Revista Colombiana de Estadística - Applied Statistics
January 2024, volume 47, issue 1, pp. 37–65
http://doi.org/10.15446/rce.v47n1.111899

Application of the New Extended Topp-Leone Distribution to Complete and Censored Data

Aplicación de la nueva distribución extendida Topp-Leone a datos completos y censurados

Violet Zivai Nyamajiwa[1,a], Regent Retrospect Musekwa[2,b], Boikanyo Makubate[3,c]

[1]Department of Mathematics and Statistical Sciences, Botswana International University of Science and Technology, Palapye, Botswana

Abstract

One of the most important applications of statistical models is in analyzing survival data. In this study, we developed the Gamma Type Two Half Logistic Topp-Leone-G model using the technique earlier proposed by Zografos and Balakrishnan. Different characteristics of the proposed distribution are obtained. In order to estimate the model parameters based on complete and censored data, the maximum likelihood estimation method is used. Through Monte Carlo simulation, the performance of the estimators is evaluated. The proposed distribution's potential significance and applicability are empirically demonstrated using actual datasets. We found that our new distribution is a very competitive model for describing both complete and censored observations in survival analysis. The work demonstrated that in certain cases, our new model performed better than other parametric models with the same number of parameters.

***Key words*:** Gamma distribution; Type II Half Logistic Top Leone-G distribution; Order Statistics; Probability Weighted Moments; Rényi entropy.

Resumen

Una de las aplicaciones más importantes de los modelos estadísticos es el análisis de datos de supervivencia. En este estudio, desarrollamos el modelo Gamma Tipo Dos Half Logistic Topp-Leone-G utilizando la técnica propuesta anteriormente por Zografos y Balakrishnan. Se obtienen diferentes características de la distribución propuesta. Para estimar los parámetros del

[a]Postgraduate Student. E-mail: nv21100064@studentmail.biust.ac.bw
[b]Postgraduate Student. E-mail: mr21100062@studentmail.biust.ac.bw
[c]Professor. E-mail: makubateb@biust.ac.bw

modelo a partir de datos completos y censurados se utiliza el método de estimación de máxima verosimilitud. A través de la simulación Monte Carlo se evalúa el desempeño de los estimadores. La importancia y aplicabilidad potencial de la distribución propuesta se demuestran empíricamente utilizando conjuntos de datos reales. Descubrimos que nuestra nueva distribución es un modelo muy competitivo para describir observaciones tanto completas como censuradas en el análisis de supervivencia. El trabajo demostró que, en ciertos casos, nuestro nuevo modelo funcionó mejor que otros modelos paramétricos con la misma cantidad de parámetros.

***Palabras clave*:** Distribución gamma; Distribución media logística superior Leone-G tipo II; Estadísticas de pedidos; Momentos ponderados de probabilidad; Entropía de Rényi.

1. Introduction

In order to get insight into the characteristics of failure times and hazard functions that may not be available with non-parametric methods, a suitable parametric model is frequently of relevance in the analysis of survival data. The Topp-Leone (Nadarajah & Kotz, 2003; Topp & Leone, 1955) and Half Logistic (Balakrishnan, 1985) distributions are some of the most popular and widely used distributions for modelling survival data. Despite this, it has been discovered that neither the Topp-Leone (TL) nor the Half Logistic (HL) provide sufficient parametric fit for those survival data with non-monotone failure rate, such as the unimodal failure rate functions. To make standard distributions (like TL, HL, and others) more adaptable for modeling lifetime data, various academics have suggested techniques for doing so (Adamidis & Loukas, 1998; Alzaatreh et al., 2013; Cordeiro et al., 2013; Makubate, Oluyede & Gabanakgosi, 2021; Marshall & Olkin, 1997; Ristić & Balakrishnan, 2012; Zografos & Balakrishnan, 2009). (Al-Shomrani et al., 2016) extended the TL distribution via (Alzaatreh et al., 2013) to obtain the Topp-Leone-G (TL-G) distribution. Extensions of the TL-G are not hard to find in the literature (Bantan et al., 2020; Chamunorwa et al., 2021; Chipepa et al., 2020; Keganne et al., 2023; Sangsanit & Bodhisuwan, 2016). In their article, Soliman et al. (2017) used (Ristić & Balakrishnan, 2012) to create an extension of HL called the Type Two Half Logistic-G (TIIHL-G) distribution. Consequently, for any continuous baseline cdf, G(x; ξ), the cdf and pdf of the Type Two Half Logistic Topp-Leone-G (TIIHLTL-G) of distribution is given by

$$G_{TIIHLTL-G}(x) = 1 - \frac{1-\left(1-\overline{G}^2(x;\xi)\right)^b}{1+\left(1-\overline{G}^2(x;\xi)\right)^b}, \tag{1}$$

and

$$g_{TIIHLTL-G}(x) = \frac{4bg(x;\xi)\overline{G}(x;\xi)\left(1-\overline{G}^2(x;\xi)\right)^{b-1}}{\left(1+\left(1-\overline{G}^2(x;\xi)\right)^b\right)^2}, \tag{2}$$

respectively, where $\overline{G}(x;\xi) = 1 - G(x;\xi)$, $b > 0$ is a shape parameter and ξ is a vector of parameters.

In this article the TIIHLTL-G distribution is extended via (Zografos & Balakrishnan, 2009). The new distribution is referred to as the Gamma Type Two Half Logistic Topp-Leone-G (GTIIHLTL-G) distribution. The GTIIHLTL-G distribution is being proposed in order to offer a model for modeling data with skewed to the right or left, reverse-J, J, and almost symmetric failure rates. Additionally, the new distribution can model data with uni-modal, declining, rising, and bathtub-like hazard rate function (hrf).

Following the notation by Zografos & Balakrishnan (2009), the class of Gamma-G (GG) distributions is defined as follows. Consider a continuous distribution $G(x;\xi)$ with density function $g(x;\xi)$, where ξ is a vector of parameters. Then, the cdf and pdf of the GG family of distributions is given by

$$F_{GG}(x) = \frac{1}{\Gamma(\delta)}\int_0^{-\ln \bar{G}(x;\xi)} w^{\delta-1}e^{-w}dw \qquad = \frac{\gamma(\delta, -\ln \bar{G}(x;\zeta))}{\Gamma(\delta)}, \tag{3}$$

and

$$f_{GG}(x) = \frac{1}{\Gamma(\delta)}[-\ln \bar{G}(x;\xi)]^{\delta-1}g(x;\xi), \tag{4}$$

respectively, where x is such that $G(x;\xi)$ is a well defined continuous cdf, $w \in (0,\infty)$, $\delta > 0$, $\Gamma(\delta) = \int_0^\infty w^{\delta-1}e^{-w}dw$ is the gamma function and $\gamma(\delta, y) = \int_0^y w^{\delta-1}e^{-w}dw$ is the incomplete gamma function. The distribution $G(x;\xi)$ is referred to as the parent distribution (Jones, 2004). In the literature, extensions of the GG distribution are readily available (Alzaatreh et al., 2014; Oluyede et al., 2018; Foya et al., 2017; Oluyede et al., 2017). Note that GG is a member of the T-X family of distributions defined by Alzaatreh et al. (2013).

The cdf of the GTIIHLTL-G is obtained by inserting equation (1) into equation (3) and is given by

$$F_{GTIIHLTL-G}(x) \quad = \quad \frac{\gamma\left(\delta, -\ln\left[\frac{1-\left(1-\bar{G}^2(x;\xi)\right)^b}{1+\left(1-\bar{G}^2(x;\xi)\right)^b}\right]\right)}{\Gamma(\delta)}. \tag{5}$$

The corresponding pdf is given by

$$\begin{aligned} f_{GTIIHLTL-G}(x) \quad &= \quad \frac{4bg(x;\xi)}{\Gamma(\delta)}\left[-\ln\left(\frac{1-\left(1-\bar{G}^2(x;\xi)\right)^b}{1+\left(1-\overline{G}^2(x;\xi)\right)^b}\right)\right]^{\delta-1} \\ &\times \quad \frac{\bar{G}(x;\xi)\left(1-\bar{G}^2(x;\xi)\right)^{b-1}}{\left(1+\left(1-\bar{G}^2(x;\xi)\right)^b\right)^2}. \end{aligned} \tag{6}$$

Note that the survival function, hazard function, reverse hazard function and the quantile function for the GTIIHLTL-G family of distributions can be easily deduced from equations (5) and (6) respectively. The pdf in (6) can be expressed as

$$f_{GTIIHLTL-G}(x) = \sum_{m=0}^{\infty} Q^*_{m+1} h^*_{m+1}(x;\xi), \tag{7}$$

where

$$\begin{aligned} Q^*_{m+1} &= \frac{b}{\Gamma(\delta)} \sum_{q,p,j,s=0}^{\infty} \frac{2^{(\delta+j+s+1)}(-1)^{p+m}}{m+1} b_{s,j} \binom{\delta-1}{j} \binom{\delta+j+s+p}{p} \\ &\times \binom{b[\delta+j+s+p]-1}{q} \binom{2q+1}{m}, \end{aligned}$$

is a constant, and

$$h^*_{m+1}(x;\xi) = (m+1) g(x;\xi) G^m(x;\xi), \tag{8}$$

is a pdf of the exponentiated-G (Exp-G) distribution with power parameter $m+1$ (see Appendix A for more details). Consequently, some important statistical properties of the GTIIHLTL-G distribution can be derived directly from those of the Exp-G distribution.

The rest of the paper is outlined as follows. In Section 2 we presented four special cases of the GTIIHLTL-G distribution. Some statistical properties of GTIIHLTL-G are discussed in Section 3. Section 4 provides details on maximum likelihood estimation (MLE) of the parameters of the model. The finite sample characteristics of the parameter estimators were examined using Monte Carlo simulations in Section 5. Application of the new model to real data sets is provided in Section 6, followed by concluding remarks in Section 7.

2. Some Special Models

This section presents some special models of GTIIHLTL-G distribution. We considered the uniform distribution, exponential distribution, Weibull distribution, and Pareto distribution as the parent distributions.

2.1. Gamma Type Two Half Logistic Topp-Leone-Uniform

Assume the parent distribution is uniform in the interval $0 < x < \theta$. In this case, we refer to the new distribution as the Gamma Type Two Half Logistic Topp-Leone-Uniform (GTIIHLTL-U) with cdf and pdf given by

$$F_{GTIIHLTL-U}(x) = \frac{\gamma\left(\delta, -\ln\left[\frac{1-\left(1-[1-(\frac{x}{\theta})]^2\right)^b}{1+\left(1-[1-(\frac{x}{\theta})]^2\right)^b}\right]\right)}{\Gamma(\delta)}, \tag{9}$$

and

$$f_{GTIIHLTL-U}(x) = \frac{4b(\frac{1}{\theta})}{\Gamma(\delta)}\left[-\ln\left(\frac{1-\left(1-\left[1-\left(\frac{x}{\theta}\right)\right]^2\right)^b}{1+\left(1-\left[1-\left(\frac{x}{\theta}\right)\right]^2\right)^b}\right)\right]^{\delta-1} \times \frac{\left(1-\frac{x}{\theta}\right)\left(1-\left[1-\frac{x}{\theta}\right]^2\right)^{b-1}}{\left(1+\left(1-\left[1-\frac{x}{\theta}\right]^2\right)^b\right)^2}, \tag{10}$$

respectively.

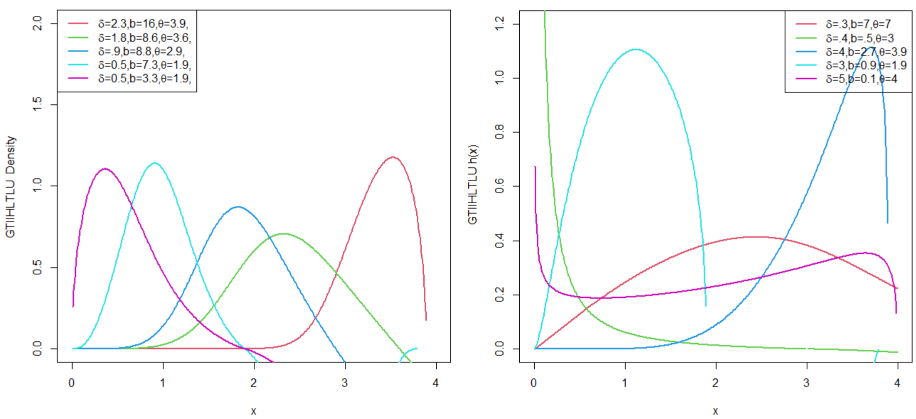

FIGURE 1: Density and Hazard Plots for the GTIIHLTL-U distribution

2.2. Gamma Type Two Half Logistic Topp-Leone-Exponential

Here we assume the parent distribution is exponential with rate $\lambda > 0$. The distribution is called the Gamma Type Two Half Logistic Topp-Leone-Exponential (GTIIHLTL-E) with cdf and pdf given by

$$F_{GTIIHLTL-E}(x) = \frac{\gamma\left(\delta, -\ln\left[\frac{1-\left(1-e^{-\lambda x}\right)^b}{1+\left(1-e^{-\lambda x}\right)^b}\right]\right)}{\Gamma(\delta)}, \tag{11}$$

and

$$f_{GTIIHLTL-E}(x) = \frac{4b\lambda e^{-2\lambda x}}{\Gamma(\delta)}\left[-\ln\left(\frac{1-\left(1-e^{-\lambda x}\right)^b}{1+\left(1-e^{-\lambda x}\right)^b}\right)\right]^{\delta-1} \times \frac{\left(1-e^{-\lambda x}\right)^{b-1}}{\left(1+\left(1-e^{-\lambda x}\right)^b\right)^2}, \tag{12}$$

respectively, where $x \geq 0$.

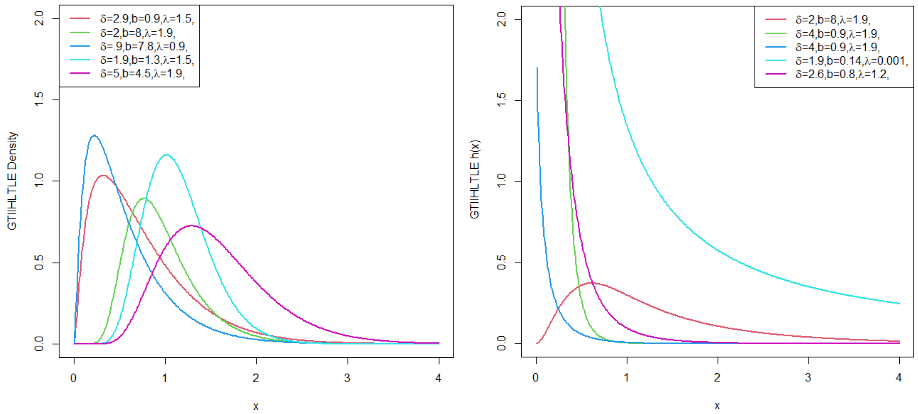

FIGURE 2: Density and Hazard Plots for the GTIIHLTL-E distribution

2.3. Gamma Type Two Half Logistic Topp-Leone-Weibull

As a third special case, assume the parent distribution is Weibull with scale parameter $\lambda > 0$ and shape parameter $\beta > 0$. Then, $G(x;\beta,\lambda) = 1 - e^{-\lambda x^{\beta}}$, where $x \geq 0$. The new distribution is called the Gamma Type Two Half Logistic Topp-Leone-Weibull (GTIIHLTL-W) distribution with cdf and pdf given by

$$F_{GTIIHLTL-W}(x) = \frac{\gamma\left(\delta, -\ln\left[\frac{1-\left(1-e^{-2\lambda x^{\beta}}\right)^{b}}{1+\left(1-e^{-2\lambda x^{\beta}}\right)^{b}}\right]\right)}{\Gamma(\delta)}, \tag{13}$$

and

$$f_{GTIIHLTL-W}(x) = \frac{4b\lambda\beta e^{-2\lambda x^{\beta}}}{\Gamma(\delta)}\left[-\ln\left(\frac{1-\left(1-e^{-2\lambda x^{\beta}}\right)^{b}}{1+\left(1-e^{-2\lambda x^{\beta}}\right)^{b}}\right)\right]^{\delta-1} \times \frac{\left(1-e^{-2\lambda x^{\beta}}\right)^{b-1}}{\left(1+\left(1-e^{-2\lambda x^{\beta}}\right)^{b}\right)^{2}}, \tag{14}$$

respectively.

2.4. Gamma Type Two Half Logistic Topp-Leone-Pareto

The final special case considers the Pareto distribution with cdf given by $G_P(x;k;\theta) = 1 - \left(\frac{\theta}{x}\right)^{k}$, where x, k and θ are positive numbers. Here the new distribution is referred to as the Gamma Type Two Half Logistic Topp-Leone-Pareto (GTIIHLTL-P) distribution with cdf and pdf given by

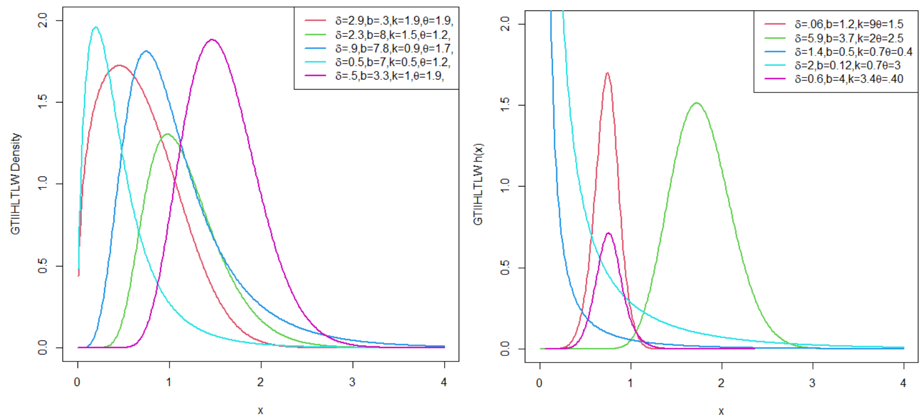

FIGURE 3: Density and Hazard Plots for the GTIIHLTL-W distribution

$$F_{GTIIHLTL-P}(x) = \frac{\gamma\left(\delta, -\ln\left[\frac{1-\left(1-\left\{\frac{\theta}{x}\right\}^{(2k)}\right)^b}{1+\left(1-\left\{\frac{\theta}{x}\right\}^{(2k)}\right)^b}\right]\right)}{\Gamma(\delta)}, \tag{15}$$

and

$$f_{GTIIHLTL-P}(x) = \frac{4bk\frac{\theta^{2k}}{x^{2k+1}}}{\Gamma(\delta)}\left[-\ln\left(\frac{1-\left(1-\left\{\frac{\theta}{x}\right\}^{(2k)}\right)^b}{1+\left(1-\left\{\frac{\theta}{x}\right\}^{(2k)}\right)^b}\right)\right]^{\delta-1} \times \frac{\left(1-\left\{\frac{\theta}{x}\right\}^{(2k)}\right)^{b-1}}{\left(1+\left(1-\left\{\frac{\theta}{x}\right\}^{(2k)}\right)^b\right)^2}, \tag{16}$$

respectively.

3. Some Statistical Properties

Some statistical properties of the new distribution including the pdf of order statistics, weighted moments and the Rényi entropy are presented in this section.

3.1. Order Statistics

Let $X_1, X_2, \ldots, X_n$ denote an independent and identically distributed random sample of size n ($n \in \mathbb{Z}^+$) from a pdf defined in (6). Furthermore, let Z_1 be the minimum of these X_i, Z_2 the next X_i in order of magnitude, ..., Z_n the largest of X_i. That is $Z_1 \leq Z_2 \leq Z_3 \leq \cdots \leq Z_n$ represent $X_1, X_2, \ldots, X_n$ when the latter are arranged in ascending order of magnitude. Then, Z_i, $i = 1, 2, 3, \ldots, n$ is called

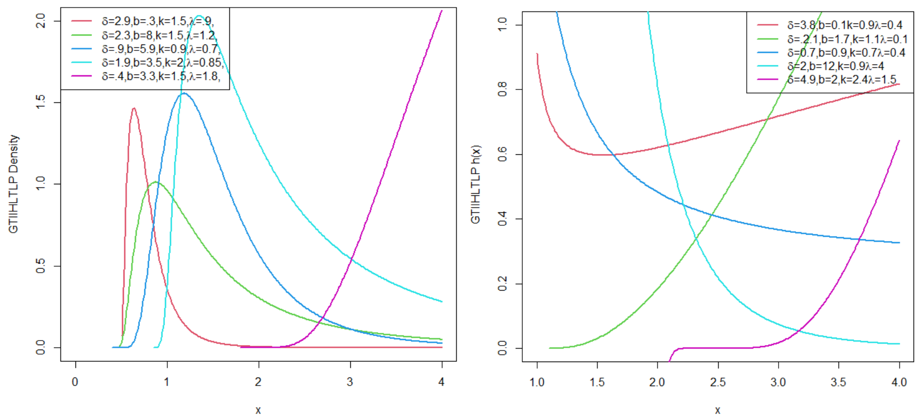

Figure 4: Density and Hazard Plots for the GTIIHLTL-P distribution

the i^{th} order statistic (Hogg et al., 2005) of the random sample $X_1, X_2, \dots, X_n$. The joint pdf of $Z_1, Z_2, Z_3, \dots, Z_n$ is

$$g(z_1, z_2, z_3, \dots, z_n) = n! f(z_1) f(z_2) f(z_3) \dots f(z_n).$$

The marginal pdf of Z_i, is given by

$$f_{i:n}(z_i) = \frac{n!}{(i-1)!(n-i)!} f(z_i) G^{i-1}(z_i)[1 - G(z_i)]^{n-i}. \tag{17}$$

As a result, the pdf of the i^{th} order statistic of the GTIIHLTL-G distribution is given by

$$\begin{aligned} f_{i:n}(z_i) &= \frac{n! f_{GTIIHLTL-G}(z_i;\xi)}{(i-1)!(n-i)!} \sum_{k=0}^{\infty} (-1)^k \binom{n-i}{k} [F_{GTIIHLTL-G}(z_i)]^{k+i-1} \\ &= \sum_{q=0}^{\infty} X^*_{w+1} g_{w+1}(z_i;\xi), \end{aligned} \tag{18}$$

where

$$\begin{aligned} X^*_{w+1} &= \frac{n!}{(i-1)!(n-i)!} \sum_{k,m,n,w,r,v,s=0}^{\infty} \left\{ \frac{b(-1)^{k+r+w} d_{m,k+i-1}}{(w+1)\Gamma(\delta)^{k+i}} 2^{\delta(k+i)+m+1+q+s} \right. \\ &\times \binom{n-i}{k} \binom{\delta(k+i)+m-1}{q} \binom{\delta(k+i)+m+q+s+v}{v} b_{s,q} \\ &\times \left. \binom{b[\delta(k+i)+m+q+s]-1+bv}{r} \binom{2r+1}{w} \right\}, \end{aligned} \tag{19}$$

is a constant and $g_{w+1}(z_i;\xi) = (w+1) g(z_i;\xi) G^w(z_i;\xi)$, is the Exp-G distribution with power parameter $w+1$ (see Appendix B for more details).

3.2. Probability Weighted Moments

The probability weighted moment (PWM) is the expectation of a function of a random variable given that the mean of the variable exists. The $(i,j)^{th}$ PWM of a random variable X for the GTIIHLTL-G family of distributions, say $\varrho_{i,j}$ is given by;

$$\varrho_{i,j} = E(X^i F(X)^j) = \int_0^\infty x^i f(x) F(x)^j dx = \nabla_1 \int_0^\infty x^i g_*(x) dx. \tag{20}$$

where $g_*(x)$ is the pdf of Exp-G distribution with some power parameter $(w+1)$,

$$\begin{aligned} \nabla_1 &= \sum_{n,w,r,v,s,q=0}^{\infty} \frac{b2^{\delta(j+1)+n+q+s+1}}{(w+1)\left[\Gamma(\delta)\right]^{j+1}} d_{n,j} b_{s,q} \binom{\delta(j+1)+n-1}{q} \binom{2r+1}{w} \\ &\times \binom{\delta(j+1)+n+q+s+v}{v} \binom{b\left[\delta(j+1)+n+q+s\right]+bv-1}{r}, \end{aligned}$$

is a constant, $d_{0,j} = c_0^{(j)}$, $d_{n,j} = (n_{c_0})^{-1} \sum_{z=1}^{\infty} [(j)z - n + z] c_z d_{n-z}, j$ and $c_n = \frac{(-1)^n}{n!(n+\delta)}$ (see Makubate, Rannona, Oluyede & Chipepa (2021) and references contained in it). As a result, the probability weighted moments of the GTIIHLTL-G family of distributions can be easily obtained from the r^{th} moments of the Exp-G family.

3.3. Rényi Entropy

Since its introduction in thermodynamics, Rényi entropy has been used in many fields, including statistics, chemistry, physics, information theory, and computer theory. The Rényi entropy generalizes a number of entropy concepts, including Hartley entropy (Aczél et al., 1974), collision entropy (Bosyk et al., 2012), min-entropy (Kim et al., 2021), and Shannon entropy (Shannon, 1948). The Rényi entropy of a random variable X with pdf $f(x)$ is expressed as

$$I_R(v) = (1-v)^{-1} \log \left[\int_{-\infty}^{\infty} f^v(x) dx \right], \tag{21}$$

where $v \geq 0$ and $v \neq 1$. The value of v plays a crucial rule in controlling the sensitivity of $I_R(v)$ to different parts of the probability distribution. As $v \to 0$, $I_R(v)$ highlights the contribution of the most likely outcomes, whereas, $I_R(v)$ increases the weight of less likely outcomes in the distribution, as v moves away from zero.

Following equation (21), the Rényi entropy of the GTIIHLTL-G distribution can be expressed as

$$\begin{aligned} f^{v}(x) &= \frac{(4b)^{v} g^{v}(x;\xi)}{(\Gamma(\delta))^{v}} \left[-\ln \left(\frac{1-\left(1-\overline{G}^{2}(x;\xi)\right)^{b}}{1+\left(1-\overline{G}^{2}(x;\xi)\right)^{b}} \right) \right]^{v(\delta-1)} \\ &\times \left(1-\overline{G}^{2}(x;\xi)\right)^{v(b-1)} \overline{G}^{v}(x;\xi) \left(1+\left(1-\overline{G}^{2}(x;\xi)\right)^{b}\right)^{-2v} \\ &= (v-1)^{-1} \log \left[\sum_{v,d=0}^{\infty} \varphi^{*}_{d+1} e^{(1-v)IREG} \right], \end{aligned} \tag{22}$$

where,

$$\begin{aligned} \varphi^{*}_{d+1} &= \frac{(4b)^{v}}{(\Gamma(\delta))^{v}} \sum_{d,z,k,j,s=0}^{\infty} \binom{v(\delta-1)}{j} b_{s,j} 2^{v(\delta-1)+j+s} (-1)^{z+d} \\ &\times \binom{v(\delta+1)+j+s+k-1}{k} \binom{b[v(\delta)+j+s+k]-v}{z} \\ &\times \binom{2z+v}{d} \left(\frac{d}{v}+1\right), \end{aligned}$$

is a constant, and $I_{REG} = \int_{-\infty}^{\infty} \left[\left(\frac{d}{v}+1\right) g(x;\xi) G(x;\xi)^{\frac{d}{v}} \right]^{v} dx$ is the Rényi entropy of the Exp-G distribution with power parameter $\frac{d}{v}+1$ (see Appendix C for the proof).

4. Parameter Estimation

In this section, the maximum likelihood estimators (MLE's) of the GTIIHLTL-G distribution's unknown parameters were generated.

4.1. Estimation of Parameters for Uncensored Data

Let $X_1, X_2, \dots, X_n$ be a random sample of size n from the GTIIHLTL-G distribution. The log-likelihood function is given by;

$$\begin{aligned} \ell &= n\log(4b) - n\log(\Gamma(\delta)) + \sum_{k=0}^{n} \log g(x;\xi) + \sum_{k=0}^{n} \log(1-G(x;\xi)) \\ &+(\delta-1)\sum_{k=0}^{n} \log\left[-\ln\left(1-(1-\overline{G}^{2}(x;\xi))^{b}\right) + \ln\left(1+(1-\overline{G}^{2}(x;\xi))^{b}\right)\right] \\ &+(b-1)\sum_{k=0}^{n} \log(1-\overline{G}^{2}(x;\xi)) - 2\sum_{k=0}^{n} \log\left(1+(1-\overline{G}^{2}(x;\xi))^{b}\right) \end{aligned}$$

The first derivatives of the log-likelihood function with respect to each parameter vector $\triangle = (\delta; b; \xi)^T$, are given in Appendix D. The MLE's are obtained by solving the system of non-linear equations

$$\left(\frac{\partial \ell}{\partial b}, \frac{\partial \ell}{\partial \delta}, \frac{\partial \ell}{\partial \xi}\right)^T = \mathbf{0}. \tag{23}$$

The non-linear equations in (23), also referred to as the score equations can be solved using either $R^{©}$, $MATLAB^{©}$, or $SAS^{©}$ software.

4.2. Estimation of Parameters for Censored Data

Here we construct the GTIIHLTL-G distribution log-likelihood functions to handle type I right censored data. When an experiment has a fixed number of participants or items and ends at a certain time, type I censorship takes place, and any subjects that are still present are right-censored. Let the lifetime of the first r failed items be $x_1, x_2, \ldots, x_r$, the censoring be $x_{r+1}, x_{r+2}, \ldots, x_n$ and $\Delta = [\delta, b, \xi]^T$ be the vector of parameters. The likelihood function of type I censoring is given by;

$$\begin{aligned}
\ell(\Delta) &= n\log(4b) - n\log(\Gamma(\delta)) + \sum_{k=0}^{n}\log g(x;\xi) + \sum_{k=0}^{n}\log(1 - G(x;\xi)) \\
&+ (\delta - 1)\sum_{k=0}^{n}\log\left[-\ln\left(1 - (1 - \bar{G}^2(x;\xi))^b\right) + \ln\left(1 + (1 - \bar{G}^2(x;\xi))^b\right)\right] \\
&+ (b - 1)\sum_{k=0}^{n}\log(1 - \bar{G}^2(x;\xi)) - 2\sum_{k=0}^{n}\log\left(1 + (1 - \bar{G}^2(x;\xi))^b\right) \\
&+ (n - r)\log\left(1 - \frac{\gamma\left(\delta, -\ln\left[\frac{1-\left(1-\bar{G}^2(x;\xi)\right)^b}{1+\left(1-\bar{G}^2(x;\xi)\right)^b}\right]\right)}{\Gamma(\delta)}\right).
\end{aligned} \tag{24}$$

The components of the of the score vector $\mathrm{U} = \left(\frac{\partial \ell}{\partial \delta}, \frac{\partial \ell}{\partial b}, \frac{\partial \ell}{\partial \xi}\right)$ are provided in Appendix D. To obtain the likelihood estimates, we therefore have to equate the score vector to zero and solve for δ, b, and ξ.

5. Monte Carlo Simulation

In this section, a simulation exercise was carried out using the R program (stats4) to evaluate the convergence of the MLEs of the GTIIHLTL-G with respect to the sample size. Using the GTIIHLTL-W distribution, 1000 samples of sizes $n = 25$, 50, 100, 200, 600 and 800 were iteratively simulated. For every one of the 1000 replications, the MLEs are determined. For the MLE of the parametor β, say

$\hat{\beta}$, the average bias (Bias) and the root mean square error (RMSE) are calculated by

$$Bias(\hat{\beta}) = \frac{\sum_{i=1}^{1000} \hat{\beta}_i}{1000} - \beta \quad \text{and} \quad RMSE(\hat{\beta}) = \sqrt{\frac{\sum_{i=1}^{1000} (\hat{\beta}_i - \beta)^2}{1000}},$$

respectively. The simulation results are shown in Table 1. It is evident from the results that the root mean square error (RMSE) and average bias values are decreasing as the sample size (n) increases, demonstrating that convergence has been achieved.

Table 1: Simulation results

		(0.9, 1.1, 1.1, 0.9)			(0.9, 0.9, 1.1, 0.9)		
parameter	Sample size	MLE	RMSE	Bias	MLE	RMSE	Bias
δ	25	1.6932	1.47150	0.7932	1.6336	1.3739	0.7336
	50	1.5545	1.29397	0.6545	1.4558	1.1508	0.5558
	100	1.4032	1.13091	0.5032	1.3303	1.0009	0.4303
	200	1.3592	1.02555	0.4592	1.3008	0.9085	0.4008
	600	1.2133	0.82677	0.3133	1.2294	0.7795	0.3294
	800	1.1693	0.76370	0.2693	1.2379	0.8095	0.3379
b	25	32.5761	234.48203	31.4761	20.2965	209.5298	19.3965
	50	3.0086	11.76476	1.9086	2.1802	8.1179	1.2802
	100	1.7898	2.03925	0.6898	1.4148	1.5343	0.5148
	200	1.4806	1.43223	0.3806	1.2025	1.1540	0.3025
	600	1.2565	0.81309	0.1565	0.9984	0.6236	0.0984
	800	1.2139	0.67768	0.1139	0.9748	0.5572	0.0748
β	25	1.2708	1.34655	0.1708	1.1467	1.0350	0.0467
	50	1.0571	0.68680	-0.0429	1.0570	0.6791	-0.0430
	100	1.0322	0.46899	-0.0678	1.0304	0.4455	-0.0696
	200	1.0188	0.33866	-0.0812	1.0219	0.3554	-0.0781
	600	1.0189	0.19861	-0.0811	1.0071	0.2074	-0.0929
	800	1.0342	0.18041	-0.0658	1.0103	0.2010	-0.0897
λ	25	1.9314	1.78175	1.0314	1.8883	1.6462	0.9883
	50	1.5737	1.18977	0.6737	1.5221	1.0823	0.6221
	100	1.3741	0.90211	0.4741	1.3336	0.8181	0.4336
	200	1.2750	0.71657	0.3750	1.2496	0.6588	0.3496
	600	1.1439	0.53848	0.2439	1.1613	0.5281	0.2613
	800	1.1080	0.49271	0.2080	1.1649	0.5496	0.2649

6. Application

In this section, we consider two real data sets and fit the GTIIHLTL-W distribution. Futhermore, the GTIIHLTL-W fits were compared with some competitive distributions with the same number of parameters, like the Top-Leone-Weibull-Poisson (TLWP) by Handique et al. (2023), the Exponentiated Log Logistic Weibull (ELLOW) by Fagbamigbe et al. (2019), the Exponentiated Weibull Lomax (EWL) by Hassan & Abd-Allah (2018), the Weibull Lomax (WL) by Tahir et al. (2015). Model performance was assessed based on goodness-of-fit statistics, namely, -2 log likelihood (-2 log L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Cramer-Von Mises (W), and Andersen-Darling (A), as described by Chen & Balakrishnan (1995).

6.1. Uncensored Data

The distributions were fitted to the data set taken from (Henze & Klar, 2002) consisting of precipitation (inches) from Jug Bridge, Maryland. Table 2 shows the MLEs parameters with standard errors in parenthesis of the models for the precipitation data. The estimated variance-covariance matrix is provided in Appendix E. Consequently, the estimated 95% confidence intervals for the model parameters are given by $\delta \in [3.3574 \pm 4.4145]$, $b \in [3950 \pm 1.9295 \times 10^{-5}]$, $\beta \in [0.2756 \pm 0.1269]$ and $\lambda \in [4.2904 \pm 0.7691]$.

TABLE 2: Parameter estimates for various models fitted for precipitation data set.

Model	Estimates			
	δ	b	β	λ
$GTIIHLTL-W$	3.3574	39507	0.2756	4.2904
	(2.2523)	(9.8449×10^{-6})	(0.0647)	(0.3924)
	α	β	b	θ
$TLWP$	0.0980	1.1842	1.9107	2.5732×10^{-8}
	(0.0469)	(0.1934)	(0.5689)	(0.0224)
	β	λ	θ	γ
$ELLOLW$	3.0800×10^{-8}	0.0580	8.1202	1.9932
	(2.2177)	(0.0109)	(3.9295×10^{-5})	(0.2437)
	a	b	α	β
WL	1.1366	4.1583	0.0553	1.1272
	(1.8234×10^{-6})	(2.3454)	(0.0179)	(2.3298)
	α	v	a	b
EWL	9.3726×10^{-4}	379.9600	83.6020	0.2297
	(6.8786×10^{-4})	(2.7469×10^{-6})	(1.2506×10^{-6})	(0.0290)

Table 3 presents the goodness-of-fit statistics, and as shown by the lowest goodness-of-fit statistics and the highest Kolmogorov-Smirnov (K-S) P value, we conclude that the proposed model provides the best overall fit to the precipitation data set.

TABLE 3: Goodness-of-fit statistics for various models fitted for precipitation data set.

	Statistics							
Model	$-2\log L$	AIC	CAIC	BIC	HQIC	W^*	A^*	P-value
$GTIIHLTL-W$	154.26	162.26	163.59	168.48	164.41	0.0324	0.2222	0.9696
$TLWP$	165.00	173.00	174.33	179.22	175.15	0.1027	0.6658	0.3243
$ELLOLW$	164.98	172.98	174.31	179.20	175.13	0.1638	1.0262	0.4499
WL	160.99	168.99	170.32	175.21	171.14	0.1087	0.7003	0.667
EWL	155.78	163.78	165.11	170.00	165.93	0.0390	0.2827	0.9429

Furthermore, Figures 5, 6 and 7 make it evidently clear that the GTIIHLTL-W distribution offers a better fit and effectively captures the precipitation data set. The histogram demonstrates how well the new model can handle extreme-tailed data sets. The Kaplan-Meier and CDF curves demonstrate that the GTIIHLTL-W model performs satisfactorily. Lastly, the fitted hrf exhibits a declining shape.

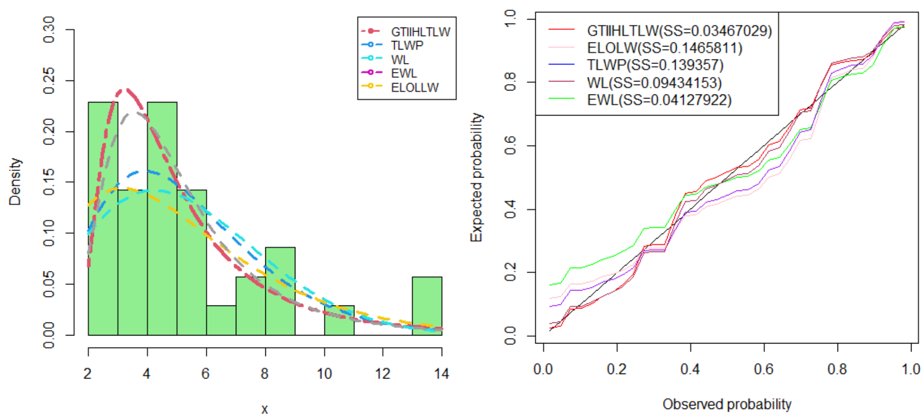

Figure 5: Fitted densities and probability plots for precipitation data

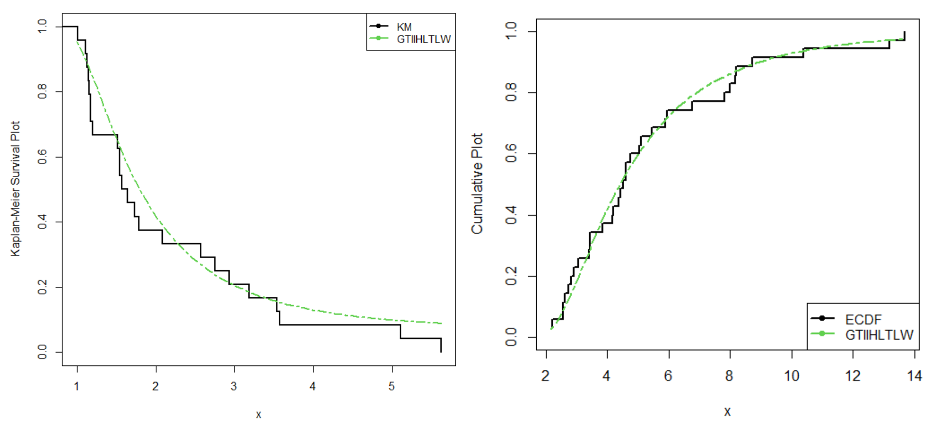

Figure 6: Fitted Kaplan-Meier and CDF plots for precipitation data

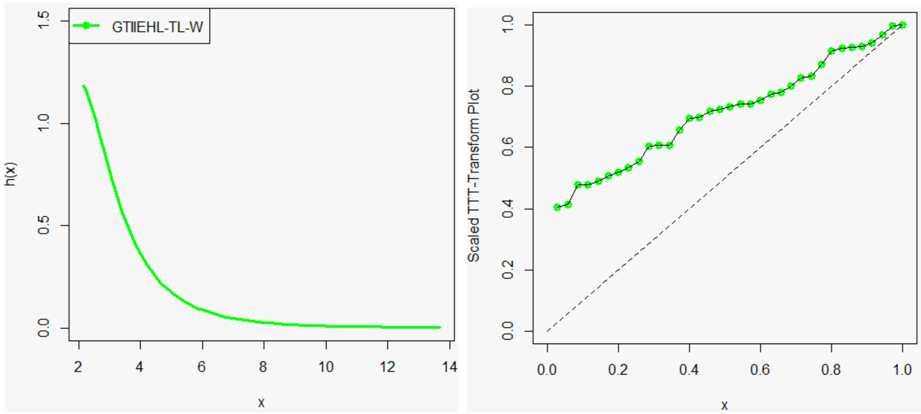

FIGURE 7: Fitted HRF and TTT plots for precipitation data

6.2. Censored Data

The Head and Neck Cancer data set taken from (Elgarhy et al., 2018) consisting of survival times (in days) for the patients in Arm A of the Head-and-Neck Cancer Trial, was used to fit the models. Table 4 shows the parameter values and standard errors in parenthesis, and Appendix E shows the estimated variance-covariance matrix for the GTIIHLTL-W model on head and neck cancer data. As a result, the estimated 95% confidence intervals for the model parameters are given by $\delta \in [0.6768 \pm 1.0028]$, $b \in [6.6670 \pm 18.5719]$, $\beta \in [0.5529 \pm 0.5373]$ and $\lambda \in [0.0410 \pm 0.1646]$.

TABLE 4: Parameter estimates for various models fitted for the cancer data set.

Model	Estimates			
	δ	b	β	λ
$GTIIHLTL-W$	0.6768	6.6670	0.5529	0.0410
	(0.5116)	(9.4754)	(0.2741)	(0.0839)
	α	β	b	θ
$TLWP$	0.0365	0.5224	1.4562	8.2074×10^{-10}
	(0.0264)	(0.1060)	(0.3525)	(0.0034)
	β	λ	θ	γ
$ELLOLW$	1.9582×10^{-6}	1.6685×10^{-3}	1.9841	1.0466
	(0.5088)	(4.7693×10^{-4})	(2.9392×10^{-6})	(0.1158)
	a	b	α	β
WL	686.3300	1.9756	0.0256	89.5330
	(7.0285×10^{-6})	(0.2163)	(8.6443×10^{-3})	(1.2162×10^{-3})
	α	v	a	b
EWL	7.3220×10^{-5}	6.3238	22.5230	0.2506
	(3.8604×10^{-5})	(4.0806×10^{-4})	(5.1841×10^{-6})	(0.0284)

Table 5, presents the goodness of fit measures, and according to the findings, we note that the GTIIHLTL-W has the highest P-value and the lowest goodness-of-fit statistics when compared to the competing models. As a result, we conclude that the GTIIHLTL-W model outperforms the competing models.

Table 5: Goodness-of-fit statistics for various models fitted for the cancer data set.

Model	Statistics							
	$-2\log\ L$	AIC	CAIC	BIC	HQIC	W^*	A^*	P-value
$GTIIHLTL-W$	571.51	579.51	580.56	586.55	582.11	0.1136	0.6371	0.5266
$TLWP$	592.50	600.50	601.55	607.55	603.10	0.1622	0.9101	0.0037
$ELLOLW$	577.79	585.79	586.84	592.83	588.39	0.2509	1.4265	0.2323
WL	572.32	580.32	581.37	587.36	582.92	0.1314	0.7332	0.4496
EWL	572.47	580.47	581.52	587.51	583.06	0.1283	0.7215	0.4705

Additionally, Figures 8, 9 and 10 demonstrate that our proposed model offers a better fit to the data than the competing distributions when dealing with right-skewed data.

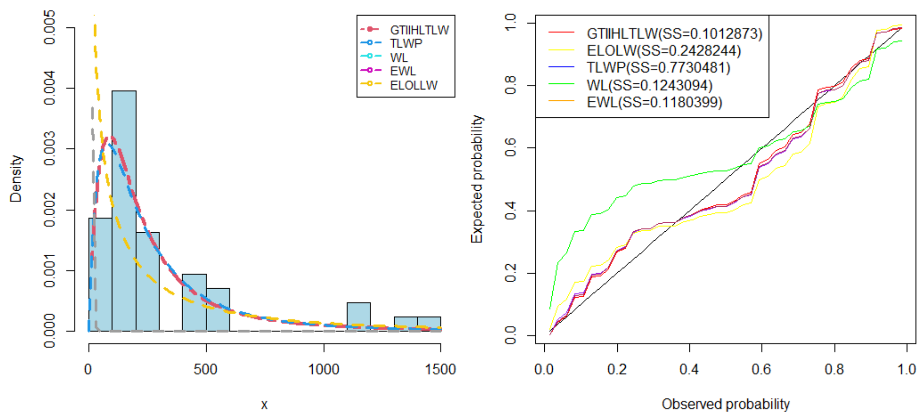

Figure 8: Fitted densities and probability plots for the cancer data

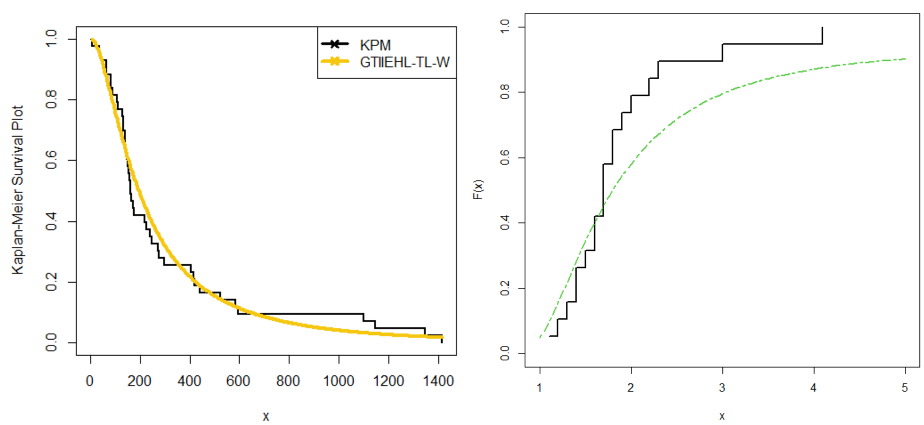

Figure 9: Fitted Kaplan-Meier and CDF plots for the cancer data

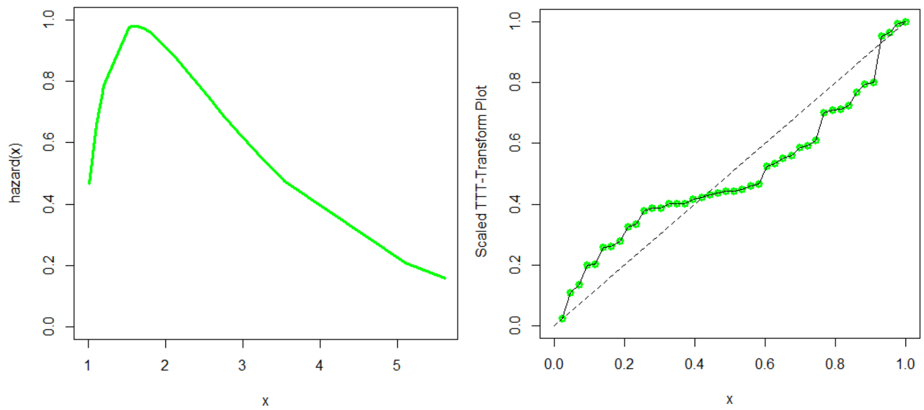

FIGURE 10: Fitted HRF and TTT plots for the cancer data

7. Concluding Remarks

The Gamma Type II Half Logistic Topp-Leone-G is a new family of distributions that has been proposed in this article. As an endless linear combination of the exponentiated-G family of distributions, this new family of distributions can be written. As a result, certain crucial statistical characteristics of the new distribution, including pdf of the i^{th} order statistic, probability weighted moments and the Rényi entropy can be readily deduced from the statistical properties of the exponentiated-G distribution. The new distribution can be used to model data with skewed to the right or left, reverse-J, J, and almost symmetric failure rates (Figures 1(a), 2(a), 3(a), 4(a)). Also, the new distribution can model data with uni-modal, declining, rising, and bathtub-like hazard rate function (Figures 1(b), 2(b), 3(b), 4(b)). The Gamma Type II Half Logistic Topp-Leone-Weibull, a particular example of the new distribution, was applied to censored and uncensored datasets, and it is clear from the findings that the proposed model outperforms numerous other models with the same number of parameters. We come to the conclusion that both complete and censored survival data can be analyzed using the Gamma Type II Half Logistic Topp-Leone-G.

[Received: October 2023 — Accepted: December 2023]

References

Aczél, J., Forte, B. & Ng, C. T. (1974), 'Why the shannon and hartley entropies are 'natural'', *Advances in Applied Probability* **6**(1), 131–146.

Adamidis, K. & Loukas, S. (1998), '*A lifetime distribution with decreasing failure rate*', *Statistics & Probability Letters* **39**(1), 35–42.

Al-Shomrani, A., Arif, O., Shawky, A., Hanif, S. & Shahbaz, M. Q. (2016), '*Toppâ*

Leone Family of Distributions: Some Properties and Application', *Pakistan Journal of Statistics and Operation Research* pp. 443–451.

Alzaatreh, A., Famoye, F. & Lee, C. (2014), '*The gamma-normal distribution: Properties and applications*', *Computational Statistics & Data Analysis* **69**, 67–80.

Alzaatreh, A., Lee, C. & Famoye, F. (2013), '*A new method for generating families of continuous distributions*', *Metron* **71**(1), 63–79.

Balakrishnan, N. (1985), '*Order statistics from the half logistic distribution*', *Journal of statistical computation and simulation* **20**(4), 287–309.

Bantan, R. A., Jamal, F., Chesneau, C. & Elgarhy, M. (2020), '*Type II Power Topp-Leone generated family of distributions with statistical inference and applications*', *Symmetry* **12**(1), 75.

Bosyk, G. M., Portesi, M. & Plastino, A. (2012), 'Collision entropy and optimal uncertainty', *Physical Review A* **85**, 012108. https://link.aps.org/doi/10.1103/PhysRevA.85.012108

Chamunorwa, S., Oluyede, B., Makubate, B. & Chipepa, F. (2021), 'The exponentiated odd Weibull-Topp-Leone-G family of distributions: Model, properties and applications'.

Chen, G. & Balakrishnan, N. (1995), '*A general purpose approximate goodness-of-fit test*', *Journal of Quality Technology* **27**(2), 154–161.

Chipepa, F., Oluyede, B., Makubate, B. et al. (2020), '*The Topp-Leone-Marshall-Olkin-G family of distributions with applications*', *International Journal of Statistics and Probability* **9**(4), 15–32.

Cordeiro, G. M., Ortega, E. M. & da Cunha, D. C. (2013), '*The exponentiated generalized class of distributions*', *Journal of Data Science* **11**(1), 1–27.

Elgarhy, M., Arslan Nasir, M., Jamal, F. & Ozel, G. (2018), '*The type II Topp-Leone generated family of distributions: Properties and applications*', *Journal of Statistics and Management Systems* **21**(8), 1529–1551.

Fagbamigbe, A. F., Basele, G. K., Makubate, B. & Oluyede, B. O. (2019), '*Application of the Exponentiated Log-Logistic Weibull Distribution to Censored Data*', *Journal of the Nigerian Society of Physical Sciences* pp. 12–19.

Foya, S., Oluyede, B. O., Fagbamigbe, A. F. & Makubate, B. (2017), '*The gamma log-logistic Weibull distribution: model, properties and application*', *Electronic Journal of Applied Statistical Analysis* **10**(1), 206–241.

Handique, L., Chakraborty, S. & Ali, M. M. (2023), '*The Topp-Leone-G Power Series Distribution*', *G Families of Probability Distributions: Theory and Practices* p. 132.

Hassan, A. S. & Abd-Allah, M. (2018), '*Exponentiated Weibull-Lomax distribution: properties and estimation*', *Journal of Data Science* **16**(2), 277–298.

Henze, N. & Klar, B. (2002), '*Goodness-of-fit tests for the inverse Gaussian distribution based on the empirical Laplace transform*', *Annals of the Institute of Statistical Mathematics* **54**, 425–444.

Hogg, R., McKean, J. & Craig, A. (2005), *Introduction to Mathematical Statistics*, Pearson education international, Pearson Education. https://books.google.co.bw/books?id=vIEZAQAAIAAJ

Jones, M. C. (2004), '*Families of distributions arising from distributions of order statistics*', *Test* **13**, 1–43.

Keganne, K., Gabaitiri, L. & Makubate, B. (2023), 'A new extension of the exponentiated generalized-G family of distributions', *Scientific African* **20**, e01719. https://www.sciencedirect.com/science/article/pii/S2468227623001758

Kim, Y., Guyot, C. & Kim, Y.-S. (2021), 'On the efficient estimation of min-entropy', *IEEE Transactions on Information Forensics and Security* **16**, 3013–3025.

Makubate, B., Oluyede, B. & Gabanakgosi, M. (2021), '*A New Lindley-Burr XII Distribution: Model, Properties and Applications*', *International Journal of Statistics and Probability* **10**(4), 33–51.

Makubate, B., Rannona, K., Oluyede, B. & Chipepa, F. (2021), 'The topp leone-g power series class of distributions with applications', *Pakistan Journal of Statistics and Operation Research* **17**, 827–846.

Marshall, A. W. & Olkin, I. (1997), '*A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families*', *Biometrika* **84**(3), 641–652.

Nadarajah, S. & Kotz, S. (2003), '*Moments of some J-shaped distributions*', *Journal of Applied Statistics* **30**(3), 311–317.

Oluyede, B. O., Makubate, B., Wanduku, D., Elbatal, I. & Sherina, V. (2017), '*The gamma-generalized inverse Weibull distribution with applications to pricing and lifetime data*', *Journal of Computations & Modelling* **7**(2), 1.

Oluyede, B., Pu, S., Makubate, B. & Qiu, Y. (2018), '*The gamma-Weibull-G family of distributions with applications*', *Austrian Journal of Statistics* **47**(1), 45–76.

Ristić, M. M. & Balakrishnan, N. (2012), '*The gamma-exponentiated exponential distribution*', *Journal of Statistical Computation and Simulation* **82**(8), 1191–1206.

Sangsanit, Y. & Bodhisuwan, W. (2016), '*The Topp-Leone generator of distributions: properties and inferences*', *Songklanakarin Journal of Science & Technology* **38**(5).

Shannon, C. E. (1948), 'A mathematical theory of communication', *The Bell System Technical Journal* **27**(3), 379–423.

Soliman, A. H., Elgarhy, M. A. E. & Shakil, M. (2017), '*Type II half logistic family of distributions with applications*', *Pakistan Journal of Statistics and Operation Research* pp. 245–264.

Tahir, M. H., Cordeiro, G. M., Mansoor, M. & Zubair, M. (2015), '*The Weibull-Lomax distribution: properties and applications*', *Hacettepe Journal of Mathematics and Statistics* **44**(2), 455–474.

Topp, C. W. & Leone, F. C. (1955), '*A family of J-shaped frequency functions*', *Journal of the American Statistical Association* **50**(269), 209–219.

Zografos, K. & Balakrishnan, N. (2009), '*On families of beta-and generalized gamma-generated distributions and associated inference*', *Statistical Methodology* **6**(4), 344–362.

Appendix A. Linear Representation of the pdf

The pdf of the-G family of distributions is expressed as an infinite linear combination of exponentiated-G densities using the Power series expansion for logarithm function.

$$\begin{aligned} f_{GTIIHLTL-G}(x;\xi) &= \left[-\ln\left(\frac{1-\left(1-\overline{G}^2(x,\xi)\right)^b}{1+\left(1-\overline{G}^2(x,\xi)\right)^b}\right)\right]^{\delta-1}\frac{4bg(x;\xi)}{\Gamma(\delta)} \\ &\quad \left(1-\overline{G}^2(x,\xi)\right)^{b-1}\overline{G}(x;\xi)\left(1+\left(1-\overline{G}^2(x;\xi)\right)^b\right)^{-2} \\ &= \left[-\ln\left(1-\left\{1-\frac{1-\left(1-\overline{G}^2(x,\xi)\right)^b}{1+\left(1-\overline{G}^2(x,\xi)\right)^b}\right\}\right)\right]^{\delta-1}\frac{4bg(x;\xi)}{\Gamma(\delta)} \\ &\quad \overline{G}(x;\xi)\left(1-\overline{G}^2(x,\xi)\right)^{b-1}\left(1+\left(1-\overline{G}^2(x;\xi)\right)^b\right)^{-2}. \end{aligned}$$

Let

$$\begin{aligned} y &= \left\{1-\frac{1-\left(1-\overline{G}^2(x,\xi)\right)^b}{1+\left(1-\overline{G}^2(x,\xi)\right)^b}\right\} \\ &= \frac{2\left(1-\overline{G}^2(x,\xi)\right)^b}{1+\left(1-\overline{G}^2(x,\xi)\right)^b} \end{aligned}$$

Using expansion $-\ln(1-y)=\sum_{k=0}^{\infty}\frac{y^{i+1}}{i+1}$ we have

$$(-\ln(1-y))^{\delta-1}=y^{\delta-1}\sum_{j=0}^{\infty}\binom{\delta-1}{j}y^j\left[\sum_{s=0}^{\infty}\frac{y^s}{s+2}\right]^j$$

Applying the results on a power series raised to a positive integer with $a_s=(s+2)^{-1}$ we obtain $\sum_{s=0}^{\infty}(a_sy_s)^j=\sum_{s=0}^{\infty}b_{s,j}y^s$, where the constants $b_{s,n}$ and $b_{0,j}$ are define as in (Makubate, Rannona, Oluyede & Chipepa, 2021).

$$\therefore (-\ln(1-y))^{\delta-1} = \sum_{j,s=0}^{\infty}\binom{\delta-1}{j}y^{\delta+j+s-1}b_{s,j} \tag{25}$$

Therefore the $f_{GTIIHLTL-G}(x;\xi)$ is represented as follows :

$$
\begin{aligned}
&= \frac{4bg(x;\xi)}{\Gamma(\delta)} \sum_{j,s=0}^{\infty} \left[\frac{2\left(1-\overline{G}^2(x,\xi)\right)^b}{1+\left(1-\overline{G}^2(x,\xi)\right)^b} \right]^{\delta+j+s-1} b_{s,j} \binom{\delta-1}{j} \overline{G}(x;\xi) \\
&\quad \left(1-\overline{G}^2(x,\xi)\right)^{b-1} \left(1+\left(1-\overline{G}^2(x;\xi)\right)^b\right)^{-2} \\
&= \frac{4bg(x;\xi)}{\Gamma(\delta)} \sum_{j,s=0}^{\infty} 2^{(\delta+j+s-1)} b_{s,j} \binom{\delta-1}{j} \overline{G}(x;\xi) \left(1-\overline{G}^2(x,\xi)\right)^{b[\delta+j+s]-1} \\
&\quad \left(1+\left(1-\overline{G}^2(x;\xi)\right)^b\right)^{-(\delta+j+s+1)} \\
&= \frac{4bg(x;\xi)}{\Gamma(\delta)} \sum_{p,j,s=0}^{\infty} 2^{(\delta+j+s-1)} b_{s,j} \binom{\delta-1}{j} \binom{\delta+j+s+p}{p} \overline{G}(x;\xi) \\
&\quad \left(1-\overline{G}^2(x,\xi)\right)^{b[\delta+j+s+p]-1} \\
&= \frac{4bg(x;\xi)}{\Gamma(\delta)} \sum_{q,p,j,s=0}^{\infty} 2^{(\delta+j+s-1)} (-1)^q b_{s,j} \binom{\delta-1}{j} \binom{\delta+j+s+p}{p} \\
&\quad \binom{b[\delta+j+s+p]-1}{q} \left(1-G(x,\xi)\right)^{2q+1} \\
&= \frac{4bg(x;\xi)}{\Gamma(\delta)} \sum_{m,q,p,j,s=0}^{\infty} 2^{(\delta+j+s-1)} (-1)^{q+m} b_{s,j} \binom{\delta-1}{j} \binom{\delta+j+s+p}{p} \\
&\quad \binom{b[\delta+j+s+p]-1}{q} \binom{2q+1}{m} G^m(x,\xi) \\
&= \sum_{m=0}^{\infty} Q^*_{m+1} h^*_{m+1}(x;\xi) \qquad (26)
\end{aligned}
$$

where

$$
h^*_{m+1}(x;\xi) = (m+1)g(x;\xi)G^m(x;\xi),
$$

and

$$
\begin{aligned}
Q^*_{m+1} = \frac{b}{\Gamma(\delta)} \sum_{q,p,j,s=0}^{\infty} \frac{2^{(\delta+j+s+1)}(-1)^{p+m}}{m+1} b_{s,j} \binom{\delta-1}{j} \binom{\delta+j+s+p}{p} \\
\binom{b[\delta+j+s+p]-1}{q} \binom{2q+1}{m}.
\end{aligned}
$$

Appendix B. The pdf of the i^{th} order statistic

Without loss of generality we let $x = z_i$. The pdf of the i^{th} order statistic of the GTIIHLTL-G distribution is given by

$$\begin{aligned} f_{i:n}(x) &= \frac{n! f_{GTIIHLTL-G}(x;\xi)}{(i-1)!(n-i)!}\sum_{k=0}^{\infty}(-1)^k\binom{n-i}{k}[F_{GTIIHLTL-G}(x)]^{k+i-1} \\ &= \frac{n!4bg(x;\xi)}{(i-1)!(n-i)![\Gamma(\delta)]^{k+i}}\sum_{k=0}^{\infty}(-1)^k\binom{n-i}{k} \\ &\quad \frac{\overline{G}(x;\xi)\left(1-\overline{G}^2(x;\xi)\right)^{b-1}}{\left(1+\left(1-\overline{G}^2(x;\xi)\right)^b\right)^2}\left[-\ln\left(\frac{1-\left(1-\overline{G}^2(x;\xi)\right)^b}{1+\left(1-\overline{G}^2(x;\xi)\right)^b}\right)\right]^{\delta-1} \\ &\quad \left[\gamma\left(\delta, -\ln\left[\frac{1-\left(1-\overline{G}^2(x;\xi)\right)^b}{1+\left(1-\overline{G}^2(x;\xi)\right)^b}\right]\right)\right]^{k+i-1} \end{aligned}$$

For the incomplete gamma function $\gamma(\delta; x)$, we apply the following power series (Oluyede et al., 2018)

$$\gamma(\delta;x) = \sum_{m=0}^{\infty}\frac{(-1)^m x^{m+\delta}}{(m+\delta)m!} = \sum_{m=0}^{\infty}\frac{(-1)^m\left(-\ln\left[\frac{1-\left(1-\overline{G}^2(x;\xi)\right)^b}{1+\left(1-\overline{G}^2(x;\xi)\right)^b}\right]\right)^{m+\delta}}{(m+\delta)m!}. \tag{27}$$

$$\begin{aligned} \Longrightarrow f_{i:n}(x) &= \frac{n!4bg(x;\xi)}{(i-1)!(n-i)![\Gamma(\delta)]^{k+i}}\sum_{k=0}^{\infty}(-1)^k\binom{n-i}{k} \\ &\quad \frac{\overline{G}(x;\xi)\left(1-\overline{G}^2(x,\xi)\right)^{b-1}}{\left(1+\left(1-\overline{G}^2(x;\xi)\right)^b\right)^2}\left[-\ln\left(\frac{1-\left(1-\overline{G}^2(x;\xi)\right)^b}{1+\left(1-\overline{G}^2(x;\xi)\right)^b}\right)\right]^{\delta-1} \\ &\quad \left[\sum_{m=0}^{\infty}\frac{(-1)^m\left(-\ln\left[\frac{1-\left(1-\overline{G}^2(x,\xi)\right)^b}{1+\left(1-\overline{G}^2(x;\xi)\right)^b}\right]\right)^{m+\delta}}{(m+\delta)m!}\right]^{k+i-1} \\ &= \frac{n!4bg(x;\xi)}{(i-1)!(n-i)![\Gamma(\delta)]^{k+i}}\sum_{k=0}^{\infty}(-1)^k\binom{n-i}{k}\frac{\overline{G}(x;\xi)\left(1-\overline{G}^2(x;\xi)\right)^{b-1}}{\left(1+\left(1-\overline{G}^2(x;\xi)\right)^b\right)^2} \\ &\quad \left(-\ln\left[\frac{1-\left(1-\overline{G}^2(x;\xi)\right)^b}{1+\left(1-\overline{G}^2(x;\xi)\right)^b}\right]\right)^{\delta(k+i)-1} \\ &\quad \left[\sum_{m=0}^{\infty}\frac{(-1)^m\left(-\ln\left[\frac{1-\left(1-\overline{G}^2(x;\xi)\right)^b}{1+\left(1-\overline{G}^2(x;\xi)\right)^b}\right]\right)^{m}}{(m+\delta)m!}\right]^{k+i-1}, \end{aligned}$$

let $c_m = \frac{(-1)^m}{m!(m+\delta)}$ and using the result on a power series raised to a positive integer, we obtain

$$\left(\sum_{m=0}^{\infty} c_m \left\{-\ln\left[\frac{1-\left(1-\overline{G}^2(x;\xi)\right)^b}{1+\left(1-\overline{G}^2(x;\xi)\right)^b}\right]\right\}^m\right)^{k+i-1} = \sum_{m=0}^{\infty} d_{m,k+i-1}\left\{-\ln\left[\frac{1-\left(1-\overline{G}^2(x;\xi)\right)^b}{1+\left(1-\overline{G}^2(x;\xi)\right)^b}\right]\right\}^m$$

where the constants $d_{0,k+i-1}$ and $d_{n,k+i-1}$ are define in (Makubate, Rannona, Oluyede & Chipepa, 2021) such that,

$$\begin{aligned}
f_{i:n}(x) &= \frac{n!4bg(x;\xi)}{(i-1)!(n-i)![\Gamma(\delta)]^{k+i}} \sum_{k,m=0}^{\infty} (-1)^k d_{m,k+i-1}\binom{n-i}{k} \\
&\quad \frac{\overline{G}(x;\xi)\left(1-\overline{G}^2(x;\xi)\right)^{b-1}}{\left(1+\left(1-\overline{G}^2(x;\xi)\right)^b\right)^2} \\
&\quad \left[-\ln\left(\frac{1-\left(1-\overline{G}^2(x;\xi)\right)^b}{1+\left(1-\overline{G}^2(x;\xi)\right)^b}\right)\right]^{\delta(k+i)-1} \left[-\ln\left(\frac{1-\left(1-\overline{G}^2(x;\xi)\right)^b}{1+\left(1-\overline{G}^2(x;\xi)\right)^b}\right)\right]^m \\
&= \frac{n!4bg(x;\xi)}{(i-1)!(n-i)![\Gamma(\delta)]^{k+i}} \sum_{k,m=0}^{\infty} (-1)^k d_{m,k+i-1}\binom{n-i}{k} \\
&\quad \frac{\overline{G}(x;\xi)\left(1-\overline{G}^2(x;\xi)\right)^{b-1}}{\left(1+\left(1-\overline{G}^2(x;\xi)\right)^b\right)^2} \\
&\quad \left[-\ln\left(\frac{1-\left(1-\overline{G}^2(x;\xi)\right)^b}{1+\left(1-\overline{G}^2(x;\xi)\right)^b}\right)\right]^{\delta(k+i)+m-1} \\
&= \frac{n!4bg(x;\xi)}{(i-1)!(n-i)![\Gamma(\delta)]^{k+i}} \sum_{k,m=0}^{\infty} (-1)^k d_{m,k+i-1}\binom{n-i}{k} \\
&\quad \frac{\overline{G}(x;\xi)\left(1-\overline{G}^2(x;\xi)\right)^{b-1}}{\left(1+\left(1-\overline{G}^2(x;\xi)\right)^b\right)^2} \\
&\quad \left[-\ln\left(1-\left\{1-\frac{1-\left(1-\overline{G}^2(x;\xi)\right)^b}{1+\left(1-\overline{G}^2(x;\xi)\right)^b}\right\}\right)\right]^{\delta(k+i)+m-1}
\end{aligned}$$

Using the Power series expansion for logarithm function that is,

$$(-\ln(1-y))^{\delta(k+i)+n-1} = \sum_{q,s=0}^{\infty}\binom{\delta(k+i)+n-1}{j} y^{\delta(k+i)+n+q+s-1} b_{s,q} \tag{28}$$

we obtain

$$
\begin{aligned}
f_{i:n}(x) =& \frac{n!4bg(x;\xi)}{(i-1)!(n-i)![\Gamma(\delta)]^{k+i}} \sum_{k,m,q,s=0}^{\infty} (-1)^k d_{m,k+i-1} \binom{n-i}{k} b_{s,q} \\
& \binom{\delta(k+i)+m-1}{q} \frac{\overline{G}(x;\xi)\left(1-\overline{G}^2(x;\xi)\right)^{b-1}}{\left(1+\left(1-\overline{G}^2(x;\xi)\right)^b\right)^2} \\
& \left(\frac{2\left(1-\overline{G}^2(x;\xi)\right)^b}{1+\left(1-\overline{G}^2(x;\xi)\right)^b}\right)^{\delta(k+i)+m+q+s-1} \\
=& \frac{n!4bg(x;\xi)}{(i-1)!(n-i)![\Gamma(\delta)]^{k+i}} \sum_{k,m,q,s=0}^{\infty} (-1)^k d_{m,k+i-1} \binom{n-i}{k} b_{s,q} \\
& \binom{\delta(k+i)+m-1}{q} 2^{\delta(k+i)+m+q+s-1} \overline{G}(x;\xi) \\
& \left(1+\left(1-\overline{G}^2(x;\xi)\right)^b\right)^{-[\delta(k+i)+m+q+s+1} \left(1-\overline{G}^2(x;\xi)\right)^{b[\delta(k+i)+m+j+s]-1} \\
=& \frac{n!4bg(x;\xi)}{(i-1)!(n-i)![\Gamma(\delta)]^{k+i}} \sum_{k,m,q,s,v=0}^{\infty} (-1)^k d_{m,k+i-1} \binom{n-i}{k} b_{s,q} \\
& \binom{\delta(k+i)+m-1}{q} 2^{\delta(k+i)+m+q+s-1} \binom{\delta(k+i)+m+q+s+v}{v} \\
& \overline{G}(x;\xi)\left(1-\overline{G}^2(x;\xi)\right)^{b[\delta(k+i)+m+q+s]-1+bv} \\
=& \frac{n!4bg(x;\xi)}{(i-1)!(n-i)![\Gamma(\delta)]^{k+i}} \sum_{k,m,q,s,v,r=0}^{\infty} (-1)^k d_{m,k+i-1} \binom{n-i}{k} \\
& \binom{\delta(k+i)+m-1}{q} b_{s,q} 2^{\delta(k+i)+m+j+s-1} \binom{\delta(k+i)+m+q+s+v}{v} (-1)^r \\
& \binom{b\left[\delta(k+i)+m+q+s\right]-1+bv}{r} \left(1-G(x;\xi)\right)^{2r+1} \\
=& \frac{n!4bg(x;\xi)}{(i-1)!(n-i)![\Gamma(\delta)]^{k+i}} \sum_{k,m,q,s,v,r,w=0}^{\infty} (-1)^k d_{m,k+i-1} \binom{n-i}{k} b_{s,q} \\
& \binom{\delta(k+i)+m-1}{q} 2^{\delta(k+i)+m+q+s-1} \binom{\delta(k+i)+m+q+s+v}{v} (-1)^{r+w} \\
& \binom{b\left[\delta(k+i)+m+q+s\right]-1+bv}{r} \binom{2r+1}{w} G^w(x;\xi)
\end{aligned}
\tag{29}
$$

Therefore, the pdf of the i^{th} order statistic from the GTIIHLTL-G is given by:

$$\begin{aligned} f_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m,w,r,v,s,q=0}^{\infty}\frac{4b(-1)^{k+r+w}d_{m,k+i-1}}{(w+1)\Gamma(\delta)^{k+i}}2^{\delta(k+i)+m-1+q+s} \\ &\binom{n-i}{k}\binom{\delta(k+i)+m-1}{q}\binom{\delta(k+i)+m+q+s+v}{v}b_{s,q} \\ &\binom{b[\delta(k+i)+m+q+s]-1+bv}{r}\binom{2r+1}{w}(w+1)g(x;\xi)G^{w}(x;\xi) \\ &= \sum_{q=0}^{\infty}X^{*}_{w+1}g_{w+1}(x;\xi) \end{aligned} \tag{30}$$

where

$$\begin{aligned} X^{*}_{w+1} &= \frac{n!}{(i-1)!(n-i)!}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m,w,r,v,s,q=0}^{\infty}\frac{b(-1)^{k+r+w}d_{m,k+i-1}}{(w+1)\Gamma(\delta)^{k+i}}2^{\delta(k+i)+m+1+q+s} \\ &\binom{n-i}{k}\binom{\delta(k+i)+m-1}{q}\binom{\delta(k+i)+m+q+s+v}{v}b_{s,q} \\ &\binom{b[\delta(k+i)+m+q+s]-1+bv}{r}\binom{2r+1}{w} \end{aligned}$$

and $g_{w+1}(x;\xi) = (w+1)g(x;\xi)G^{w}(x;\xi)$ is the Exp-G distribution with power parameter $w+1$.

Appendix C. Rényi Entropy of the GTIIHLTL-G

We obtained the Rényi entropy for the GTIIHLTL-G distribution from equation (6) as follows:

$$\begin{aligned} f^{v}(x) &= \frac{(4b)^{v}g^{v}(x;\xi)}{(\Gamma(\delta))^{v}}\left[-\ln\left(\frac{1-\left(1-\overline{G}^{2}(x;\xi)\right)^{b}}{1+\left(1-\overline{G}^{2}(x;\xi)\right)^{b}}\right)\right]^{v(\delta-1)} \\ &\left(1-\overline{G}^{2}(x;\xi)\right)^{v(b-1)}\overline{G}^{v}(x;\xi)\left(1+\left(1-\overline{G}^{2}(x;\xi)\right)^{b}\right)^{-2v} \end{aligned}$$

Using the following expansion,

$$(-\ln(1-y))^{v(\delta-1)} = \sum_{j,s=0}^{\infty}\binom{v(\delta-1)}{j}y^{v(\delta-1)+j+s}b_{s,j},$$

we obtain

$$\begin{aligned}
f^v(x) &= \frac{(4b)^v g^v(x;\xi)}{(\Gamma(\delta))^v} \sum_{j,s=0}^{\infty} \binom{v(\delta-1)}{j} b_{s,j} \left[\frac{2\left(1-\overline{G}^2(x;\xi)\right)^b}{1+\left(1-\overline{G}^2(x;\xi)\right)^b} \right]^{v(\delta-1)+j+s} \\
&\quad \left(1-\overline{G}^2(x;\xi)\right)^{v(b-1)} \overline{G}^v(x;\xi) \left(1+\left(1-\overline{G}^2(x;\xi)\right)^b\right)^{-2v} \\
&= \frac{(4b)^v g^v(x;\xi)}{(\Gamma(\delta))^v} \sum_{j,s=0}^{\infty} \binom{v(\delta-1)}{j} b_{s,j} 2^{v(\delta-1)+j+s} \overline{G}^v(x;\xi) \\
&\quad \left(1-\overline{G}^2(x;\xi)\right)^{b[v(\delta)+j+s]-v} \left(1+\left(1-\overline{G}^2(x;\xi)\right)^b\right)^{-[v(\delta+1)+j+s]} \\
&= \frac{(4b)^v g^v(x;\xi)}{(\Gamma(\delta))^v} \sum_{k,j,s=0}^{\infty} \binom{v(\delta-1)}{j} b_{s,j} 2^{v(\delta-1)+j+s} \binom{v(\delta+1)+j+s+k-1}{k} \\
&\quad \overline{G}^v(x;\xi) \left(1-\overline{G}^2(x;\xi)\right)^{b[v(\delta)+j+s+k]-v} \\
&= \frac{(4b)^v g^v(x;\xi)}{(\Gamma(\delta))^v} \sum_{z,k,j,s=0}^{\infty} \binom{v(\delta-1)}{j} b_{s,j} 2^{v(\delta-1)+j+s} (-1)^z \\
&\quad \binom{v(\delta+1)+j+s+k-1}{k} \binom{b[v(\delta)+j+s+k]-v}{z} (1-G(x;\xi))^{2z+v} \\
&= \frac{(4b)^v g^v(x;\xi)}{(\Gamma(\delta))^v} \sum_{d,z,k,j,s=0}^{\infty} \binom{v(\delta-1)}{j} b_{s,j} 2^{v(\delta-1)+j+s} (-1)^{z+d} \\
&\quad \binom{v(\delta+1)+j+s+k-1}{k} \binom{b[v(\delta)+j+s+k]-v}{z} \binom{2z+v}{d} G^d(x;\xi).
\end{aligned}$$

Consequently, we can write the Rényi entropy of the GTIIHLTL-G family of distributions in terms of the Rényi entropy of the Exp-G distribution as follows:

$$I_R(v) = (v-1)^{-1} \log \left[\sum_{v,d=0}^{\infty} \varphi^*_{d+1} e^{(1-v)IREG} \right],$$

where

$$\begin{aligned}
\varphi^*_{d+1} &= \frac{(4b)^v}{(\Gamma(\delta))^v} \sum_{d,z,k,j,s=0}^{\infty} \binom{v(\delta-1)}{j} b_{s,j} 2^{v(\delta-1)+j+s} (-1)^{z+m} \\
&\quad \binom{v(\delta+1)+j+s+k-1}{k} \binom{b[v(\delta)+j+s+k]-v}{z} \\
&\quad \binom{2z+v}{d} \left(\frac{d}{v}+1\right)
\end{aligned}$$

and $I_{REG} = \int_{-\infty}^{\infty} \left[\left(\frac{d}{v}+1\right) g(x;\xi) G(x;\xi)^{\frac{d}{v}}\right]^v dx$ is the Rényi entropy of the Exp-G distribution with power parameter $\frac{d}{v}+1$.

Appendix D. Estimation of Parameters for Uncensored and Censored Data

$$\frac{\partial \ell}{\partial \delta} = \frac{n\Gamma'(\delta)}{\Gamma(\delta)} + \sum_{k=0}^{n} \log\left[-\ln\left(1-(1-\overline{G}^2(x;\xi))^b\right) + \ln\left(1+(1-\overline{G}^2(x;\xi))^b\right)\right]$$

$$\begin{aligned}\frac{\partial \ell}{\partial \xi_z} &= \frac{g'(\xi_z)}{g(x;\xi)} + \sum_{k=0}^{n} \frac{-G'(\xi_z)}{(1-G(x;\xi))} + (b-1)\sum_{k=0}^{n} \frac{2\overline{G}(x;\xi)G'(\xi_z)}{(1-\overline{G}^2(x;\xi))} \\ &\quad -2\sum_{k=0}^{n} \frac{2b(1-\overline{G}^2(x;\xi))^{b-1}\overline{G}(x;\xi)G'(\xi_z)}{[1+(1-\overline{G}^2(x;\xi))^b]}\end{aligned}$$

$$+(\delta-1)\sum_{k=0}^{n}\left[\frac{4b(1-\overline{G}^2(x;\xi))^{b-1}(\overline{G}(x;\xi))G'(\xi_z)}{\left(-\ln\left(\frac{1-(1-\overline{G}^2(x;\xi))^b}{1+(1-\overline{G}^2(x;\xi))^b}\right)\right)\left(1-(1-\overline{G}^2(x;\xi))^b\right)\left(1+(1-\overline{G}^2(x;\xi))^b\right)}\right]$$

$$\frac{\partial \ell}{\partial b} = \frac{n}{b} + \sum_{k=0}^{n}(1-\overline{G}^2(x;\xi)) - 2\sum_{k=0}^{n}\frac{(1-\overline{G}^2(x;\xi))^b \log(1-\overline{G}^2(x;\xi))}{[1+(1-\overline{G}^2(x;\xi))^b]}$$

$$+(\delta-1)\sum_{k=0}^{n}\left[\frac{2(1-\overline{G}^2(x;\xi))^b\log(1-\overline{G}^2(x;\xi))}{[1-(1-\overline{G}^2(x;\xi))^b][1+(1-\overline{G}^2(x;\xi))^b]\left(-\ln\left(\frac{1-(1-\overline{G}^2(x;\xi))^b]}{1+(1-\overline{G}^2(x;\xi))^b}\right)\right)}\right]$$

Estimation of Parameters for Censored Data

$$\begin{aligned}U_\delta &= \frac{n\Gamma'(\delta)}{\Gamma(\delta)} + \sum_{k=0}^{n}\log\left[-\ln\left(1-(1-\overline{G}^2(x;\xi))^b\right)+\ln\left(1+(1-\overline{G}^2(x;\xi))^b\right)\right] \\ &\quad +(n-r)\left(\frac{\frac{\Gamma'(\delta)\gamma(\delta)}{\Gamma(\delta)}-\gamma}{\left(1-\gamma\left(\delta,-\ln\left[\frac{1-\left(1-\overline{G}^2(x;\xi)\right)^b}{1+\left(1-\overline{G}^2(x;\xi)\right)^b}\right]\right)\right)}\right)\end{aligned}$$

$$U_b = \frac{n}{b} + \sum_{k=0}^{n}(1-\overline{G}^2(x;\xi)) - 2\sum_{k=0}^{n}\frac{(1-\overline{G}^2(x;\xi))^b \log(1-\overline{G}^2(x;\xi))}{[1+(1-\overline{G}^2(x;\xi))^b]}$$

$$\begin{aligned}&+(\delta-1)\sum_{k=0}^{n}\left[\frac{2(1-\overline{G}^2(x;\xi))^b\log(1-\overline{G}^2(x;\xi))}{[1-(1-\overline{G}^2(x;\xi))^b][1+(1-\overline{G}^2(x;\xi))^b]\left(-\ln\left(\frac{1-(1-\overline{G}^2(x;\xi))^b]}{1+(1-\overline{G}^2(x;\xi))^b}\right)\right)}\right] \\ &+(n-r)\sum_{k=0}^{n}\left[\frac{2(1-\overline{G}^2(x;\xi))^b\log(1-\overline{G}^2(x;\xi))\Gamma(\delta)}{[1-(1-\overline{G}^2(x;\xi))^b][1+(1-\overline{G}^2(x;\xi))^b]\left[\Gamma(\delta)-\left(\delta,-\ln\left(\frac{1-(1-\overline{G}^2(x;\xi))^b]}{1+(1-\overline{G}^2(x;\xi))^b}\right)\right)\right]}\right]\end{aligned}$$

$$\begin{aligned}U_{\xi_z} &= \frac{g'(\xi_z)}{g(x;\xi)} + \sum_{k=0}^{n} \frac{-G'(\xi_z)}{(1-G(x;\xi))} + (b-1)\sum_{k=0}^{n} \frac{2\overline{G}(x;\xi)G'(\xi_z)}{(1-\overline{G}^2(x;\xi))} \\ &\quad -2\sum_{k=0}^{n} \frac{2b(1-\overline{G}^2(x;\xi))^{b-1}\overline{G}(x;\xi)G'(\xi_z)}{[1+(1-\overline{G}^2(x;\xi))^b]}\end{aligned}$$

$$\begin{aligned}&+(\delta-1)\sum_{k=0}^{n}\left[\frac{4b(1-\overline{G}^2(x;\xi))^{b-1}(\overline{G}(x;\xi))G'(\xi_z)}{\left(-\ln\left(\frac{1-(1-\overline{G}^2(x;\xi))^b}{1+(1-\overline{G}^2(x;\xi))^b}\right)\right)\left(1-(1-\overline{G}^2(x;\xi))^b\right)\left(1+(1-\overline{G}^2(x;\xi))^b\right)}\right] \\ &(n-r)\sum_{k=0}^{n}\left[\frac{4(1-\overline{G}^2(x;\xi))^{b-1}(\overline{G}(x;\xi))G'(\xi_z)\Gamma(\delta)}{[1-(1-\overline{G}^2(x;\xi))^b][1+(1-\overline{G}^2(x;\xi))^b]\left[\Gamma(\delta)-\left(\delta,-\ln\left(\frac{1-(1-\overline{G}^2(x;\xi))^b]}{1+(1-\overline{G}^2(x;\xi))^b}\right)\right)\right]}\right]\end{aligned}$$

Appendix E. Variance-Covariance Matrices

The estimated variance-covariance matrix for GTIIHLTLW model on precipitation data is given by

$$\begin{bmatrix} 5.0728 & -2.1543\times10^{-5} & 0.1260 & 0.7756 \\ -2.1543\times10^{-5} & 9.6921\times10^{-11} & -4.6701\times10^{-7} & -3.7323\times10^{-6} \\ 0.1260 & -4.6701\times10^{-7} & 4.1932\times10^{-3} & 0.0137 \\ 0.7756 & -3.7323\times10^{-6} & 0.0137 & 0.1539 \end{bmatrix}$$

The estimated variance-covariance matrix for GTIIHLTLW model on the head and neck cancer data is given by:

$$\begin{bmatrix} 0.2618 & -3.5537 & 0.06697 & -0.0186 \\ -3.5537 & 89.7845 & -2.4078 & 0.7319 \\ 0.0669 & -2.4078 & 0.0751 & -0.0229 \\ -0.0186 & 0.7319 & -0.0229 & 0.0070 \end{bmatrix}$$