta > 0$, we obtain the cdf of the Erf-W model, given by

$$F(x) = \text{Erf}\left[\exp\left(\alpha x^{\beta}\right) - 1\right]. \tag{3}$$

The pdf associated with the new model arises from the derivation of Equation (3) and can be expressed as

$$f(x) = \frac{2}{\sqrt{\pi}}\, \alpha\, \beta\, x^{\beta-1}\, \exp[\alpha x^{\beta} - (e^{\alpha x^{\beta}} - 1)^2], \tag{4}$$

where $x \geqslant 0$, $\alpha > 0$, $\beta > 0$. Figure 2 shows plots of X's density. It reveals that the new model is flexible and accommodates symmetrical and asymmetrical shapes.



(a) Fixed $\alpha = 0.5$ and selected values of β (b) Fixed $\beta = 1.5$ and selected values of α

(c) Selected values for α e β

FIGURE 1: Plots of the Erf-W density function for some parameter values

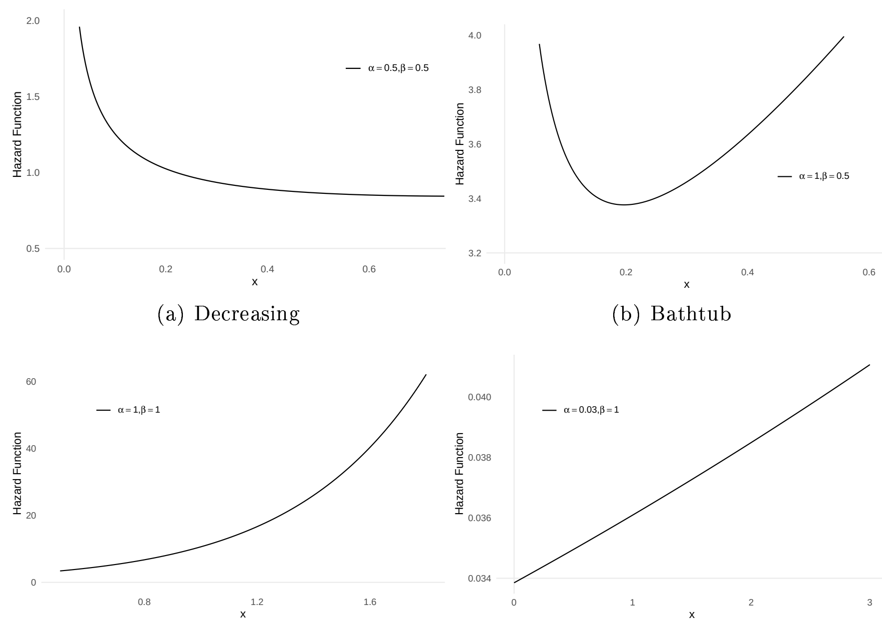
Two other functions that uniquely characterize the random variable X and are of particular importance in reliability studies and survival analysis are the survival (sf) and hazard rate functions (hrf). For the Erf-W model, the sf and hrf are given, respectively, by

$$S(x) = 1 - \text{Erf}\left[\exp\left(\alpha x^{\beta}\right) - 1\right] \tag{5}$$

and

$$h(x) = \frac{2\,\alpha\,\beta\,x^{\beta-1}\,\exp[\alpha x^{\beta} - (e^{\alpha x^{\beta}} - 1)^2]}{\sqrt{\pi}\,\{1 - \text{Erf}\,[\exp\,(\alpha x^{\beta}) - 1]\}}. \tag{6}$$

Plots of the Erf-W hrf for selected parameter values are shown in Figure 2. It reveals that the hrf of the proposed model has monotonic and non-monotonic forms such as decreasing, increasing, and bathtub, representing a gain in terms of the base Weibull model.



(a) Decreasing (b) Bathtub

(c) Increasing (d) Monotone increasing

FIGURE 2: Plots of the Erf-W hrf for selected parameter values

2.1. Additional Characterization

In this section, we present some properties of the new model.

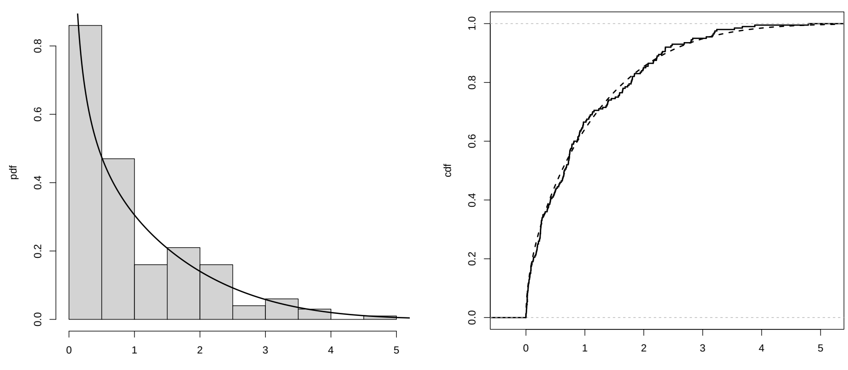
2.1.1. Quantile Function

The quantile function (qf), obtained as the inverse of the cdf, is crucial for applications. Based on the qf, it is possible to generate occurrences of the distribution, compute the median, and measures of asymmetry and kurtosis, among others. By inverting Equation (3), we bring the qf of the Erf-W distribution as

$$Q(\tau) = \left\{ \frac{1}{\alpha} \log \left[\frac{1}{\sqrt{2}} \Phi^{-1} \left(\frac{\tau + 1}{2} \right) + 1 \right] \right\}^{\frac{1}{\beta}}, \tag{7}$$

where $\tau \in (0, 1)$ and $\Phi(\cdot)$ indicates the cumulative distribution function of the standard normal distribution, while $\Phi^{-1}(\cdot)$ corresponds to the qf associated with the same distribution.

To illustrate the usability of qf in generating random numbers, consider Equation (7) to generate sixty occurrences of the Erf-W(0.5, 0.5) distribution. Figure 3 below shows the plots of the pdf, histogram, exact, and empirical cdfs for simulated data. It is possible to infer that there is a good fit of the data to the model, which corroborates the use of Equation (7) in the simulation studies performed in Section 4.3.



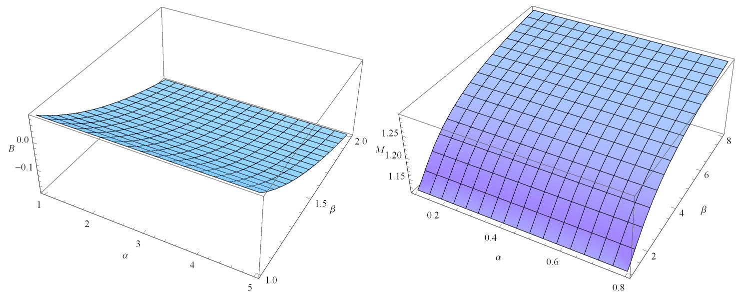
(a) Histogram and exact pdf (b) Empirical and exact cdfs

FIGURE 3: Plots of the Erf-W(0.5, 0.5) pdf, histogram, exact and empirical cdfs for simulated data with $n = 60$

As a second illustration, we use the qf of X to determine the Bowley's skewness Kenney & Keeping (1962) (B) and Moors's kurtosis Moors (1988) (M). These measures are given by

$$B = \frac{Q(3/4) + Q(1/4) - 2Q(1/2)}{Q(3/4) - Q(1/4)} \quad \text{and} \quad M = \frac{Q(3/8) - Q(1/8) + Q(7/8) - Q(5/8)}{Q(6/8) - Q(2/8)},$$

where $Q(\tau)$, $\tau \in (0, 1)$, is the qf of the Erf-W model defined in Equation (7). Figure 4 shows the plots of Bowley's skewness and Moors's kurtosis. These graphs reveal that the β parameter significantly influences both measures.



(a) Bowley skewness (b) Moors kurtosis

Figure 4: Plots of the Bowley skewness and Moors kurtosis of X

2.1.2. Expansion for Density, Moments and Moment Generating Function

In proposing new probability distributions, it is common practice to derive simplified expressions for the density and cumulative functions. The purpose of these simplifications is to facilitate the analysis and calculation of essential properties inherent to these models. In this context, a significant advance consists of representing the density and cumulative functions of the new distribution in terms of the pdf and cdf of the *Exponentiated*-G (Exp-G for short) generator. The Exp-G family has been well established in statistical literature since the 1950s, initially developed by the work of Lehmann (1953). The elegant and intuitive method of generating new probability distributions by raising the standard cumulative distribution function to a power $a > 0$ has promoted the creation of several exponentialized distributions, particularly in the last 20 years. We recommend consulting Tahir & Nadarajah (2015)'s seminal paper for a comprehensive review of these works. In particular, for an exhaustive study of Erf-W, we recommend the research by Nadarajah et al. (2013). Using results from Fernández & De Andrade (2020), the cdf of the Erf-W distribution can be expressed as

$$f(x) = \sum_{k,m=0}^{\infty} a_{k,m}\, h_{m+2k+1}(x), \tag{8}$$

where $h_{m+2k+1}(x) = (m+2k+1)\, g(x)\, G(x)^{m+2k}$ is the exponentiated Weibull (Exp-W) pdf with power parameter $m+2k+1$ and $a_{k,m}$ defined by the recursive formula below

$$a_{k,m} = \frac{2\,(-1)^k}{\sqrt{\pi}\, k!\,(2k+1)} \frac{1}{m} \sum_{j=1}^{m} [2j(k+1)-m] d_{2k+1,m},$$

where $d_{2k+1,0} = 1, d_{2k+1,m} = \dfrac{1}{m}\displaystyle\sum_{j=1}^{m}\big[2j(k+1)-m\big]\, d_{2k+1,m-j}, m \geq 1.$

The result presented by Equation (8) is the most important of this Subsection. It reveals that the density of the new model can be presented as a sum of Exp-W densities with power parameter $m+2k+1$. Thus, several properties such as moments and function generating, reliability entropy, among others, can be obtained from those defined for the Exp-W model. In practical terms, this means that, without loss of precision, we can use the version of the density presented in (8) instead of the version presented in Equation (4) to obtain the properties of the proposed distribution.

As a first application of Equation (8), we obtain expressions for the moments and moment generating function of the Erf-W distribution. After an algebraic manipulation, we obtain

$$\mathbb{E}(X) = \sum_{k,m=0}^{\infty} b_{m,k} \mathbb{E}_{m+2k+1}(X) \tag{9}$$

and

$$\mathbb{M}(t) = \sum_{k,m=0}^{\infty} b_{m,k}\, \mathrm{M}_{m+2k+1}(t), \tag{10}$$

where $\mathbb{E}_{m+2k+1}(X)$ and $\mathrm{M}_{m+2k+1}(t)$ are, respectively, the nth-moment and moment-generating function of Exp-W distribution, which can be obtained by Nadarajah et al. (2013).

At first, formulas such as those presented in expressions (8), (9), and (10) may scare away applied researchers not interested in theoretical mathematical formulations. But we are convinced that our formulas are manageable and useful for practical purposes. To illustrate this, in Table 1, we use the procedures we present for moments, quantile, skewness, and kurtosis to compute the first three moments, the theoretical median (Med), plus the B and M measures for selected parameter values of the Erf-W model.

TABLE 1: Three ordinary first moments, median, Bowley skewness and Moors kurtosis of X, for selected parameter values

α	β	E(X)	E(X^2)	E(X^3)	Med	B	M
2.5	0.5	0.0379	0.0031	0.0003	0.0243	−0.4521	1.2129
2.0	1.0	0.2066	0.0592	0.0200	0.1953	−0.0900	1.1202
1.5	1.5	0.4019	0.1952	0.1052	0.4073	−0.0824	1.1521
1.0	2.0	0.6040	0.4133	0.3048	0.6245	−0.1814	1.1829
0.5	2.5	0.8707	0.8284	0.8343	0.9054	−0.2455	1.2067
0.1	3.0	1.5124	2.4447	4.1325	1.5740	−0.2903	1.2248
0.5	0.5	0.9472	1.9134	5.3183	0.6083	−0.4521	1.2129
3.0	3.0	0.7867	0.2535	0.1378	0.5066	−0.2903	1.2248

2.1.3. Reliability

Type $\mathrm{P}(X_1 > X_2)$ probabilities frequently emerge in applied studies, particularly in works related to lifetime. Here, we derive the reliability, referred to as R, when $X_1 \sim$ Erf-W(α_1, β_1) and $X_2 \sim$ Erf-W(α_2, β_2) are two independent stochastic variables. Let $f_1(x)$ represent the probability density function of X_1, and $F_2(x)$ represent the cumulative distribution function of X_2. The reliability can be expressed as $R = \mathrm{P}(X_1 > X_2) = \int_0^\infty f_1(x)\, F_2(x)\, \mathrm{d}x$, and utilizing Equation (8) yields

$$R = \sum_{j,k=0}^{\infty} \mathcal{I}_{j,k}\,(j+1)\alpha_1\beta_1 \int_0^\infty x^{\beta_1-1}\left[1-\exp\left(-\alpha_1 x^{\beta_1}\right)\right]^j \times \left[1-\exp\left(-\alpha_2 x^{\beta_2}\right)\right]^{k+1}\mathrm{d}x, \tag{11}$$

where $\mathcal{I}_{j,k} = \sum_{m,n=1}^{\infty}(-1)^{j+k+m+n+2}\binom{b_1}{m}\binom{m\,a_1}{j+1}\binom{b_2}{n}\binom{na_2}{k+1}$.

Despite the integral in the above form not having a solution in closed-form, there are no difficulties with its use in practical situations, with the upper limits on the sums being replaced by one hundred for approximations with up to four decimal places of precision. Table 2 presents a small numerical study intending to compute the reliability in the context of the Erf-W model for selected parameter values.

Table 2: Reliability for the Erf-W distribution and selected parameter values

$\alpha_1=\alpha_2=0.5$			$\alpha_1=\alpha_2=1.5$			$\alpha_1=\alpha_2=3.0$		
β_1	β_2	R	β_1	β_2	R	β_1	β_2	R
0.1	2.0	0.4218	2.5	0.5	0.9773	4.0	1.0	0.9949
0.3	1.5	0.4116	2.0	1.0	0.8050	3.5	2.0	0.8561
0.5	0.5	0.5000	1.5	1.5	0.5000	3.0	3.0	0.5000
1.5	0.3	0.5884	1.0	2.0	0.1949	2.0	3.5	0.1439
2.0	0.1	0.6261	0.5	2.5	0.0227	1.5	4.0	0.0050
$\beta_1=\beta_2=0.5$			$\beta_1=\beta_2=1.5$			$\beta_1=\beta_2=3.0$		
α_1	α_2	R	α_1	α_2	R	α_1	α_2	R
0.1	2.0	0.9763	2.5	0.5	0.0982	4.0	1.0	0.1242
0.3	1.5	0.9017	2.0	1.0	0.2581	3.5	2.0	0.2963
0.5	0.5	0.5000	1.5	1.5	0.5000	3.0	3.0	0.5000
1.5	0.3	0.0982	1.0	2.0	0.7419	2.0	3.5	0.7037
2.0	0.1	0.0236	0.5	2.5	0.9017	1.5	4.0	0.8758

2.1.4. Entropy

In information theory, the entropy of a random variable X measures the amount of uncertainty associated with it. Two entropy measures that are already classic in the studies of new probability distributions are the Rényi and Shannon measures. The Rényi ($\mathrm{E_R}(f)$) and Shannon ($\mathrm{E_S}(f)$) entropies of a density $f(x)$ are given by

$$\mathrm{E_R}(f) = \frac{1}{(1-\rho)}\log\left(\int_0^\infty f(x)^\rho \mathrm{d}x\right) \quad \text{and} \quad \mathrm{E_S}(f) = -\int_0^\infty f(x)\log f(x) dx.$$

For X with density (4), the Rényi entropy collapses

$$\mathrm{E_R}(f) = \frac{1}{(1-\rho)}\log\left\{\int_0^\infty \left[\frac{2}{\sqrt{\pi}}\,\alpha\,\beta\,x^{\beta-1}\,\exp[\alpha x^\beta - (e^{\alpha x^\beta}-1)^2]\right]^\rho \mathrm{d}x\right\}, \tag{12}$$

where $\rho > 0$ and $\rho \neq 1$.

Consider the expansion (witch is holds for $|z| < 1$ and $\rho > 0$),

$$(1-z)^{-\rho} = \sum_{j=1}^{\infty} w_j z_j, \quad w_j = \frac{\Gamma(\rho + j)}{j!\,\Gamma(\rho)}. \tag{13}$$

Applied the Taylor expansion thrice and (13) in Equation (12), an expression to the Erf-W Rényi entropy is given by

$$\mathrm{E}_{\mathrm{R}}(f) = \frac{1}{1-\rho} \log \left[\frac{2^{\rho}}{\pi^{\rho/2}} \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=0}^{2k+j} \binom{2k+j}{\ell} \Gamma\left[\frac{1+\rho(\beta-1)}{\beta}\right] \frac{(-1)^{\ell}\,(-\rho)^{k}\,(\alpha\beta)^{\rho}\,w_j}{k!\,\beta\,[\alpha(\ell+\rho)]^{\frac{1+\rho+\rho\beta}{\beta}}} \right],$$

where $w_j = \dfrac{\Gamma[2(\rho+1)+j]}{j!\,\Gamma[2(\rho+1)]}$.

The Shannon entropy can be obtained from the Rényi entropy doing $\delta \uparrow 1$. For X with density (4), the Shannon entropy reduces to

$$\begin{aligned}\mathrm{E}_{\mathrm{S}}(f) = &-2\log(2) + \frac{1}{2}\log(\pi) - E\big[\log(X)\big] + E\left\{\left[\mathrm{e}^{-\alpha x^{\beta}} - 1\right]^2\right\} \\ &- 2E\big[\log(1 - \alpha\beta x^{\beta-1}\mathrm{e}^{-\alpha x^{\beta}})\big].\end{aligned}$$

Table 3 presents a small numerical study in which we compute the entropy of Rényi, considering some values of ρ and for Shannon entropy, both for selected values of the parameters α and β. Interestingly, both entropies approach in numerical values for $\rho = 0.8$, illustrating the limit situation in which the Shannon entropy can be obtained from the entropy of Rényi.

Table 3: Rényi and Shannon entropy for Erf-W random variable with selected parameter values

		Rényi entropy				
α	β	$\rho = 0.2$	$\rho = 0.8$	$\rho = 2.0$	$\rho = 4.0$	Shannon entropy
0.1	2.5	−0.9478	−0.7303	−0.5645	−0.455	−0.6896
0.5	2.0	−0.4319	−0.2514	−0.1148	−0.0197	−0.2185
0.8	1.8	−0.2334	−0.0643	−0.0591	−0.1467	−0.0347
1.0	1.5	−0.1467	−0.0148	−0.1182	−0.1922	−0.0401
1.5	1.0	−0.1888	−0.4107	−0.5154	−0.5716	−0.4395
1.8	0.8	−0.4806	−0.7989	−0.9843	−1.1871	−0.8452

3. Regression Model for Quantiles of Erf-W Distribution

Let $\boldsymbol{Y} = (Y_1, \ldots, Y_n)^{\top}$ be independent random variables, where each Y_t has pdf Erf-W, given by Equation (4). To introduce the regression structure, consider the quantile μ_t, for $t = 1, \ldots, n$. Taking $\mu_t = Q(\tau)$ in the Equation (7) we have

$$\mu_t = \left\{ \frac{1}{\alpha} \log \left[\frac{1}{\sqrt{2}} \Phi^{-1} \left(\frac{\tau+1}{2} \right) + 1 \right] \right\}^{\frac{1}{\beta}}.$$

Rewriting the expression in terms of β we have

$$\beta = \frac{1}{\log(\mu_t)} \log \left\{ \frac{1}{\alpha} \log \left[\frac{1}{\sqrt{2}} \Phi^{-1} \left(\frac{\tau+1}{2} \right) + 1 \right] \right\}, \tag{14}$$

which is held under the condition $\alpha < \log \left[\frac{1}{\sqrt{2}} \Phi^{-1} \left(\frac{\tau+1}{2} \right) + 1 \right]$.

Considering the reparameterization, given by Equation (14), the regression structure for quantile modeling can be expressed by

$$g(\mu_t) = \boldsymbol{x}_t^\top \boldsymbol{\beta} = \eta_t, \tag{15}$$

where $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_k)^\top$ is the vector of unknown parameters ($\boldsymbol{\beta} \in \mathbb{R}^k$), and $\boldsymbol{x}_t = (x_{t1}, \ldots, x_{tk})^\top$ are observations of k covariates ($k < n$), which are assumed to be fixed and known, and $g(.)$ is a strictly monotonic twice differentiable link function, such that $g : \mathbb{R} \to \mathbb{R}_+$. Among the possible link functions that satisfy these conditions, we have the logarithmic function $[g(\mu_t) = \log(\mu_t)]$. If the model includes the intercept, we have $x_{t1} = 1$ for all $t = 1, \ldots, n$.

In Section 4, we present the maximum likelihood estimator (MLEs) for the parameters of the quantile regression model based on the Erf-W distribution, with α and β fixed and τ known. The respective score vector is expressed.

4. Estimation and Inference

In this Section, we present procedures for obtaining estimators for the parameters of the unconditional and regression models. For both models, the maximum likelihood (ML) method is considered. Additionally, the maximum product of spacing (MPS) method is addressed for the absolute model.

4.1. Unconditional Model

Two methods are considered: ML and MPS.

4.1.1. Maximum Likelihood

Let $\boldsymbol{X} = (X_1, \ldots, X_n)^\top$ be a random sample of size n from the Erf-W(α, β) distribution. The likelihood function of the vector of parameters $\boldsymbol{\theta} = (\alpha, \beta)^\top$ corresponding to the sample can be written as

$$L(\theta) = \prod_{t=1}^{n} \left(\frac{2}{\sqrt{\pi}} \alpha\beta \right)^n x_t^{\beta-1} \exp \left\{ \alpha x_t^\beta - \left[e^{\alpha x_t^\beta} - 1 \right]^2 \right\}$$

Thus, the log-likelihood function associated with the Erf-W model can be expressed as

$$\ell(\theta) = n\log(2\alpha\beta) - \frac{n}{2}\log(\pi) + (\beta-1)\sum_{t=0}^{n}\log(x_t) + \alpha\sum_{t=0}^{n}x_t^{\beta} - \sum_{t=0}^{n}\left[\exp\left(\alpha x_t^{\beta}\right)-1\right]^2.$$

The components of the score vector proceed by partial derivation of the log-likelihood concerning the parameter vector and are given by

$$\mathrm{U}_\alpha(\theta) = \frac{\partial}{\partial\alpha}\ell(\boldsymbol{\theta}) = \frac{n}{\alpha} - 2\sum_{t=0}^{n}\varphi_t x_t^{\beta} + \sum_{t=0}^{n}x_t^{\beta}$$

and

$$\mathrm{U}_\beta(\theta) = \frac{\partial}{\partial\beta}\ell(\boldsymbol{\theta}) = \frac{n}{\beta} - 2\alpha\sum_{t=0}^{n}\varphi_t x_t^{\beta}\log(x_t) + \alpha\sum_{t=0}^{n}x_t^{\beta}\log(x_t) + \sum_{t=0}^{n}\log(x_t),$$

where $\kappa_t = \exp\left(\alpha x_i^{\beta}\right)$ and $\varphi_t = (\kappa_t - 1)\kappa_t$, with $t = 1, \ldots, n$.

The EMVs of the vector of parameters $\boldsymbol{\theta} = (\alpha, \beta)^\top$, denoted by $\widehat{\boldsymbol{\theta}}_{EMV} = (\widehat{\alpha}_{EMV}, \widehat{\beta}_{EMV})^\top$, are obtained by solving the system of simultaneous equations

$$\begin{cases} \mathrm{U}_\alpha(\boldsymbol{\theta})\big|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}} = 0, \\ \mathrm{U}_\beta(\boldsymbol{\theta})\big|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}} = 0. \end{cases} \tag{16}$$

The nonlinear Equation in (16) cannot be solved analytically and require statistical software with iterative numerical techniques. Fortunately, computational problems like this are far from a challenge since several programming languages and free software have maximization routines. For example, there exist many maximization methods in R (R Core Team, 2022) scripts like NR (Newton-Raphson), BFGS (Broyden-Fletcher-Goldfarb-Shanno), BHHH (Berndt-Hall-Hall-Hausman), SANN (Simulated-Annealing), NM (Nelder-Mead), and L-BFGS-B, among others. In the R language, the main optimization routines involve the functions: `optim()` (package `stats`), `nloptr()` (package `nloptr`), `GenSA()` (package `GenSA`), among others. In Python scripts (Python Software Foundation, 2024), there are the options `minimize()`, `brute()`, `differential_evolution()` from the `scipy.optimize` module.

The negative elements of the observation matrix $J(\boldsymbol{\theta})$, necessary for hypothesis testing and confidence intervals, have been omitted but can be obtained from the authors upon request. For large sample size, the distribution of $(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})$ can be approximated to a bi-variate normal distribution with zero means and variance-covariance matrix $J(\boldsymbol{\theta})^{-1}$. Some asymptotic properties of $\hat{\boldsymbol{\theta}}$ can be based on this normal approximation.

4.1.2. Maximum Product of Spacings

The ML method is generally the first choice of statisticians and applied researchers when the goal is to determine adequate estimates for unknown population parameters. Nevertheless, the statistical theory of estimation offers a wide variety of methods that can be used to obtain estimators with desirable optimality properties and that perform well in practical situations. One of these methods is called MPS, which considers the maximization of the geometric mean of spacing in the data, evaluated by the accumulated distribution. These method provide estimators that may be even more adequate than the ML method in some contexts. In addition, it has desirable optimality properties such as consistency and efficiency. Several works have considered the MPS in different contexts, and here, we refer to some of them: Cheng & Amin (1979), Cheng & Amin (1983), Ranneby (1984), Wong & Li (2006), Singh et al. (2014), and Singh et al. (2016), to mention a few.

Let $\boldsymbol{X} = (X_1, \ldots, X_n)^\top$, a random sample whose random variables have distribution Erf-W(α, β), where your function of cumulative distribution is given by the Equation (3), $(X_{(1)}, \ldots, X_{(n)})$ the order statistics and $\boldsymbol{\theta} = (\alpha, \beta)^\top$ the array of parameters. define

$$D_t(\boldsymbol{\theta}) = F_X(x_{(t)}) - F_X(x_{(t-1)}), \text{ for } t = 1, \ldots, n+1,$$

the uniform distances of a random sample with distribution Erf-W(α, β), where $F_X(x_{(0)}) = 0$ and $F_X(x_{(n+1)}) = 1$. Note that $\sum_{t=1}^{n+1} D_t(\boldsymbol{\theta}) = 1$. The MPSEs of $\boldsymbol{\theta} = (\alpha, \beta)^\top$ is defined by

$$\widehat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}\in\boldsymbol{\Theta}}\{\mathrm{G}(\boldsymbol{\theta})\} = \arg\max_{\boldsymbol{\theta}\in\boldsymbol{\Theta}}\left\{\left[\prod_{t=1}^{n+1} D_t(\boldsymbol{\theta})\right]^{\frac{1}{n+1}}\right\},$$

where $\mathrm{G}(\boldsymbol{\theta})$ is the geometric mean with $\boldsymbol{\Theta} = \mathbb{R}_+^2$ the parametric space.

Equivalently,

$$\widehat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}\in\boldsymbol{\Theta}}\{\mathrm{H}(\boldsymbol{\theta})\} = \arg\max_{\boldsymbol{\theta}\in\boldsymbol{\Theta}}\left\{\frac{1}{n+1}\sum_{t=1}^{n+1}\log(D_t(\boldsymbol{\theta}))\right\},$$

where $\mathrm{H}(\boldsymbol{\theta})$ is the logarithm of the geometric mean.

Alternatively, the MPSEs of the parameter vector $\boldsymbol{\theta} = (\alpha, \beta)^\top$ can be obtained by solving the following system of equations

$$\begin{cases} \mathrm{V}_\alpha(\boldsymbol{\theta})\big|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}} = 0, \\ \mathrm{V}_\beta(\boldsymbol{\theta})\big|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}} = 0, \end{cases}$$

where

$$\mathrm{V}_\alpha(\boldsymbol{\theta}) = \frac{\partial}{\partial\alpha}\mathrm{H}(\boldsymbol{\theta}) = \frac{1}{n+1}\sum_{t=1}^{n+1}\frac{1}{D_t(\boldsymbol{\theta})}\Big[\nabla_1(\boldsymbol{\theta}|x_{(t)}) - \nabla_1(\boldsymbol{\theta}|x_{(t-1)})\Big],$$

$$\mathrm{V}_\beta(\boldsymbol{\theta}) = \frac{\partial}{\partial\beta}\mathrm{H}(\boldsymbol{\theta}) = \frac{1}{n+1}\sum_{t=1}^{n+1}\frac{1}{D_t(\boldsymbol{\theta})}\Big[\nabla_2(\boldsymbol{\theta}|x_{(t)}) - \nabla_2(\boldsymbol{\theta}|x_{(t-1)})\Big].$$

We define

$$\frac{\partial}{\partial \theta_j} F_X(x) = \frac{\partial}{\partial \theta_j} \left[2F_Z \left(\mathrm{T}(\boldsymbol{\theta}) \right) - 1 \right],$$

with $\mathrm{T}(\boldsymbol{\theta}) = \sqrt{2}\Big\{ \exp\left(\alpha x^{\beta} \right) - 1 \Big\}$, $j = 1, 2$.

By Leibniz's rule,

$$\frac{\partial}{\partial \theta_j} F_X(x) = \lim_{a \to -\infty} \frac{\partial}{\partial \theta_j} \left[2 \int_a^{\mathrm{T}(\boldsymbol{\theta})} f_Z(z) dz - 1 \right] = 2 f_Z \left(\mathrm{T}(\boldsymbol{\theta}) \right) \frac{\partial}{\partial \theta_j} \left[\mathrm{T}(\boldsymbol{\theta}) \right], \quad (17)$$

with Z denoting a normal standard radon variable. Therefore, by the Equation (17), we have

$$\begin{aligned} \nabla_1(\boldsymbol{\theta}|x_{(t)}) &= \frac{\partial}{\partial \alpha} F_X(x_{(t)}) = x_{(t)}^{\beta} \varphi_t(\boldsymbol{\theta}), \\ \nabla_2(\boldsymbol{\theta}|x_{(t)}) &= \frac{\partial}{\partial \beta} F_X(x_{(t)}) = \alpha x_{(t)}^{\beta} \log\left(x_{(t)} \right) \varphi_t(\boldsymbol{\theta}), \end{aligned} \quad (18)$$

with $\varphi_t(\boldsymbol{\theta}) = \frac{2}{\sqrt{\pi}} \exp\left(-\Big\{ \exp\left(\alpha x_{(t)}^{\beta} \right) - 1 \Big\}^2 \right) \exp\left(\alpha x_{(t)}^{\beta} \right)$, $t = 1, \ldots, n$.

Note that if $x_{(t+k)} = x_{(t+k-1)} = \ldots = x_{(t)}$, we have $D_{t+k}(\boldsymbol{\theta}) = D_{t+k-1}(\boldsymbol{\theta}) = \ldots = D_t(\boldsymbol{\theta}) = 0$, which indicates that the estimator MPS is highly responsive to very close observations. Hence, when $D_t(\boldsymbol{\theta}) = 0$ for a given $t = 1, \ldots, n+1$, we need to replace $D_t(\boldsymbol{\theta})$ with $f_X(x_{(t)})$, where $f_X(\cdot)$ represents the pdf of the distribution Erf-W(α, β) as defined in Equation (4). In addition to point estimation, inferences via hypothesis tests and asymptotic confidence intervals are built under the assumption of asymptotic normality of the MLEs, which is verified for the model Erf-W(α, β) through simulation studies.

4.2. Quantile Regression Model

Let $\boldsymbol{\theta} = (\alpha, \boldsymbol{\beta}^{\top})^{\top}$ be the vector of regression parameters. Using the reparameterization given by Equation (14), the log-likelihood function is given by

$$\begin{aligned} \ell(\boldsymbol{\theta}; \boldsymbol{x}) = \sum_{t=1}^{n} \ell_t(\boldsymbol{\theta}, x_t) = \log\left(2\alpha\beta \right) - \frac{1}{2} \log\left(\pi \right) + \left(\beta - 1 \right) \log\left(x_t \right) + \alpha \left(x_t \right)^{\beta} \\ - \left[\exp\left(\alpha x_t^{\beta} \right) - 1 \right]^2, \end{aligned}$$

for $t = 1, \ldots, n$. By the reparametrization given in Equation (14) and μ_t satisfying Equation (15), β is defined as

$$\beta = \frac{1}{\log(\mu_t)} \log \left\{ \frac{1}{\alpha} \log \left[\frac{1}{\sqrt{2}} \Phi^{-1} \left(\frac{\tau + 1}{2} \right) + 1 \right] \right\}.$$

Analogously to what happens with the unconditional model, the EMV's of $\boldsymbol{\theta} = (\alpha, \boldsymbol{\beta}^\top)^\top$ does not have a closed form and need to be obtained through iterative numeric routines. For this, it is often helpful to provide the analytic derivatives of the likelihood concerning each element of the parameter vector. The coordinates of the score vector about the α parameter are given by

$$\begin{aligned} \mathrm{U}_\alpha(\boldsymbol{\theta}) = {} & x_t^{\rho_t}(1-\sigma_t) - 2x_t^{\rho_t}(1-\sigma_t)\left[\exp\left(\alpha x_t^{\rho_t}\right) - 1\right]\exp\left(\alpha x_t^{\rho_t}\right) \\ & + \alpha^{-1}\left(1 - \frac{1}{\rho_t \log(\mu_t)}\right) - \alpha^{-1}\sigma_t, \end{aligned}$$

where $\rho_t := \frac{\log(\frac{\psi}{\alpha})}{\log(\mu_t)}$, $\sigma_t := \dfrac{\log(x_t)}{\log(\mu_t)}$ and $\psi := \log\left[\frac{1}{\sqrt{2}}\Phi^{-1}\left(\frac{\tau+1}{2}\right) + 1\right]$, for $t = 1, \ldots, n$.

The coordinates of the score vector concerning the parameters β_j, $j = 1, \ldots, k$, are given by

$$\begin{aligned} \mathrm{U}_{\beta_j}(\boldsymbol{\theta}) = \sum_{t=1}^{n} \frac{1}{g'(\mu_t)\,\mu_t} \Big\{ & 2\alpha\rho_t\sigma_t x_t^{\rho_t}\left[\exp\left(\alpha x_t^{\rho_t}\right) - 1\right]\exp\left(\alpha x_t^{\rho_t}\right) \\ & - \rho_t\sigma_t(\alpha x_t^{\rho_t} + 1) - \frac{1}{\log(\mu_t)} \Big\} x_{tj}, \end{aligned}$$

where $g'(\cdot)$ denotes the first derivative of link function.

The MLEs for the Erf-W quantile regression model parameters have an asymptotic normal distribution. This is a desirable feature, which facilitates the construction of confidence intervals and asymptotic hypothesis tests, which work-well even on moderately sized samples, as evidenced in Section 4.3.

4.3. Monte Carlo Simulation Study

Simulation studies are precious in statistics, as they allow studying the probabilistic behavior underlying some population quantity of interest in controlled scenarios. In this Section, we present Monte Carlo simulation studies conducted to evaluate the performance of estimators for parameters of unconditional regression models and Erf-W.

4.3.1. Unconditional Model

We explore, through a simulation analysis, the performance of the MLEs and the MPSEs for the parameters of the Erf-W. For the data generation process, we use the qf (7) with sample sizes ranging from 20 to 200 and select specific values for α and β. The simulation process involves $10,000$ Monte Carlo replications conducted using the `R` software and employing the *conjugate gradient* (CG) optimization method within the `maxLik` script. The outcomes of these new simulations are presented in Tables 4 and 5, which include mean, bias, mean square error (MSE), asymmetry, kurtosis, confidence intervals with covers 95% and 99% (CR(95%) and CR(99%), respectively), also hypothesis tests with significance levels 1% and 5% (HT(1%) and HT(5%), respectively). In general, for all scenarios considered and both estimators, the estimates approach the actual values of the parameters as the sample size is reduced. The accuracies of confidence intervals and hypothesis tests accompany this result. The asymmetry and kurtosis evidence the tendency towards asymptotic normality.

Table 4: MLEs for Erf-W distribution, fixed selected α and β parameter values and $n \in \{20, 100, 150, 200\}$

		Mean	Bias	MSE	Asym.	Kurtosis	CR(95%)	CR(99%)	HT(5%)	HT(1%)
$n = 20$	$\alpha = 0.4$	−0.3973	−0.0027	−0.0052	−0.0689	−3.0627	−0.9301	−0.9814	−0.0186	−0.0699
	$\beta = 0.7$	−0.7617	−0.0617	−0.0304	−0.9083	−4.5651	−0.9521	−0.9916	−0.0084	−0.0479
	$\alpha = 1.1$	−1.1947	−0.0947	−0.0602	−2.1360	−12.082	−0.9736	−0.9982	−0.0018	−0.0264
	$\beta = 0.9$	−0.9817	−0.0817	−0.0540	−1.0559	−5.5006	−0.9487	−0.9900	−0.0100	−0.0513
	$\alpha = 1.6$	−1.8168	−0.2168	−0.2753	−2.3230	−13.874	−0.9810	−0.9964	−0.0036	−0.0190
	$\beta = 1.3$	−1.4155	−0.1155	−0.1047	−0.7941	−3.9940	−0.9536	−0.9930	−0.0070	−0.0464
	$\alpha = 2.8$	−3.4747	−0.6747	−2.8817	−4.6068	−58.828	−0.9730	−0.9892	−0.0108	−0.0270
	$\beta = 3.5$	−3.8165	−0.3165	−0.7810	−0.9064	−4.4604	−0.9529	−0.9925	−0.0075	−0.0471
	$\alpha = 4.1$	−5.4878	−1.3878	−14.073	−6.3966	−91.420	−0.9636	−0.9851	−0.0149	−0.0364
	$\beta = 2.2$	−2.3973	−0.1973	−0.3013	−0.9342	−4.8078	−0.9566	−0.9924	−0.0076	−0.0434
$n = 100$	$\alpha = 0.4$	−0.3990	−0.0010	−0.0009	−0.0255	−3.1142	−0.9461	−0.9876	−0.0124	−0.0539
	$\beta = 0.7$	−0.7121	−0.0121	−0.0042	−0.4113	−3.3509	−0.9490	−0.9897	−0.0103	−0.0510
	$\alpha = 1.1$	−1.1149	−0.0149	−0.0051	−0.6316	−3.7964	−0.9529	−0.9917	−0.0083	−0.0471
	$\beta = 0.9$	−0.9140	−0.0140	−0.0069	−0.3478	−3.1808	−0.9501	−0.9901	−0.0099	−0.0499
	$\alpha = 1.6$	−1.6354	−0.0354	−0.0215	−0.6963	−3.8290	−0.9525	−0.9936	−0.0064	−0.0475
	$\beta = 1.3$	−1.3211	−0.0211	−0.0140	−0.3644	−3.2296	−0.9531	−0.9910	−0.0090	−0.0469
	$\alpha = 2.8$	−2.9077	−0.1077	−0.1658	−0.8905	−4.4839	−0.9606	−0.9922	−0.0078	−0.0394
	$\beta = 3.5$	−3.5617	−0.0617	−0.1068	−0.3791	−3.2098	−0.9491	−0.9896	−0.0104	−0.0509
	$\alpha = 4.1$	−4.2912	−0.1912	−0.5656	−1.0359	−5.3217	−0.9551	−0.9878	−0.0122	−0.0449
	$\beta = 2.2$	−2.2361	−0.0361	−0.0416	−0.3787	−3.4196	−0.9495	−0.9884	−0.0116	−0.0505
$n = 150$	$\alpha = 0.4$	−0.3994	−0.0006	−0.0006	−0.0392	−3.1172	−0.9494	−0.9880	−0.0120	−0.0506
	$\beta = 0.7$	−0.7077	−0.0077	−0.0027	−0.3232	−3.1604	−0.9494	−0.9905	−0.0095	−0.0506
	$\alpha = 1.1$	−1.1100	−0.0100	−0.0031	−0.4909	−3.4293	−0.9540	−0.9930	−0.0070	−0.0460
	$\beta = 0.9$	−0.9095	−0.0095	−0.0045	−0.3157	−3.1229	−0.9493	−0.9908	−0.0092	−0.0507
	$\alpha = 1.6$	−1.6220	−0.0220	−0.0130	−0.5878	−3.5625	−0.9553	−0.9929	−0.0071	−0.0447
	$\beta = 1.3$	−1.3137	−0.0137	−0.0093	−0.3318	−3.2286	−0.9499	−0.9899	−0.0101	−0.0501
	$\alpha = 2.8$	−2.8659	−0.0659	−0.0949	−0.7475	−4.2316	−0.9594	−0.9930	−0.0070	−0.0406
	$\beta = 3.5$	−3.5397	−0.0397	−0.0662	−0.3432	−3.3061	−0.9541	−0.9911	−0.0089	−0.0459
	$\alpha = 4.1$	−4.2251	−0.1251	−0.3386	−0.7589	−4.0188	−0.9543	−0.9914	−0.0086	−0.0457
	$\beta = 2.2$	−2.2224	−0.0224	−0.0267	−0.2975	−3.0848	−0.9504	−0.9907	−0.0093	−0.0496
$n = 200$	$\alpha = 0.4$	−0.3997	−0.0003	−0.0005	−0.0027	−2.9180	−0.9495	−0.9893	−0.0107	−0.0505
	$\beta = 0.7$	−0.7057	−0.0057	−0.0020	−0.2659	−3.2060	−0.9525	−0.9897	−0.0103	−0.0475
	$\alpha = 1.1$	−1.1080	−0.0080	−0.0023	−0.4164	−3.3079	−0.9511	−0.9918	−0.0082	−0.0489
	$\beta = 0.9$	−0.9072	−0.0072	−0.0032	−0.2878	−3.2278	−0.9527	−0.9915	−0.0085	−0.0473
	$\alpha = 1.6$	−1.6172	−0.0172	−0.0094	−0.4895	−3.5060	−0.9529	−0.9917	−0.0083	−0.0471
	$\beta = 1.3$	−1.3108	−0.0108	−0.0068	−0.2678	−3.0955	−0.9520	−0.9907	−0.0093	−0.0480
	$\alpha = 2.8$	−2.8498	−0.0498	−0.0694	−0.6131	−3.7767	−0.9544	−0.9903	−0.0097	−0.0456
	$\beta = 3.5$	−3.5281	−0.0281	−0.0490	−0.2517	−3.1447	−0.9539	−0.9897	−0.0103	−0.0461
	$\alpha = 4.1$	−4.1969	−0.0969	−0.2416	−0.6438	−3.8426	−0.9541	−0.9887	−0.0113	−0.0459
	$\beta = 2.2$	−2.2190	−0.0190	−0.0200	−0.2499	−3.1199	−0.9488	−0.9907	−0.0093	−0.0512

TABLE 5: MPSEs for Erf-W distribution, fixed selected α and β parameter values and $n \in \{20, 100, 150, 200\}$

		Mean	Bias	MSE	Asym.	Kurtosis	CR(95%)	CR(99%)	HT(1%)	HT(5%)
$n = 20$	$\alpha = 0.4$	−0.4088	−0.0088	−0.0043	−0.0214	−3.2336	−0.9468	−0.9851	−0.0149	−0.0532
	$\beta = 0.7$	−0.6593	−0.0407	−0.0219	−0.9117	−4.4910	−0.8920	−0.9576	−0.0424	−0.1080
	$\alpha = 1.1$	1.0563	−0.0437	−0.0274	−1.6458	−8.2122	−0.9321	−0.9866	−0.0134	−0.0679
	$\beta = 0.9$	−0.8485	−0.0515	−0.0355	−0.9235	−4.6874	−0.8940	−0.9572	−0.0428	−0.1060
	$\alpha = 1.6$	−1.5143	−0.0857	−0.1244	−2.2725	−12.916	−0.8665	−0.9481	−0.0519	−0.1335
	$\beta = 1.3$	−1.2180	−0.0820	−0.0761	−0.9572	−4.7490	−0.8798	−0.9505	−0.0495	−0.1202
	$\alpha = 2.8$	−2.6712	−0.1288	−1.0736	−3.0523	−20.630	−0.8043	−0.8953	−0.1047	−0.1957
	$\beta = 3.5$	−3.3115	−0.1885	−0.5686	−1.0145	−5.1716	−0.8541	−0.9445	−0.0555	−0.1459
	$\alpha = 4.1$	−3.8922	−0.2078	−3.6033	−3.2007	−19.454	−0.7688	−0.8596	−0.1404	−0.2312
	$\beta = 2.2$	−1.9725	−0.1275	−0.1988	−0.9606	−4.5444	−0.8340	−0.9280	−0.0720	−0.1660
$n = 100$	$\alpha = 0.4$	−0.4039	−0.0039	−0.0009	−0.0160	−3.0053	−0.9439	−0.9893	−0.0107	−0.0561
	$\beta = 0.7$	−0.6801	−0.0199	−0.0041	−0.3650	−3.3041	−0.9175	−0.9733	−0.0267	−0.0825
	$\alpha = 1.1$	−1.0791	−0.0209	−0.0046	−0.6257	−3.9388	−0.9346	−0.9809	−0.0191	−0.0654
	$\beta = 0.9$	−0.8763	−0.0237	−0.0066	−0.3072	−3.3550	−0.9246	−0.9745	−0.0255	−0.0754
	$\alpha = 1.6$	−1.5550	−0.0450	−0.0188	−0.7041	−3.9916	−0.9090	−0.9681	−0.0319	−0.0910
	$\beta = 1.3$	−1.2631	−0.0369	−0.0143	−0.3913	−3.1771	−0.9219	−0.9768	−0.0232	−0.0781
	$\alpha = 2.8$	−2.6958	−0.1042	−0.1323	−0.9113	−4.4659	−0.8871	−0.9549	−0.0451	−0.1129
	$\beta = 3.5$	−3.4072	−0.0928	−0.1017	−0.4396	−3.3626	−0.9107	−0.9741	−0.0259	−0.0893
	$\alpha = 4.1$	−3.9176	−0.1824	−0.4277	−0.9086	−4.7134	−0.8747	−0.9405	−0.0595	−0.1253
	$\beta = 2.2$	−2.0410	−0.0590	−0.0364	−0.3101	−3.1937	−0.9048	−0.9700	−0.0300	−0.0952
$n = 150$	$\alpha = 0.4$	−0.403	−0.0035	−0.0006	−0.0173	−3.0208	−0.9494	−0.9894	−0.0106	−0.0506
	$\beta = 0.7$	−0.6832	−0.0168	−0.0027	−0.3225	−3.2319	−0.9206	−0.9778	−0.0222	−0.0794
	$\alpha = 1.1$	−1.0832	−0.0168	−0.0030	−0.5241	−3.4070	−0.9363	−0.9854	−0.0146	−0.0637
	$\beta = 0.9$	−0.8803	−0.0197	−0.0046	−0.3295	−3.2135	−0.9217	−0.9784	−0.0216	−0.0783
	$\alpha = 1.6$	−1.5658	−0.0342	−0.0119	−0.5447	−3.3591	−0.9234	−0.9803	−0.0197	−0.0766
	$\beta = 1.3$	−1.2713	−0.0287	−0.0095	−0.3458	−3.1912	−0.9228	−0.9777	−0.0223	−0.0772
	$\alpha = 2.8$	−2.7097	−0.0903	−0.0856	−0.6930	−3.8534	−0.8969	−0.9647	−0.0353	−0.1031
	$\beta = 3.5$	−3.4221	−0.0779	−0.0679	−0.3272	−3.1969	−0.9181	−0.9789	−0.0211	−0.0819
	$\alpha = 4.1$	−3.9340	−0.1660	−0.2816	−0.7224	−3.8093	−0.8867	−0.9562	−0.0438	−0.1133
	$\beta = 2.2$	−2.0506	−0.0494	−0.0245	−0.3237	−3.1098	−0.9125	−0.9754	−0.0246	−0.0875
$n = 200$	$\alpha = 0.4$	−0.4023	−0.0023	−0.0005	−0.0665	−2.9118	−0.9499	−0.9908	−0.0092	−0.0501
	$\beta = 0.7$	−0.6879	−0.0121	−0.0021	−0.3630	−3.2404	−0.9244	−0.9828	−0.0172	−0.0756
	$\alpha = 1.1$	−1.0857	−0.0143	−0.0022	−0.4371	−3.2317	−0.9329	−0.9866	−0.0134	−0.0671
	$\beta = 0.9$	−0.8832	−0.0168	−0.0034	−0.2493	−3.0695	−0.9282	−0.9804	−0.0196	−0.0718
	$\alpha = 1.6$	−1.5722	−0.0278	−0.0092	−0.5411	−3.4336	−0.9174	−0.9799	−0.0201	−0.0826
	$\beta = 1.3$	−1.2773	−0.0227	−0.0071	−0.3822	−3.3574	−0.9260	−0.9832	−0.0168	−0.0740
	$\alpha = 2.8$	−2.7252	−0.0748	−0.0640	−0.5565	−3.5563	−0.9070	−0.9653	−0.0347	−0.0930
	$\beta = 3.5$	−3.4335	−0.0665	−0.0505	−0.2106	−3.0227	−0.9213	−0.9778	−0.0222	−0.0787
	$\alpha = 4.1$	−3.9693	−0.1307	−0.2131	−0.6848	−3.8461	−0.8943	−0.9593	−0.0407	−0.1057
	$\beta = 2.2$	−2.0605	−0.0395	−0.0182	−0.2799	−3.0969	−0.9130	−0.9781	−0.0219	−0.0870

4.3.2. Quantile Regression Model

For the regression model, we considered the ML method and studied the estimation performance based on 10,000 Monte Carlo replications, using the `R` software and employing the CG optimization method within the `maxLik` script. The simulation considered a model with an intercept and two covariates, both occurrences of the standard uniform distribution. Therefore, the regression parameters β_0, β_1, and β_2. The experiment counted with fixed $\alpha = 0.1$, $\beta_1 = 0.5$, $\beta_2 = 0.3$, $\beta_3 = 0.2$, $\tau \in \{0.2, 0.5, 0.9\}$ paramter values and $n \in \{50, 100, 200\}$. The outcomes of these new simulations are presented in Tables 6 and, in general, across all the scenarios considered, the estimates converge to fixed values of the parameters. The accuracies of confidence intervals and hypothesis tests reflect this outcome. The asymmetry and kurtosis provide evidence of the tendency towards asymptotic normality.

Table 6: MLEs for Erf-W quantile regression model for fixed $\alpha = 0.1,\ \beta_1 = 0.5,\ \beta_2 = 0.3,\ \beta_3 = 0.2,\ \tau \in \{0.2, 0.5, 0.9\}$ and $n \in \{50, 100, 200\}$

		Mean	Bias	MSE	Asym.	Kurtosis	CR(95%)	CR(99%)	HT(5%)	HT(1%)
$\tau = 0.2$										
$n = 50$	α	−0.0901	−0.0099	−0.0007	−0.3739	−3.2425	−0.8743	−0.9384	−0.1257	−0.0616
	β_1	−0.5648	−0.0648	−0.0438	−0.0977	−3.1976	−0.9095	−0.9629	−0.0905	−0.0371
	β_2	−0.3511	−0.0511	−0.0267	−0.3304	−3.2285	−0.9213	−0.9700	−0.0787	−0.0300
	β_3	−0.2331	−0.0331	−0.0248	−0.4277	−3.7487	−0.9129	−0.9641	−0.0871	−0.0359
$n = 100$	α	−0.0928	−0.0072	−0.0004	−0.4201	−3.2351	−0.9128	−0.9676	−0.0872	−0.0324
	β_1	−0.5488	−0.0488	−0.0226	−0.1895	−3.1260	−0.9378	−0.9813	−0.0622	−0.0187
	β_2	−0.3369	−0.0369	−0.0153	−0.3379	−3.3206	−0.9351	−0.9804	−0.0649	−0.0196
	β_3	−0.2243	−0.0243	−0.0089	−0.3892	−3.2069	−0.9346	−0.9788	−0.0654	−0.0212
$n = 200$	α	−0.0922	−0.0078	−0.0002	−0.2926	−2.8249	−0.8362	−0.9502	−0.1638	−0.0498
	β_1	−0.5552	−0.0552	−0.0143	−0.1433	−2.9299	−0.8946	−0.9722	−0.1054	−0.0278
	β_2	−0.3416	−0.0416	−0.0090	−0.2547	−3.0567	−0.8940	−0.9696	−0.1060	−0.0304
	β_3	−0.2274	−0.0274	−0.0058	−0.3936	−3.1424	−0.9086	−0.9743	−0.0914	−0.0257
$\tau = 0.5$										
$n = 50$	α	−0.0932	−0.0068	−0.0007	−0.3943	−3.1780	−0.8949	−0.9495	−0.1051	−0.0505
	β_0	−0.4937	−0.0063	−0.0073	−0.1366	−3.3466	−0.9103	−0.9674	−0.0897	−0.0326
	β_1	−0.3071	−0.0071	−0.0143	−0.0201	−3.3858	−0.9214	−0.9770	−0.0786	−0.0230
	β_2	−0.2090	−0.0090	−0.0115	−0.0344	−3.6255	−0.9211	−0.9744	−0.0789	−0.0256
$n = 100$	α	−0.0967	−0.0033	−0.0004	−0.1857	−3.0313	−0.9211	−0.9712	−0.0789	−0.0288
	β_0	−0.4976	−0.0024	−0.0026	−0.0878	−3.1957	−0.9295	−0.9792	−0.0705	−0.0208
	β_1	−0.3031	−0.0031	−0.0046	−0.0260	−3.1022	−0.9359	−0.9851	−0.0641	−0.0149
	β_2	−0.2033	−0.0033	−0.0044	−0.0146	−3.2816	−0.9357	−0.9849	−0.0643	−0.0151
$n = 200$	α	−0.0983	−0.0017	−0.0002	−0.1605	−3.0805	−0.9366	−0.9806	−0.0634	−0.0194
	β_1	−0.4987	−0.0013	−0.0012	−0.0618	−3.0835	−0.9462	−0.9865	−0.0538	−0.0135
	β_2	−0.3021	−0.0021	−0.0022	−0.0448	−3.1196	−0.9441	−0.9867	−0.0559	−0.0133
	β_3	−0.2015	−0.0015	−0.0018	−0.0098	−3.1569	−0.9470	−0.9891	−0.0530	−0.0109
$\tau = 0.9$										
$n = 50$	α	−0.0938	−0.0062	−0.0007	−0.3584	−3.0734	−0.8960	−0.9539	−0.1040	−0.0461
	β_1	−0.4875	−0.0125	−0.0072	−0.1808	−4.2975	−0.9073	−0.9602	−0.0927	−0.0398
	β_2	−0.3049	−0.0049	−0.0125	−0.0110	−3.6870	−0.9150	−0.9696	−0.0850	−0.0304
	β_3	−0.1978	−0.0022	−0.0122	−0.0557	−4.4265	−0.9194	−0.9733	−0.0806	−0.0267
$n = 100$	α	−0.0968	−0.0032	−0.0004	−0.2043	−3.0268	−0.9225	−0.9715	−0.0775	−0.0285
	β_1	−0.4925	−0.0075	−0.0022	−0.2209	−3.3343	−0.9259	−0.9738	−0.0741	−0.0262
	β_2	−0.3034	−0.0034	−0.0039	−0.0848	−3.2769	−0.9294	−0.9815	−0.0706	−0.0185
	β_3	−0.2002	−0.0002	−0.0047	−0.0515	−3.2009	−0.9352	−0.9847	−0.0648	−0.0153
$n = 200$	α	−0.0981	−0.0019	−0.0002	−0.1687	−3.0738	−0.9355	−0.9801	−0.0645	−0.0199
	β_1	−0.4965	−0.0035	−0.0009	−0.2142	−3.1728	−0.9391	−0.9819	−0.0609	−0.0181
	β_2	−0.3016	−0.0016	−0.0019	−0.0070	−3.0793	−0.9448	−0.9877	−0.0552	−0.0123
	β_3	−0.2006	−0.0006	−0.0019	−0.0012	−3.1231	−0.9442	−0.9876	−0.0558	−0.0124

5. Applications to actual data

This section analyzes the models proposed by this research from an applied perspective. To this end, accurate data sets are compared to competing models well-established in the literature, thus establishing an effective counterpoint.

5.1. Unconditional Model

This subsection presents an application to actual data, aiming to evaluate the adjustment power of the Erf-W distribution and compare it to other bi-parametric models. The dataset considered comes from the *Federal Trade Commission*, an independent US government agency whose primary mission is to promote consumer protection. These are 346 observations and consist of nicotine measurements made of various brands of cigarettes in the year 1998, which can be accessed at `http://www.ftc.gov/reports/tobacco` and `http://pw1.netcom.com/rdavis2/smoke.html`. Table 7 below presents descriptive statistics of the considered data. It is essential to highlight that the data consist of observations ranging from 0.1 to 2, with a mean of 0.8526.

Table 7: Descriptive statistics for nicotine levels data set

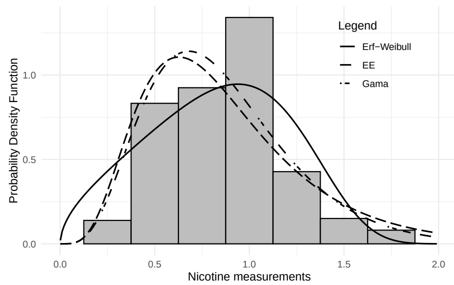
Statistics	
n	346
Mean	0.8526
Median	0.9
Variance	0.11201
Min	0.1
Max	2
Asymmetry	0.17219
Kurtosis	0.31555

We consider the gamma, exponentiated exponential (EE), and Gumbel models for comparative purposes. Table 8 provides the MLEs for the parameters of the fitted models with their respective standard errors, in addition to the values of the statistics Cramér-von Mises, Kolmogorov-Smirnov (KS), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and Corrected Akaike Information Criterion (CAIC). In general, the smallest values of these statistics point to the best model to fit the data. According to the goodness of fit statistics, the Erf-W model can be chosen as the best model for the nicotine levels data set. A visual comparison of the fit presented in Figure 5 also favors the model proposed in this research.

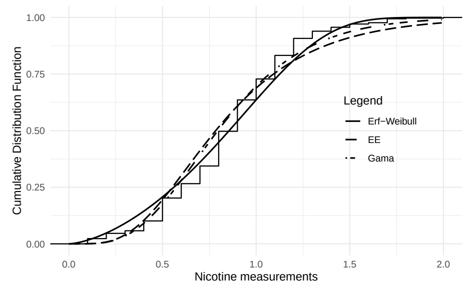
Table 8: MLEs (and their standard errors in parentheses), Cramér-von Mises, KS, AIC, BIC and CAIC statistics for nicotine levels in cigarettes data set

Distribution	$\widehat{\alpha}$	$\widehat{\beta}$	Cramér-von Mises	KS	AIC	BIC	CAIC
Erf-W	0.4957	1.5408	0.6417	0.1061	258.1295	265.8224	258.1645
	(0.0170)	(0.0652)	–	–	–	–	–
EE	5.5471	2.7317	1.4491	0.1721	302.4433	310.1362	302.4783
	(0.5165)	(0.1283)	–	–	–	–	–
Gamma	4.9423	5.7993	1.0943	0.1610	273.6732	281.3660	273.7081
	(0.3488)	(0.4327)	–	–	–	–	–
Gumbel	0.6864	0.3225	0.9911	0.1511	266.9117	274.6046	266.9467
	(0.0184)	(0.0126)	–	–	–	–	–

A second application reinforces the model's flexibility proposed in this research and its superior fit compared to Gumbel, Gamma, and EE distributions. The dataset used in the second application includes 63 measurements of the resistances of 1.5 cm glass fibers, initially collected by researchers at the UK National Physics Laboratory and later explored by Bourguignon et al. (2014).



(a) Estimated pdf of the Erf-W model



(b) Estimated cdf of the Erf-W model

Figure 5: Estimated pdf and cdf of the Erf-W model for nicotine levels in cigarettes data set

Table 9 presents descriptive statistics, which summarize the empirical distribution of the data. The average resistance is 1.5068, with a slightly higher median at 1.59, indicating an asymmetric distribution, as suggested by the asymmetry value of −0.8786. The variability of the data measured by the standard deviation is 0.3241 and a variance of 0.1051. Resistance values vary between a minimum of 0.55 and a maximum of 2.24, with kurtosis of 0.8002, pointing to a less flat distribution than the normal model. Unfortunately, there is no information about the unit of measurement resistances.

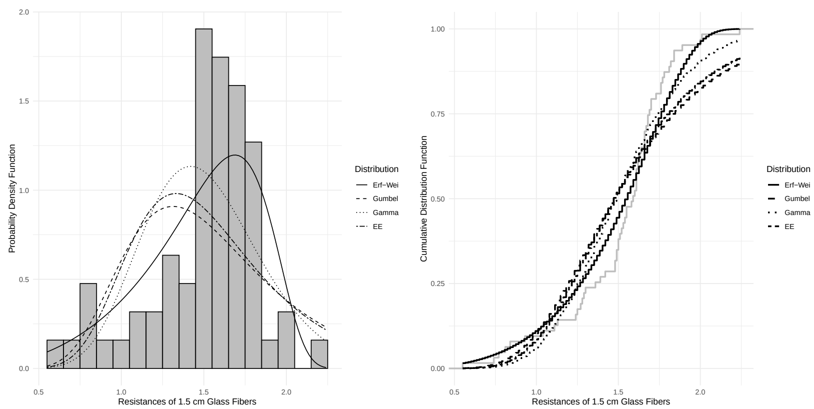
Table 9: Descriptive statistics for resistances of 1.5 cm glass fibers data set

Statistics	
n	63
Mean	1.5068
Median	1.59
Standard Deviation	0.3241
Variance	0.1051
Min	0.55
Max	2.24
Skewness	−0.8786
Kurtosis	0.8002

Table 10 presents the MLEs (and their standard errors in parentheses), Cramér-von Mises, KS, AIC, BIC and CAIC statistics for resistances of 1.5 cm glass fibers dataset. Based on these statistics, it is evident that the model proposed in this research is superior to competing distributions in terms of fit for the data set under analysis. All goodness-of-fit statistics present the lowest values for the Erf-W model, indicating that it can be chosen as the best model. Graphs presented in Figure 6 corroborate the choice of model Erf-W as the most appropriate to the data.

TABLE 10: MLEs (and their standard errors in parentheses), Cramér-von Mises, KS, AIC, BIC and CAIC statistics for resistances of 1.5 cm glass fibers data set

Distribution	$\widehat{\alpha}$	$\widehat{\beta}$	Cramér-von Mises	KS	AIC	BIC	CAIC
Erf-W	0.0953	3.2536	0.1959	0.1491	37.5309	41.8172	67.7438
	(0.0211)	(0.3159)	–	–	–	–	–
EE	2.4187	23.9533	0.7557	0.2222	67.5438	71.8300	302.4783
	(0.1840)	(5.4163)	–	–	–	–	–
Gamma	17.4561	11.5859	0.5684	0.2165	51.9031	56.1894	52.1031
	(2.2692)	(1.5548)	–	–	–	–	–
Gumbel	1.3318	0.3749	0.7545	0.2242	65.0380	69.3243	65.2380
	(0.0503)	(0.0321)	–	–	–	–	–



(a) Estimated pdf of the Erf-W model (b) Estimated cdf of the Erf-W model

FIGURE 6: Estimated pdf and cdf of the Erf-W model for resistances of 1.5 cm glass fibers data set

5.1.1. Quantile Regression Model

This Section presents an application to actual data for the Erf-W quantile regression model. The data refer to the student grade point average (GPA) (Y) of 120 students randomly selected from the new first-year class of a university. The data are available on the website `https://users.stat.ufl.edu/~rrandles/sta4210/Rclassnotes/data/textdatasets/KutnerData/Chapter%20%203%20Data%20Sets/CH03PR03.txt`. The purpose of this study is to explain GPA performance based on three other variables: ACT score (X_1) and intelligence test score ($X2$).

Table 11 presents a descriptive analysis of the data, and Figure 5.1.1 presents the dispersion, empirical densities and correlation for student's grade point average data set.

TABLE 11: Descriptive statistics for student's grade point average data set

Statistics	y	x_1	x_2
Mean	−3.0740	−24.725	−117.97
Median	−3.0775	−25.000	−117.50
Mode	−3.7500	−24.000	−113;110
Variance	−0.4152	−19.999	−150.32
Min	−0.5000	−14.000	−75.000
Max	−4.0000	−35.000	−149.00
Asymmetry	−0.8228	−0.1346	−0.1740
Kurtosis	−1.0286	−0.5862	−0.3520

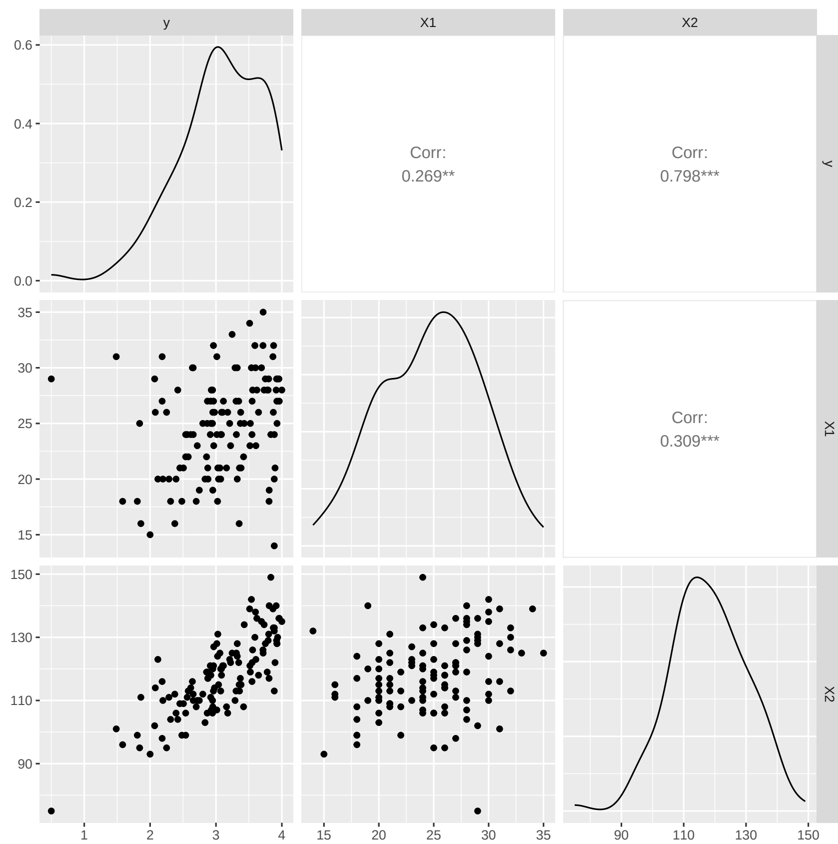


FIGURE 7: Dispersion, densities and correlation for student's grade point average data set

Table 12 shows the MLEs of the Erf-W quantile regression model for students' grade point average data set, with their standard errors and p-values. Based on this information, it can be seen that all estimates are significant at the $\alpha = 1\%$ level. To analyze the goodness of fit, the graphs of the diagnostic analysis are shown in Figure 8. These graphs reinforce that the proposed model can be used to explain the phenomenon under investigation.

Table 12: MLEs (and their standard errors in parentheses) of the Erf-W quantile regression model for student's grade point average data set

Parameters	Estimate	Std. Error	z-value	$Pr(>\|z\|)$
α	−0.0020	−0.0001	−40.327	−0.0000
β_1	−0.1854	−0.0420	−4.4160	−0.0000
β_2	−0.0042	−0.0016	−2.6922	−0.0071
β_3	−0.0089	−0.0000	−3935.84	−0.0000

Log-likelihood: −81.4425
AIC:172.8849 BIC:185.9108

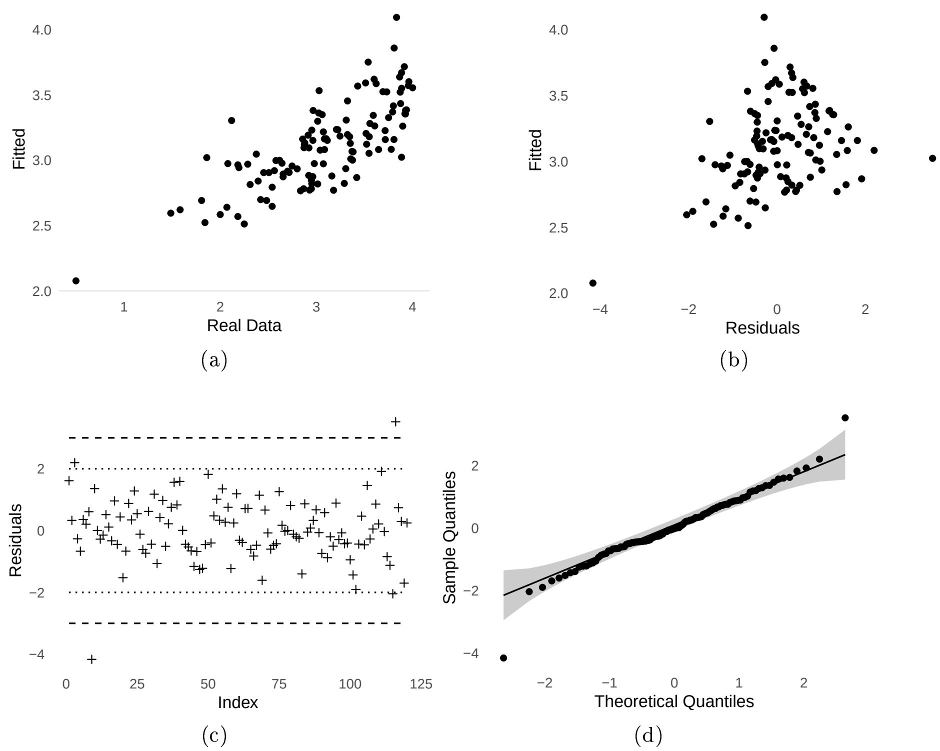


Figure 8: Plots of diagnostics for fitted Erf-W quantile regression model

To establish a comparison with the regression model proposed in this research, we considered the gamma regression model, which is part of the Generalized Additive Models for Location, Scale, and Shape (GAMLSS) family of models. GAMLSS extends generalized additive models (GAMs) and linear models (GLMs). For the computational implementation in `R` language, the `gamlss` package developed by

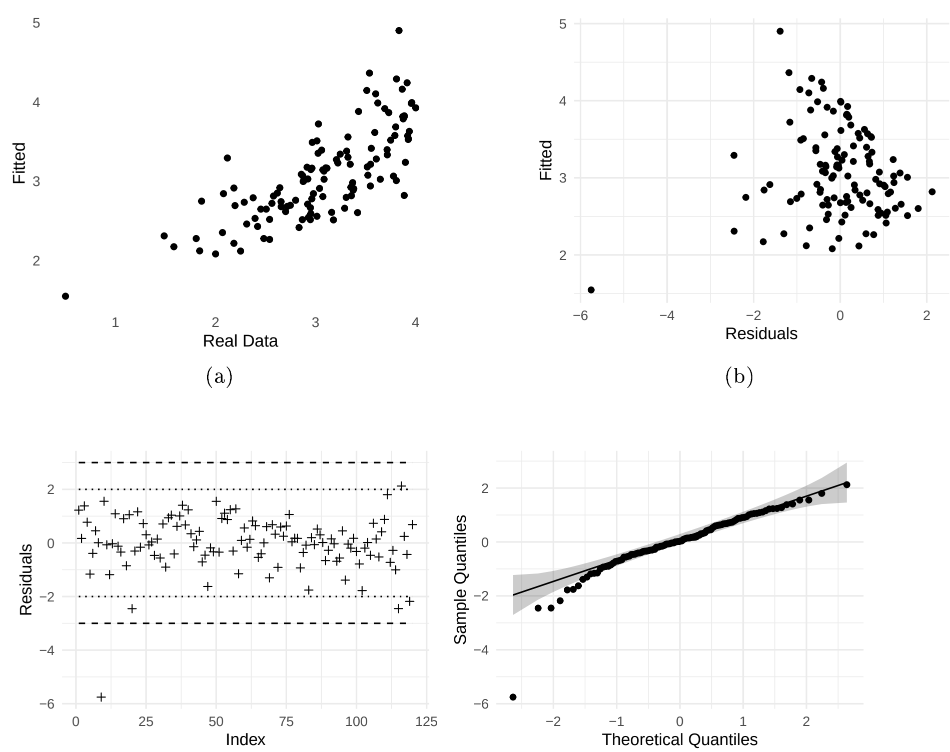
Stasinopoulos & Rigby (2007) was used. We recommend reading the research presented by these authors, not only for an introduction to the package's features but also for an excellent overview of the GAMLSS models.

Table 13 presents the MLEs (and their standard errors in parentheses) of the gamma regression model for the student's grade point average data set and the AIC and BIC statistics values. These measures are bigger than the corresponds to the Erf-W quantile regression model. The graphs of the diagnostic analysis are presented in Figure 9.

TABLE 13: MLEs (and their standard errors in parentheses) of the gamma regression model for student's grade point average data set

Parameters	Estimate	Std. Error	t value	$Pr(>\|t\|)$
σ	-1.8102	0.0659	-27.470	0.0000
β_1	-0.6907	0.1597	-4.3240	0.0000
β_2	-0.0013	0.0036	-0.3460	0.7300
β_3	-0.0155	0.0014	-11.018	0.0000

Log-likelihood: -84.85953
AIC: 177.7191 BIC: 188.869



(a) (b)

(c) (d)

FIGURE 9: Plots of diagnostics for fitted gamma regression model

In general, the smallest values of AIC an BIC statistics point to the best model to fit the data. According to the goodness of fit statistics, the Erf-W quantile regression model can be chosen as the best model.

6. Conclusions

This research presents the Erf-Weibull model, which results from altering the standard Weibull distribution through the Gauss function error. For the new model, properties such as moments, moment-generating function, reliability, and entropy, among others, are maintained. Two parameter definition methods are presented and evaluated numerically. A regression framework for modeling the quantiles of the distribution is managed and assessed in practice via applications to actual data and simulation studies. We hope researchers from different knowledge areas can use the new model.

Acknowledgements

The authors are grateful to the editor-in-chief of the Colombian statistics journal, as well as the anonymous referees for their contributions to improving this paper.

[Received: July 2023 — Accepted: February 2024]

References

Alexander, C., Cordeiro, G., Ortega, E. & Sarabia, J. (2012), 'Generalized beta-generated distributions', *Computational Statistics & Data Analysis* **56**, 1880–1897.

Bourguignon, M., Silva, R. B. & Cordeiro, G. M. (2014), 'The weibull-G family of probability distributions', *Journal of Data Science* **12**(1), 53–68.

Cheng, R. & Amin, N. (1979), 'Maximum product-of-spacings estimation with applications to the lognormal distribution', *Math Report* **791**.

Cheng, R. & Amin, N. (1983), 'Estimating parameters in continuous univariate distributions with a shifted origin', *Journal of the Royal Statistical Society: Series B (Methodological)* **45**(3), 394–403.

Cordeiro, G. M. & de Castro, M. (2011), 'A new family of generalized distributions', *Journal of Statistical Computation and Simulation* **81**, 883–893.

Cordeiro, G. M., Ortega, E. M. & da Cunha, D. C. (2013), 'The exponentiated generalized class of distributions', *Journal of Data Science* **11**(1), 1–27.

Eugene, N., Lee, C. & Famoye, F. (2002), 'Beta-normal distribution and its applications', *Communications in Statistics - Theory and Methods* **31**, 497–512.

Fernández, L. M. Z. & De Andrade, T. A. (2020), 'The erf-G family: New unconditioned and log-linear regression models', *Chilean Journal of Statistics (ChJS)* **11**(1).

Kenney, J. F. & Keeping, E. S. (1962), *Mathematics of statistics*, Princeton, NJ: Van Nostrand.

Lehmann, E. L. (1953), 'The power of rank tests', *Annals of Mathematical Statistics* **24**, 23–43.

Marshall, A. N. & Olkin, I. (1997), 'A new method for adding a parameter to a family of distributions with applications to the exponential and Weibull families', *Biometrika* **84**, 641–652.

Moors, J. J. (1988), 'A quantile alternative for kurtosis', *J. Royal Statist. Soc. D* **37**, 25–32.

Nadarajah, S., Cordeiro, G. M. & Ortega, E. M. M. (2013), 'The exponentiated Weibull distribution: A survey', *Statistical Papers* **54**, 839–877.

Nadarajah, S., Cordeiro, G. M. & Ortega, E. M. M. (2015), 'The Zografos-Balakrishnan-G family of distributions: Mathematical properties and applications', *Communications in Statistics - Theory and Methods* **44**, 186–215.

Python Software Foundation (2024), 'Python language reference', `https://www.python.org`.

R Core Team (2022), *R: A Language and Environment for Statistical Computing*, R Foundation for Statistical Computing, Vienna, Austria. https://www.R-project.org/

Ranneby, B. (1984), 'The maximum spacing method. an estimation method related to the maximum likelihood method', *Scandinavian Journal of Statistics* pp. 93–112.

Rinne, H. (2008), *The Weibull Distribution: A Handbook*, 1st edn, Chapman and Hall/CRC, New York. https://doi.org/10.1201/9781420087444

Ristić, M. M. & Balakrishnan, N. (2012), 'The gamma-exponentiated exponential distribution', *Journal of Statistical Computation and Simulation* **82**, 1191–1206.

Singh, R. K., Singh, S. K. & Singh, U. (2016), 'Maximum product spacings method for the estimation of parameters of generalized inverted exponential distribution under progressive type II censoring', *Journal of Statistics and Management Systems* p. 219–245.

Singh, U., Singh, S. K. & Singh, R. K. (2014), 'A comparative study of traditional estimation methods and maximum product spacings method in generalized inverted exponential distribution', *Journal of Statistics Applications & Probability* pp. 153–169.

Stasinopoulos, D. M. & Rigby, R. A. (2007), 'Generalized additive models for location scale and shape (GAMLSS) in R', *Journal of Statistical Software* **23**(7), 1–46. https://doi.org/10.18637/jss.v023.i07

Tahir, M. H. & Nadarajah, S. (2015), 'Parameter induction in continuous univariate distributions: Well-established G families', *Anais da Academia Brasileira de Ciências* **87**, 539–568.

Wong, T. S. T. & Li, W. K. (2006), 'A note on the estimation of extreme value distributions using maximum product of spacings', *IMS Lecture Notes-Monograph Series* pp. 272–283.

Zografos, K. & Balakrishnan, N. (2009), 'On families of Beta- and generalized Gamma-generated distributions and associated inference', *Statistical Methodology* **6**, 344–362.