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A Binary-Type Exponential Estimator for Estimating Finite Population Mean

Un estimador exponencial de tipo binario para estimar la media de una población finita

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Abstract

This paper suggests binary-type exponential estimator for estimating finite population mean under simple random sampling using auxiliary information. The proposed estimator includes number of estimators as its member which has been listed. Mathematical expression for the bias and mean square error of proposed estimator have been derived up to first order approximation. An empirical study was carried out using some existing population data sets and it was found that the proposed estimator performed better than other existing estimators considered in the study including linear regression estimator.

***Key words*:** Auxiliary variable; Bias; Binary-type Estimator; Exponential; Mean square error.

Resumen

Este artículo sugiere un estimador exponencial de tipo binario para estimar la media de una población finita bajo muestreo aleatorio simple utilizando información auxiliar. El estimador propuesto incluye el número de estimadores como miembros que se han enumerado. Se han obtenido expresiones matemáticas para el sesgo y el error cuadrático medio del estimador propuesto hasta una aproximación de primer orden. Se llevó a cabo un estudio empírico utilizando algunos parámetros de conjuntos de datos de población existentes y se encontró que el estimador propuesto funcionó mejor que otros estimadores existentes considerados en el estudio, incluido el estimador de regresión lineal.

***Palabras clave*:** Variable auxiliar; Sesgo; Estimador de tipo binario; Exponencial; Error cuadrático medio.

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1. Introduction

In survey sampling, the utilization of auxiliary information is frequently acknowledged to increase the precision of the estimators of population characteristics. Ratio, product, difference and regression methods of estimation are good examples in this context. If the correlation between study variable Y and the auxiliary variable X is positive and high, the ratio method of estimation introduced by Cochran (1940) is preferred. On the other hand if the correlation between Y and X is negative and high, the product method of estimation suggested by Murthy (1964) can be utilized. Hansen et al. (1953) suggested difference estimator for finite population mean which turned into regression estimator by replacing the optimum value of constant term with its estimate. In the sequence of suggesting modification over classical estimators, Rao (1991) and Gupta et al. (2012) suggested difference-type and difference-cum-ratio type estimator respectively and found to be more efficient than classical regression estimator. Bahl & Tuteja (1991) proposed new ratio and product type exponential estimators for estimating the population mean and showed that these estimators are better than classical ratio and product estimator. Following Bahl & Tuteja (1991) and Rao (1991), Grover & Kaur (2011) proposed an improved estimator for finite population mean. Shabbir et al. (2014) suggested a new difference-cum-exponential type estimator and found that it performed better than regression estimator, Bahl & Tuteja (1991) and Rao (1991), Grover & Kaur (2011) estimators. Singh & Audu (2015) proposed a class of exponential ratio-product type estimator which performs better than the classical ratio and product type estimators of finite population mean. Uraiwan & Nuanpan (2019) modified two families of ratio-type estimators by adjusting the Khoshnevisan et al. (2007) and Kumar (2013) estimators then they suggested a combined family of ratio-type estimator by taking linear combination of the two modified families of ratio-type estimators. Uraiwan & Nuanpan (2019) estimator was equally efficient as linear regression estimator. Singh & Dahiru (2021) suggested first time a binary-type estimator of population mean of study variable which generates the number of ratio-type estimators and thus, enable to make a unified study of several estimators and found to be more efficient than regression estimator.In this paper, our objective is to suggest an improved binary-type estimator of population mean under simple random sampling by utilizing the information on single auxiliary variable. The outline of the paper is as follows: Section 1 is devoted to introduction of work carried out. In Section 2, we have explained the usual notation required to denote the characteristic and other parameters. In Section 3, we have explained briefly the existing estimators which are relevant for present study. Section 4 is devoted to proposed estimator with its properties. In Section 5, we have considered real population data sets to show the application and performance of proposed estimator. In Section 6, we have discussed the results obtained by calculating MSE values and PREs using R software 4.1.3 and conclusion is stated in Section 7.

2. Notations

Let Y_i and X_i denote the values of characteristic under study and auxiliary characteristic respectively for the ith unit in the population of size $N(i = 1, 2, \ldots, N)$. Let

$\bar{Y} = \frac{1}{N}\sum\limits_{i=1}^{N} Y_i$; The population mean of characteristic under study.

$\bar{X} = \frac{1}{N}\sum\limits_{i=1}^{N} X_i$; The population mean of auxiliary variable.

$S_Y^2 = \frac{1}{N-1}\sum\limits_{i=1}^{N}\left(Y_i - \bar{Y}\right)^2$; The population mean square of characteristic under study.

$S_X^2 = \frac{1}{N-1}\sum\limits_{i=1}^{N}\left(X_i - \bar{X}\right)^2$; The population mean square of auxiliary variable.

$S_{YX} = \frac{1}{N-1}\sum\limits_{i=1}^{N}\left(Y_i - \bar{Y}\right)\left(X_i - \bar{X}\right)$; The population covariance of study and auxiliary variables.

$\rho = \frac{S_{YX}}{S_Y S_X}$; The population correlation coefficient between study and auxiliary variables.

Consider a sample of size n is drawn according to simple random sampling without replacement from the population of size N. Let $\bar{y} = \frac{1}{n}\sum\limits_{i=1}^{n} y_i$ be the sample mean of study variable and $\bar{x} = \frac{1}{n}\sum\limits_{i=1}^{n} x_i$ be the sample mean of auxiliary variable, $s_x^2 = \frac{1}{n-1}\sum\limits_{i=1}^{n}\left(x_i - \bar{x}\right)^2$ is the sample variance of the auxiliary variable X and $s_{yx} = \frac{1}{n-1}\sum\limits_{i=1}^{n}\left(y_i - \bar{y}\right)\left(x_i - \bar{x}\right)$ is the sample covariance between X and Y.

3. Related Existing Estimators

In this section, several existing estimators which are relevant for the present study have been considered.

3.1. Sample Mean Estimator

Well known sample mean estimator, is denoted by $\bar{y}$, and is an unbiased estimator of population mean with variance

$$Var\left(\bar{y}\right) = \theta\,\bar{Y}^2\,C_Y^2 \quad (1)$$

where $\theta = \frac{1}{n} - \frac{1}{N}$

3.2. Conventional Ratio and Product Estimators

Ratio and product estimators denoted by $\bar{y}_r$ and $\bar{y}_p$ respectively as suggested by Cochran (1940) and Murthy (1964) by using information on single auxiliary variable, are respectively given in equation 2 as,

$$\bar{y}_r = \frac{\bar{y}}{\bar{x}}\bar{X}\,;\ \bar{x} \neq 0 \ \text{ and } \ \bar{y}_p = \frac{\bar{y}\,\bar{x}}{\bar{X}}\,;\ \bar{X} \neq 0\,;\ \bar{X} \ \text{ is known} \tag{2}$$

The MSE of ratio and product estimators, up to first order of approximation, are respectively given in equations 3 and 4 respectively as;

$$MSE\left(\bar{y}_r\right) = \theta\,\bar{Y}^2\left(C_Y^2 + C_X^2 - 2\rho C_Y C_X\right) \tag{3}$$

$$MSE\left(\bar{y}_p\right) = \theta\,\bar{Y}^2\left(C_Y^2 + C_X^2 + 2\rho C_Y C_X\right) \tag{4}$$

Conventional ratio and product estimators will be more efficient than sample mean estimator if $\rho > \frac{1}{2}\frac{C_X}{C_Y}$ and $\rho < -\frac{1}{2}\frac{C_X}{C_Y}$ respectively.

3.3. Classical Difference and Regression Estimators

The classical difference estimator $\bar{y}_d$ is an unbiased estimator of population mean $\bar{Y}$ given in equation 5 and its variance is minimum for $\beta = \frac{S_{YX}}{S_X^2}$ given in equation 6

$$\bar{y}_d = \bar{\text{y}} + \beta\left(\bar{X} - \bar{x}\right) \tag{5}$$

$$Var\left(\bar{y}_d\right) = \theta\overline{\text{Y}}^2 C_Y^2\left(1 - \rho^2\right) \tag{6}$$

The classical regression estimator $\bar{y}_l$ is given in equation 7. It is a bias estimator of population mean and its mean square error is given in equation 8. It is well known fact that the regression estimator is always more efficient than sample mean, ratio and product estimators.

$$\bar{y}_l = \bar{\text{y}} + \widehat{\beta}\left(\bar{X} - \bar{x}\right) \tag{7}$$

Where $\widehat{\beta} = \frac{s_{yx}}{s_x^2}$ the sample regression coefficient between Y and X.

$$MSE_{.}\left(\bar{y}_l\right) = \theta\overline{\text{Y}}^2 C_Y^2\left(1 - \rho^2\right) \tag{8}$$

3.4. Bahl and Tuteja Estimators

Bahl & Tuteja (1991) suggested first time ratio-type and product-type exponential estimators for population mean denoted by $\bar{y}_{\text{Re}}$ and $\bar{y}_{\text{Pe}}$ respectively given in equation 9 and their MSE in equation 10 and equation 11 respectively. The exponential-type estimators were found to be more efficient than conventional ratio and product estimators respectively.

$$\bar{y}_{\mathrm{Re}} = \bar{y}\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \text{ and } \bar{y}_{Pe} = \bar{y}\exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}}\right) \tag{9}$$

$$MSE\left(\bar{y}_{\mathrm{Re}}\right) = \theta\,\bar{Y}^2\left(C_Y^2 + \frac{1}{4}C_X^2 - \rho C_Y C_X\right) \tag{10}$$

$$MSE\left(\bar{y}_{Pe}\right) = \theta\,\bar{Y}^2\left(C_Y^2 + \frac{1}{4}C_X^2 + \rho C_Y C_X\right) \tag{11}$$

3.5. Rao Estimator

Rao (1991) introduced the difference type estimator with two unknown constants to improve the efficiency of the classical difference estimator. The estimator is given in equation 12 as,

$$t_{Rao} = k_1\bar{y} + k_2\left(\bar{X}-\bar{x}\right) \tag{12}$$

where k_1 and k_2 are constants.The optimum values k_1 and k_2 are $k_{1opt} = \frac{1}{1+\theta(1-\rho^2)C_Y^2}$ and $k_{2opt} = k_{1opt}\frac{\bar{Y}\rho C_Y}{\bar{X}C_X}$. The minimum mean square error of t_{Rao} up to the first order of approximation at optimum values of k_1 and k_2 is given by equation 13. It was found that t_{Rao} was more efficient than linear regression estimator for minimum value of k_1 and k_2.

$$MSE_{\min}\left(t_{Rao}\right) = \bar{Y}^2\left\{1 - \frac{1}{1+\theta\left(1-\rho^2\right)C_Y^2}\right\} \tag{13}$$

3.6. Grover and Kaur Estimator

Grover & Kaur (2011) suggested an improved estimator of population mean given in equation 14 as,

$$t_{gk} = \left[m_1\bar{y} + m_2\left(\bar{X}-\bar{x}\right)\right]\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{14}$$

where m_1 and m_2 are constants. The minimum mean square error of the generalized exponential estimator t_{gk}, up to the first order of approximation, at the optimum value of m_1 and m_2 i.e.
$m_{1opt} = \frac{1-\frac{1}{8}\theta C_X^2}{1+\theta(1-\rho^2)C_Y^2}$ and $m_{2opt} = \frac{\bar{Y}}{\bar{X}}\left\{\frac{1}{2} - m_{1opt}\left(1-\rho\frac{C_Y}{C_X}\right)\right\}$ is given in equation 15 as,

$$MSE_{\min}\left(t_{gk}\right) = MSE\left(\bar{y}_l\right) - \frac{\theta^2\bar{Y}^2\left\{C_X^2 + 8\left(1-\rho^2\right)C_Y^2\right\}^2}{64\left\{1+\theta\left(1-\rho^2\right)C_Y^2\right\}}$$

The minimum MSE of t_{gk} can also be expressed as,

$$MSE_{\min}\left(t_{gk}\right) = MSE\left(\bar{y}_l\right) - \left(M_1 + M_2\right) \tag{15}$$

where $M_1 = \frac{\frac{\{MSE(\bar{y}_l)\}^2}{\bar{Y}^2}}{1+\frac{MSE(\bar{y}_l)}{\bar{Y}^2}}$ *and* $M_2 = \frac{\theta C_X^2\left\{MSE(\bar{y}_l)+\frac{1}{16}\theta\bar{Y}^2C_X^2\right\}}{4.\left\{1+\frac{MSE(\bar{y}_l)}{\bar{Y}^2}\right\}}$

Thus, it is clear that t_{gk} is always more efficient than linear regression estimator.

3.7. Gupta et al (2012) Estimator

Gupta et al. (2012) suggested a difference-cum-ratio estimator given in equation 16 as,

$$t_{Gupta} = \left[d_1\bar{y} + d_2\left(\bar{X} - \bar{x}\right)\right]\left(\frac{\bar{X}}{\bar{x}}\right) \tag{16}$$

where d_1 and d_2 are suitably chosen constants. The minimum mean square error of t_{Gupta} up to first order of approximation, at the optimum value of d_1 and d_2 i.e., $d_{1opt} = \frac{1-\theta C_X^2}{1-\theta\left\{C_X^2-(1-\rho^2)C_Y^2\right\}}$ and $d_{2opt} = \frac{\bar{Y}}{\bar{X}}\left\{1 + d_{1opt}\left(\rho\frac{C_Y}{C_X} - 2\right)\right\}$ is given in equation 17. Gupta et al. (2012) showed that the estimator t_{Gupta} was more efficient than linear regression estimator.

$$MSE_{\min}\left(t_{Gupta}\right) = \bar{Y}^2\left\{\frac{\theta C_Y^2\left(1-\rho^2\right)\left(1-\theta C_X^2\right)}{\theta C_Y^2\left(1-\rho^2\right) + \left(1-\theta C_X^2\right)}\right\} \tag{17}$$

3.8. Shabbir et al (2014) Estimator

Following Bahl & Tuteja (1991), Rao (1991) and Grover & Kaur (2011), a new difference-cum-exponential type estimator was suggested by Shabbir et al. (2014) given in equation 18 as,

$$t_{Shab} = \left[\frac{\bar{y}}{2}\left\{\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) + \exp\left(\frac{\bar{x}-\bar{X}}{\bar{X}+\bar{x}}\right)\right\} + \tau_1\left(\bar{X}-\bar{x}\right) + \tau_2\bar{y}\right]\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \tag{18}$$

where τ_1 and τ_1 are constants.The minimum MSE of t_{Shab} up to first order of approximation, at the optimum value of τ_1 and τ_1 i.e.,

$$\tau_{1opt} = \frac{\bar{Y}\left[-4\rho C_Y + C_X\left\{2 - \theta C_X^2 + \theta\rho C_Y C_X + 2\theta\left(-1+\rho^2\right)C_Y^2\right\}\right]}{4\bar{X}C_X\left\{-1+\theta\left(-1+\rho^2\right)C_Y^2\right\}}$$

and $\tau_{2opt} = \frac{\theta\left\{C_X^2-4\left(-1+\rho^2\right)C_Y^2\right\}}{4\left\{-1+\theta\left(-1+\rho^2\right)C_Y^2\right\}}$ is given in equation 19. Shabbir et al. (2014) found that their estimator performed better than the sample mean, ratio, product, and regression estimators as well as Bahl & Tuteja (1991), Rao (1991) and Grover & Kaur (2011) estimators.

$$MSE_{\min}\left(t_{Shab}\right) = MSE\left(\bar{y}_l\right) - \left(T_1 + T_2\right) \tag{19}$$

where $T_1 = \frac{\theta^2\bar{Y}^2\left\{C_X^2+8\left(1-\rho^2\right)C_Y^2\right\}^2}{64\left\{1+\theta(1-\rho^2)C_Y^2\right\}}$ and $T_1 = \frac{\theta^2\bar{Y}^2C_X^2\left\{3C_X^2+16\left(1-\rho^2\right)C_Y^2\right\}}{64\left\{1+\theta(1-\rho^2)C_Y^2\right\}}$

3.9. Uraiwan and Nuanpan (2019) Estimator

Uraiwan & Nuanpan (2019) modified two families of ratio-type estimators and by adjusting the Khoshnevisan et al. (2007) and Kumar (2013) estimators then Uraiwan & Nuanpan (2019) suggested a combined family of ratio-type estimator by taking linear combination of the two modified families of ratio-type estimators given in equation 20 as;

$$t_{RC} = \alpha t_R + (1-\alpha)\, t_{\text{Reg}} \tag{20}$$

where $t_R = \bar{y}\left(\frac{a\bar{X}+c}{a\bar{x}+c}\right)$ and $t_{\text{Reg}} = \left\{\bar{y} + b\left(\bar{X}-\bar{x}\right)\right\}\left(\frac{d\bar{X}+h}{d\bar{x}+h}\right)$ Here a, c ,d and h are either real numbers or functions of known parameters of an auxiliary variable while α is an unknown constant to be determined to minimize the mean square error of t_{RC}. For optimum value of α i.e., $\alpha_{opt} = \frac{\rho C_Y - (\lambda_2 + bR)C_X}{(\lambda_1 - \lambda_2 - bR)C_X}$, $\lambda_1 = \frac{a\bar{X}}{a\bar{X}+c}$, $\lambda_2 = \frac{d\bar{X}}{d\bar{X}+h}$ and $R = \frac{\bar{X}}{\bar{Y}}$. Now, the minimum mean square error of t_{RC} up to first order of approximation is given in equation 21 as,

$$MSE_{\min}\left(t_{\text{RC}}\right) = \bar{\text{Y}}^2 \theta C_Y^2 \left(1-\rho^2\right) \tag{21}$$

which is the MSE of linear regression estimator.

3.10. Singh and Dahiru (2021) Estimator

Singh & Dahiru (2021) proposed a new binary-type estimator of finite population mean in equation 22 as;

$$t_{BT} = \left[p_1 \overline{y}_\beta + p_2 \overline{y}\left\{\frac{(\alpha+\beta)\overline{X} + g\delta\overline{x}}{(\alpha+g\delta)\overline{X} + \beta\overline{x}}\right\}\right] \tag{22}$$

where p_1 and p_2 are constants. α , β and δ take values 0 and 1 and $g = \frac{n}{N-n}$ .The minimum MSE of estimator t_{BT} up to first order of approximation, at optimum values of p_1 and p_2, i.e., $p_{1opt} = \frac{(B_2 - B_3 B_4)}{(B_1 B_2 - B_3^2)}$ and $p_{2opt} = \frac{(B_1 B_4 - B_3)}{(B_1 B_2 - B_3^2)}$ is given in equation 23. Singh & Dahiru (2021) showed that the estimator t_{BT} was more efficient than the sample mean estimator, ratio, product estimators, and regression estimator.

$$MSE_{\min}(t_{BT}) = \overline{Y}^2\left[1 - \frac{\left(B_2 - 2B_3B_4 + B_1B_4^2\right)}{(B_1B_2 - B_3^2)}\right] \tag{23}$$

where $B_1 = \left[1 + \theta\left\{C_Y^2 + k^2 C_X^2 - 2k\rho C_X C_Y\right\}\right]$,

$$B_2 = \left[1 + \theta\left\{C_Y^2 + (\ell_1 - \ell_2)^2 C_X^2 + 2\left(\ell_2^2 - \ell_1\ell_2\right)C_X^2 + 4\left(\ell_1 - \ell_2\right)\rho C_X C_Y\right\}\right],$$

$$B_3 = \left[1 + \theta\left\{C_Y^2 + \left(\ell_2^2 - \ell_1\ell_2\right)C_X^2 - k\left(\ell_1 - \ell_2\right)C_X^2 + 2\left(\ell_1 - \ell_2\right)\rho C_X C_Y - k\rho C_X C_Y\right\}\right],$$

$$B_4 = \left[1 + \theta\left\{\left(\ell_2^2 - \ell_1\ell_2\right)C_X^2 + \left(\ell_1 - \ell_2\right)\rho C_X C_Y\right\}\right]$$

4. Proposed Estimator

Motivated by Rao (1991), Grover & Kaur (2011), Shabbir et al. (2014) and Singh & Dahiru (2021) we proposed a family of binary-type exponential estimator for estimating population mean as,

$$t_{Be} = \left[w_1 \bar{y}_\beta + w_2 \bar{y} \left\{ \frac{\psi\left\{\varphi_1\left(\alpha,\beta,\delta\right)\right\}}{\psi\left\{\varphi_2\left(\alpha,\beta,\delta\right)\right\}} \right\} \right] \exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}} \right) \tag{24}$$

where α , β and δ take values 0 and 1, $\varphi_1\left(\alpha,\beta,\delta\right) = \frac{g\delta}{\alpha+\beta+g\delta}$, $\varphi_2\left(\alpha,\beta,\delta\right) = \frac{\beta}{\alpha+\beta+g\delta}$, $\psi\left\{\varphi_i\left(\alpha,\beta,\delta\right)\right\} = \varphi_i\left(\alpha,\beta,\delta\right) + \left\{1-\varphi_i\left(\alpha,\beta,\delta\right)\right\} \frac{\bar{X}}{\bar{x}}$.

For different values of α , β and δ, proposed estimator exhibits different binary-type estimators that are given below in Table 1.

Table 1: Members of proposed estimator

Estimators	α	β	δ
$t_{Be1} = \left[w_1\bar{y}_\beta + w_2\bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)\right]\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)$	0	1	0
$t_{Be2} = \left[w_1\bar{y}_\beta + w_2\bar{y}\left(\frac{\bar{x}}{\bar{X}}\right)\right]\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)$	0	0	1
$t_{Be3} = \left[w_1\bar{y}_\beta + w_2\bar{y}\left(\frac{\bar{X}+g\,\bar{x}}{g\bar{X}+\bar{x}}\right)\right]\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)$	0	1	1
$t_{Be4} = \left[w_1\bar{y}_\beta + w_2\bar{y}\left(\frac{2\bar{X}}{\bar{X}+\bar{x}}\right)\right]\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)$	1	1	0
$t_{Be5} = \left[w_1\bar{y}_\beta + w_2\bar{y}\right]\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)$	1	0	0
$t_{Be6} = \left[w_1\bar{y}_\beta + w_2\bar{y}\left(\frac{\bar{X}+g\,\bar{x}}{(1+g)\bar{X}}\right)\right]\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)$	1	0	1

4.1. Properties of Proposed Estimator

To study the properties of estimator, in this section, we have obtained the bias and MSE of the proposed estimator t_{Be}, let us define the sample means as,

$$\bar{y} = \bar{Y}\left(1+e_0\right), \;\; \bar{x} = X\left(1+e_1\right) \tag{25}$$

Therefore,

$$E\left(e_0\right) = E\left(e_1\right) = 0, \quad E\left(e_0^2\right) = \theta C_Y^2, \; E\left(e_1^2\right) = \theta C_X^2, \; E\left(e_0 e_1\right) = \theta\rho C_Y C_X \tag{26}$$

Expressing the estimator t_{Be} in terms of e's from equation 25 and solving it, we have,

$$t_{Be} = \left[w_1 \overline{Y}\left(1+e_0-ke_1\right) + w_2\overline{Y} \left\{ \begin{array}{l} 1+e_0+\left(c-d\right)e_1 + \\ \left(c-d\right)e_0e_1 + \left(d^2-cd\right)e_1^2 \end{array} \right\} \right] \left(1-\frac{1}{2}e_1+\frac{3}{8}e_1^2\right)$$

where $c = \frac{g\delta}{\alpha+g\delta+\beta}, d = \frac{\beta}{\alpha+g\delta+\beta}$ and $k = \rho\frac{C_Y}{C_X}$.

$$t_{Be} - \bar{Y} = \overline{Y}\begin{bmatrix} w_1 + w_2 - 1 + w_1e_0 + w_2e_0 - kw_1e_1 - \frac{1}{2}w_1e_1 - \frac{1}{2}w_2e_1 + \\ w_2\left(c-d\right)e_1 - \frac{1}{2}w_1e_0e_1 - \frac{1}{2}w_2e_0e_1 + w_2\left(c-d\right)e_0e_1 + \\ w_2\left(d^2-cd\right)e_1^2 + \frac{1}{2}w_1ke_1^2 - \frac{1}{2}w_2\left(c-d\right)e_1^2 + \frac{3}{8}\left(w_1+w_2\right)e_1^2 \end{bmatrix} \tag{27}$$

By taking expectation both the sides, and substituting the values from equation 26, bias of the proposed estimator is obtained as;

$$\text{Bias}\left(t_{Be}\right) = \overline{Y}\left[w_1 + w_2 - 1 + \theta\begin{Bmatrix} w_2\left(d^2-cd\right)C_X^2 + \frac{1}{2}w_1kC_X^2 \\ -\frac{1}{2}w_2\left(c-d\right)C_X^2 + \frac{3}{8}\left(w_1+w_2\right)C_X^2 \\ +w_2\left(c-d\right)\rho C_XC_Y - \frac{1}{2}\left(w_1+w_2\right)\rho C_XC_Y \end{Bmatrix}\right] \tag{28}$$

Now, MSE of proposed estimator t_{Be} up to first order of approximation is obtained by squaring and taking expectation both sides of equation equation 27. Thus we have

$$MSE\left(t_{Be}\right) = \overline{Y}^2E\begin{bmatrix} w_1 + w_2 - 1 + w_1e_0 + w_2e_0 - kw_1e_1 - \frac{1}{2}w_1e_1 - \frac{1}{2}w_2e_1 + \\ w_2\left(c-d\right)e_1 - \frac{1}{2}w_1e_0e_1 - \frac{1}{2}w_2e_0e_1 + w_2\left(c-d\right)e_0e_1 + \\ w_2\left(d^2-cd\right)e_1^2 + \frac{1}{2}w_1ke_1^2 - \frac{1}{2}w_2\left(c-d\right)e_1^2 + \frac{3}{8}\left(w_1+w_2\right)e_1^2 \end{bmatrix}^2$$

By solving and substituting the values from equation 26, we have

$$MSE(t_{Be}) = \overline{Y}^2\left[1 + w_1^2A_1 + w_2^2A_2 + 2w_1w_2A_3 - 2w_1A_4 - 2w_2A_5\right] \tag{29}$$

where $A_1 = 1 + \theta\left\{C_Y^2 + \left(k+1\right)^2C_X^2 - 2\left(k+1\right)\rho C_XC_Y\right\}$,

$$A_2 = +\theta\left[C_Y^2 + \left\{\left(c-d\right)-1\right\}^2C_X^2 + 2\left(d^2-cd\right)C_X^2 + 2\left\{2\left(c-d\right)-1\right\}\rho C_XC_Y\right],$$

$$A_3 = 1 + \theta\Big[C_Y^2 + \left\{d^2 - cd - \left(c-d\right)\left(k+1\right) + k + 1\right\}C_X^2 + 2\left\{\left(c-d\right) - \frac{1}{2}k - 1\right\}\rho C_XC_Y\Big],$$

$$A_4 = 1 + \theta\left\{\left(\frac{3}{8} + \frac{k}{2}\right)C_X^2 - \frac{1}{2}\rho C_XC_Y\right\},$$

$$A_5 = 1 + \theta\left[\left\{d^2 - cd - \frac{1}{2}\left(c-d\right) + \frac{3}{8}\right\}C_X^2 + \left\{\left(c-d\right) - \frac{1}{2}\right\}\rho C_XC_Y\right].$$

Partially Differentiating equation 29 with respect to w_1 and w_2. Setting $\frac{\partial[MSE(t_{Be})]}{\partial w_i} = 0$ for $i = 1, 2$, the optimum values of w_1 and w_2 are obtained as; $w_{1opt} = \frac{A_3A_5 - A_2A_4}{A_3^2 - A_1A_2}$ and $w_{2opt} = \frac{A_3A_4 - A_1A_5}{A_3^2 - A_1A_2}$ Substituting the optimum values of w_1 and w_2 in equation 29. On simplification we get the minimum MSE of proposed estimator t_{Be} as,

$$MSE_{\min.}(t_{Be}) = \overline{Y}^2\left(1 + \frac{A_1A_5^2 + A_2A_4^2 - 2A_3A_4A_5}{A_3^2 - A_1A_2}\right) \tag{30}$$

5. Empirical Study of the Proposed Estimator

In this section, the performance of members of the proposed estimator and other estimators, considered in section 2, have been evaluated numerically by taking four real population data sets based on high, moderate, low and negative correlation between study and auxiliary variable given in Table 2 below.

- Population I: [Source: Murthy (1977), pp. 228] Y: Output and X: Number of workers.
- Population II: [Source: Kalidar & Cingi (2005)] Y: The apple production amount in 1999 and X: Number of apple trees in 1999 in Black sea region of Turkey.
- Population III: [Source: Srianstava et al. (1989)] Y: The measurement of weight children and X: The mid-arm circumference of children.
- Population IV: [Source: Maddala (1977), pp. 282] Y: The consumption per capita and X: The deflated prices of veal.

Table 2: Summary of population data sets

Population I						
N	n	$\bar{Y}$	$\bar{X}$	C_Y	C_X	ρ_{YX}
80	10	51.8264	2.8513	0.3542	0.9484	0.915
Population II						
204	50	966	26441	2.4739	1.7171	0.71
Population III						
55	30	17.08	16.92	0.12688	0.07	0.54
Population IV						
16	4	7.6375	75.4343	0.2278	0.0986	-0.6823

The MSEs and Percentage relative efficiencies (PREs) of members of proposed estimator and other estimators as well as classical estimators have been calculated based on population data sets I-IV given in Table 2.

6. Results and Discussions

It is observed from Table 3 that all members t_{Be1} - t_{Be6} of proposed estimator performed better than $\bar{y}$, $\bar{y}_r$, $\bar{y}_p$, $\bar{y}_l$, t_{btr}, t_{btp} and t_{Rao} for all population I-IV with the exception that estimator t_{Rao} is equally efficient for population IV even though t_{Be3} and t_{Be6} are more efficient than t_{Rao} for population IV. It is also seen that the estimators t_{Be1} and t_{Be3} are more efficient than all the estimators considered in the study for population I and t_{Be2} is more efficient for population II while t_{Be2} and t_{Be6} are found to be more efficient than all the estimators considered in the study for population III and population IV. Therefore among all the estimators considered here proposed estimator is preferable due to having higher PRE.

Table 3: MSEs and PREs of Members of Proposed and Classical Estimators

Estimators	Population I	Population II	Population III	Population IV	
$\bar{y}$	MSE	29.48542	86226.17	0.0711571	0.5675592
	PRE	100	100	100	100
$\bar{y}_R$	MSE	96.40182	42781.39	0.05041742	1.009117
	PRE	30.58595	201.5506	141.1359	56.24314
$\bar{y}_P$	MSE	385.3576	212750.8	0.1352137	0.338662
	PRE	7.651443	40.52918	52.62565	167.5887
$\bar{y}_l$	MSE	4.799489	42759.56	0.05040768	0.3033415
	PRE	614.345	201.6536	141.1632	187.1024
t_{btr}	MSE	10.09505	54118.79	0.0553726	0.7617556
	PRE	292.0779	159.3276	128.5059	74.5067
t_{btp}	MSE	154.5729	139103.5	0.09777077	0.426528
	PRE	19.07541	61.98705	72.77951	133.0649
t_{Rao}	MSE	4.790928	40886.05	0.05039897	0.3017728
	PRE	615.4427	210.8938	141.1876	188.0754
t_{gk}	MSE	4.437168	40403.41	0.05040765	0.3033344
	PRE	664.5098	213.4131	141.1633	187.1068
t_{Gupta}	MSE	4.790198	40802.76	0.05039897	0.3017693
	PRE	615.5365	211.3243	141.1876	188.0772
t_{Shab}	MSE	3.564417	39865.51	0.050397	0.3014851
	PRE	827.2156	216.2926	141.1931	188.2545
t_{BT1}	MSE	4.790198	40802.76	0.05039897	0.3017693
	PRE	615.5365	211.3243	141.1876	188.0772
t_{BT2}	MSE	4.790884	40865.44	0.05039897	0.3017506
	PRE	615.4483	211.0002	141.1876	188.0888
t_{BT3}	MSE	4.79023	40880.38	0.05039897	0.3017713
	PRE	615.5324	210.9231	141.1876	188.0759
t_{BT4}	MSE	4.790756	40865.9	0.05039897	0.3017715
	PRE	615.4647	210.9978	141.1876	188.0758
t_{BT5}	MSE	4.790928	40886.05	0.05039897	0.3017722
	PRE	615.4427	210.8938	141.1876	188.0754
t_{BT6}	MSE	4.790922	40882.93	0.05039897	0.3017719
	PRE	615.4434	210.9099	141.1876	188.0755
t_{Be1}	MSE	2.102657	41891.63	0.05040155	0.3021512
	PRE	1402.293	210.9099	141.1804	187.8395
t_{Be2}	MSE	3.585475	39367.38	0.05039596	0.2996873
	PRE	822.3573	219.0295	141.196	189.3838
t_{Be3}	MSE	2.277392	40866.97	0.05039783	0.3019345
	PRE	1294.701	210.9923	141.1908	187.9743
t_{Be4}	MSE	4.606772	41232.18	0.05039988	0.3019034
	PRE	640.045	209.1235	141.185	187.9936
t_{Be5}	MSE	4.437168	40403.41	0.05039801	0.3016316
	PRE	664.5098	213.4131	141.1903	188.163
t_{Be6}	MSE	4.186299	40024.05	0.05039658	0.3014585
	PRE	704.3314	215.4359	141.1943	188.2711

7. Conclusions

In the present paper, we have suggested a binary-type exponential estimator of the population mean of a study variable using single auxiliary variable which includes six estimators as its particular case. Expressions for the bias and the mean square error of suggested estimator have been derived up to first order of approximation and minimum mean square error has also been obtained. By numerical comparison it is found that some of the members of proposed estimator are more efficient than the sample mean estimator, ratio and product estimators, linear regression estimator, Rao (1991) estimator, Grover & Kaur (2011), Gupta et al. (2012) estimator, Shabbir et al. (2014) estimator and Singh & Dahiru (2021) estimators. Therefore, the proposed estimator is recommended for estimating the finite population mean in simple random sampling in order to get more efficient estimation.

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References

Bahl, S. & Tuteja, R. (1991), 'Ratio and product type exponential estimators', *Journal of Information and Optimization Sciences* **12**(1), 159–164.

Cochran, W. G. (1940), 'The estimation of the yields of cereal experiments by sampling for the ratio of grain to total produce', *The Journal of Agricultural Science* **30**(2), 262–275.

Grover, L. K. & Kaur, P. (2011), 'An improved estimator of the finite population mean in simple random sampling', *Model Assisted Statistics and Applications* **6**(1), 47–55.

Gupta, S., Shabbir, J., Sousa, R. & Real, P. C. (2012), 'Estimation of the mean of a sensitive variable in the presence of auxiliary information', *Communications in Statistics-Theory and Methods* **41**(13), 2394–2404.

Hansen, M. H., Hurwitz, W. N. & Madow, W. G. (1953), *Sample Survey Methods and Theory*, John Wiley and Sons, New York.

Kalidar, C. & Cingi, H. (2005), 'A new estimator using two auxiliary variables', *Applied Mathematics and Computation* **162**(2), 901–908.

Khoshnevisan, M., Singh, R., Chauhan, P., Sawan, N. & Smarandache, F. (2007), 'A General Family of estimators for Estimating Population Mean Using Known Value of Some Population Parameter(s)', *Far East Journal of Theoretical Statistics* **22**, 181–191.

Kumar, S. (2013), 'Improved Estimators in Finite Population Surveys: Theory and Application', *Journal of Applied Modern Statistical Method* **12**(1), 120–127.

Maddala, G. S. (1977), *Econometrics Economics handbook series*, McGraw Hills Publication Company, New York.

Murthy, M. N. (1964), 'Product method of estimation', *Sankhya A* **26**(1), 69–74.

Murthy, M. N. (1977), *Sampling: Theory and Methods*, Statistical Pub. Society, India.

Rao, T. J. (1991), 'On certain methods of improving ratio and regression estimators', *Communications in Statistics-Theory and Methods* **20**(10), 3325–3340.

Shabbir, J., Haq, A. & Gupta, S. (2014), 'A new difference –cum exponential type estimator of finite population mean in simple random sampling', *Revista Colombiana de Estadistica* **37**(1), 199–211.

Singh, R. V. K. & Audu, A. (2015), 'Improve exponential ratio-product type estimators for finite population mean', *International Journal of Engineering Science and Innovative Technology* **4**(3), 317–322.

Singh, R. V. K. & Dahiru, S. (2021), 'A binary-type estimator for estimating population mean in simple random sampling', *Advances and Applications in Mathematical Sciences* **21**(1), 1–13.

Srianstava, R. S., Srivastava, S. & Khare, B. (1989), 'Chain ratio type estimator for ratio of two population means using auxiliary characters', *Communications in Statistics-Theory and Methods* **18**(10), 3917–3926.

Uraiwan, J. & Nuanpan, L. (2019), 'A combined family of ratio estimators for population mean using an auxiliary variable in simple random sampling', *Journal of Mathematical and Fundamental Sciences* **51**(1), 1–12.