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Multivariate Normality Tests for Serially Correlated Data

Pruebas de normalidad multivariadas para datos correlacionados en serie

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Abstract

We extend univariate normality tests for time-dependent observations to their multivariate versions using orthogonalization or empirical standardization of the data. This extension allows us to assess the multivariate normality of serially correlated data. The proposed test statistics asymptotically follow the χ^2 distribution, which allows for readily applicable tests. A comprehensive Monte Carlo study indicates that the proposed tests exhibit good size control and high empirical power. Furthermore, we provide empirical illustrations of all the extended tests using West German macroeconomic data (Lütkepohl, 2005).

***Key words*:** Macroeconomic data; Monte Carlo; Multivariate normality; Orthogonalization; Time series; χ^2-distribution.

Resumen

Extendemos las pruebas de normalidad univariadas para observaciones dependientes del tiempo a sus versiones multivariadas usando ortogonalización o estandarización empírica de los datos. Esta extensión nos permite evaluar la normalidad multivariada. de datos correlacionados en serie. Las estadísticas de prueba propuestas siguen asintóticamente la distribución χ^2, que permite pruebas fácilmente aplicables. Un comprensivo Estudio de Montecarlo indica que las pruebas propuestas presentan buen tamaño control y alto poder empírico. Además, proporcionamos ilustraciones empíricas. de todas las pruebas ampliadas utilizando datos macroeconómicos de Alemania Occidental (Lütkepohl, 2005).

***Palabras clave*:** Datos macroeconómicos; Monte Carlo; Normalidad multivariada; Ortogonalización; χ^2-distribución; Series de tiempo.

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1. Introduction

The normality assumption is a fundamental concept in statistics and is commonly utilized by researchers in both theoretical and applied settings. For instance, normality is typically a prerequisite in the analysis of variance, and without this assumption holding, the results of the analysis of variance are not trustworthy (Khan & Rayner, 2003).

Estimation methods such as maximum likelihood, generalized least squares, and weighted least squares of structural equation models require the normality assumption to obtain the same fit function and estimates of unknown parameters. In the case of non-normalities, the fit function and estimates obtained by weighted least squares differ from those obtained by maximum likelihood and generalized least squares. Consequently, most fit functions do not converge to the population as the sample size increases (Andreassen et al., 2006). The normality assumption plays an important role in determining sufficient and necessary conditions to obtain the finite sample distributions of least square estimates. Statistical models, including linear regression and other modeling techniques, often assume normally distributed residuals to make accurate forecasts or predictions. Loy et al. (2016) stated that prediction intervals may be worthless or inaccurate if residuals are not normally distributed. Lütkepohl (2005) pointed out that normality is required for the underlying data generation process when setting up forecast intervals. Generally, models with non-normally distributed residuals do not provide a good representation of the data generation process. Yuan et al. (2005) and Andreassen et al. (2006) have discussed the consequences of the normality assumption in both simple independent and identically distributed, henceforth IID, models to more complicated structural equation models.

Given the previous examples, it is not surprising that the literature on testing normality has a history of more than eight decades. For book-level treatments, see Thode (2002) on testing normality and D'Agostino & Stephens (1986) on determining goodness-of-fit. A wide range of methods to test the normality of IID data has been proposed and discussed in the literature. The most popular method especially in economics for testing the normality of univariate data based on skewness and kurtosis was proposed by Jarque & Bera (1987), henceforth JB. Generalized versions of the JB test statistic have also been popularly applied to test multivariate normality in IID settings (Mardia, 1970; Koizumi et al., 2009; Kim, 2016). JB popularity is due to its ease of implementation and standard asymptotic behavior, i.e., it only involves calculating averages and the test statistic has a χ^2 distribution asymptotically. From a theoretical perspective, the JB test has a nice LM-based interpretation and is flexible and popular enough to be extended to the outside of the IID case. For example, Lobato & Velasco (2004) and Bai & Ng (2005) modified the JB test to the univariate time series case. Horváth et al. (2020) and Chen & Genton (2023) extended the JB test statistic to the univariate and multivariate spatial grid cases, respectively.

Relatively less work has been done to generalize the JB test for multivariate time series data compared to the work done for univariate time series data. A generalized version of the JB test is provided in Lütkepohl (2005) for assessing the multivariate normality of the errors of a vector autoregressive (VAR) process. The generalized test statistic is developed based on the standardized residuals of the VAR model, and it has been proved that the test statistic follows a χ^2 distribution under a Gaussian white noise process. Kilian & Demiroglu (2000), through a comprehensive Monte Carlo study, pointed out that Lütkepohl (2005)'s test suffers from severe size distortions even for large samples in the presence of autoregressive persistence. However, Kilian & Demiroglu (2000) developed a bootstrap version of the JB test statistic for testing the multivariate normality of a VAR process and a vector error-correction (VEC) process. Their Monte Carlo results revealed that the bootstrap test works very well, even for processes with roots close to unity. Recently, Elbouch et al. (2022); Olivier et al. (2022) extended the Mardia (1970)'s multivariate kurtosis test to multivariate time series, to assess the multivariate normality of time-dependence data, using random projection.

In practice, Lütkepohl (2005)'s test may not be reliable due to its poor finite-sample properties in the presence of serial correlation, and the practical implementation of the bootstrap version of the JB test may be challenging because it requires knowledge of the model structure. Furthermore, it is well documented that the kurtosis test exhibits slow convergence of sample kurtosis to the normal asymptotic distribution, even with a large number of observations and a white noise process (Lobato & Velasco, 2004; Bai & Ng, 2005). However, the multivariate kurtosis test for time series data introduced by Elbouch et al. (2022); Olivier et al. (2022) may not be accurate for small and moderate sample sizes.

In light of these issues and drawing inspiration from Kim (2016) and Villasenor Alva & Estrada (2009), this study extends the univariate normality tests proposed by Lobato & Velasco (2004) and Bai & Ng (2005) to the multivariate time series case using orthogonalization or empirical standardization of the data. Under the null hypothesis of normality, the proposed tests asymptotically follow the χ^2 distribution. Therefore, their practical implementation is more straightforward and easier than the bootstrap version of the JB test, as they do not require knowledge of the model structure and its estimation. Additionally, the proposed tests are robust against serial dependence and address both skewness and kurtosis. However, we can conclude that the proposed tests are more general than the bootstrap testing procedure (Kilian & Demiroglu, 2000), Lütkepohl (2005)'s test, and the multivariate kurtosis test (Elbouch et al., 2022; Olivier et al., 2022). Extensive Monte Carlo simulations reveal that the proposed test statistics have good size control and high empirical power. Moreover, the study provides an empirical exercise for illustrative purposes using quarterly, seasonally adjusted time series data from Lütkepohl (2005).

The organization of the rest of the study is as follows: Section 2 summarizes the setup of univariate normality tests for time series data. Section 3 presents the set of proposed normality tests for multivariate time series observations. Section 4 provides Monte Carlo experiments to document the finite-sample properties of the proposed tests. Section 5 includes an empirical exercise for illustrative purposes, and Section 6 contains the concluding remarks. Monte Carlo results are presented in Appendix A, while the graphical presentation of the proposed test statistics is given in Appendix B. Finally, data on West German fixed investment, disposable income, and consumption expenditures is provided in Appendix C.

2. Normality Tests for Univariate Time Series

Let X be a stochastic process with continuous cumulative distribution function (CDF) G_X and $X_1, \ldots, X_T$ be a random sample of size T from X satisfies $\mathbb{E}\left(X_t^{16}\right) < \infty$. Let $\widehat{\mu}_k = \frac{1}{T}\sum_{i=1}^{T}\left(X_i - \overline{X}\right)^k$ be the kth sample moment, where $\overline{X} = \frac{1}{T}\sum_{i=1}^{T} X_i$ is the sample mean.

In the same spirit as the JB test, Lobato & Velasco (2004) developed the test statistic under time-dependence to test the null hypothesis that $H_0 : X_t \sim N(\mu, \sigma^2)$, where $N(\mu, \sigma^2)$ denotes the normal distribution with an unknown mean μ and unknown variance σ^2, against the alternative hypothesis that H_0 does not hold. The test statistic is defined as follows:

$$\mathrm{G} = \frac{T\widehat{\mu}_3^2}{6\widehat{F}^{(3)}} + \frac{T(\widehat{\mu}_4 - 3\widehat{\mu}_2^2)^2}{24\widehat{F}^{(4)}}, \tag{1}$$

with

$$\widehat{F}^{(k)} = \sum_{j=1-T}^{T-1} \widehat{\gamma}(j)^k \quad k = 3, 4$$

where

$$\widehat{\gamma}(j) = \frac{1}{T}\sum_{t=1}^{T-|j|}(X_t - \overline{X})(X_{t+|j|} - \overline{X}),$$

for $j = 0, \pm 1, \pm 2, \ldots, \pm(T-1)$, $\widehat{\gamma}(j)$ is a consistent estimator of the population jth order autocovariance. Furthermore, $6\widehat{F}^{(3)}$ and $24\widehat{F}^{(4)}$ are consistent estimators, under normality, of the asymptotic variance of $\sqrt{T}\widehat{\mu}_3$ and $\sqrt{T}(\widehat{\mu}_4 - 3\widehat{\mu}_2^2)$, respectively, and they account for the serial correlation in the observations.

Bai & Ng (2005) modified the JB test statistic for time series observations to test H_0. In contrast to Lobato & Velasco (2004), they accounted for serial correlation in data using a nonparametric long-run variance estimator. Their test statistic is defined as follows:

$$\begin{aligned}\mathrm{BN} &= \frac{T}{\widehat{\omega}_3}\left(\frac{\widehat{\mu}_3}{\widehat{\mu}_2^{3/2}}\right)^2 + \frac{T}{\widehat{\omega}_4}\left(\frac{\widehat{\mu}_4}{\widehat{\mu}_2^2} - 3\right)^2 \\ &= \frac{T\widehat{\tau}^2}{\widehat{\omega}_3} + \frac{T(\widehat{\kappa} - 3)^2}{\widehat{\omega}_4},\end{aligned} \tag{2}$$

where $\widehat{\omega}_3$ and $\widehat{\omega}_4$ are consistent estimators of the long-run variance of $\sqrt{T}\widehat{\tau}$ and $\sqrt{T}(\widehat{\kappa} - 3)$, respectively. These estimators are computed using an automatic lag selection procedure introduced by Newey & West (1994). Moreover, $\widehat{\tau}$ and $\widehat{\kappa}$ represent the sample counterparts of the skewness coefficient τ and kurtosis coefficient κ, respectively. Under the null of normality, the G and BN test statistics asymptotically follow a χ^2 distribution with 2 degrees of freedom.

In addition to H_0, we can use the first components of G and BN to test the null hypothesis that the skewness is zero. Namely, the skewness test statistics are fined as:

$$\text{GS} = \frac{T\widehat{\mu}_3^2}{6\widehat{F}^{(3)}}, \tag{3}$$

$$\text{BS} = \frac{T\widehat{\tau}^2}{\widehat{\omega}_3}, \tag{4}$$

and they asymptotically follow a χ^2 distribution with 1 degree of freedom. We have not considered the second components of BN and G to test the null hypothesis that the kurtosis is 3. This is because the kurtosis test has an extremely slow convergence to a normal asymptotic distribution, and the sample estimate of kurtosis significantly deviates from its true value in the presence of serial correlation, even with a large number of observations Bai & Ng (2005).

It is worth noting that BN only requires finite sample moments up to the 8^{th} order, while the G test demands moments up to the 16^{th} order, especially to establish the consistency of $\widehat{F}^{(3)}$ and $\widehat{F}^{(4)}$. Additionally, the BN test statistic is not exclusively formulated under the null of normality, unlike the G test statistic, which is developed assuming normality. However, the BN test statistic can be used to detect deviations from distributions other than the normal distribution, making it suitable as a goodness-of-fit test rather than merely a normality test. A main advantage of G over BN is that it does not require any kernel smoothing or truncation.

3. Proposed Normality Tests for Multivariate Time Series

Let $\mathbf{X} = (X_{1t}, X_{2t}, \ldots, X_{mt})$ be a m-dimensional strictly stationary random vector with a cumulative distribution function $\mathbf{G_X}$ and $\{\mathbf{X}_1, \ldots, \mathbf{X}_T\}$ is a set of dependent observations of size T from $\mathbf{X}$ with sample mean vector $\overline{\mathbf{X}} = \frac{1}{T}\sum_{t=1}^{T}\mathbf{X_t}$ and sample covariance matrix $\mathbf{S} = \frac{1}{T}\sum_{t=1}^{T}\left(\mathbf{X_t} - \overline{\mathbf{X}}\right)\left(\mathbf{X_t} - \overline{\mathbf{X}}\right)'$. Our objective is to test the null hypothesis that $\boldsymbol{H_0} : \mathbf{X_t} \sim N_m(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $N_m(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ represents the m-variate normal distribution with unknown mean vector $\boldsymbol{\mu}$ and unknown covariance matrix $\boldsymbol{\Sigma}$, against the alternative hypothesis that $\boldsymbol{H_0}$ does not hold.

To formulate the multivariate normality tests for time-dependence observations, we define scaled residuals, following the steps given in Villasenor Alva & Estrada (2009), as follows:

$$\mathbf{Z}_t^* = \mathbf{S}^{-1/2}\left(\mathbf{X}_t - \overline{\mathbf{X}}\right), \quad t = 1, \ldots, T$$

where $\mathbf{S}^{-1/2}$ represents the unique symmetric square root matrix of $\mathbf{S}$ and satisfies $\mathbf{S}^{-1/2\prime}\mathbf{S}\mathbf{S}^{-1/2} = \boldsymbol{I}$. Additionally, we assume that $T \geq m+1$ so that $\mathbf{S}$ is invertible with probability one (Chen & Genton, 2023). Under the null of multivariate normality, the scaled residuals $\mathbf{Z}_t^*$ asymptotically follow a multivariate standard normal distribution, denoted as $N_m(\mathbf{0}, \boldsymbol{I})$. Here, $\mathbf{0}$ is the null vector of order $m \times 1$, while $\boldsymbol{I}$ is identity matrix of order $m \times m$. This implies that the components

$\{Z^*_{1t}, \dots, Z^*_{mt}\}$ of $\mathbf{Z}^*_t$ are approximately independent and each follows a univariate standard normal distribution (Villasenor Alva & Estrada, 2009).

It is worth noting that each coordinate of the standardized data, i.e., $\{Z^*_{c1}, \dots, Z^*_{cT}\}$, where $c = 1, 2, \dots, m$, exhibits serial correlation. This is in contrast to the setup given in Villasenor Alva & Estrada (2009) and Kim (2016), where each coordinate of the standardized data is serially independent. Therefore, the JB test, Lütkepohl (2005)'s test, and Mardia (1970)'s multivariate test are not valid in our setup, as they have been developed for IID settings.

To account for the serial correlation, we compute and aggregate the univariate G, GS, BN, and BS test statistics for each coordinate of the transformed data. This approach is inspired by Kim (2016). Finally, we propose the following test statistics to test $\boldsymbol{H_0}$:

$$\mathrm{G_M} = \sum_{c=1}^{m} \mathrm{G}\,(c)\,, \quad \mathrm{GS_M} = \sum_{c=1}^{m} \mathrm{GS}\,(c)\,, \tag{5}$$

and

$$\mathrm{BN_M} = \sum_{c=1}^{m} \mathrm{BN}\,(c)\,, \quad \mathrm{BS_M} = \sum_{c=1}^{m} \mathrm{BS}\,(c) \tag{6}$$

where $\mathrm{G}\,(c)$, $\mathrm{GS}\,(c)$, $\mathrm{BN}\,(c)$, and $\mathrm{BS}\,(c)$ are the respective test statistics in equations (1), (3), (2), and (4) for each coordinate $\{Z^*_{c1}, \dots, Z^*_{cT}\}$, where $c = 1, 2, \dots, m$. Given that the scaled residuals are approximately independent under $\boldsymbol{H_0}$, and the square roots of the first and second components of G and BN converge to the standard normal distribution, the test statistics $\mathrm{G_M}$ and $\mathrm{BN_M}$ asymptotically follow a χ^2 distribution with $2m$ degrees of freedom under $\boldsymbol{H_0}$. Meanwhile, $\mathrm{GS_M}$ and $\mathrm{BS_M}$ asymptotically follow a χ^2 distribution with m degrees of freedom under the null hypothesis that the skewness is zero.

Mathematically, it is tedious and challenging to derive the asymptotic distributions of the proposed test statistics; however, we used Monte Carlo simulations to justify the asymptotic distribution of each test statistic. Monte Carlo simulations are commonly used to compute the empirical distribution of a statistic, particularly when the exact distribution is unknown or mathematically difficult to find. For this purpose, we computed 10 000 values of $\mathrm{G_M}$, $\mathrm{BN_M}$, $\mathrm{GS_M}$, and $\mathrm{BS_M}$ for sample sizes of 2000 and 3000. These samples were generated from models M1, M2, and M3, where the error terms ϵ_{1t}, ϵ_{2t}, and ϵ_{3t} are independently obtained from a standard normal distribution (the details of each model are given in Section 4). Furthermore, we generated 10 000 random numbers from a χ^2 distribution with degrees of freedom 2, 3, 4, and 6, which is our theoretical distribution. To compare the empirical distribution of each test statistic with its respective theoretical distribution, we plotted the computed values of $\mathrm{G_M}$, $\mathrm{BN_M}$, $\mathrm{GS_M}$, and $\mathrm{BS_M}$ against their respective theoretical random numbers. These graphical results are provided in Appendix B for each sample size and models M1, M2, and M3. Figures B1 to B24 clearly show that the empirical distributions of $\mathrm{G_M}$ and $\mathrm{BN_M}$ closely follow a χ^2 distribution with $2m$ degrees of freedom, while the empirical distributions of $\mathrm{GS_M}$ and $\mathrm{BS_M}$ closely follow a χ^2 distribution with m degrees of freedom. A similar pattern has been observed when samples are generated from models M4 and M5; however, results can be provided upon request.

4. Monte Carlo Experiments

4.1. Monte Carlo Design

We now document the finite-sample performance of the proposed tests $\mathrm{G_M}$ and $\mathrm{BN_M}$, along with their skewness counterparts $\mathrm{GS_M}$ and $\mathrm{BS_M}$, respectively. The performance of each test is evaluated based on the empirical size and power of the test. The empirical size of the test is the probability of rejecting a null hypothesis that is actually true in the population, while the empirical power of the test is the probability of rejecting a null hypothesis that is actually false in the population. Let ϵ_{mt}, where $m \in \{1, 2, 3, 4\}$, be the IID error terms, we consider the following stochastic processes:

$$\mathrm{M}_1 : \begin{pmatrix} X_{1,t} \\ X_{2,t} \end{pmatrix} = \begin{bmatrix} 0.70 & 0.20 \\ 0.20 & 0.70 \end{bmatrix} \begin{pmatrix} X_{1,t-1} \\ X_{2,t-1} \end{pmatrix} + \begin{pmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{pmatrix},$$

$$\mathrm{M}_2 : \begin{pmatrix} X_{1,t} \\ X_{2,t} \end{pmatrix} = \begin{bmatrix} 0.40 & -0.10 \\ -0.20 & 0.60 \end{bmatrix} \begin{pmatrix} X_{1,t-1} \\ X_{2,t-1} \end{pmatrix} + \begin{bmatrix} -0.40 & 0.60 \\ -0.20 & 0.20 \end{bmatrix} \begin{pmatrix} X_{1,t-2} \\ X_{2,t-2} \end{pmatrix} + \begin{pmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{pmatrix},$$

$$\mathrm{M}_3 : \begin{pmatrix} X_{1,t} \\ X_{2,t} \\ X_{3,t} \end{pmatrix} = \begin{bmatrix} 0.50 & 0.20 & 0.10 \\ 0.40 & 0.30 & 0.20 \\ 0.20 & 0.60 & -0.10 \end{bmatrix} \begin{pmatrix} X_{1,t-1} \\ X_{2,t-1} \\ X_{3,t-1} \end{pmatrix} + \begin{pmatrix} \epsilon_{1t} \\ \epsilon_{2t} \\ \epsilon_{3t} \end{pmatrix},$$

$$\mathrm{M}_4 : \begin{cases} X_{1,t} = 0.40 X_{1,t-1} + \epsilon_{1t}, \\ X_{2,t} = 0.20 X_{2,t-1} + \epsilon_{2t}, \\ X_{3,t} = 0.80 X_{3,t-1} + \epsilon_{3t}, \end{cases} \quad \text{and} \quad \mathrm{M}_5 : \begin{cases} X_{1,t} = 0.40 X_{1,t-1} + \epsilon_{1t}, \\ X_{2,t} = 0.20 X_{2,t-1} + \epsilon_{2t}, \\ X_{3,t} = 0.80 X_{3,t-1} + \epsilon_{3t}, \\ X_{4,t} = 0.50 X_{4,t-1} + \epsilon_{4t}, \end{cases}$$

for $t = 1, 2, \dots, (100+T)$. M_1 and M_2 represent the bivariate vector autoregressive process with lag 1 and 2 respectively, while M_3 represents the trivariate vector autoregressive process with lag 1. M_4 and M_5 are the combination of three and four independent AR(1) processes respectively. The sample size T takes on three values: 100, 500, and 1000;[1] the first 100 observations are omitted for each experiment to eliminate the start-up effects.

[1] Unlike the Monte Carlo comparisons of the finite sample performance of IID-based normality tests, the Monte Carlo designs in the time series case feature at least 100 observations, see, for example, Bai & Ng (2005); Lobato & Velasco (2004); Psaradakis & Vávra (2018).

To estimate the size of the test, the error terms ϵ_{1t}, ϵ_{2t}, ϵ_{3t}, and ϵ_{4t} are independently generated from a standard normal distribution, while to estimate the raw power of the test, we generate ϵ_{1t}, ϵ_{2t}, ϵ_{3t}, and ϵ_{4t} independently from the following distributions: (a) Uniform distribution, labeled by $U(0,1)$, (b) Beta(2,2), (c) Student t-distribution with 1 degree of freedom, labeled by t_1, (d) Student t-distribution with 5 degrees of freedom, labeled by t_5, (e) Student t-distribution with 10 degrees of freedom, labeled by t_{10}, (f) Laplace(0,1), (g) Logistic(0,1), (h) Chi-square distribution with 1 degree of freedom, labeled by χ_1^2, (i) Chi-square distribution with 5 degrees of freedom, labeled by χ_5^2, (j) Standard log-normal distribution, labeled by LN(0,1), (k) Pareto (6,1), and (l) generalized lambda distribution (GLD) with following quantile function:

$$F_{\epsilon}^{-1}(u) = \theta_1 + \frac{u^{\theta_3} - (1-u)^{\theta_4}}{\theta_2},$$

where $u \sim U(0,1)$. The values of the unknown parameters θ_1, θ_2, θ_3, and θ_4, to generate the alternative asymmetric distributions, are obtained from Bai & Ng (2005) and given in Table 1. These distributions cover interval-based distributions, non-normal symmetric distributions, and asymmetric distributions. We fixed the significance level at $\alpha = 0.05$, and the number of replications to 10000 in the Monte Carlo experiments. To estimate the long-run covariance matrix of BN_M and BN_M, we set the user inputs as follows. We use the Bartlett kernel introduced by Newey & West (1987), with the truncation lag selected using the automatic procedure of Newey & West (1994). Similar to Bai & Ng (2005), we have not applied prewhitening. Monte Carlo outputs are given in Appendix A.

Table 1: Generalized lambda distributions

Distributions	θ_1	θ_2	θ_3	θ_4	τ	κ
A1	0	-1	-0.10	-0.18	2	21.2
A2	0	-1	-0.001	-0.130	3.16	23.8
A3	0	-1	-0.0001	-0.1700	3.8	40.7

Note: A1-A3 are asymmetric distributions. τ and κ represent the skewness coefficient and kurtosis coefficient, respectively.

4.2. Simulation Results

To demonstrate the incompatibility of classical multivariate normality tests with time series data, we compute the empirical sizes of well-known IID multivariate tests proposed by Mardia (1970) (coded as MS and MK), Koizumi et al. (2009) (coded as KJM_M), and Kim (2016) (coded as JB_M and RJB_M). The results are presented in Table A1, revealing significant upward size distortions. This implies that these tests substantially deviate from their asymptotic distributions and may lead to misleading outcomes in the presence of serial correlation.

Table A2 presents the empirical sizes of the proposed test statistics. The results reveal that the empirical sizes of G_M and GS_M are very close to the nominal size. The BN_M and BS_M tests show upward size distortions, however, these size distortions are relatively smaller than the size distortions of $\mathrm{JB_M}$, $\mathrm{RJB_M}$, MS, MK, and $\mathrm{KJB_M}$.

Table A3 presents the empirical powers of the $\mathrm{GS_M}$, $\mathrm{G_M}$, $\mathrm{BS_M}$, and $\mathrm{BN_M}$ tests when the error terms ϵ_{1t}, ϵ_{2t}, ϵ_{3t}, and ϵ_{4t} are generated from interval-based distributions, specifically $U(0,1)$ and Beta(2,2). It is observed that the $\mathrm{BN_M}$ test outperforms its competitors.

Table A4 provides the empirical powers of $\mathrm{GS_M}$, $\mathrm{G_M}$, $\mathrm{BS_M}$, and $\mathrm{BN_M}$ when ϵ_{1t}, ϵ_{2t}, ϵ_{3t}, and ϵ_{4t} are obtained from non-normal symmetric distributions, particularly t_1, t_5, t_{10}, Laplace(0,1), and Logistic(0,1). Results indicate that the $\mathrm{G_M}$ test is the most powerful test, closely followed by the $\mathrm{GS_M}$ test. The $\mathrm{BS_M}$ test performs worse than all other tests.

The empirical powers of $\mathrm{GS_M}$, $\mathrm{G_M}$, $\mathrm{BS_M}$, and $\mathrm{BN_M}$ against asymmetric distributions, χ^2_1, Pareto(6,1), LN(0,1), A1, A2, and A3, are presented in Table A5. It is observed that $\mathrm{GS_M}$ and $\mathrm{G_M}$ performed well compared to $\mathrm{BS_M}$ and $\mathrm{BN_M}$. In the case of a small sample, the $\mathrm{BS_M}$ test has slightly higher power than $\mathrm{BN_M}$, while for a large sample, both tests have more or less equal power.

5. Empirical Illustration

For illustrative purposes, we consider quarterly seasonally adjusted data on West German fixed investment, disposable income, and consumption expenditures in billions of Deutsche Marks (DM) from 1960Q1 to 1982Q4. These variables are obtained from datasets[2] provided by Applied Time Series Econometrics (Lütkepohl, 2005) and presented in Table C1 (see Appendix C), where X, Y, and Z represent fixed investment, disposable income, and consumption expenditure, respectively. We utilize the GS and BS test statistics to examine symmetry in X, Y, and Z, while $\mathrm{GS_M}$ and $\mathrm{BS_M}$ are applied to test their joint symmetry. To assess individual normality of X, Y, and Z, we employ G and BN tests. Meanwhile, the $\mathrm{G_M}$ and $\mathrm{BN_M}$ statistics are used to evaluate the joint normality of the data.

We take the first difference of the logarithm of each variable before computing the test statistics. The results of applying these test statistics along with sample skewness and kurtosis are provided in Tables 2 and 3. The first two columns of Table 2 contain the values of the univariate sample skewness $\widehat{\tau}$ and kurtosis $\widehat{\kappa}$, respectively, while the last four columns provide the values of the univariate test statistics GS, BS, G, and BN, respectively. In Table 3, the first two columns present the values of the multivariate sample skewness $b_{1,2}$ and kurtosis $b_{2,3}$, respectively, while the last four columns report the values of the multivariate test statistics $\mathrm{GS_M}$, $\mathrm{BS_M}$, $\mathrm{G_M}$, and $\mathrm{BN_M}$, respectively. We reject individual symmetry, joint symmetry, individual normality, and joint normality when the calculated test statistics exceed their critical values at a significance level of 0.05.

[2] `http://www.jmulti.de/data_atse.html`

Table 2: Application of univariate normality tests to West German macroeconomic data.

Time Series	$\hat{\tau}$	$\hat{\kappa}$	GS	BS	G	BN
X	0.357	6.207	2.749	0.404	45.009	2.829
Y	-0.532	5.007	2.741	0.893	8.867	2.183
Z	-0.351	3.110	0.004	0.002	1.002	0.768

The 5% critical values are as follows: 3.8415 for GS and BS, 5.992 for G and BN.

Table 3: Application of multivariate normality tests to West German macroeconomic data.

Time Series	$b_{1,3}$	$b_{2,3}$	$\mathrm{GS_M}$	$\mathrm{BS_M}$	$\mathrm{G_M}$	$\mathrm{BN_M}$
All 3 series	1.290	20.070	5.494	1.301	54.877	5.781

The 5% critical values are as follows: 7.815 for $\mathrm{GS_M}$ and $\mathrm{BS_M}$, and 12.592 for $\mathrm{G_M}$ and $\mathrm{BN_M}$.

The results indicate that the GS and BS tests fail to reject symmetry in X, Y, and Z. Similar to GS and BS, the $\mathrm{GS_M}$ and $\mathrm{BS_M}$ tests fail to provide evidence against joint symmetry in the data. The G test rejects normality in X and Y, while it fails to find evidence against normality in Z. In contrast, the BN test fails to reject normality for all X, Y, and Z. In terms of multivariate normality, the $\mathrm{G_M}$ test rejects joint normality in the data. Unlike $\mathrm{G_M}$, the $\mathrm{BN_M}$ test fails to reject joint normality in the data.

6. Conclusions

This study generalized univariate time series normality tests G, $\mathrm{GS_M}$, $\mathrm{BS_M}$, and BN to the multivariate time series case using orthogonalization or empirical standardization of the data. Under the null hypothesis of multivariate normality, the generalized test statistics asymptotically follow the χ^2 distribution, providing readily applicable multivariate normality testing procedures to assess the normality of vector autoregressive processes. Extensive Monte Carlo experiments revealed that in terms of test size, the $\mathrm{GS_M}$ and $\mathrm{G_M}$ test statistics have more accurate sizes than $\mathrm{BS_M}$ and $\mathrm{BN_M}$. In terms of test power, $\mathrm{BN_M}$ outperformed its competitors against interval-based distributions, while $\mathrm{G_M}$ and $\mathrm{GS_M}$ are the most powerful tests against non-normal symmetric and asymmetric distributions, except in a few cases where $\mathrm{BN_M}$ performed better than $\mathrm{GS_M}$. We believe that in multivariate time series settings, these experiments would be useful for practitioners in their empirical work of detecting non-normalities.

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Appendix A

TABLE A1: Empirical size of JB_M, RJB_M, MS, MK, KJB_M under time series settings, $\alpha = 0.05$

Model	T	JB_M	RJB_M	MS	MK	KJB_M
M1	100	0.085	0.113	0.188	0.094	0.170
	500	0.209	0.208	0.371	0.155	0.385
	1000	0.268	0.256	0.416	0.191	0.453
M2	100	0.069	0.096	0.080	0.064	0.067
	500	0.107	0.110	0.106	0.111	0.119
	1000	0.122	0.118	0.119	0.118	0.138
M3	100	0.057	0.080	0.070	0.069	0.065
	500	0.072	0.072	0.132	0.083	0.130
	1000	0.072	0.067	0.142	0.094	0.145
M4	100	0.087	0.122	0.093	0.063	0.081
	500	0.169	0.169	0.179	0.077	0.173
	1000	0.205	0.192	0.197	0.083	0.194
M5	100	0.086	0.123	0.106	0.096	0.100
	500	0.177	0.179	0.232	0.080	0.230
	1000	0.205	0.191	0.246	0.080	0.247

Table A2: Empirical size of the multivariate normality test (time series case), $\alpha = 0.05$

Model	T	GS_M	G_M	BS_M	BN_M
M1	100	0.030	0.027	0.084	0.084
	500	0.041	0.039	0.066	0.090
	1000	0.051	0.049	0.070	0.096
M2	100	0.043	0.037	0.076	0.094
	500	0.048	0.048	0.061	0.096
	1000	0.054	0.054	0.061	0.086
M3	100	0.045	0.048	0.065	0.102
	500	0.050	0.054	0.058	0.094
	1000	0.049	0.051	0.053	0.089
M4	100	0.040	0.038	0.083	0.108
	500	0.047	0.049	0.059	0.095
	1000	0.055	0.052	0.061	0.091
M5	100	0.037	0.040	0.082	0.115
	500	0.049	0.053	0.064	0.103
	1000	0.052	0.057	0.059	0.098

Table A3: Empirical power of the multivariate normality test (time series case) against interval based distributions, $\alpha = 0.05$

		U(0,1)				Beta(2,2)			
Model	T	GS_M	G_M	BS_M	BN_M	GS_M	G_M	BS_M	BN_M
M1	100	0.007	0.004	0.070	0.153	0.015	0.009	0.077	0.119
	500	0.016	0.011	0.063	0.486	0.020	0.011	0.065	0.298
	1000	0.016	0.091	0.058	0.695	0.022	0.036	0.060	0.444
M2	100	0.011	0.003	0.070	0.253	0.016	0.008	0.066	0.174
	500	0.010	0.126	0.060	0.768	0.016	0.035	0.064	0.492
	1000	0.009	0.612	0.061	0.949	0.017	0.209	0.061	0.711
M3	100	0.002	0.002	0.063	0.731	0.006	0.003	0.066	0.424
	500	0.003	0.976	0.056	1.000	0.007	0.471	0.058	0.963
	1000	0.002	1.000	0.059	1.000	0.007	0.968	0.055	0.999
M4	100	0.002	0.047	0.070	0.998	0.004	0.006	0.077	0.816
	500	0.002	1.000	0.061	1.000	0.004	0.999	0.058	1.000
	1000	0.002	1.000	0.057	1.000	0.005	1.000	0.057	1.000
M5	100	0.001	0.041	0.072	1.000	0.002	0.006	0.075	0.865
	500	0.001	1.000	0.059	1.000	0.001	1.000	0.058	1.000
	1000	0.001	1.000	0.057	1.000	0.002	1.000	0.060	1.000

TABLE A4: Empirical power of the multivariate normality test (time series case) against symmetric distributions, $\alpha = 0.05$

	Model	M1			M2			M3			M4			M5		
	T	100	500	1000	100	500	1000	100	500	1000	100	500	1000	100	500	1000
t_1	GS_M	0.958	0.997	0.999	0.975	0.998	0.999	0.998	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000
	G_M	0.997	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	BS_M	0.066	0.010	0.005	0.064	0.009	0.005	0.029	0.005	0.005	0.056	0.005	0.002	0.059	0.004	0.003
	BN_M	0.193	0.089	0.059	0.192	0.068	0.054	0.194	0.072	0.062	0.211	0.061	0.043	0.282	0.085	0.063
t_5	GS_M	0.172	0.305	0.360	0.257	0.421	0.497	0.450	0.680	0.758	0.523	0.743	0.812	0.597	0.830	0.888
	G_M	0.216	0.611	0.847	0.341	0.822	0.973	0.641	0.995	1.000	0.741	1.000	1.000	0.819	1.000	1.000
	BS_M	0.091	0.064	0.056	0.076	0.052	0.050	0.067	0.042	0.040	0.087	0.049	0.047	0.093	0.052	0.047
	BN_M	0.071	0.113	0.255	0.074	0.194	0.477	0.087	0.496	0.875	0.148	0.566	0.833	0.171	0.719	0.934
t_{10}	GS_M	0.067	0.094	0.103	0.101	0.138	0.149	0.164	0.234	0.262	0.207	0.309	0.341	0.240	0.364	0.394
	G_M	0.075	0.162	0.249	0.123	0.290	0.453	0.250	0.640	0.871	0.323	0.826	0.975	0.369	0.885	0.992
	BS_M	0.090	0.066	0.062	0.076	0.056	0.058	0.068	0.054	0.052	0.083	0.056	0.054	0.093	0.058	0.055
	BN_M	0.078	0.061	0.071	0.072	0.065	0.113	0.071	0.110	0.340	0.087	0.224	0.567	0.100	0.263	0.672
Laplace(0,1)	GS_M	0.137	0.208	0.228	0.232	0.315	0.338	0.420	0.543	0.568	0.501	0.647	0.678	0.587	0.732	0.752
	G_M	0.179	0.573	0.831	0.328	0.848	0.984	0.691	0.999	1.000	0.852	1.000	1.000	0.899	1.000	1.000
	BS_M	0.105	0.073	0.066	0.086	0.058	0.055	0.074	0.050	0.045	0.099	0.059	0.057	0.107	0.060	0.054
	BN_M	0.082	0.181	0.455	0.089	0.365	0.791	0.148	0.889	0.998	0.318	0.919	0.993	0.379	0.979	1.000
Logistic(0,1)	GS_M	0.067	0.104	0.109	0.120	0.150	0.156	0.193	0.262	0.279	0.239	0.345	0.369	0.280	0.399	0.420
	G_M	0.076	0.194	0.301	0.143	0.369	0.567	0.309	0.782	0.959	0.396	0.932	0.998	0.454	0.968	1.000
	BS_M	0.091	0.068	0.065	0.078	0.056	0.057	0.069	0.052	0.051	0.086	0.058	0.055	0.089	0.057	0.055
	BN_M	0.074	0.068	0.096	0.073	0.087	0.181	0.070	0.223	0.615	0.101	0.419	0.846	0.112	0.509	0.925

TABLE A5: Empirical power of the multivariate normality test (time series case) against asymmetric distributions, $\alpha = 0.05$

	Model	M1			M2			M3			M4			M5		
	T	100	500	1000	100	500	1000	100	500	1000	100	500	1000	100	500	1000
χ_1^2	GS_M	0.908	1.000	1.000	0.987	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	G_M	0.839	1.000	1.000	0.971	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	BS_M	0.738	0.999	1.000	0.813	0.996	1.000	0.987	1.000	1.000	0.982	1.000	1.000	0.998	1.000	1.000
	BN_M	0.606	0.998	1.000	0.780	0.997	1.000	0.980	1.000	1.000	0.973	1.000	1.000	0.997	1.000	1.000
LN(0,1)	GS_M	0.965	1.000	1.000	0.996	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	G_M	0.941	1.000	1.000	0.990	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	BS_M	0.589	0.909	0.960	0.623	0.839	0.903	0.886	0.983	0.994	0.903	0.983	0.995	0.970	0.999	1.000
	BN_M	0.513	0.914	0.960	0.651	0.880	0.929	0.886	0.982	0.993	0.889	0.982	0.994	0.964	0.998	1.000
Pareto(6,1)	GS_M	0.901	1.000	1.000	0.982	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	G_M	0.848	1.000	1.000	0.966	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	BS_M	0.634	0.964	0.989	0.700	0.930	0.968	0.945	0.996	0.999	0.942	0.998	1.000	0.988	1.000	1.000
	BN_M	0.512	0.961	0.987	0.683	0.948	0.977	0.933	0.997	1.000	0.929	0.997	0.999	0.983	1.000	1.000
A1	GS_M	0.471	0.957	0.998	0.652	0.995	1.000	0.917	1.000	1.000	0.960	1.000	1.000	0.985	1.000	1.000
	G_M	0.477	0.976	1.000	0.666	0.998	1.000	0.943	1.000	1.000	0.978	1.000	1.000	0.992	1.000	1.000
	BS_M	0.226	0.707	0.925	0.282	0.795	0.934	0.465	0.976	0.997	0.517	0.971	0.997	0.643	0.994	1.000
	BN_M	0.174	0.749	0.955	0.258	0.867	0.961	0.473	0.988	0.998	0.580	0.981	0.998	0.709	0.997	1.000
A2	GS_M	0.863	1.000	1.000	0.974	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	G_M	0.804	1.000	1.000	0.947	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	BS_M	0.630	0.979	0.996	0.719	0.960	0.986	0.960	0.998	1.000	0.953	0.999	1.000	0.991	1.000	1.000
	BN_M	0.500	0.975	0.995	0.683	0.970	0.988	0.939	0.998	1.000	0.939	0.999	1.000	0.988	1.000	1.000
A3	GS_M	0.905	1.000	1.000	0.984	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	G_M	0.859	1.000	1.000	0.968	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	BS_M	0.633	0.962	0.989	0.698	0.928	0.965	0.944	0.996	0.999	0.940	0.998	1.000	0.988	1.000	1.000
	BN_M	0.513	0.959	0.986	0.683	0.945	0.975	0.927	0.995	0.999	0.927	0.997	0.999	0.984	1.000	1.000

Appendix B

Figure B1: Empirical distribution of G_M vs theoretical distribution (χ^2_4), Model M1 & ($T = 2000$)

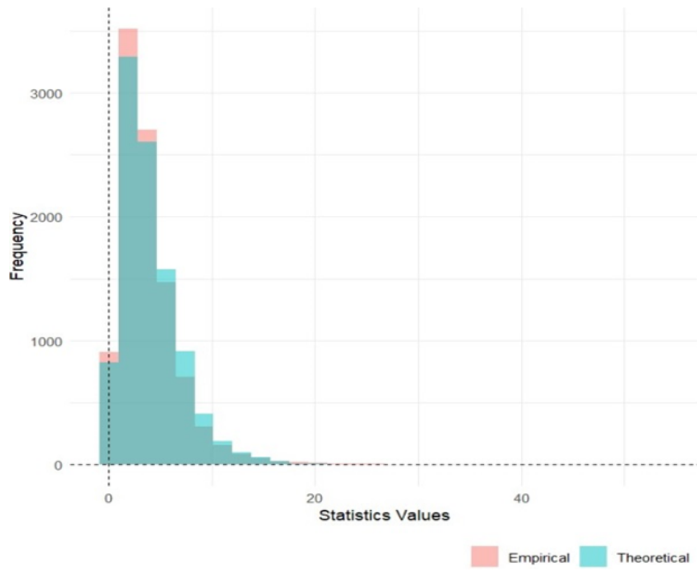


Figure B2: Empirical distribution of G_M vs theoretical distribution (χ^2_4), Model M1 & ($T = 3000$)

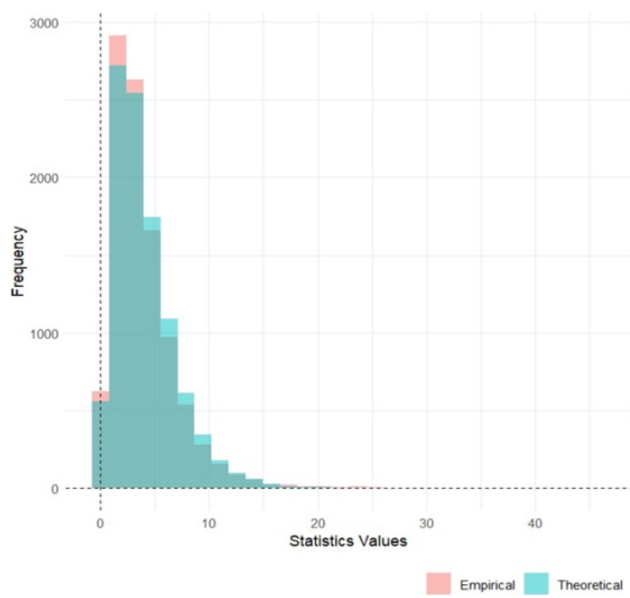


Figure B3: Empirical distribution of GS_M vs theoretical distribution (χ^2_2), Model M1 & ($T = 2000$)

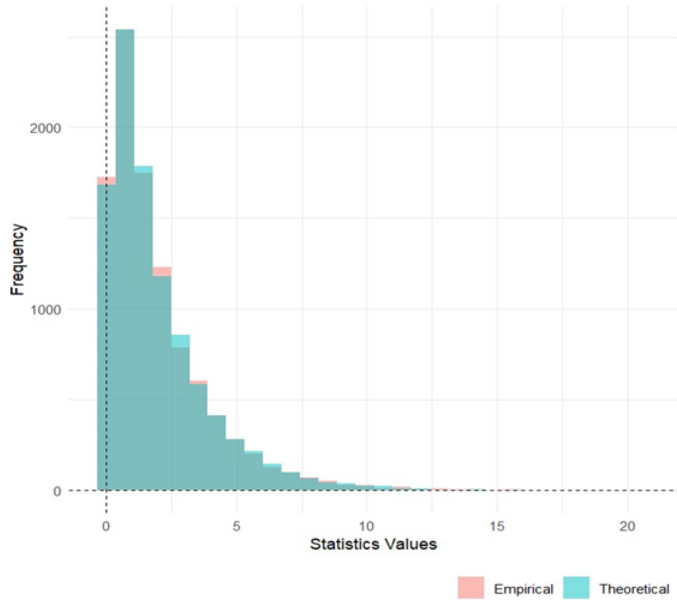


Figure B4: Empirical distribution of GS_M vs theoretical distribution (χ^2_2), Model M1 & ($T = 3000$)

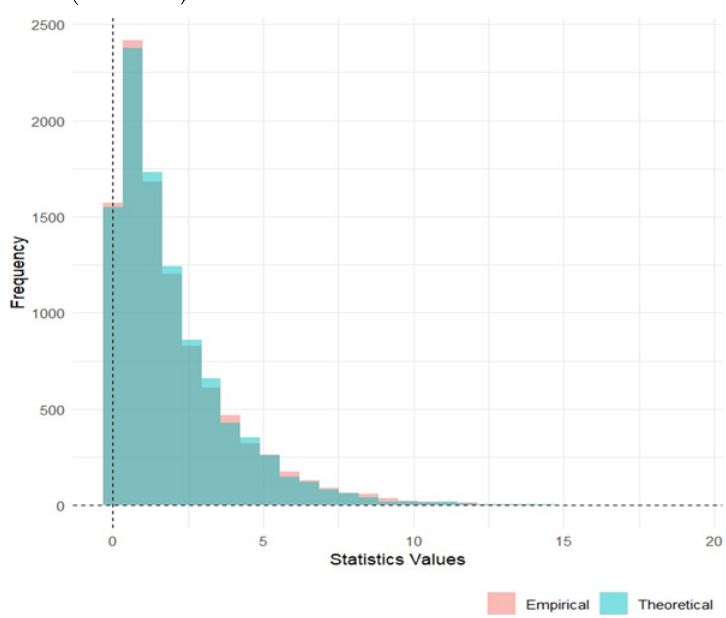


Figure B5: Empirical distribution of $\mathrm{BN_M}$ vs theoretical distribution (χ_4^2), Model M1 & ($T = 2000$)

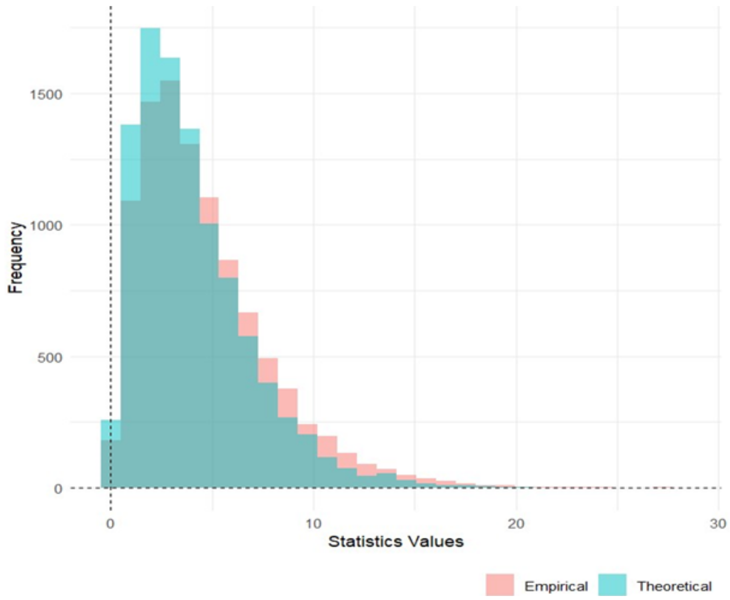


Figure B6: Empirical distribution of $\mathrm{BN_M}$ vs theoretical distribution (χ_4^2), Model M1 & ($T = 3000$)

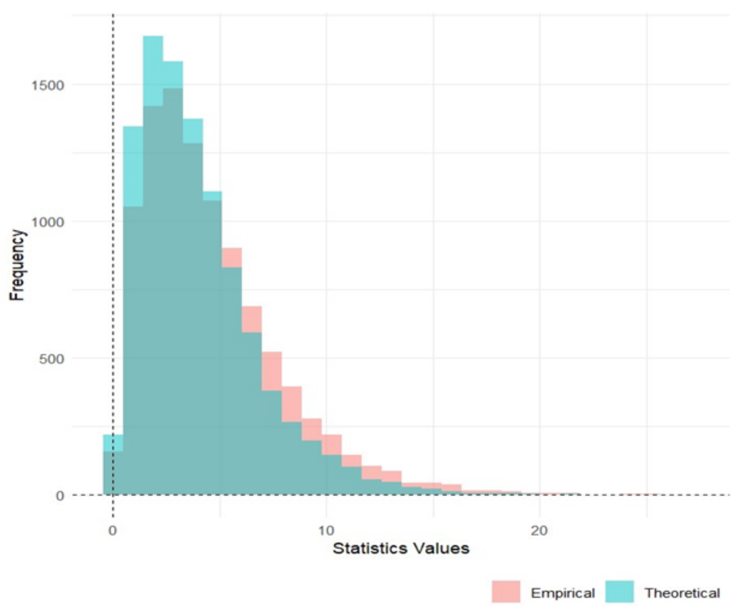


FIGURE B7: Empirical distribution of $\mathrm{BS_M}$ vs theoretical distribution (χ^2_2), Model M1 & ($T = 2000$)

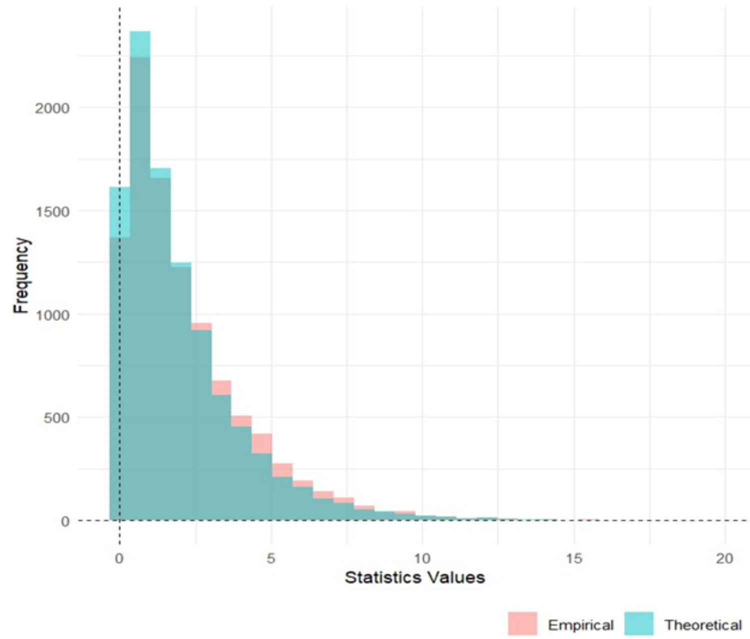


FIGURE B8: Empirical distribution of $\mathrm{BS_M}$ vs theoretical distribution (χ^2_2), Model M1 & ($T = 3000$)

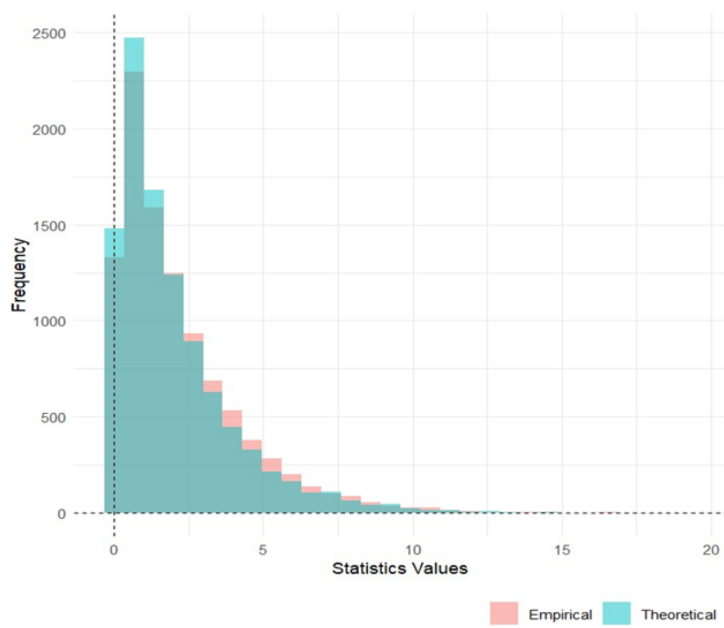


FIGURE B9: Empirical distribution of G_M vs theoretical distribution (χ^2_4), Model M2 & ($T = 2000$)

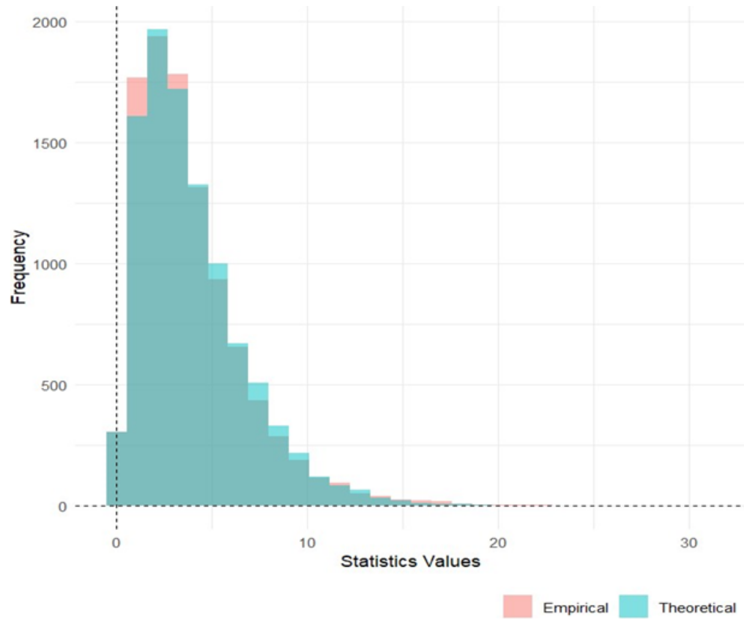


FIGURE B10: Empirical distribution of G_M vs theoretical distribution (χ^2_4), Model M2 & ($T = 3000$)

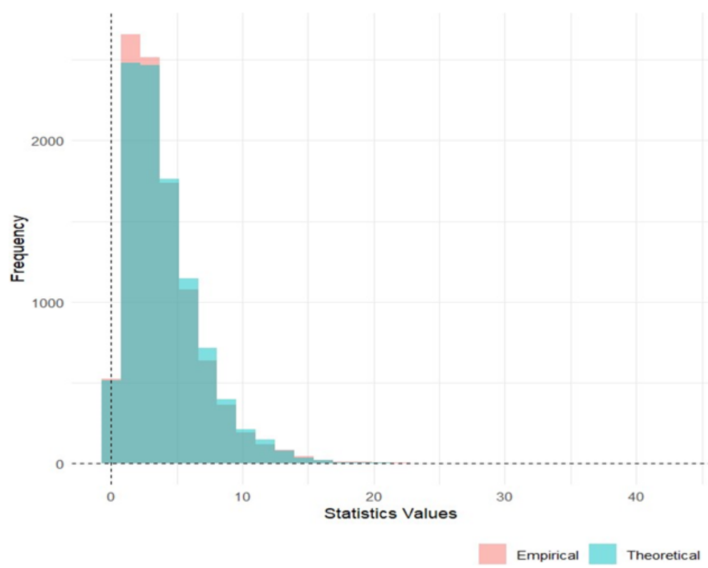


Figure B11: Empirical distribution of GS_M vs theoretical distribution (χ^2_2), Model M2 & ($T = 2000$)

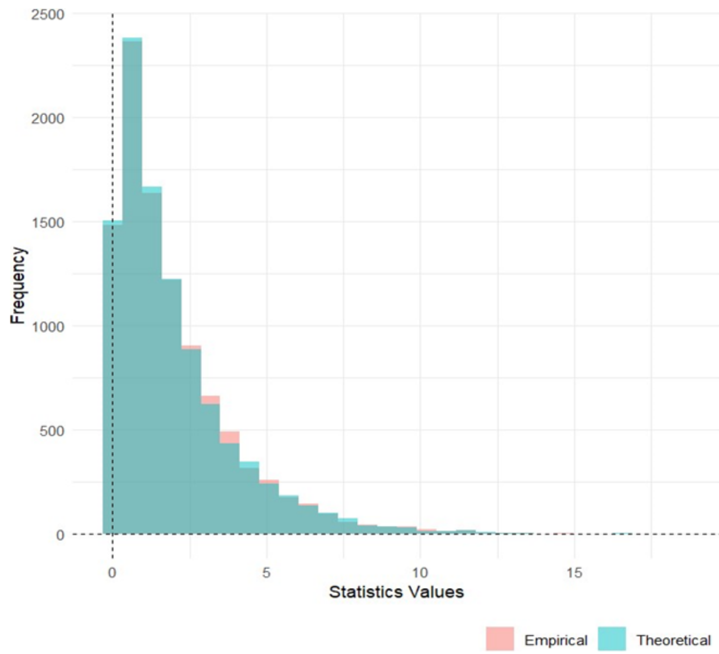


Figure B12: Empirical distribution of GS_M vs theoretical distribution (χ^2_2), Model M2 & ($T = 3000$)

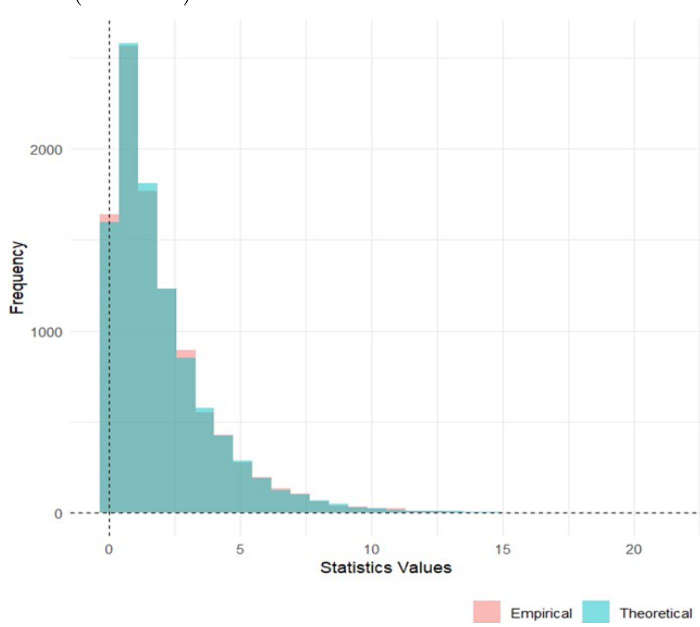


Figure B13: Empirical distribution of $\mathrm{BN_M}$ vs theoretical distribution (χ^2_4), Model M2 & ($T = 2000$)

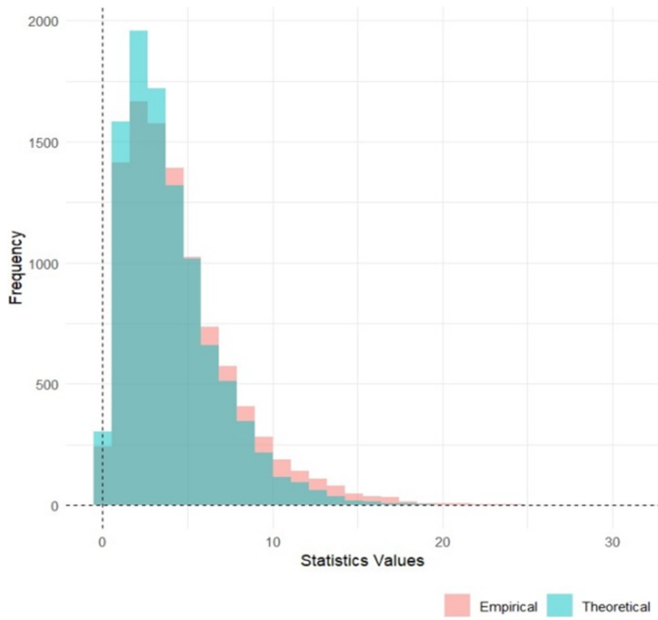


Figure B14: Empirical distribution of $\mathrm{BN_M}$ vs theoretical distribution (χ^2_4), Model M2 & ($T = 3000$)

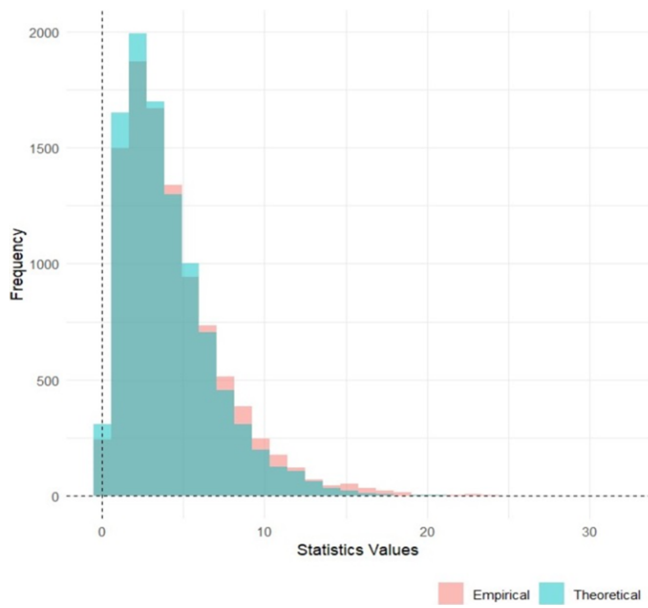


FIGURE B15: Empirical distribution of BS_M vs theoretical distribution (χ^2_2), Model M2 & ($T = 2000$)

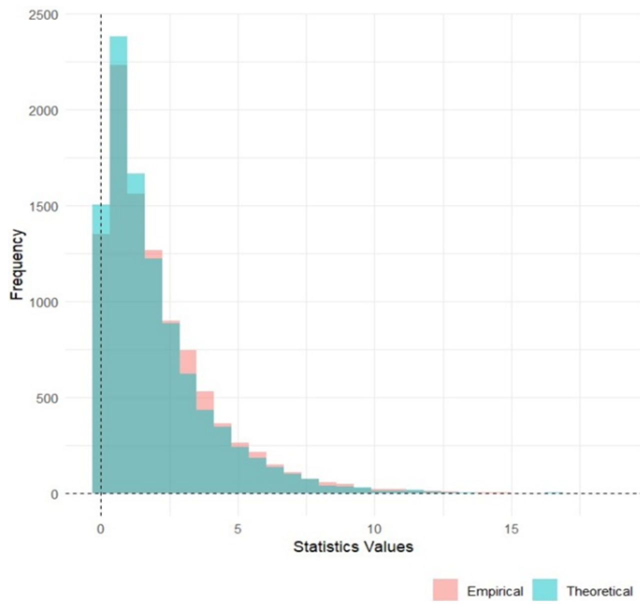


FIGURE B16: Empirical distribution of BS_M vs theoretical distribution (χ^2_2), Model M2 & ($T = 3000$)

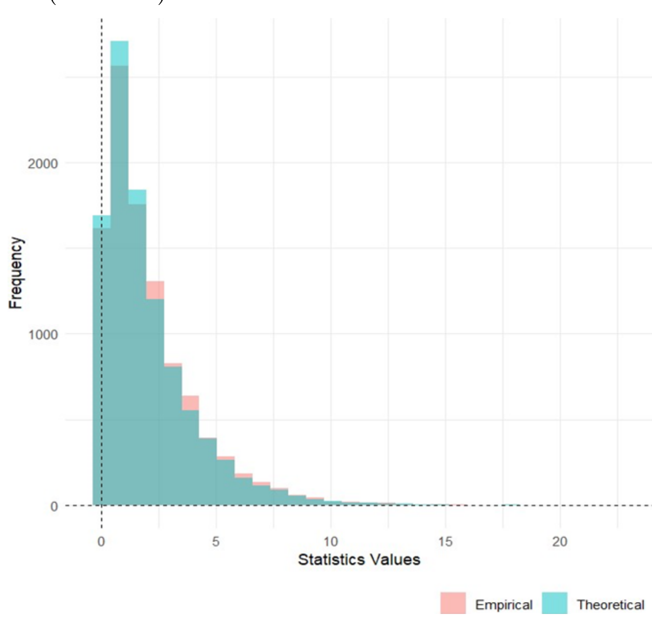


Figure B17: Empirical distribution of G_M vs theoretical distribution (χ_6^2), Model M3 & ($T = 2000$)

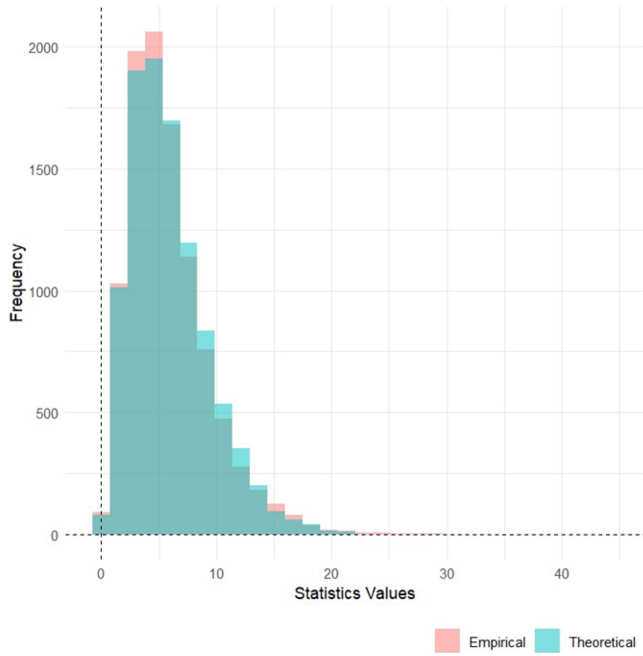


Figure B18: Empirical distribution of G_M vs theoretical distribution (χ_6^2), Model M3 & ($T = 3000$)

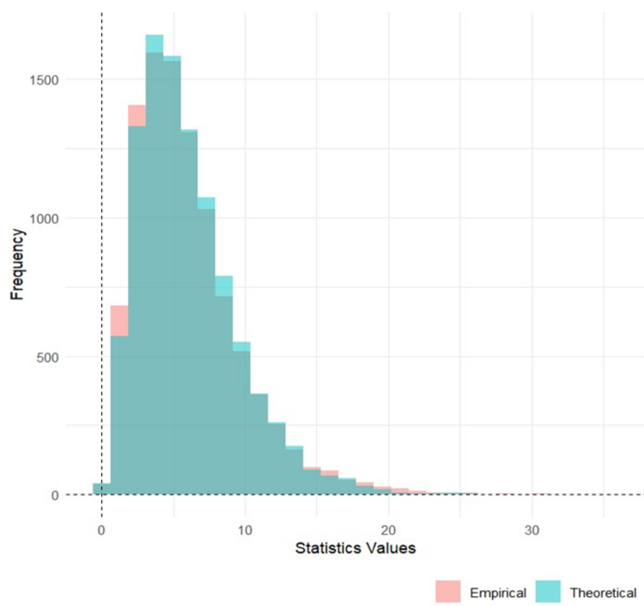


FIGURE B19: Empirical distribution of GS_M vs theoretical distribution (χ^2_3), Model M3 & ($T = 2000$)

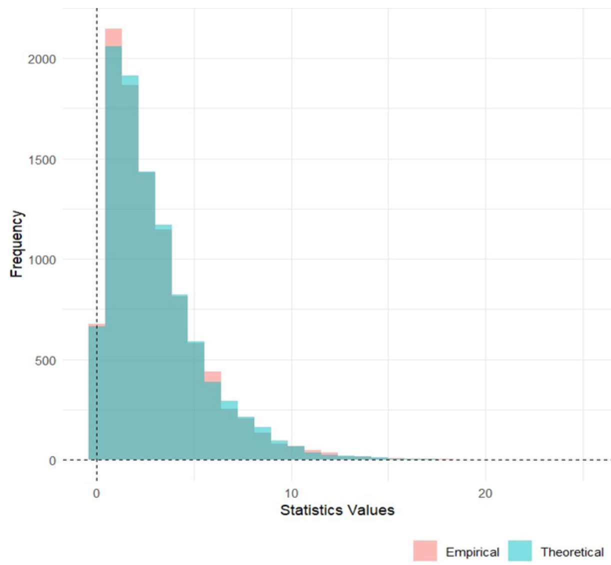


FIGURE B20: Empirical distribution of GS_M vs theoretical distribution (χ^2_3), Model M3 & ($T = 3000$)

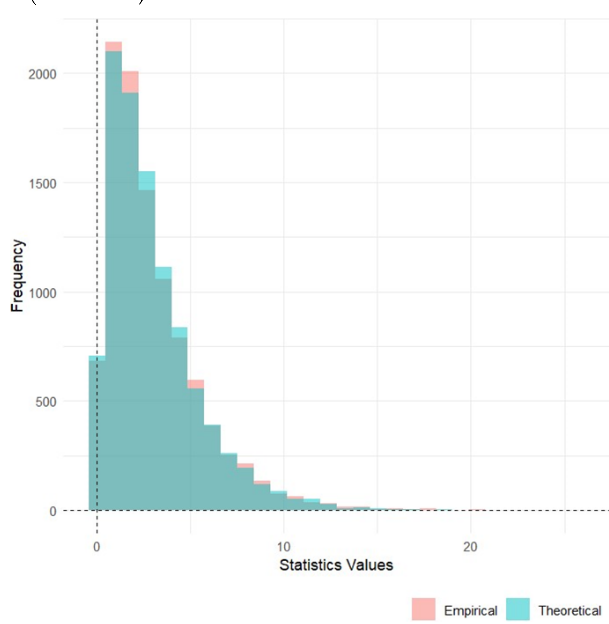


Figure B21: Empirical distribution of $\mathrm{BN_M}$ vs theoretical distribution (χ_6^2), Model M3 & ($T = 2000$)

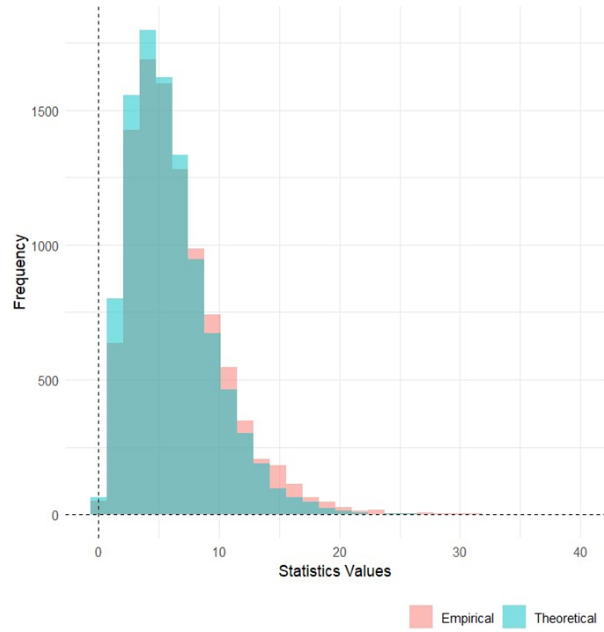


Figure B22: Empirical distribution of $\mathrm{BN_M}$ vs theoretical distribution (χ_6^2), Model M3 & ($T = 3000$)

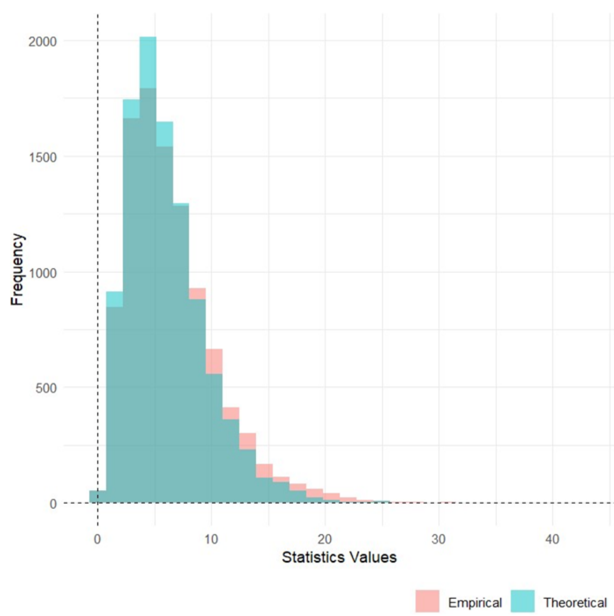


FIGURE B23: Empirical distribution of $\mathrm{BS_M}$ vs theoretical distribution (χ^2_3), Model M3 & ($T = 2000$)

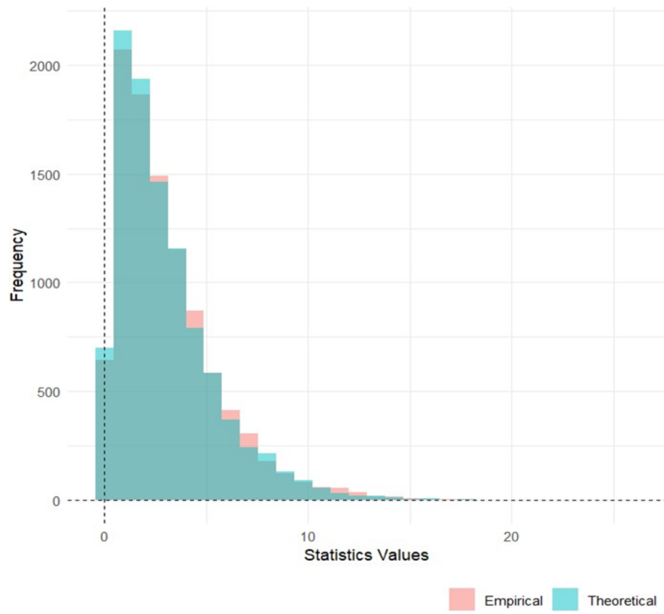
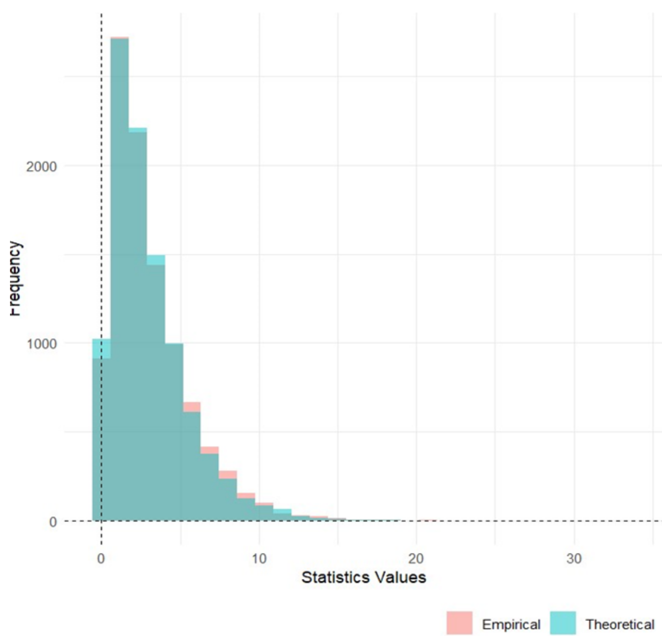


FIGURE B24: Empirical distribution of $\mathrm{BS_M}$ vs theoretical distribution (χ^2_3), Model M3 & ($T = 3000$)



Appendix C

Table C1: West German fixed investment, disposable income, and consumption expenditures in billions of Deutsche Marks (DM)

Period	X	Y	Z	Period	X	Y	Z	Period	X	Y	Z
1960Q1	180	451	415	1967Q4	301	812	715	1975Q3	519	1756	1485
1960Q2	179	465	421	1968Q1	280	837	724	1975Q4	538	1780	1516
1960Q3	185	485	434	1968Q2	289	853	746	1976Q1	549	1807	1549
1960Q4	192	493	448	1968Q3	303	876	758	1976Q2	570	1831	1567
1961Q1	211	509	459	1968Q4	322	897	779	1976Q3	559	1873	1588
1961Q2	202	520	458	1969Q1	315	922	798	1976Q4	584	1897	1631
1961Q3	207	521	479	1969Q2	339	949	816	1977Q1	611	1910	1650
1961Q4	214	540	487	1969Q3	364	979	837	1977Q2	597	1943	1685
1962Q1	231	548	497	1969Q4	371	988	858	1977Q3	603	1976	1722
1962Q2	229	558	510	1970Q1	375	1025	881	1977Q4	619	2018	1752
1962Q3	234	574	516	1970Q2	432	1063	905	1978Q1	635	2040	1774
1962Q4	237	583	525	1970Q3	453	1104	934	1978Q2	658	2070	1807
1963Q1	206	591	529	1970Q4	460	1131	968	1978Q3	675	2121	1831
1963Q2	250	599	538	1971Q1	475	1137	983	1978Q4	700	2132	1842
1963Q3	259	610	546	1971Q2	496	1178	1013	1979Q1	692	2199	1890
1963Q4	263	627	555	1971Q3	494	1211	1034	1979Q2	759	2253	1958
1964Q1	264	642	574	1971Q4	498	1256	1064	1979Q3	782	2276	1948
1964Q2	280	653	574	1972Q1	526	1290	1101	1979Q4	816	2318	1994
1964Q3	282	660	586	1972Q2	519	1314	1102	1980Q1	844	2369	2061
1964Q4	292	694	602	1972Q3	516	1346	1145	1980Q2	830	2423	2056
1965Q1	286	709	617	1972Q4	531	1385	1173	1980Q3	853	2457	2102
1965Q2	302	734	639	1973Q1	573	1416	1216	1980Q4	852	2470	2121
1965Q3	304	751	653	1973Q2	551	1436	1229	1981Q1	833	2521	2145
1965Q4	307	763	668	1973Q3	538	1462	1242	1981Q2	860	2545	2164
1966Q1	317	766	679	1973Q4	532	1493	1267	1981Q3	870	2580	2206
1966Q2	314	779	686	1974Q1	558	1516	1295	1981Q4	830	2620	2225
1966Q3	306	808	697	1974Q2	524	1557	1317	1982Q1	801	2639	2235
1966Q4	304	785	688	1974Q3	525	1613	1355	1982Q2	824	2618	2237
1967Q1	292	794	704	1974Q4	519	1642	1371	1982Q3	831	2628	2250
1967Q2	275	799	699	1975Q1	526	1690	1402	1982Q4	830	2651	2271
1967Q3	273	799	709	1975Q2	510	1759	1452				