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The Zografos-Balakrishnan Type-I Heavy-Tailed-G Family of Distributions with Applications

La familia de distribuciones Zografos-Balakrishnan tipo I de colas pesadas G con aplicaciones

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Abstract

We propose a new family of distributions called the Zografos-Balakrishnan type-I heavy-tailed-G (ZBTIHT-G) distributions. A special model of the proposed family, namely Zografos-Balakrishnan type-I heavy-tailed-Weibull (ZBTIHT-W) model is thoroughly studied. Statistical properties of the new family of distributions including, among others, the hazard rate function, quantile function, moments, distribution of order statistics and Rényi entropy are presented. The maximum likelihood method of estimation is used for estimating the model parameters and Monte Carlo simulation is conducted to examine the performance of the estimators of the model parameters. The flexibility and importance of the new family of distributions are demonstrated by means of applications to real data sets.

***Key words*:** Baseline Distribution; Heavy-Tailed; Actuarial Measures; Estimation; Simulations; Zografos-Balakrishnan.

Resumen

Proponemos una nueva familia de distribuciones Zografos-Balakrishnan tipo I de colas pesadas G con aplicaciones (ZBTIHT-G). Un modelo especial de la familia propuesta Zografos-Balakrishnan tipo I-Weibull de cola pesada (ZBTIHT-W) está profundamente estudiada. Propiedades estadísticas de la nueva familia de distribuciones que incluyen, entre otras, la función de tasa de riesgo. Se presenta la función cuantil, momentos, distribución de estadísticas de orden y entropía de Rényi. Se utiliza el método de estimación

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de máxima verosimilitud para estimar los parámetros del modelo y se realiza una simulación de Monte Carlo para examinar el desempeño de los estimadores de los parámetros del modelo. La flexibilidad e importancia de la nueva familia de distribuciones son demostradas mediante aplicaciones a conjuntos de datos reales.

***Palabras clave*:** Distribución de referencia; Cola pesada; Estimación; Medidas actuariales; Simulaciones; Zografos-Balakrishnan.

1. Introduction

New models in statistical distribution theory are sometimes generated by introducing additional parameters to existing distributions, resulting in flexible and generalized families of distributions. Some recent generators of distributions include gamma-G (type 1) by Zografos & Balakrishnan (2009), gamma-G (type 2) by Ristić & Balakrishnan (2012), transform-transfomer (T-X) by Alzaatreh et al. (2013) and Marshall-Olkin-G by Marshall & Olkin (1997). Examples of recently generalized families of distributions include the Gompertz-Topp-Leone-G by Rannona et al. (2022), transmuted alpha power-G by Eghwerido et al. (2021), Gompertz-G by Alizadeh et al. (2017), generalized odd Lindley-G by Afify et al. (2019) and Type II generalized Topp-Leone family of distributions by Hassan et al. (2019). Zografos & Balakrishnan (2009) introduced a gamma-generator, gamma-G (type 1) named Zografos-Balakrishnan-G (ZB-G) distribution with cumulative distribution function (cdf) and probability density function (pdf) given as

$$F_{ZB-G}(x;\delta,\zeta)=\frac{1}{\Gamma(\delta)}\int_0^{-\log(1-G(x;\zeta))}t^{\delta-1}e^{-t}dt=\frac{\gamma\left(\delta,-\log(1-G(x;\zeta))\right)}{\Gamma(\delta)} \tag{1}$$

and

$$f_{ZB-G}(x;\delta,\zeta)=\frac{1}{\Gamma(\delta)}\big[-\log(1-G(x;\zeta))\big]^{\delta-1}g(x;\zeta), \tag{2}$$

respectively, where $\delta>0$, $\Gamma(\delta)=\int_0^\infty e^{-x}x^{\delta-1}dx$, $G(x;\zeta)$ is the baseline distribution, ζ is a vector of parameters and $\gamma(\delta,x)=\int_0^x t^{\delta-1}e^{-t}dt$ is the incomplete gamma function. Recent distributions in which the Zografos and Balakrishnan transformation was applied to include the Zografos-Balakrishnan log-logistic distribution by Ramos et al. (2013), gamma-generalized Inverse Weibull distribution by Oluyede et al. (2017), gamma-Weibull-G distributions by Oluyede et al. (2018), Zografos-Balakrishnan odd log-logistic generalized half-normal distribution by Mozafari et al. (2019), gamma generalized Lindley-log logistic distribution by Moakofi et al. (2020) and gamma power half-logistic distribution by Arshad et al. (2023). Zhao et al. (2020) developed the type-I heavy-tailed (TIHT) distribution with cdf and pdf given by

$$F_{HT-G}(x;\theta,\psi)=1-\left(\frac{1-G(x;\psi)}{1-(1-\theta)G(x;\psi)}\right)^{\theta}, \tag{3}$$

and

$$f_{HT-G}(x;\theta,\psi) \quad = \quad \frac{\theta^2 g(x;\psi)\left(1-G(x;\psi)\right)^{\theta-1}}{\left(1-(1-\theta)G(x;\psi)\right)^{\theta+1}}, \tag{4}$$

respectively, for $\theta > 0$, $x \in \mathbb{R}$ and parameter vector ψ, where $G(x;\psi)$ is the baseline cdf. Examples of recently developed heavy-tailed distributions include the Weibull-Lomax distribution by Tahir et al. (2015), heavy-tailed log-logistic distribution by Teamah et al. (2021), a new family of heavy-tailed distributions by Ahmad et al. (2022), heavy-tailed beta-power transformed-Weibull distribution by Zhao et al. (2021) and the heavy-tailed exponential distribution by Afify et al. (2020).

In this paper, we propose a new family of heavy-tailed distributions which has more flexibility when fitted to real life data. Its pdf can be almost symmetrical, right-skewed as well as left-skewed. The hazard rate function has shapes that include decreasing, increasing, uni-modal, bathtub, and upside down bathtub, which are applicable in real life situations.

The rest of the paper is organized as follows. In Section 2, we introduce the distribution and its sub-families. In Section 3, mathematical and statistical properties are explored including expansion of the pdf, quantile function, moments, generating function and Rényi entropy. The estimation of the parameters are obtained using different methods of estimation in Section 4. Some special cases are given in Section 5. Also, in this Section, we plot the density function, hazard rate function and present 3D plots of skewness and kurtosis. Comparison of different estimation methods are presented in Section 6 where Monte Carlo simulation study is employed to examine the bias and mean square error of the estimates. Risk measures and numerical studies of actuarial measures are done in Section 7. Applications of a member of the new family of distributions to real data sets are given in Section 8 and conclusions are in Section 9.

2. The New Generalized Distribution

We take the baseline cdf in Equation 1 as the TIHT-G family of distributions to obtain the new family of distributions herein referred to as the ZBTIHT-G distribution. The cdf and pdf of the Zografos-Balakrishnan type-I heavy-tailed-G (ZBTIHT-G) family of distributions are given by

$$\begin{aligned} F(x;\theta,\delta,\zeta) &= \frac{1}{\Gamma(\delta)}\int_0^{-\log\left(\left[\frac{1-G(x;\zeta)}{1-(1-\theta)G(x;\zeta)}\right]^{\theta}\right)} t^{\delta-1}e^{-t}dt \\ &= \frac{\gamma\left(\delta, -\log\left(\left[\frac{1-G(x;\zeta)}{1-(1-\theta)G(x;\zeta)}\right]^{\theta}\right)\right)}{\Gamma(\delta)}, \end{aligned} \tag{5}$$

and

$$f(x;\theta,\delta,\zeta) = \frac{\left[-\log\left(\left[\frac{1-G(x;\zeta)}{1-(1-\theta)G(x;\zeta)}\right]^{\theta}\right)\right]^{\delta-1}\theta^2 g(x;\zeta)\left[1-G(x;\zeta)\right]^{\theta-1}}{\Gamma(\delta)\left[1-(1-\theta)G(x;\zeta)\right]^{\theta+1}}, \quad (6)$$

respectively, for $x, \theta, \delta > 0$, and parameter vector ζ. The hazard rate function (hrf) is given by

$$\begin{aligned} h(x;\theta,\delta,\zeta) &= \frac{\left[-\log\left(\left[\frac{1-G(x;\zeta)}{1-(1-\theta)G(x;\zeta)}\right]^{\theta}\right)\right]^{\delta-1}\theta^2 g(x;\zeta)\left[1-G(x;\zeta)\right]^{\theta-1}}{\Gamma(\delta)\left[1-(1-\theta)G(x;\zeta)\right]^{\theta+1}} \\ &\times \left[1-\frac{\gamma\left(\delta,-\log\left(\left[\frac{1-G(x;\zeta)}{1-(1-\theta)G(x;\zeta)}\right]^{\theta}\right)\right)}{\Gamma(\delta)}\right]^{-1} \end{aligned} \quad (7)$$

for $x, \theta, \delta > 0$, and parameter vector ζ.

3. Some Statistical Properties

Statistical properties of the ZBTIHT-G family of distributions are explored in this section. The statistical properties considered include expansion of the density function as well as the quantile function, moments, generating function, probability weighted moments, Rényi entropy, and distribution of the order statistics.

3.1. Linear Representation of the Density Function

Expansion of the pdf of the ZBTIHT-G family of distributions is considered in this subsection. Let $y = 1-\left[\frac{1-G(x;\zeta)}{1-(1-\theta)G(x;\zeta)}\right]^{\theta}$, $0 < y < 1$. Using the following series expansion: $-\log(1-y) = \sum_{i=0}^{\infty}\frac{y^{i+1}}{i+1}$, we have

$$\left[-\log(1-y)\right]^{\delta-1} = y^{\delta-1}\sum_{j=0}^{\infty}\binom{\delta-1}{j}y^j\left[\sum_{k=0}^{\infty}\frac{y^k}{k+2}\right]^j.$$

Let $a_k = (k+2)^{-1}$, and applying the results of a power series raised to a positive integer (Gradshteyn & Ryzhik, 2014), that is, $\left(\sum_{k=0}^{\infty}a_k y^k\right)^j = \sum_{k=0}^{\infty}b_{k,j}y^k$, where $b_{k,j} = (ka_0)^{-1}\sum_{l=0}^{\infty}\left[l(i+1)-k\right]a_l b_{k-1,j}$ and $b_{0,j} = a_0^j$, we can write the pdf of ZBTIHT-G distribution as

$$
\begin{aligned}
f(x;\theta,\delta,\zeta) &= \sum_{j,k=0}^{\infty}\binom{\delta-1}{j}b_{k,j}y^{k+j+\delta-1}\frac{\theta^2 g(x;\zeta)\left[1-G(x;\zeta)\right]^{\theta-1}}{\Gamma(\delta)\left[1-(1-\theta)G(x;\zeta)\right]^{\theta+1}}\\
&= \sum_{j,k=0}^{\infty}\binom{\delta-1}{j}b_{k,j}\left(1-\left[\frac{1-G(x;\zeta)}{1-(1-\theta)G(x;\zeta)}\right]^{\theta}\right)^{k+j+\delta-1}\\
&\times \frac{\theta^2 g(x;\zeta)\left[1-G(x;\zeta)\right]^{\theta-1}}{\Gamma(\delta)\left[1-(1-\theta)G(x;\zeta)\right]^{\theta+1}}.
\end{aligned}
$$

The pdf of ZBTIHT-G family of distributions can be written as (see appendix for details)

$$
\begin{aligned}
f(x;\theta,\delta,\zeta) &= \sum_{j,k,m,n,p,q,r=0}^{\infty} b_{k,j}\frac{(-1)^{m+n+p+q+r}}{\Gamma(\delta)}\binom{\delta-1}{j}\binom{k+j+\delta-1}{m}\binom{\theta m+\theta-1}{n}\\
&\times \binom{-(\theta m+\theta+1)}{p}\binom{n+p}{q}(1-\theta)^p\theta^2 g(x;\zeta)G^r(x;\zeta)\\
&= \sum_{r=0}^{\infty}U_{r+1}h_{r+1}(x;\zeta), \qquad (8)
\end{aligned}
$$

where

$$
\begin{aligned}
U_{r+1} &= \sum_{j,k,m,n,p,q=0}^{\infty} b_{k,j}\frac{(-1)^{m+n+p+q+r}}{(r+1)\,\Gamma(\delta)}\binom{\delta-1}{j}\binom{k+j+\delta-1}{m}\binom{\theta m+\theta-1}{n}\\
&\times \binom{-(\theta m+\theta+1)}{p}\binom{n+p}{q}(1-\theta)^p\theta^2, \qquad (9)
\end{aligned}
$$

and $h_{r+1}(x;\zeta) = (r+1)\,g(x;\zeta)G^r(x;\zeta)$ is the exponentiated-G (exp-G) density with power parameter $(r+1)$ and parameter vector ζ. The new density can be expressed as an infinite linear combination of exp-G densities and the properties of exp-G family are well established. Consequently, the mathematical and statistical properties of the new family of distributions follows directly from those of exp-G family of distributions.

3.2. Quantile Function

The quantile function of the ZBTIHT-G family of distributions is obtained by inverting the non-linear equation

$$
F_{ZBTIHT-G}(x;\theta,\delta,\zeta) = \frac{\gamma\left(\delta,-\log\left(\left[\frac{1-G(x;\zeta)}{1-(1-\theta)G(x;\zeta)}\right]^{\theta}\right)\right)}{\Gamma(\delta)} = u
$$

for $0 \leq u \leq 1$. Note that

$$
\frac{1-G(x;\zeta)}{1-(1-\theta)G(x;\zeta)} = \left[\exp\left(-\gamma^{-1}\Big(\delta,(1-u)\Gamma(\delta)\Big)\right)\right]^{\frac{1}{\theta}},
$$

which simplifies to

$$G(x;\zeta) \quad = \quad \frac{1-\left[\exp\left(-\gamma^{-1}\Big(\delta,(1-u)\Gamma(\delta)\Big)\right)\right]^{\frac{1}{\theta}}}{1-(1-\theta)\left[\exp\left(-\gamma^{-1}\Big(\delta,(1-u)\Gamma(\delta)\Big)\right)\right]^{\frac{1}{\theta}}}.$$

Thus, the quantile function of the ZBTIHT-G family of distributions is given by

$$Q_G(u;\theta,\delta,\zeta) = G^{-1}\left(\frac{1-\left[\exp\left(-\gamma^{-1}\Big(\delta,u\Gamma(\delta)\Big)\right)\right]^{\frac{1}{\theta}}}{1-(1-\theta)\left[\exp\left(-\gamma^{-1}\Big(\delta,u\Gamma(\delta)\Big)\right)\right]^{\frac{1}{\theta}}}\right). \tag{10}$$

Quantiles are obtained using equation (10) via a specified baseline cdf G using R software.

3.3. Moments and Probability Weighted Moments

Moment, moment generating functions and probability weighted moments (PWMs) of the ZBTIHT-G family of distributions are presented in this section. Using Equation (A1), we can obtain the p^{th} moment of the ZBTIHT-G family of distributions as follows

$$E(X^p) \quad = \quad \int_{-\infty}^{\infty} x^p f(x;\theta,\delta,\zeta)dx = \sum_{r=0}^{\infty} U_{r+1} E(Y_{r+1}^p), \tag{11}$$

where $E(Y_{r+1}^p)$ is the p^{th} moment of Y_{r+1} which follows exp-G distribution with power parameter $(r+1)$ and U_{r+1} is defined as Equation (A2). The moment generating function (MGF) is given by

$$M_X(t) = E(e^{tX}) = \sum_{r=0}^{\infty} U_{r+1} E(e^{tY_{r+1}}),$$

where $E(e^{tY_{r+1}})$ is the moment generating function of the exp-G distribution with power parameter $(r+1)$ and U_{r+1} is given by Equation (A2). The PWMs of a random variable X with cdf F(x) and pdf f(x) are defined by

$$\omega_{a,r} = E\Big(X^a[F(X)]^r\Big) = \int_{-\infty}^{\infty} x^a [F(x)]^r f(x)dx. \tag{12}$$

The PWMs of the ZBTIHT-G family of distributions is given by

$$\omega_{a,r} = \sum_{v=0}^{\infty} \tau_{v+1} \int_{-\infty}^{\infty} x^a h_{v+1}(x;\zeta)dx,$$

where $h_{v+1}(x;\zeta) = (v+1)g(x;\zeta)G^v(x;\zeta)$ is the exp-G density with power parameter $(v+1)$ and parameter vector ζ, and

$$\begin{aligned}\tau_{v+1} &= \sum_{m,n,p,q,r,s,t=0}^{\infty} \frac{(-1)^{m+q+r+s+t+v}\, d_{m+\delta,r} b_{p,n} \theta^2 (1-\theta)^s}{(v+1)\,(\Gamma(\delta))^{r+1}} \binom{m+2\delta-1}{n} \\ &\times \binom{p+n+m+2\delta-1}{q} \binom{\theta q+\theta-1}{r} \binom{-(\theta q+\theta+1)}{s} \binom{r+s}{t} \binom{t}{v}.\end{aligned}$$

Details of derivations of PWMs of the ZBTIHT-G family of distributions are placed in the appendix.

3.4. Rényi Entropy

Rényi entropy of the ZBTIHT-G family of distributions is given in this section. Rényi et al. (1961) is a measure of uncertainty associated with a random variable X and is defined as

$$H_R(v) = \frac{1}{1-v} log\left(\int_0^\infty f^v(x)dx\right)$$

for $v > 0$, $v \neq 1$. From Equation (6), $f^v_{ZBTIHT-G}(x;\theta,\delta,\zeta) = f^v(x)$ can be written as

$$f^v(x) = \frac{\left[-\log\left(\left[\frac{1-G(x;\zeta)}{1-(1-\theta)G(x;\zeta)}\right]^{\theta}\right)\right]^{v(\delta-1)} \theta^{2v} g^v(x;\zeta)\,[1-G(x;\zeta)]^{v(\theta-1)}}{(\Gamma(\delta))^v\,[1-(1-\theta)G(x;\zeta)]^{v(\theta+1)}}.$$

After some simplifications (see appendix for details), Rényi entropy for the ZBTIHT-G family of distributions reduces to

$$H_R(v) = \frac{1}{1-v} log\left[\sum_{q=0}^{\infty} \Phi_q e^{(1-v)I_{REG}}\right], \tag{13}$$

where

$$\begin{aligned}\Phi_q &= \sum_{j,k,l,m,n,p=0}^{\infty} \frac{(-1)^{l+m+n+p+q}}{(\Gamma(\delta))^v \left(\frac{q}{v}+1\right)^v} \binom{v(\delta-1)}{j} \binom{k+j+\delta-1}{l} \binom{\theta l+v(\theta-1)}{m} \\ &\times \binom{-(\theta l+v(\theta+1))}{n} \binom{m+n}{p} \binom{p}{q} (1-\theta)^n \theta^{2v},\end{aligned}$$

and $I_{REG} = \frac{1}{1-v} \log\left(\int_0^\infty \left[\left(\frac{q}{v}+1\right) g(x;\zeta)\big(G(x;\zeta)\big)^{\frac{q}{v}}\right]^v dx\right)$ is the Rényi entropy of the exp-G distribution with power parameter $\frac{q}{v}$. The details leading to equation (13) are placed in the appendix.

3.5. Distribution of Order Statistics

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed ZBTIHT-G random variables. The pdf of the i^{th} order statistic, $X_{i:n}$, can be written as

$$f_{i:n}(x) = \frac{n!f(x)}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}[F(x)]^{j+i-1}.$$

Using the result form the PWMs, with $r = i + j - 1$, the pdf of the i^{th} order statistic for the ZBTIHT-G family of distributions is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\sum_{v=0}^{\infty}(-1)^j\binom{n-i}{j}C_{v+1}h_{v+1}(x;\zeta), \tag{14}$$

where $h_{v+1}(x;\zeta) = (v+1)g(x;\zeta)G^v(x;\zeta)$ and

$$\begin{aligned} C_{v+1} &= \sum_{m,n,p,q,r,s,t=0}^{\infty} \frac{(-1)^{m+q+r+s+t+v}\, d_{m+\delta,j+i-1}b_{p,n}}{(\Gamma(\delta))^{j+i}\,(v+1)}\binom{m+2\delta-1}{n}\binom{p+n+m+2\delta-1}{q} \\ &\times \binom{\theta q+\theta-1}{r}\binom{-(\theta q+\theta+1)}{s}\binom{r+s}{t}\binom{t}{v}(1-\theta)^s\theta^2. \end{aligned}$$

4. Methods of Estimation

We present several methods of estimation in this section.

4.1. Maximum Likelihood Estimation

Suppose $X_1, X_2, \ldots, X_n$ is a random sample obtained from the ZBTIHT-G family with pdf given by Equation (6). Let $\Delta = (\theta, \delta, \zeta)^T$ be the vector of model parameters. The log-likelihood function $\ell_n = \ell_n(\Delta)$ based on a random sample of size n is given by

$$\begin{aligned} \ell_n(\Delta) &= (\delta-1)\sum_{i=1}^{n}\log\left[-\log\left(\left[\frac{1-G(x_i;\zeta)}{1-(1-\theta)\,G(x_i;\zeta)}\right]^{\theta}\right)\right] + 2n\log(\theta) \\ &+ \sum_{i=1}^{n}\log[g(x_i;\zeta)] + (\theta-1)\sum_{i=1}^{n}\log[1-G(x_i;\zeta)] + n\log[\Gamma(\delta)] \\ &- (\theta+1)\sum_{i=1}^{n}\log[1-(1-\theta)\,G(x_i;\zeta)]. \end{aligned} \tag{15}$$

The maximum likelihood estimates of the parameters, denoted by $\hat{\Delta}$ is obtained by solving the non-linear equations $\left(\frac{\partial \ell_n}{\partial\theta}, \frac{\partial \ell_n}{\partial\delta}, \frac{\partial \ell_n}{\partial\zeta_k}\right)^T = \mathbf{0}$ using a numerical method such as Newton-Raphson procedure. Components of the score vector are given in the Appendix A.2.

4.2. Anderson-Darling (AD) and Right-Tail Anderson Darling (RT-AD) Estimation

Let $x_{1:n}, x_{2:n}, \ldots, x_{n:n}$ be the order statistics of a random sample from the ZBTIHT-G family of distributions. The Anderson-Darling estimates (ADEs) of the ZBTIHT-G parameters are obtained by minimizing

$$A(\theta,\delta,\zeta) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log F(x_{i:n};\theta,\delta,\zeta) + \log(1-F(x_{i:n};\theta,\delta,\zeta))\right].$$

The ADEs are obtained by solving the non-linear equations

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_s(x_{i:n};\theta,\delta,\zeta)}{F(x_{i:n};\theta,\delta,\zeta)} - \frac{\Delta_s(x_{n+1-i:n});\theta,\delta,\zeta}{F(x_{n+1-i:n};\theta,\delta,\zeta)}\right] = 0,$$

$s = 1, 2, 3$, where

$$\begin{aligned} \Delta_1(x_{i:n};\theta,\delta,\zeta) &= \frac{\partial}{\partial\theta}F(x_{i:n};\theta,\delta,\zeta) \\ &= \sum_{j=0}^{\infty}\frac{(-1)^j\left[-\log\left(A^\theta\right)\right]^{j+\delta-1}(-\log A)\frac{\partial}{\partial\theta}A}{\Gamma(\delta)j!}, \end{aligned} \tag{16}$$

$$\begin{aligned} \Delta_2(x_{i:n};\theta,\delta,\zeta) &= \frac{\partial}{\partial\delta}F(x_{i:n};\theta,\delta,\zeta) \\ &= \sum_{j=0}^{\infty}\frac{(-1)^j j!\Gamma(\delta)\left(\Gamma(\delta)(j+\delta)\log\left[-\log\left(A^\theta\right)\right]\right) - (j+\delta)\Gamma'(\delta) - \Gamma(\delta)}{\left(\Gamma(\delta)(j+\delta)j!\right)^2}, \end{aligned} \tag{17}$$

$$\begin{aligned} \Delta_3(x_{i:n};\theta,\delta,\zeta) &= \frac{\partial}{\partial\zeta_k}F(x_{i:n};\theta,\delta,\zeta) \\ &= \sum_{j=0}^{\infty}\frac{(-1)^j\left[-\log\left(A^\theta\right)\right]^{j+\delta-1}\theta A^{\theta-1}\frac{\partial}{\partial\zeta_k}A}{A^\theta j!\Gamma(\delta)}, \end{aligned} \tag{18}$$

and

$$A = \frac{1-G(x;\zeta)}{1-(1-\theta)G(x;\zeta)}.$$

To get RT-ADEs, we minimise

$$R(\theta,\delta,\zeta) = \frac{n}{2} - 2\sum_{i=1}^{n}F(x_{i:n};\theta,\delta,\zeta) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log S(x_{n+1-i:n});\theta,\delta,\zeta)$$

with respect to θ, δ and ζ. The RT-ADEs may also be obtained by solving the non-linear equation

$$-2\sum_{i=1}^{n}\Delta_s(x_{i:n};\theta,\delta,\zeta) + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\Delta_s(x_{n+1-i:n};\theta,\delta,\zeta)}{S(x_{n+1-i:n};\theta,\delta,\zeta)} = 0,$$

$s = 1, 2, 3$, where $\Delta_s(x_{i:n};\theta,\delta,\zeta)$ are given in Equations (16), (17) and (18).

4.3. Cramér-von Mises Estimation

To obtain the Cramér-von Mises estimators, the difference between the estimates of the cdf and the empirical cdf are used. The Cramér-von Mises (CVM) estimates are obtained by minimizing

$$C(\theta,\delta,\zeta) \quad = \quad -\frac{1}{12n}+\sum_{i=1}^{n}\left[F(x_{i:n};\theta,\delta,\zeta)-\frac{2i-1}{2n}\right]^2$$

with respect to θ,δ and ζ. The CVM estimates can be obtained by solving the non-linear equation

$$2\sum_{i=1}^{n}\left[F(x_{i:n};\theta,\delta,\zeta)-\frac{2i-1}{2n}\right]\Delta_s(x_{i:n};\theta,\delta,\zeta)=0,$$

$s=1,2,3$, where $\Delta_s(x_{i:n};\theta,\delta,\zeta)$ are given in Equations (16), (17) and (18).

4.4. Ordinary and Weighted Least-Squares Estimation

To get ordinary least-squares estimates, we minimize

$$OLS(\theta,\delta,\zeta) \quad = \quad \sum_{i=1}^{n}\left[F(x_{i:n};\theta,\delta,\zeta)-\frac{i}{n+1}\right]^2$$

with respect to θ,δ and ζ, and solving the non-linear equation

$$2\sum_{i=1}^{n}\left[F(x_{i:n};\theta,\delta,\zeta)-\frac{i}{n+1}\right]\Delta_s(x_{i:n},\theta,\delta,\zeta)=0,$$

$s=1,2,3$, where $\Delta_s(x_{i:n};\theta,\delta,\zeta)$ are given in equations (16), (17) and (18). The weighted least-squares estimates are obtained by minimizing

$$WLS(\theta,\delta,\zeta) \quad = \quad \sum_{i=1}^{n}\frac{(n+1)^2\,(n+2)}{i\,(n-i+1)}\left[F(x_{i:n};\theta,\delta,\zeta)-\frac{i}{n+1}\right]^2,$$

with respect to θ,δ and ζ, and solving the non-linear equation

$$2\sum_{i=1}^{n}\frac{(n+1)^2\,(n+2)}{i\,(n-i+1)}\left[F(x_{i:n};\theta,\delta,\zeta)-\frac{i}{n+1}\right]\Delta_s(x_{i:n};\theta,\delta,\zeta)=0,$$

$s=1,2,3$, and $\Delta_s(x_{i:n};\theta,\delta,\zeta)$ are given in Equations (16), (17) and (18).

4.5. Maximum Product of Spacing

Let $D_i(\theta,\delta,\zeta)=F(x_{i:n};\theta,\delta,\zeta)-F(x_{i-1:n},\theta,\delta,\zeta)$, for $i=1,2,\ldots,n+1$, be the uniform spacing of a random sample from the ZBTIHT-G family of distributions, where $F(x_0;\theta,\delta,\zeta)=0$, $F(x_{n+1},\theta,\delta,\zeta)=1$ and $\sum_{i=1}^{n} D_i(\theta,\delta,\zeta)=1$. The maximum product of spacing estimates can be obtained by maximizing the geometric mean of the spacing

$$MPS(\theta,\delta,\zeta) \quad = \quad \left[\prod_{i=1}^{n+1} D_i(\theta,\delta,\zeta)\right]^{\frac{1}{n+1}},$$

with respect to θ,δ and ζ, or maximizing the logarithm of the geometric mean of sample spacings

$$MP(\theta,\delta,\zeta) \quad = \quad \frac{1}{n+1}\sum_{i=1}^{n+1}\log\Big(D_i(\theta,\delta,\zeta)\Big).$$

The following non-linear equation can be solved to obtain the maximum product of spacing estimates

$$\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\theta,\delta,\zeta)}\left[\Delta_s(x_{i:n};\theta,\delta,\zeta)-\Delta_s(x_{i-1:n};\theta,\delta,\zeta)\right]=0,$$

$s=1,2,3$, and $\Delta_s(x_{i:n};\theta,\delta,\zeta)$ are given in Equations (16), (17) and (18).

5. Some Special Models

We present some special cases of the ZBTIHT-G family of distributions, particularly when the cdf $G(x;\zeta)$ is specified to be Weibull, Burr XII and Rayleigh distributions.

5.1. Zografos-Balakrishnan Type-I Heavy-Tailed-Weibull (ZBTIHT-W) Distribution

Suppose we take the baseline distribution to be the one parameter Weibull distribution with the cdf and pdf given by $G(x;\lambda)=1-e^{-x^{\lambda}}$ and $g(x;\lambda)=\lambda x^{\lambda-1}e^{-x^{\lambda}}$, respectively, for $\lambda>0$. From Equations (5) and (6), we obtain the cdf and pdf of the ZBTIHT-W distribution as

$$F(x;\theta,\delta,\lambda) \quad = \quad \frac{\gamma\left(\delta,-\log\left(\left[\frac{e^{-x^{\lambda}}}{1-(1-\theta)\left[1-e^{-x^{\lambda}}\right]}\right]^{\theta}\right)\right)}{\Gamma(\delta)}$$

and

$$f(x;\theta,\delta,\lambda) \quad = \quad \frac{\left[-\log\left(\left[\frac{e^{-x^{\lambda}}}{1-(1-\theta)\left[1-e^{-x^{\lambda}}\right]}\right]^{\theta}\right)\right]^{\delta-1}\theta^2\lambda x^{\lambda-1}e^{-\theta x^{\lambda}}}{\Gamma(\delta)\left[1-(1-\theta)\left[1-e^{-x^{\lambda}}\right]\right]^{\theta+1}},$$

respectively, for $x,\theta,\delta,\lambda>0$.

Plots of the pdf and hrf of the ZBTIHT-W distribution are given in Figure 1. The pdf of the ZBTIHT-W distribution takes several shapes including reverse-J, right-skewed, left-skewed as well as almost symmetric, whereas the hrf displays upside down bathtub, decreasing and increasing shapes. Tables 1 and 2 gives the quantiles and moments of the ZBTIHT-W distribution for selected parameter values. Figure 2 shows 3D plots of skewness and kurtosis, we notice that when δ is fixed, skewness and kurtosis of the ZBTIHT-W distribution increase as θ and λ increase. Furthermore, when we fix θ, the skewness and kurtosis increase as λ and δ increase. An increase in skewness and kurtosis means that the data has a more concentrated region indicating heavy-tails. Thus, the ZBTIHT-W distribution can model heavy-tailed data.

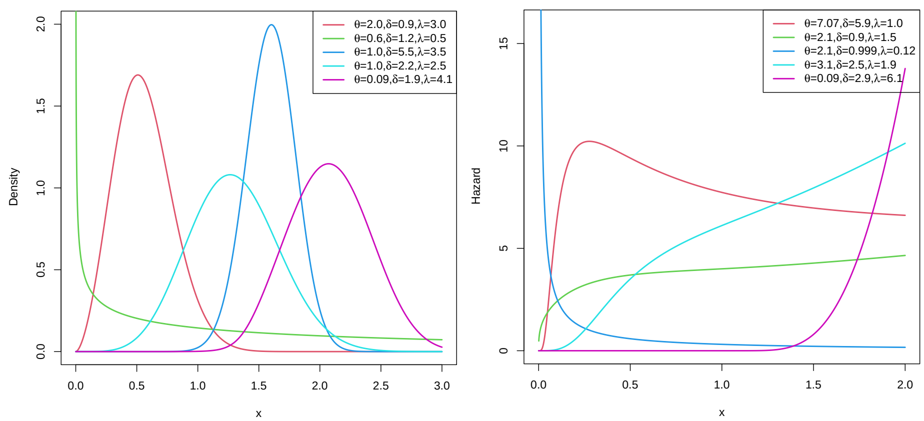

Figure 1: Plots of the Density and Hazard Rate Functions for the ZBTIHT-W Distribution

Table 1: Quantiles for ZBTIHT-W Distribution

	(θ,δ,λ)				
u	$(1, 1.5, 1.3)$	$(1.7, 1.1, 1.8)$	$(1, 0.5, 2.5)$	$(2, 0.3, 2.9)$	$(9.9, 1, 1.5)$
0.1	0.3897	0.1835	0.1441	0.0390	0.0107
0.2	0.5874	0.2738	0.2508	0.0876	0.0151
0.3	0.7704	0.3539	0.3510	0.1379	0.0230
0.4	0.9494	0.4309	0.4543	0.1918	0.0282
0.5	1.1395	0.5118	0.5531	0.2517	0.0392
0.6	1.3470	0.6014	0.6605	0.3177	0.0436
0.7	1.5938	0.7020	0.7801	0.3931	0.0554
0.8	1.9110	0.8362	0.9241	0.4834	0.0659
0.9	2.4007	1.0419	1.1283	0.6165	0.0899

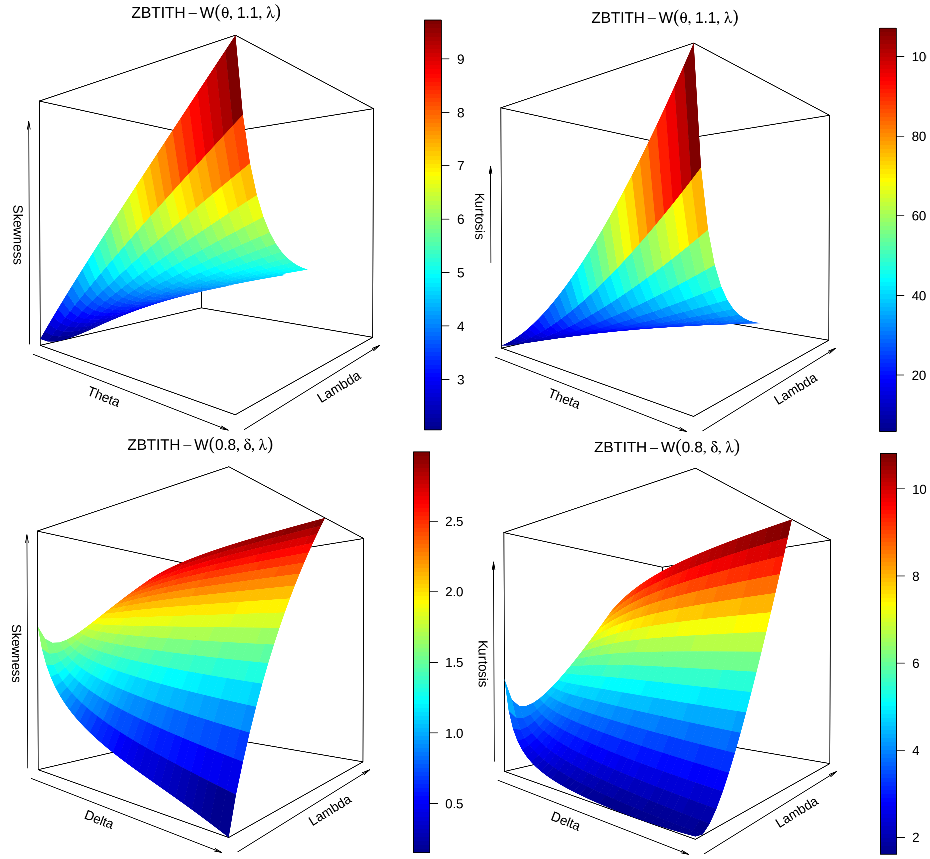

FIGURE 2: Plots of Skewness and Kurtosis for ZBTIHT-W Distribution

TABLE 2: Moments for ZBTIHT-W Distribution

	$(\theta, \delta, \lambda)$				
	$(2, 1.5, 1.3)$	$(2.7, 1.1, 1.8)$	$(3.4, 0.5, 2.5)$	$(4, 1.3, 2.9)$	$(1.9, 1, 1.5)$
E(X)	0.3763	0.3334	0.2347	0.4053	0.3484
$E(X^2)$	0.2121	0.1509	0.0792	0.1856	0.1829
$E(X^3)$	0.1403	0.0829	0.0332	0.0939	0.1156
$E(X^4)$	0.1023	0.0521	0.0163	0.0517	0.0818
$E(X^5)$	0.0796	0.0360	0.0090	0.0306	0.0622
$E(X^6)$	0.0647	0.0267	0.0055	0.0193	0.0497
SD	0.2655	0.1995	0.1553	0.1461	0.2480
CV	0.7056	0.5984	0.6619	0.3606	0.7118
CS	0.3940	0.7647	0.8791	0.4324	0.5911
CK	2.2599	3.2338	3.7323	3.1813	2.5491

5.2. Zografos-Balakrishnan Type-I Heavy-Tailed-Burr XII (ZBTIHT-BXII) Distribution

Suppose we take the baseline distribution to be the Burr XII distribution with the cdf and pdf given by $G(x;c,k) = 1-(1+x^c)^{-k}$ and $g(x;c,k) = ckx^{c-1}(1+x^c)^{-k-1}$, respectively, for $c,k>0$. From Equations (5) and (6), we obtain the cdf and pdf of the ZBTIHT-BXII distribution as

$$F(x;\theta,\delta,c,k) = \frac{\gamma\left(\delta, -\log\left(\left[\frac{(1+x^c)^{-k}}{1-(1-\theta)\left[1-(1+x^c)^{-k}\right]}\right]^{\theta}\right)\right)}{\Gamma(\delta)}$$

and

$$f(x;\theta,\delta,c,k) = \frac{\left[-\log\left(\left[\frac{(1+x^c)^{-k}}{1-(1-\theta)\left[1-(1+x^c)^{-k}\right]}\right]^{\theta}\right)\right]^{\delta-1}\theta^2 ckx^{c-1}(1+x^c)^{-\theta k-1}}{\Gamma(\delta)\left[1-(1-\theta)\left[1-(1+x^c)^{-k}\right]\right]^{\theta+1}},$$

respectively, for $x,\theta,\delta,c,k>0$. Note that when $c=1$, and $k=1$, we obtain the Zografos-Balakrishnan Heavy-Tailed-Lomax (ZBHT-Lom) distribution and Zografos-Balakrishnan Heavy-Tailed-Log-logistic (ZBHT-LLoG) distribution, respectively.

Plots of the pdf and hrf of the ZBTIHT-BXII distribution are given in Figure 3. The pdf of the ZBTIHT-BXII distribution takes several shapes including reverse-J, right-skewed, left-skewed as well as almost symmetric, whereas the hrf displays upside down bathtub, bathtub followed by upside down bathtub, decreasing and increasing shapes. Tables 3 and 4 gives the quantiles and moments of the ZBTIHT-BXII distribution for selected parameter values. Figure 4 shows 3D plots of skewness and kurtosis, we notice that the new distribution can model both negative and positively skewed data, furthermore, when fixing δ and θ, kurtosis of the ZBTIHT-BXII distribution increase as c and k increase. Moreover, when we fix θ and k, there are varying levels of kurtosis as δ and c increase.

Table 3: Quantiles for ZBTIHT-BXII Distribution

			(θ,δ,c,k)		
u	$(2.5,1.9,3.2,0.8)$	$(1.7,1.1,1.8,0.4)$	$(1,0.5,2.5,1.1)$	$(2,0.3,2.9,2.1)$	$(9.9,1,1.5,2.6)$
0.1	0.4969	0.3203	0.1367	0.0303	0.0050
0.2	0.5872	0.4905	0.2425	0.0682	0.0095
0.3	0.6541	0.6560	0.3449	0.1048	0.0123
0.4	0.7174	0.8424	0.4468	0.1495	0.0163
0.5	0.7845	1.0609	0.5562	0.1965	0.0208
0.6	0.8552	1.3446	0.6802	0.2486	0.0257
0.7	0.9399	1.7595	0.8305	0.3032	0.0269
0.8	1.0504	2.4785	1.0449	0.3796	0.0351
0.9	1.2343	4.2848	1.4263	0.4875	0.0463

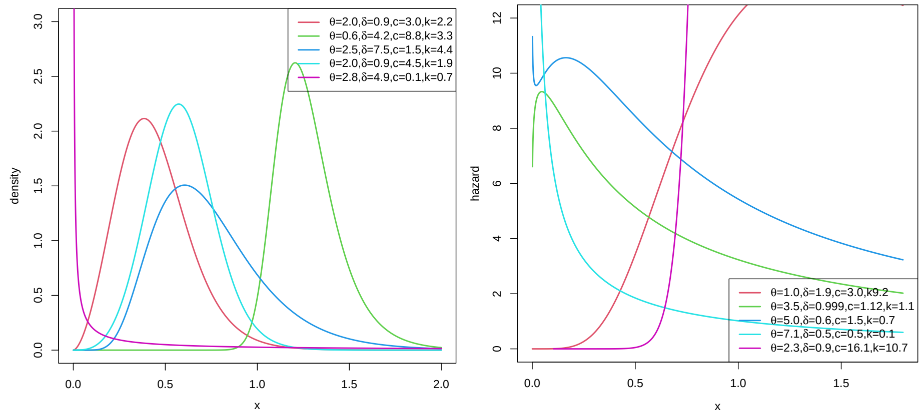

FIGURE 3: Plots of the Density and Hazard Rate Functions for the ZBTIHT-BXII Distribution

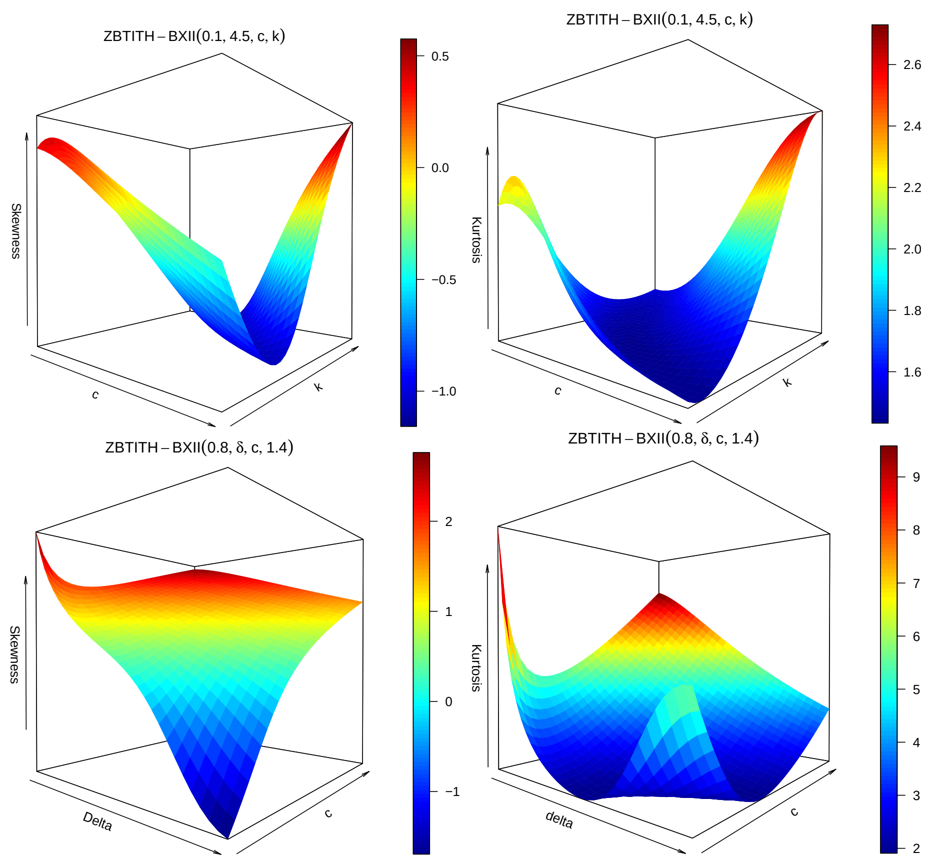

FIGURE 4: Plots of Skewness and Kurtosis for ZBTIHT-BXII Distribution

Table 4: Moments for ZBTIHT-BXII Distribution

	(θ, δ, c, k)				
	$(2, 1.5, 1.3, 1.7)$	$(2.7, 1.1, 1.8, 1.2)$	$(3.4, 0.5, 2.5, 0.8)$	$(2, 1.3, 2.9, 3.1)$	$(1.9, 1, 1.5, 1.1)$
E(X)	0.3702	0.2885	0.3310	0.1395	0.4965
$E(X^2)$	0.3702	0.2885	0.3310	0.1395	0.4965
$E(X^3)$	0.3702	0.2885	0.3310	0.1395	0.4965
$E(X^4)$	0.3702	0.2885	0.3310	0.1395	0.4965
$E(X^5)$	0.3702	0.2885	0.3310	0.1395	0.4965
$E(X^6)$	0.3702	0.2885	0.3310	0.1395	0.4965
SD	0.4829	0.4531	0.4706	0.3464	0.5000
CV	1.3042	1.5704	1.4218	2.4839	1.0070
CS	0.5375	0.9336	0.7184	2.0814	0.0139
CK	1.2889	1.8716	1.5161	5.3321	1.0002

5.3. Zografos-Balakrishnan Type-I Heavy-Tailed-Rayleigh (ZBTIHT-R) Distribution

Suppose we take the baseline distribution to be the Rayleigh distribution with the cdf and pdf given by $G(x;\lambda) = 1 - \exp\left(-\frac{x^2}{2\lambda^2}\right)$ and $g(x;\lambda) = \frac{x}{\lambda^2}\exp\left(-\frac{x^2}{2\lambda^2}\right)$, respectively, for $\lambda > 0$. From Equations (5) and (6), we obtain the cdf and pdf of the ZBTIHT-R distribution as

$$F(x;\theta,\delta,\lambda) = \frac{\gamma\left(\delta, -\log\left(\left[\frac{\exp\left(-\frac{x^2}{2\lambda^2}\right)}{1-(1-\theta)\left(1-\exp\left(-\frac{x^2}{2\lambda^2}\right)\right)}\right]^{\theta}\right)\right)}{\Gamma(\delta)}$$

and

$$f(x;\theta,\delta,\lambda) = \frac{\left[-\log\left(\left(\frac{\exp\left(-\frac{x^2}{2\lambda^2}\right)}{1-(1-\theta)\left(1-\exp\left(-\frac{x^2}{2\lambda^2}\right)\right)}\right)^{\theta}\right)\right]^{\delta-1}\theta^2\frac{x}{\lambda^2}\left[\exp\left(-\frac{x^2}{2\lambda^2}\right)\right]^{\theta}}{\Gamma(\delta)\left[1-(1-\theta)\left(1-\exp\left(-\frac{x^2}{2\lambda^2}\right)\right)\right]^{\theta+1}},$$

respectively, for $\theta, \delta, \lambda > 0$.

Plots of the pdf and hrf of the ZBTIHT-R distribution are given in Figure 5. The pdf takes several shapes including reverse-J, J, and right-skewed, whereas the hrf displays bathtub, bathtub followed by upside down bathtub, decreasing and increasing shapes. Tables 5 and 6 gives the quantiles and moments of the ZBTIHT-R distribution for selected parameter values. Figure 6 shows 3D plots of skewness and kurtosis. We notice that when δ is fixed, skewness and kurtosis of the ZBTIHT-R distribution increase as θ and λ increase. Furthermore, when we fix λ, the skewness and kurtosis increase as θ and δ increase. An increase in skewness and kurtosis means that the ZBTIHT-R distribution can model heavy tailed data.

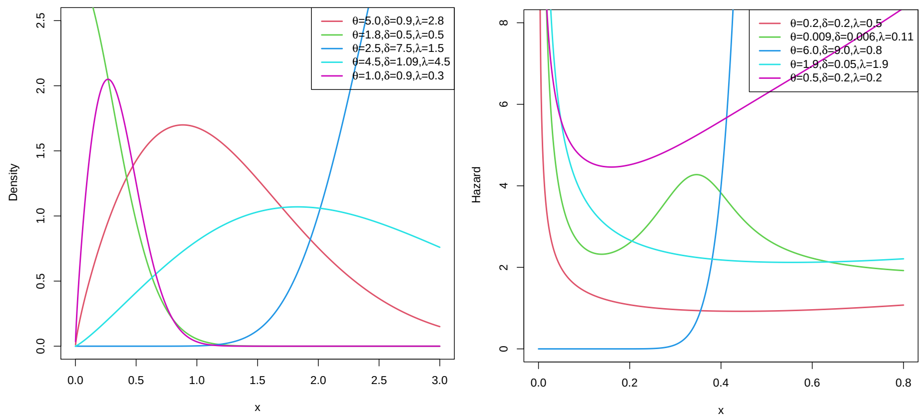

FIGURE 5: Plots of the density and hazard rate functions for the ZBTIHT-R distribution

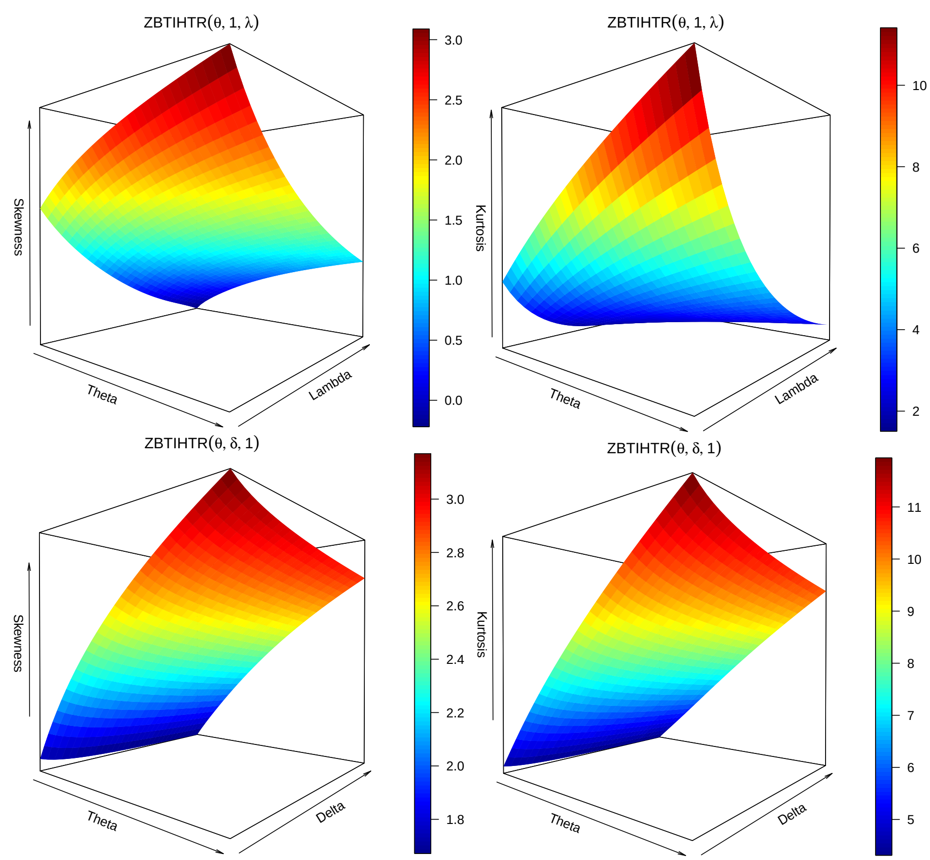

FIGURE 6: Plots of Skewness and Kurtosis for ZBTIHT-R distribution

Table 5: Quantiles for ZBTIHT-R distribution

	$(\theta, \delta, \lambda)$				
u	$(1, 1.5, 1.3)$	$(1.7, 1.1, 1.8)$	$(1, 0.5, 2.5)$	$(2, 0.3, 2.9)$	$(9.9, 1, 1.5)$
0.1	0.9937	0.5573	0.3142	0.0371	0.0720
0.2	1.3032	0.7970	0.6333	0.1186	0.1041
0.3	1.5504	1.0000	0.9651	0.2315	0.1287
0.4	1.7773	1.1932	1.3117	0.3781	0.1550
0.5	2.0000	1.3936	1.6860	0.5569	0.1814
0.6	2.2336	1.6071	2.1044	0.7773	0.2107
0.7	2.4889	1.8527	2.5901	1.0559	0.2422
0.8	2.8029	2.1671	3.2062	1.4309	0.2805
0.9	3.2498	2.6400	4.1135	2.0323	0.3442

Table 6: Moments for ZBTIHT-R distribution

	$(\theta, \delta, \lambda)$				
	$(2, 1.5, 1.3)$	$(2.7, 1.1, 1.8)$	$(3.4, 0.5, 2.5)$	$(4, 1.3, 2.9)$	$(1.9, 1, 1.5)$
E(X)	0.4079	0.5060	0.4464	0.4432	0.3784
$E(X^2)$	0.3118	0.3695	0.2856	0.3387	0.2724
$E(X^3)$	0.2512	0.2895	0.2084	0.2734	0.2116
$E(X^4)$	0.2097	0.2372	0.1635	0.2289	0.1725
$E(X^5)$	0.1797	0.2006	0.1343	0.1967	0.1453
$E(X^6)$	0.1571	0.1737	0.1138	0.1724	0.1254
SD	0.3814	0.3369	0.2939	0.3772	0.3594
CV	0.9351	0.6658	0.6584	0.8512	0.9496
CS	0.0962	-0.3234	0.1499	-0.0526	0.2313
CK	1.3298	1.7367	1.8406	1.3654	1.4815

6. Simulation Study

In order to evaluate the consistency of maximum likelihood estimates of the ZBTIHT-W distribution, Monte Carlo simulation study was conducted. To generate random numbers, the quantile function was used and different sample sizes ($n = 50$, 100, 200, 400, 800 and 1200) were generated via the R package. The packages used include `stats4`, `bbmle`, `stats`, `rootSolve` and `numDeriv`. The random samples were generated for different sets of parameters being:

- $\delta = 0.1, \theta = 9.4, \lambda = 9.4$
- $\delta = 0.6, \theta = 0.6, \lambda = 3.0$
- $\delta = 3.8, \theta = 1.0, \lambda = 3.8$
- $\delta = 3.0, \theta = 0.6, \lambda = 4.6$

We simulate $N = 3000$ samples for the true parameter values given in Table 10 and 11. The tables list the mean MLEs of the model parameters along with the respective average bias (ABias) and root mean square errors (RMSEs). The ABias and RMSE for the estimated parameter, say $\hat{\theta}$ are given by:

$$ABias(\hat{\theta}) = \frac{\sum_{i=1}^{N} \hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{N} (\hat{\theta}_i - \theta)^2}{N}},$$

respectively.

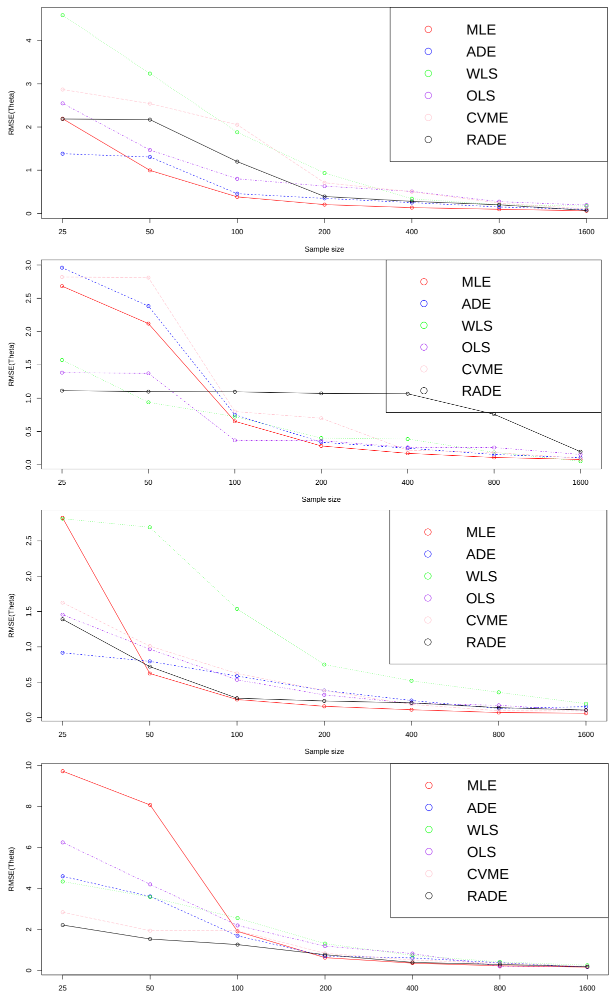

FIGURE 7: Line graphs for RMSE for different estimation methods

Table 7: Simulation results $(0.8, 0.8, 0.8)$

		RMSE						ABias					
Sample size	Parameter	MLE	WLS	ADE	CVM	RADE	OLS	MLE	WLS	ADE	CVM	RADE	OLS
	25	2.1928_3	1.3820_1	4.5918_6	2.5485_4	2.8689_5	2.1861_2	0.6306_1	0.9252_2	2.7960_6	1.5451_3	1.8659_4	-2.1814_5
	50	0.9958_1	1.3053_2	3.2381_6	1.4676_3	2.5422_5	2.1717_4	0.2286_1	0.3344_2	1.0455_3	1.4293_4	1.8283_5	-2.1661_6
	100	0.3816_1	0.4554_2	1.8786_5	0.7998_3	2.0529_6	1.1987_4	0.0940_1	0.2436_2	0.8764_4	0.6427_3	1.5217_6	-1.1788_5
δ	200	0.2026_1	0.3463_2	0.9339_6	0.6305_4	0.7089_5	0.3901_3	0.0348_1	0.1548_3	0.6948_6	0.5989_5	0.0948_2	-0.1898_4
	400	0.1328_1	0.2497_2	0.3381_4	0.5076_6	0.5003_5	0.2781_3	0.0301_1	0.0470_2	0.0803_4	0.3670_6	0.0696_3	-0.1777_5
	800	0.0943_1	0.1477_2	0.1986_3	0.2709_6	0.2388_5	0.1988_4	0.0213_1	0.0250_2	0.0790_4	0.1329_5	0.0391_3	-0.1675_6
	1600	0.0598_1	0.0921_3	0.1681_5	0.1929_6	0.1034_4	0.0680_2	0.0207_4	0.0124_1	0.0173_3	0.0942_6	0.0156_2	-0.0638_5
$\sum Ranks$		9	14	35	32	35	22	10	14	30	32	25	36
	25	2.6367_4	2.9407_5	6.8828_6	2.4653_3	2.4256_2	1.6740_1	0.8820_1	-2.3602_5	3.4239_6	1.1163_2	1.1947_3	-1.6655_4
	50	1.8076_2	2.4247_5	4.1276_6	2.2335_4	2.1839_3	1.2995_1	0.4779_1	-1.6087_5	3.3899_6	1.1042_2	1.2129_4	-1.1986_3
	100	0.6564_1	1.4142_3	1.6255_4	1.8455_5	1.9161_6	1.1984_2	0.1635_1	-0.8338_3	1.4582_6	0.9650_4	1.0666_5	-0.6956_2
θ	200	0.2833_1	0.4378_2	0.9803_4	1.8396_6	1.6916_5	0.6770_3	0.0706_1	-0.4278_3	0.8706_4	0.9390_6	0.8719_5	-0.3668_2
	400	0.1720_1	0.4344_3	0.8052_4	1.5916_6	1.5155_5	0.3783_2	0.0251_1	-0.3028_3	0.7586_5	0.7497_4	0.8179_6	-0.1670_2
	800	0.1100_1	0.3385_3	0.5671_4	0.8962_5	1.0003_6	0.2301_2	0.0155_1	-0.2322_3	0.4598_4	0.7140_5	0.8394_6	-0.0718_2
	1600	0.0803_2	0.1347_3	0.3467_5	0.2986_4	0.3566_6	0.0740_1	0.0077_1	-0.1336_3	0.2894_5	0.2475_4	0.2983_6	-0.0378_2
$\sum Ranks$		12	24	33	33	33	12	7	25	26	27	35	17
	25	2.8265_6	0.9168_1	2.8146_5	1.4561_3	1.6255_4	1.3912_2	0.4772_1	0.5641_2	1.7580_6	0.7780_3	0.9462_4	1.1143_5
	50	0.6232_1	0.7950_3	2.6927_6	0.9676_4	1.0108_5	0.7204_2	0.1621_1	0.4593_2	1.5080_6	0.4771_3	0.5309_4	-0.5312_5
	100	0.2547_1	0.5873_4	1.5376_6	0.5337_3	0.6235_5	0.2730_2	0.0599_2	0.3433_3	1.3777_6	0.4591_5	0.3744_4	-0.0590_1
λ	200	0.1584_1	0.3832_5	0.7496_6	0.3206_3	0.3824_4	0.2341_2	0.0438_1	0.2343_5	0.7251_6	0.2101_4	0.1944_3	-0.0519_2
	400	0.1099_1	0.2413_5	0.5196_6	0.2019_3	0.1845_2	0.2074_4	0.0276_1	0.1585_4	0.3446_6	0.1758_5	0.0772_3	-0.0300_2
	800	0.0707_1	0.1281_3	0.3565_6	0.1726_5	0.0961_2	0.1397_4	0.0269_1	0.0934_4	0.2843_6	0.1561_5	0.0460_3	-0.0277_2
	1600	0.0602_1	0.1537_5	0.1957_6	0.0983_3	0.0933_2	0.1054_4	0.0304_3	0.0599_4	0.1386_6	0.0853_5	0.0109_1	-0.0197_2
$\sum Ranks$		12	21	41	24	24	20	10	24	42	30	22	19
Overall $\sum Ranks$		33	59	109	89	92	54	27	63	98	89	82	72

Table 8: Simulation results $(0.8, 1.6, 1.6)$

		RMSE						ABias					
Sample size	Parameter	MLE	WLS	ADE	CVM	RADE	OLS	MLE	WLS	ADE	CVM	RADE	OLS
	25	2.6828_4	2.9599_6	1.5734_3	1.3835_2	2.8200_5	1.1134_1	0.9280_1	-1.3897_5	0.9605_2	-1.3693_4	-1.8132_6	-1.0084_3
	50	2.1202_4	2.3836_5	0.9382_1	1.3733_3	2.8103_6	1.0986_2	0.5489_1	-0.8014_4	0.7812_3	-0.6602_2	-1.8064_6	-0.9491_5
	100	0.6532_2	0.7521_4	0.7235_3	0.3667_1	0.7998_5	1.0962_6	0.1643_1	-0.3503_2	0.6198_4	-0.3628_3	-0.7981_5	-0.9563_6
δ	200	0.2832_1	0.3390_2	0.4015_4	0.3624_3	0.6993_5	1.0721_6	0.0707_1	-0.3386_2	0.3880_4	-0.3605_3	-0.6989_5	-0.9243_6
	400	0.1722_1	0.2491_3	0.3868_5	0.2623_4	0.2183_2	1.0670_6	0.0247_1	-0.2429_3	0.3545_5	-0.2616_4	-0.1717_2	-0.9058_6
	800	0.1101_1	0.1553_2	0.1867_4	0.2597_5	0.1857_3	0.7590_6	0.0153_1	-0.1455_4	0.1443_3	-0.1595_5	-0.0856_2	-0.6449_6
	1600	0.0803_2	0.1064_4	0.0531_1	0.1557_5	0.0981_3	0.1968_6	0.0075_1	-0.0464_3	0.0175_2	-0.0554_4	-0.0798_5	-0.3218_6
$\sum Ranks$		15	26	21	27	29	33	7	23	23	25	31	38
	25	0.5600_1	0.9458_4	0.7198_3	0.9769_5	0.6033_2	2.0574_6	-0.0571_1	-0.6529_5	-0.3927_4	-0.0902_2	-0.3825_3	1.0938_6
	50	0.4233_1	0.8006_4	0.5780_3	0.8234_5	0.5396_2	1.2980_6	-0.0873_1	-0.2518_3	-0.3471_5	-0.0896_2	-0.3346_4	1.0299_6
	100	0.3196_1	0.6904_5	0.3901_2	0.7588_6	0.4110_3	0.4764_4	-0.0567_1	-0.1699_3	-0.3397_6	-0.0732_2	-0.3066_4	0.3189_5
θ	200	0.2235_1	0.5808_5	0.2357_2	0.6043_6	0.2712_3	0.4448_4	-0.0197_1	-0.0736_3	-0.1629_4	-0.0593_2	-0.2667_6	0.2358_5
	400	0.1548_1	0.3618_4	0.1929_2	0.4632_6	0.2149_3	0.3885_5	-0.0127_1	-0.0596_3	-0.0835_5	-0.0451_2	-0.1992_6	0.0659_4
	800	0.0752_1	0.3362_6	0.1566_2	0.2995_5	0.1892_3	0.2771_4	-0.0029_1	-0.0352_4	-0.0403_5	-0.0268_3	-0.0889_6	0.0267_2
	1600	0.0552_1	0.0996_3	0.0580_2	0.2223_6	0.1323_4	0.1711_5	-0.0008_1	-0.0090_2	-0.0122_3	-0.0176_5	-0.0426_6	0.0152_4
$\sum Ranks$		7	31	16	39	20	34	7	23	32	18	35	32
	25	9.7185_6	4.5911_4	4.3368_3	6.2438_5	2.8380_2	2.2119_1	2.9062_2	-3.5574_5	3.0313_3	-3.2120_4	-3.8340_6	-1.7360_1
	50	8.0701_6	3.6096_4	3.5797_3	4.1978_5	1.9376_2	1.5308_1	1.7371_2	-2.5894_6	2.2328_5	-2.1924_4	-1.8312_3	-1.2243_1
	100	1.9024_3	1.6860_2	2.5523_6	2.1926_5	1.9350_4	1.2585_1	0.4420_1	-0.6722_2	1.1232_5	-0.8897_4	-0.7387_3	-1.1357_6
λ	200	0.6228_1	0.6989_2	1.3057_5	1.1853_6	0.8334_4	0.7659_3	0.1726_1	-0.4312_3	-1.1016_6	-0.8248_5	-0.6329_4	-0.2561_2
	400	0.3550_1	0.6074_4	0.7314_5	0.8218_6	0.5026_3	0.3886_2	0.0598_1	-0.3794_3	0.6513_5	-0.7868_6	-0.4019_4	-0.1270_2
	800	0.2224_2	0.3740_5	0.4138_6	0.1946_1	0.3476_4	0.2886_3	0.0349_1	-0.2733_6	0.2520_5	-0.1930_3	-0.2475_4	-0.0715_2
	1600	0.1611_1	0.1679_2	0.2503_6	0.1754_3	0.1844_5	0.1805_4	0.0186_2	-0.1676_6	0.1466_5	-0.0497_3	-0.1438_4	-0.0126_1
$\sum Ranks$		20	23	34	31	24	15	10	31	34	29	28	15
$Overall \sum Ranks$		42	80	71	97	73	82	24	77	89	72	94	85

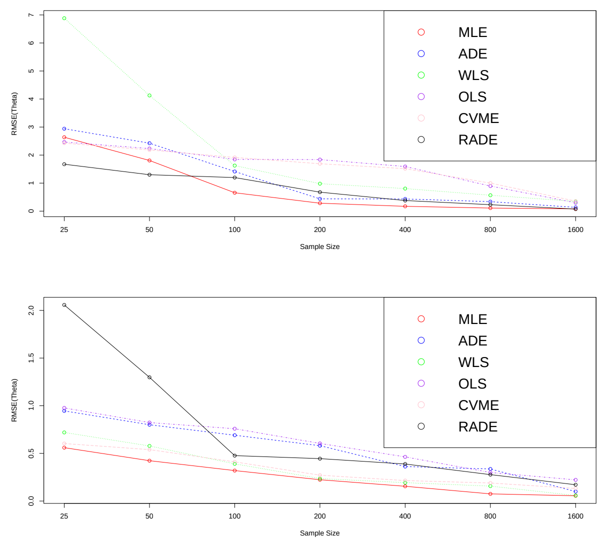

Figure 8: Line graphs for RMSE for different estimation methods

Simulation study of different methods of estimation was done and results are shown in Tables 7 and 8. Ranks were used in order to check which method of estimation performed better than others. The method of estimation with the lowest total sum of ranks was considered the best method. From Tables 9, we can conclude that MLE method performed better than other methods of estimation as it had the lowest value of total sum of ranks. As a result, this method will be employed when doing applications of the ZBTIHT-W distribution.

Table 9: Simulation results

	ABias & RMSE	
Method of estimation	Total $\sum Ranks$	Overall rank
MLE	126	1
WLS	279	2
ADE	386	6
CVM	348	5
RADE	341	4
OLS	293	3

The results in Tables 10 and 11 show that the mean estimates of the parameters are approaching the true parameter value as the sample size increases. In addition, the RMSEs and the average bias decrease with increasing sample size thereby suggesting consistency of the MLE of the model parameters.

Table 10: Simulation Results

		(0.1, 9.4, 9.4)			(0.6, 0.6, 3.0)		
Parameter	Sample size	Mean	RMSE	ABias	Mean	RMSE	ABias
θ	50	0.3474	0.6753	0.2474	0.7152	0.4573	0.1152
	100	0.2504	0.4590	0.1504	0.6348	0.2301	0.0348
	200	0.2037	0.3387	0.1037	0.6189	0.1491	0.0189
	400	0.1524	0.2118	0.0524	0.6071	0.1005	0.0071
	800	0.1344	0.1527	0.0344	0.6068	0.0707	0.0068
	1200	0.1242	0.1099	0.0242	0.6055	0.0572	0.0055
δ	50	12.2491	7.5567	2.8491	0.7906	0.7544	0.1906
	100	11.1904	5.5371	1.7904	0.6572	0.2718	0.0572
	200	10.7075	4.3472	1.3075	0.6275	0.1556	0.0275
	400	10.0668	2.8263	0.6668	0.6100	0.1002	0.0100
	800	9.8723	2.1713	0.4723	0.6090	0.0709	0.0090
	1200	9.7563	1.6927	0.3563	0.6067	0.0563	0.0067
λ	50	9.0033	1.7004	-0.3967	3.0918	0.8017	0.0918
	100	9.0792	1.3714	-0.3208	3.0752	0.5463	0.0752
	200	9.1104	1.1400	-0.2896	3.0310	0.3712	0.0310
	400	9.2170	0.8479	-0.1830	3.0209	0.2616	0.0209
	800	9.2392	0.6659	-0.1608	3.0010	0.1812	0.0010
	1200	9.2686	0.5603	-0.1314	3.0010	0.1488	0.0010

Table 11: Simulation Results

		(3.8, 1.0, 3.8)			(3.0, 0.6, 4.6)		
Parameter	Sample size	Mean	RMSE	ABias	Mean	RMSE	ABias
θ	50	5.2261	3.6797	1.4261	3.6372	1.9096	0.6372
	100	4.2469	1.5139	0.4469	3.2242	0.5760	0.2242
	200	3.9409	0.3524	0.1409	3.0995	0.2915	0.0995
	400	3.8651	0.2181	0.0651	3.0521	0.1921	0.0521
	800	3.8287	0.1415	0.0287	3.0240	0.1249	0.0240
	1200	3.8228	0.1120	0.0228	3.0206	0.1008	0.0206
δ	50	3.4798	9.7512	2.4798	1.3476	4.6870	0.7476
	100	1.5373	3.9579	0.5373	0.6893	0.5149	0.0893
	200	1.1147	0.4723	0.1147	0.6330	0.2036	0.0330
	400	1.0437	0.2632	0.0437	0.6089	0.1283	0.0089
	800	1.0341	0.1796	0.0341	0.6079	0.0897	0.0079
	1200	1.0256	0.1416	0.0256	0.6052	0.0716	0.0052
λ	50	4.0615	1.9135	0.2615	4.8771	2.0364	0.2771
	100	3.9764	1.2271	0.1764	4.8654	1.3917	0.2654
	200	3.8462	0.7191	0.0462	4.7272	0.9050	0.1272
	400	3.8326	0.5000	0.0326	4.6888	0.6242	0.0888
	800	3.7921	0.3418	-0.0079	4.6295	0.4230	0.0295
	1200	3.7921	0.2815	-0.0079	4.6237	0.3484	0.0237

7. Actuarial Measures

Risk measures are calculated in this section, and these includes value at risk (VaRq), tail value at risk ($TVaR_q$), tail variance (TV_q) and tail variance premium (TVP_q) for the ZBTIHT-G family of distributions. These risk measures play a very important role in portfolio optimization under uncertainty.

7.1. Value at Risk Measure

Let X follow the ZBTIHT-G family of distributions with pdf (6), then the VaR_q, where q is a specified level of significance, is given by

$$VaR_q = x_q = G^{-1}\left(\frac{1-\left[\exp\left(-\gamma^{-1}\Big(\delta, q\Gamma(\delta)\Big)\right)\right]^{\frac{1}{\theta}}}{1-(1-\theta)\left[\exp\left(-\gamma^{-1}\Big(\delta, q\Gamma(\delta)\Big)\right)\right]^{\frac{1}{\theta}}}\right),$$

for $0 \leq q \leq 1$.

7.2. Tail Value at Risk Measure

This measure is used to determine the expected value of loss given that an event outside a given probability level has occurred. Let X have ZBTIHT-G pdf, then using Equations (A1) and (A2), $TVaR_q$ of X is computed as

$$\begin{aligned} TVaR_q(X) &= \frac{1}{1-q}\int_{VaR_q}^{\infty} x f(x;\theta,\delta,\zeta)dx \\ &= \frac{1}{1-q}\sum_{r=0}^{\infty}\int_{VaR_q}^{\infty} xU_{r+1}h_{r+1}(x;\zeta)dx, \end{aligned}$$

where U_{r+1} is defined as Equation (A2) and $h_{r+1}(x;\zeta) = (r+1)\,g(x;\zeta)G^r(x;\zeta)$ is the exp-G density with power parameter $(r+1)$ and parameter vector ζ.

7.3. Tail Variance Measure

Tail variance (TV_q) is an actuarial measure that pays attention to the tail variance beyond VaR_q. TV_q is given as

$$\begin{aligned} TV_q &= E\left(X^2|X > x_q\right) - (TVaR_q)^2 \\ &= \frac{1}{1-q}\int_{VaR_q}^{\infty} x^2 f(x;\theta,\delta,\zeta)dx - (TVaR_q)^2 \\ &= \frac{1}{1-q}\sum_{l=0}^{\infty}\int_{VaR_q}^{\infty} x^2U_{r+1}h_{r+1}(x;\zeta)dx - (TVaR_q)^2, \end{aligned}$$

where U_{r+1} is defined as Equation (A2) and $h_{r+1}(x;\zeta) = (r+1)\,g(x;\zeta)G^r(x;\zeta)$ is the exp-G density with power parameter $(r+1)$ and parameter vector ζ.

7.4. Tail Variance Premium Measure

TVP_q is one of the important actuarial measures and is given by

$$TVP_q = TVaR_q + \delta\left(TV_q\right),$$

for $0 < \delta < 1$.

7.5. Numerical Study of Actuarial Measures

In this section, numerical study of actuarial measures is done for the ZBTIHT-W, type-I heavy-tailed-Weibull (TIHT-W), and Weibull distributions for different sets of parameters. Firstly, a random sample of size $n = 100$ is generated from these distributions, and the maximum likelihood method of estimation is used to estimate the model parameters. Secondly, a repetition of 1000 iterations are made in order to find the values of the risk measures for the distributions. The model with the highest values of risk measures has the heavier tail.

Simulated results of VaR, TVaR, TV and TVP are shown in Table 12. These results show that the ZBTIHT-W distribution has higher values of risk measures as compared to the TIHT-W and Weibull distributions, respectively. Therefore, it is evident that ZBTIHT-W has a heavier tail and it can be used effectively to model heavy-tailed data.

Table 12: Simulation results of VaR, TVaR, TV and TVP

Distribution	Level of Significance	VaR	TVaR	TV	TVP
ZB-TIHT-W	0.7	1.7684	2.6056	2.0394	4.0332
	0.75	1.8813	2.7623	2.2952	4.4836
$\delta = 0.5$	0.8	2.0250	2.9653	2.6549	5.0893
$\theta = 1.8$	0.85	2.2222	3.2479	3.2068	5.9737
$\lambda = 6.3$	0.9	2.5281	3.6909	4.1912	7.4629
	0.95	3.1448	4.5915	6.6417	10.9011
	0.7	0.1602	0.5804	0.4922	0.9250
TIHT-W	0.75	0.2062	0.6601	0.5499	1.0725
$\alpha = 0.6$	0.8	0.2702	0.7661	0.6270	1.2677
$\theta = 0.1$	0.85	0.3653	0.9167	0.7376	1.5436
$\gamma = 6.3$	0.9	0.5242	1.1565	0.9169	1.9818
	0.95	0.8665	1.6449	1.2951	2.8752
	0.7	1.0298	1.1209	0.0048	1.1243
	0.75	1.0530	1.1368	0.0043	1.1400
Weibull	0.8	1.0781	1.1547	0.0037	1.1577
$\lambda = 6.3$	0.85	1.1065	1.1756	0.0031	1.1783
	0.9	1.1409	1.2019	0.0026	1.2042
	0.95	1.1894	1.2406	0.0019	1.2424

8. Applications

In this section, we present examples to illustrate the flexibility and usefulness of the ZBTIHT-W distribution for data modelling. Several goodness-of-fit statistics are used to compare ZBTIHT-W distribution to other equi-parameter models. These include: -2log-likelihood statistic ($-2\ln(L)$), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), where $L = L(\hat{\Delta})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters. We also used the Cramér-von Mises (W^*), Anderson-Darling (A^*), the Kolmogorov-Smirnov (K-S) goodness-of-fit statistic as well as its associated p-value, and Sum of Squares (SS) from the probability plots to assess goodness-of-fit. The Sum of Squares (SS) described by Chambers (2018) is given by $SS = \sum_{j=1}^{n}\left[F_{ZBTIHT-W}(x_{(j)}) - \left(\frac{j-0.375}{n+0.25}\right)\right]^2$, where $j = 1, 2, \ldots, n$, and $x_{(j)}$ are ordered values of the observed data. In general, the smaller the values of goodness-of-fit statistics and the highest p-value for the K-S statistic, the better the fit. Furthermore, graphs including the Kaplan-Meier (KM) survival plot, empirical cumulative distribution function (ECDF), scaled total time test (TTT) plot and probability plot are presented.

The ZBTIHT-W distribution was fitted to data sets and these fits were compared to several models including the alpha power-Weibull (AP-W) distribution by Nassar et al. (2017), the type I heavy-tailed Weibull (TIHT-W) distribution by Zhao et al. (2020), the heavy-tailed beta power transformed Weibull (HTBPT-W) distribution by Zhao et al. (2021), half-logistic generalized Weibull (HLG-W) distribution by Anwar et al. (2018), Zografos-Balakrishnan Burr XII (ZBBXII) distribution by Altun et al. (2018) and type I half logistic-Weibull (TIHLW) distribution by Cordeiro et al. (2016). The pdfs are given in the appendix.

8.1. Growth Hormone Data

The data consists of the estimated time since growth hormone medication until the children reached the target age. The data was analysed by Oluyede et al. (2020). The data is presented in the appendix.

From the values of the goodness-of-fit statistics, p-value of the K-S statistic and the plots in Figures 9, 10 and 11, we can conclude that the ZBTIHT-W distribution provide a better fit compared to the other non-nested models. The hazard rate plot shows that the risk of reaching the target age is rising over time. Profile plots given in Figure 12 were done to ascertain that indeed the likelihood estimates in the application are maximum. It is notable from the asymptotic confidence intervals for each parameter and Table 13 that the shape parameters of the ZBTIHT-W distribution are significant looking at the parameter estimates as well as their standard errors. Therefore, all the parameters contribute to the flexibility of the ZBTIHT-W distribution. The estimated variance-covariance matrix for growth hormone data is given by

$$\begin{bmatrix} 0.0079 & -0.0395 & 0.0036 \\ -0.0395 & 0.1981 & -0.0179 \\ 0.0036 & -0.0179 & 0.0021 \end{bmatrix},$$

and the 95% asymptotic confidence intervals for the parameters δ, θ and λ are: 56.1553 ± 0.1738, 12.1818 ± 0.8725 and 0.5000 ± 0.0904, respectively.

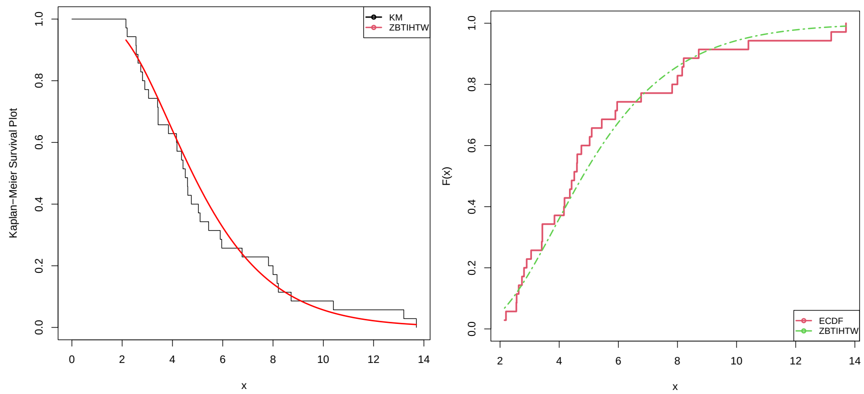

FIGURE 9: Kaplan-Meier survival plot and ECDF plot of the ZBTIHT-W distribution for growth hormone data

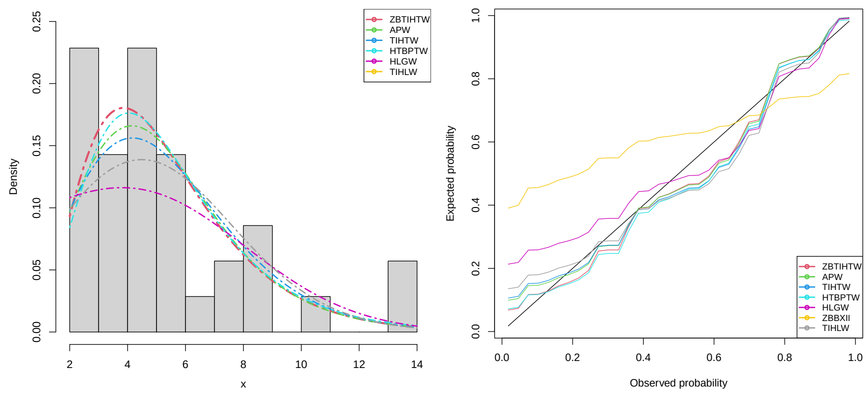

FIGURE 10: Fitted densities and expected probability plot for growth hormone data

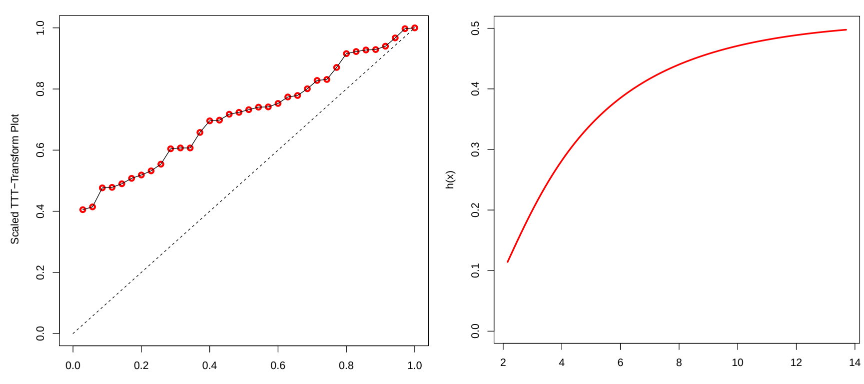

FIGURE 11: TTT plot and hazard rate function plot for growth hormone data

Table 13: Parameter estimates and goodness-of-fit statistics for various models for growth hormone data

Distribution	Estimates			-2log(L)	AIC	CAIC	BIC	W^*	A^*	K-S	P-value	SS
	δ	θ	λ									
ZBTIHT-W	56.1553	12.1818	0.5000	158.6243	164.6243	165.3985	169.2904	0.0794	0.5274	0.1087	0.8024	0.0774
	(0.0887)	(0.4451)	(0.0461)									
	α	λ	β									
AP-W	0.0430	2.4447	0.0049	161.6842	167.6842	168.4584	172.3502	0.1181	0.7528	0.1139	0.7547	0.0876
	(0.0792)	(0.3039)	(0.0041)									
	α	θ	γ									
TIHT-W	2.2899	2.7832	0.0025	162.8076	168.8076	169.5818	173.4737	0.1337	0.8451	0.1276	0.6195	0.1079
	(0.2809)	(2.1016)	(0.0041)									
	α	γ	β									
HTBPT-W	0.9189	0.5845	0.0002	159.5670	165.567	166.3412	170.2330	0.0836	0.5559	0.1241	0.6536	0.1052
	(0.1485)	(0.2305)	(0.0007)									
	w	λ	γ									
HLG-W	3.8793	1.0000	0.0452	172.3854	178.3854	179.1596	183.0514	0.2170	1.3305	0.2133	0.0827	0.2454
	(9.3011)	(0.3383)	(0.0992)									
	a	α	β									
ZBBXII	1.0000	3.2169×10^{1}	2.0116×10^{-2}	208.5027	214.5035	215.2777	219.1695	0.0334	0.2362	0.3972	0.2388×10^{-5}	1.5163
	(2.1048×10^{-1})	(5.5321×10^{-6})	(5.4303×10^{-3})									
	λ	δ	γ									
TIHLW	0.2832	0.2702	1.6584	166.6978	172.6978	173.472	177.3639	0.1861	1.1442	0.1416	0.4846	0.1491
	(0.0627)	(0.0657)	(0.2149)									

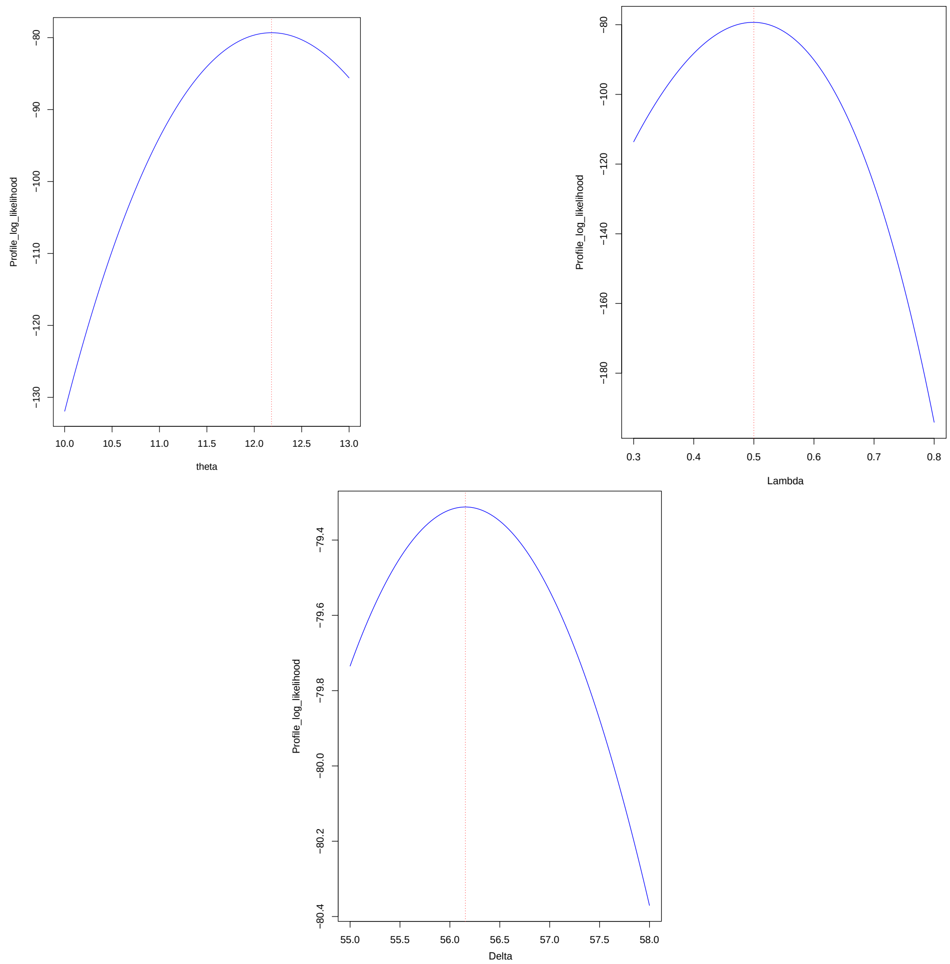

FIGURE 12: Profile log-likelihood plots of the ZBTIHT-W distribution for growth hormone data

8.2. Endurance of Deep Groove Ball Bearings Data

The data represents the number of million revolutions before failure for each of the 23 ball bearings in the life test, that is, endurance of deep groove ball bearings. The data was analyzed by Kang & Han (2009).

TABLE 14: Parameter estimates and goodness-of-fit statistics for various models for endurance of deep groove ball bearings data

Distribution	Estimates			-2log(L)	AIC	CAIC	BIC	W^*	A^*	K−S	p-value	SS
	δ	θ	λ									
ZBTIHT-W	18.0039	2.0139	0.5002	225.9471	231.9471	233.2102	235.3536	0.0327	0.1905	0.1086	0.9489	0.0296
	(0.0620)	(0.3865)	(0.0540)									
	α	θ	γ									
TIHT-W	2.1030	1.0023	9.4429×10^{-5}	227.3812	233.3814	234.6446	236.7879	0.0619	0.3476	0.1510	0.6706	0.0530
	(8.6057×10^{-9})	(2.7756×10^{-9})	(1.9700×10^{-5})									
	α	γ	β									
HTBPT-W	1.8754	0.0003	0.3986	227.8108	233.8108	235.0740	237.2173	0.0644	0.3600	0.1369	0.7821	0.0478
	(0.2366)	(0.0003)	(0.1798)									
	w	λ	γ									
HLG-W	1.5652×10^{1}	1.0000	7.3589×10^{-4}	231.4687	237.4684	238.7316	240.8749	0.0952	0.5526	0.1618	0.5838	0.0977
	(3.6310×10^{-4})	(1.7549×10^{-1})	(6.2382×10^{-4})									
	a	α	β									
ZBBXII	1.0001	1.5919×10^{1}	1.5130×10^{-2}	302.3810	308.3799	309.6431	311.7864	0.0329	0.2392	0.5106	1.2360×10^{-5}	1.7087
	(2.5969×10^{-1})	(1.0377×10^{-5})	(5.0381×10^{-3})									
	λ	δ	γ									
TIHLW	0.0439	0.0141	1.7664	228.1509	234.1509	235.4140	237.5573	0.0732	0.4094	0.1546	0.6415	0.0584
	(0.0056)	(0.0175)	(0.2920)									

From the values of the goodness-of-fit statistics, p-value of the K-S statistic and the plots in Figures 13, 14 and 15, we conclude that the ZBTIHTW distribution provide a better fit compared to the other models. The hazard rate plot shows that the risk of bearing failure is rising over time. Profile plots given in Figure 16 were done to ascertain that indeed the estimates in the application are maximum. The estimated variance-covariance matrix for endurance of deep groove ball bearings data is

$$\begin{bmatrix} 0.0038 & -0.0240 & 0.0033 \\ -0.0240 & 0.1494 & -0.0203 \\ 0.0033 & -0.0203 & 0.0029 \end{bmatrix},$$

and the 95% asymptotic confidence intervals for the parameters δ, θ and λ are: 18.0039 ± 0.1215, 2.0139 ± 0.7576 and 0.5002 ± 0.1058, respectively.

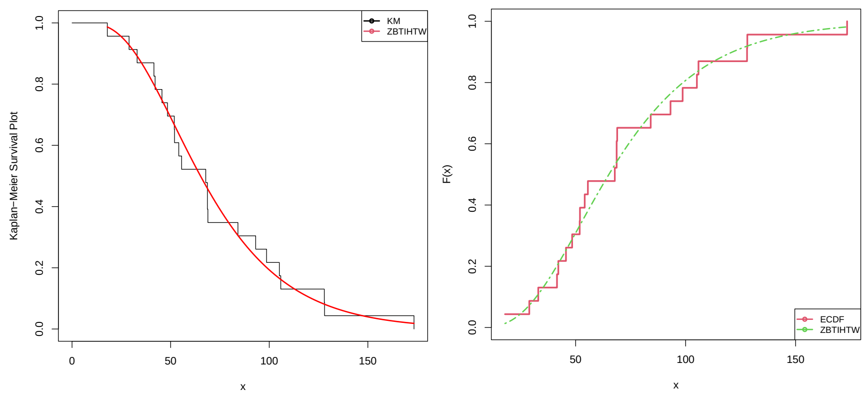

FIGURE 13: Kaplan-Meier survival plot and ECDF plot of the ZBTIHT-W distribution for endurance of deep groove ball bearings data

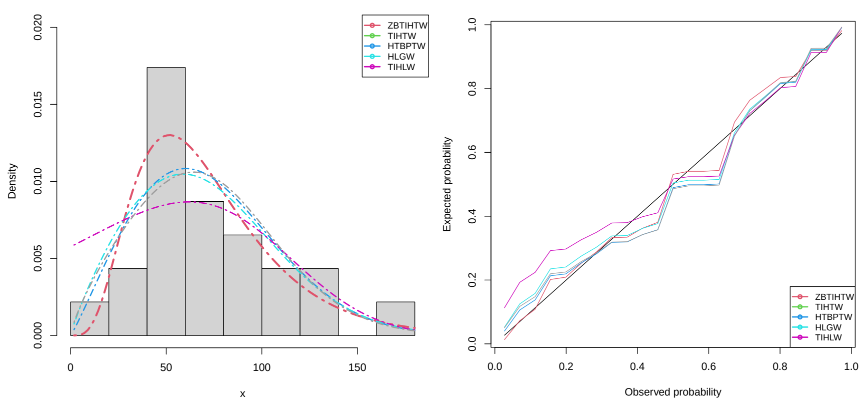

FIGURE 14: Fitted densities and expected probability plot endurance of deep groove ball bearings data

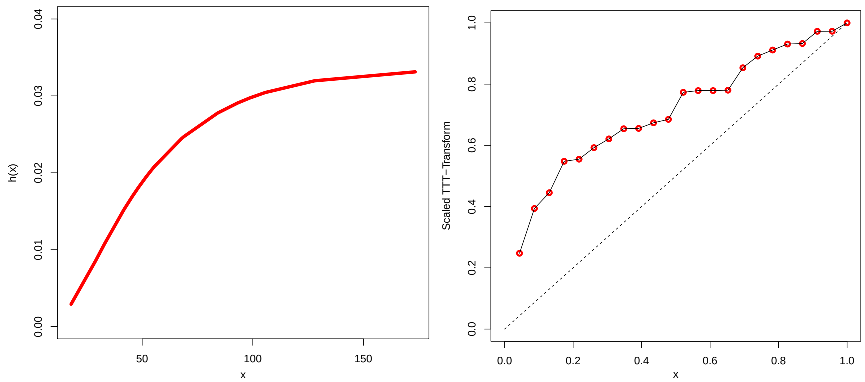

FIGURE 15: TTT plot and hazard rate function plot endurance of deep groove ball bearings data

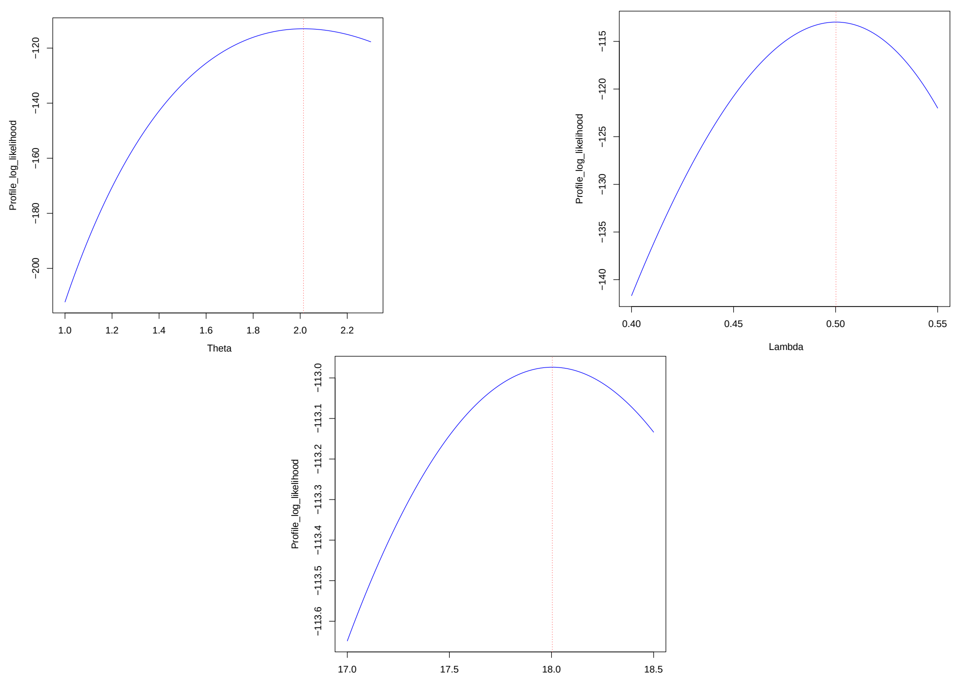

FIGURE 16: Profile log-likelihood plots of the ZBTIHT-W distribution for endurance of deep groove ball bearings data

8.3. Remission Times Data

The data represents remission times (months) of 128 bladder cancer patients. The data was analysed by Chamunorwa et al. (2021). The data is given in the appendix.

TABLE 15: Parameter estimates and goodness-of-fit statistics for various models for remission times data

Distribution	Estimates			-2log(L)	AIC	CAIC	BIC	W^*	A^*	K−S	p-value	SS
	δ	θ	λ									
ZBTIHT-W	63.0373	15.3667	0.2500	825.2752	831.2752	831.4687	839.8313	0.0747	0.5130	0.0513	0.8896	0.0672
	(0.0422)	(0.2008)	(0.0141)									
	α	θ	γ									
TIHT-W	1.1030	1.2552	0.0565	826.9924	832.9924	833.1859	841.5485	0.1151	0.6941	0.0684	0.5863	0.1353
	(0.6050)	(3.5044)	(0.3502)									
	α	γ	β									
HTBPT-W	1.0478	0.0939	1.0000	828.1738	834.1738	834.3673	842.7298	0.1314	0.7865	0.0700	0.5570	0.1498
	(0.0676)	(0.0191)	(0.7280)									
	w	λ	γ									
HLG-W	0.7022	1.0000	0.2918	826.105	832.105	832.2986	840.6611	0.0974	0.5785	0.0806	0.3768	0.1320
	(0.2131)	(0.1691)	(0.0855)									
	a	α	β									
ZBBXII	1.0000	2.3345	0.2338	907.0311	913.0312	913.2247	921.5873	0.7498	4.5550	0.2507	2.068×10^{-7}	2.7051
	(0.1224)	(0.3939)	(0.0604)									
	λ	δ	γ									
TIHLW	0.6111	0.3297	0.8880	830.1918	836.1918	836.3854	844.7479	0.1660	0.9780	0.0765	0.4420	0.1598
	(81.6756)	(44.0643)	(0.0603)									

From the values of the goodness-of-fit statistics, p-value of the K-S statistic and the plots in Figures 17, 18 and 19, we conclude that the ZBTIHTW distribution provides a better fit than the other models. The hazard rate plot increases then decreases, indicating that initially there is a higher risk of experiencing cancer recurrence but with time the risk decreases. Profile plots given in Figure 20 were done to ascertain that indeed the estimates in the application are maximum. The estimated variance-covariance matrix for remission times data is

$$\begin{bmatrix} 0.0018 & -0.0085 & 0.0004 \\ -0.0085 & 0.0403 & -0.0021 \\ 0.0004 & -0.0021 & 0.0002 \end{bmatrix},$$

and the 95% asymptotic confidence intervals for the parameters δ, θ and λ are: 63.0373 ± 0.0826, 15.3667 ± 0.3936 and 0.2500 ± 0.0276, respectively.

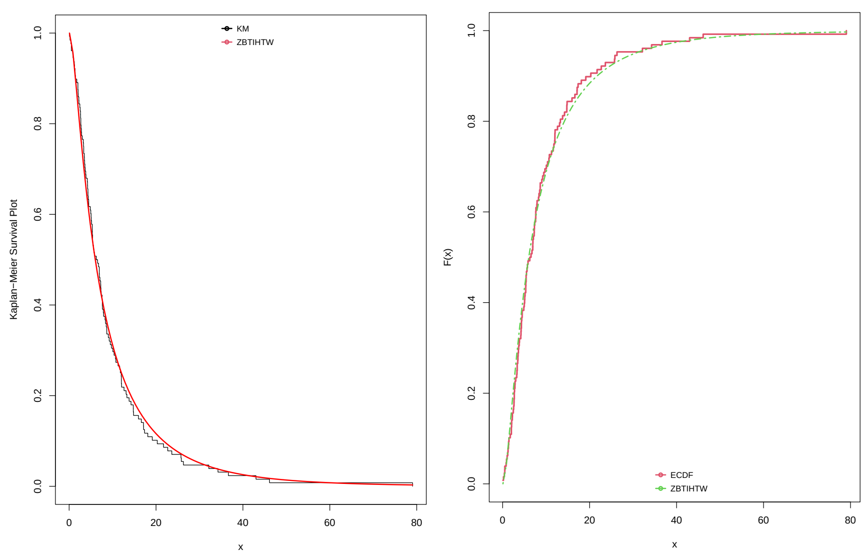

FIGURE 17: Kaplan-Meier survival plot and ECDF plot of the ZBTIHT-W distribution for remission times data

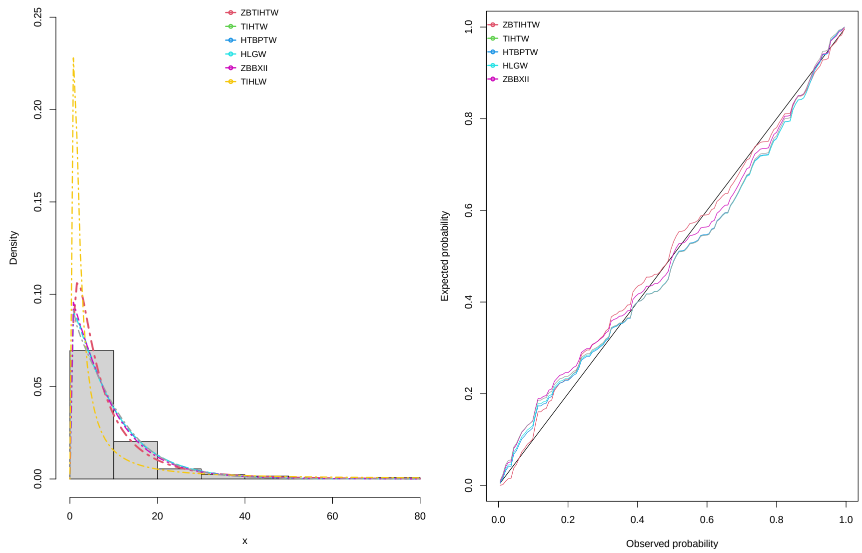

FIGURE 18: Fitted densities and expected probability plot for remission times data

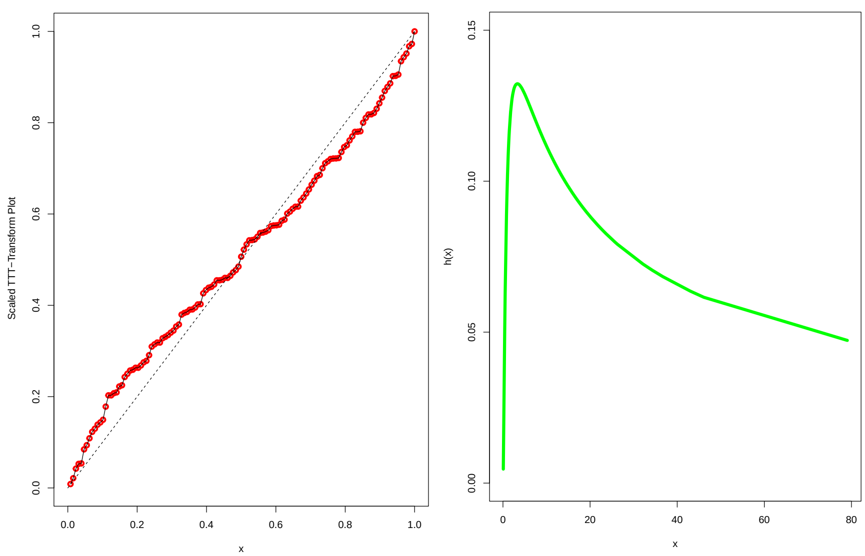

FIGURE 19: TTT plot and hazard rate function plot for remission times data

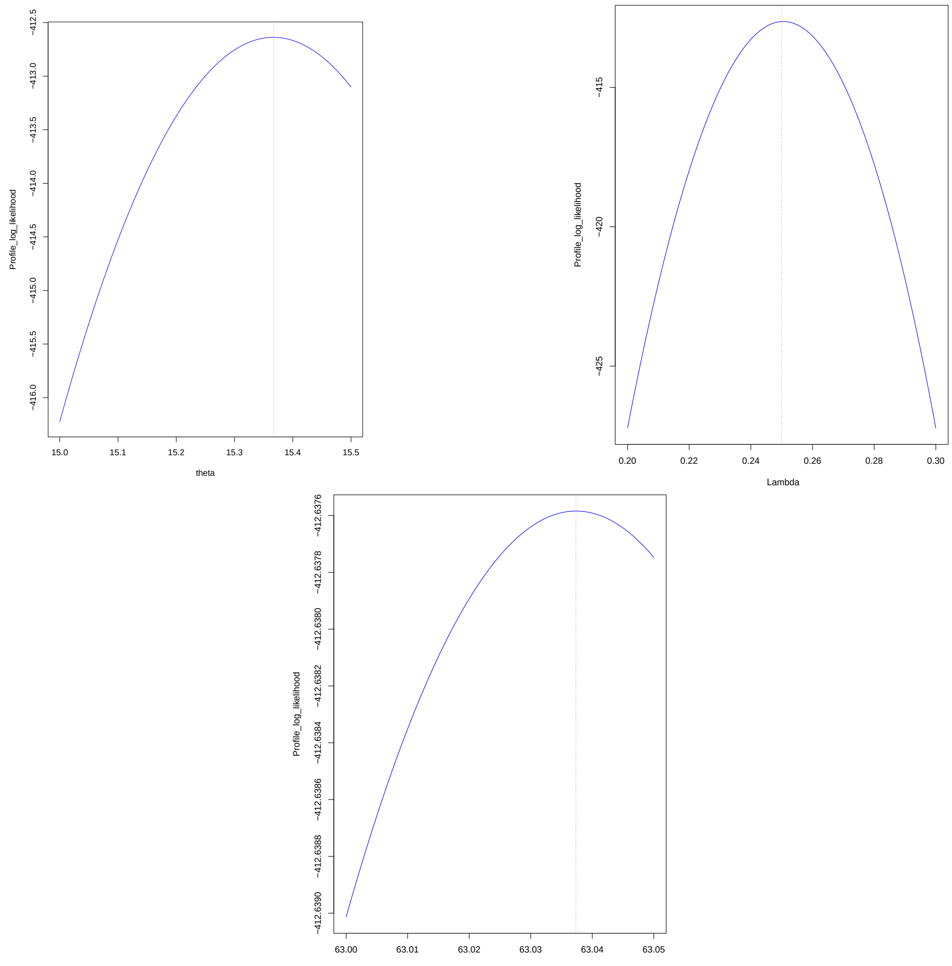

Figure 20: Profile log-likelihood plots of the ZBTIHT-W distribution for remission times data

9. Conclusions

A new generalized family of distributions called the Zografos-Balakrishnan type-I heavy-tailed-G (ZBTIHT-G) distribution was developed and presented. The density of the new family of distributions can be expressed as an infinite linear combination of exponentiated-G densities. This allows us to obtain the mathematical and statistical properties of the ZBTIHT-G family of distributions. The ZBTIHT-W distribution has 3 shape parameters which were significant based on the parameter estimates and their standard errors as well as the asymptotic confidence interval of the estimates, hence we can conclude that the parameters contribute to the flexibility of the newly developed family of distributions. The behaviour of the hazard rate functions of the new family of distribution can be monotonic as well as non-monotonic. We also obtain closed form expressions for the moments, distribution of order statistics, probability weighted moments and Rényi entropy. The method of maximum likelihood estimation (MLE) was used to estimate the model parameters since it performed better than other estimation methods as it had the lowest sum of ranks. Performance of the special case of the ZBTIHT-G distribution was examined by conducting various simulations for different sizes. Finally, the special case Zografos-Balakrishnan type-I heavy-tailed-Weibull (ZBTIHT-W) distribution was fitted to real data sets to illustrate the applicability of the proposed family of distributions.

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Appendix A. Appendix

Appendix A.1. Linear Representation of the Density Function

Expansion of the pdf of the ZBTIHT-G family of distributions is considered in this subsection. Let $y = 1 - \left[\frac{1-G(x;\zeta)}{1-(1-\theta)G(x;\zeta)}\right]^{\theta}$, $0 < y < 1$. Using the following series expansion: $-\log(1-y) = \sum_{i=0}^{\infty} \frac{y^{i+1}}{i+1}$, we have

$$\Big[-\log(1-y)\Big]^{\delta-1} \quad = \quad y^{\delta-1}\sum_{j=0}^{\infty}\binom{\delta-1}{j}y^{j}\left[\sum_{k=0}^{\infty}\frac{y^{k}}{k+2}\right]^{j}.$$

Let $a_k = (k+2)^{-1}$, and applying the results of a power series raised to a positive integer (Gradshteyn & Ryzhik, 2014), that is, $\left(\sum_{k=0}^{\infty} a_k y^k\right)^j = \sum_{k=0}^{\infty} b_{k,j} y^k$, where $b_{k,j} = (ka_0)^{-1} \sum_{l=0}^{\infty} [l(i+1)-k]\, a_l b_{k-1,j}$ and $b_{0,j} = a_0^j$, we can write the pdf of ZBTIHT-G distribution as

$$\begin{aligned} f(x;\theta,\delta,\zeta) &= \sum_{j,k=0}^{\infty} \binom{\delta-1}{j} b_{k,j} y^{k+j+\delta-1} \frac{\theta^2 g(x;\zeta)\,[1-G(x;\zeta)]^{\theta-1}}{\Gamma(\delta)\,[1-(1-\theta)G(x;\zeta)]^{\theta+1}} \\ &= \sum_{j,k=0}^{\infty} \binom{\delta-1}{j} b_{k,j} \left(1-\left[\frac{1-G(x;\zeta)}{1-(1-\theta)G(x;\zeta)}\right]^{\theta}\right)^{k+j+\delta-1} \\ &\times \frac{\theta^2 g(x;\zeta)\,[1-G(x;\zeta)]^{\theta-1}}{\Gamma(\delta)\,[1-(1-\theta)G(x;\zeta)]^{\theta+1}}. \end{aligned}$$

Considering

$$\left(1-\left[\frac{1-G(x;\zeta)}{1-(1-\theta)G(x;\zeta)}\right]^{\theta}\right)^{k+j+\delta-1} = \sum_{m=0}^{\infty} (-1)^m \binom{k+j+\delta-1}{m} \frac{[1-G(x;\zeta)]^{\theta m}}{[1-(1-\theta)G(x;\zeta)]^{\theta m}},$$

we have

$$f(x;\theta,\delta,\zeta) = \sum_{j,k,m=0}^{\infty} \frac{(-1)^m}{\Gamma(\delta)} \binom{\delta-1}{j} \binom{k+j+\delta-1}{m} b_{k,j} \frac{\theta^2 g(x;\zeta)\,[1-G(x;\zeta)]^{\theta m+\theta-1}}{[1-(1-\theta)G(x;\zeta)]^{\theta m+\theta+1}}.$$

Using the generalized binomial expansions:

$$[1-G(x;\zeta)]^{\theta m+\theta-1} = \sum_{n=0}^{\infty} (-1)^n \binom{\theta m+\theta-1}{n} G^n(x;\zeta),$$

$$[1-(1-\theta)G(x;\zeta)]^{-(\theta m+\theta+1)} = \sum_{p=0}^{\infty} (-1)^p \binom{-(\theta m+\theta+1)}{p} (1-\theta)^p G^p(x;\zeta),$$

$$[1-(1-G(x;\zeta))]^{n+p} = \sum_{q=0}^{\infty} (-1)^q \binom{n+p}{q} [1-G(x;\zeta)]^q,$$

and

$$[1-G(x;\zeta)]^q = \sum_{r=0}^{\infty} (-1)^r \binom{q}{r} G^r(x;\zeta),$$

the pdf of ZBTIHT-G family of distributions can be written as

$$\begin{aligned} f(x;\theta,\delta,\zeta) &= \sum_{j,k,m,n,p,q,r=0}^{\infty} b_{k,j} \frac{(-1)^{m+n+p+q+r}}{\Gamma(\delta)} \binom{\delta-1}{j} \binom{k+j+\delta-1}{m} \binom{\theta m+\theta-1}{n} \\ &\times \binom{-(\theta m+\theta+1)}{p} \binom{n+p}{q} (1-\theta)^p \theta^2 g(x;\zeta) G^r(x;\zeta) \\ &= \sum_{r=0}^{\infty} U_{r+1} h_{r+1}(x;\zeta), \end{aligned} \tag{A1}$$

where

$$\begin{aligned} U_{r+1} &= \sum_{j,k,m,n,p,q=0}^{\infty} b_{k,j} \frac{(-1)^{m+n+p+q+r}}{(r+1)\,\Gamma(\delta)} \binom{\delta-1}{j} \binom{k+j+\delta-1}{m} \binom{\theta m+\theta-1}{n} \\ &\times \binom{-(\theta m+\theta+1)}{p} \binom{n+p}{q} (1-\theta)^p \theta^2, \end{aligned} \tag{A2}$$

and $h_{r+1}(x;\zeta) = (r+1)\, g(x;\zeta) G^r(x;\zeta)$ is the exponentiated-G (exp-G) density with power parameter $(r+1)$ and parameter vector ζ. The new density can be expressed as an infinite linear combination of exp-G densities. Consequently, the mathematical and statistical properties of the new family of distributions follows directly from those of exp-G family of distributions.

Components of the score vector are given as follows:

$$\begin{aligned} \frac{\partial \ell_n}{\partial \theta} &= (\delta-1) \sum_{i=1}^{n} \frac{\log\left[\frac{1-G(x_i;\zeta)}{1-(1-\theta)G(x_i;\zeta)}\right] + \frac{\theta G(x_i;\zeta)}{1-(1-\theta)G(x_i;\zeta)}}{\theta \log\left(\left[\frac{1-G(x_i;\zeta)}{1-(1-\theta)G(x_i;\zeta)}\right]\right)} \\ &+ \frac{2n}{\theta} + \sum_{i=1}^{n} \log\left[1-G(x_i;\zeta)\right] - \sum_{i=1}^{n} \log\left[1-(1-\theta)\,G(x_i;\zeta)\right] \\ &- \theta \sum_{i=1}^{n} \frac{G(x_i;\zeta)}{1-(1-\theta)\,G(x_i;\zeta)}, \end{aligned}$$

$$\frac{\partial \ell_n}{\partial \delta} = \sum_{i=1}^{n} \log\left[-\log\left(\left[\frac{1-G(x_i;\zeta)}{1-(1-\theta)\,G(x_i;\zeta)}\right]^{\theta}\right)\right] + \frac{n\Gamma'(\delta)}{\Gamma(\delta)},$$

and

$$\begin{aligned} \frac{\partial \ell_n}{\partial \zeta_k} &= (\delta-1) \sum_{i=1}^{n} \frac{\frac{\partial}{\partial \zeta_k}\left[-\log\left(\left[\frac{1-G(x_i;\zeta)}{1-(1-\theta)G(x_i;\zeta)}\right]^{\theta}\right)\right]}{\left[-\log\left(\left[\frac{1-G(x_i;\zeta)}{1-(1-\theta)G(x_i;\zeta)}\right]^{\theta}\right)\right]} + \sum_{i=1}^{n} \frac{\frac{\partial}{\partial \zeta_k} g(x_i;\zeta)}{g(x_i;\zeta)} \\ &+ (\theta-1) \sum_{i=1}^{n} \frac{\frac{\partial}{\partial \zeta_k}\left[1-G(x_i;\zeta)\right]}{\left[1-G(x_i;\zeta)\right]} - (\theta+1) \sum_{i=1}^{n} \frac{\frac{\partial}{\partial \zeta_k}\left[1-(1-\theta)\,G(x_i;\zeta)\right]}{\left[1-(1-\theta)\,G(x_i;\zeta)\right]}. \end{aligned}$$

Appendix A.2. Derivation of PWMs

Let $y = 1 - \left[\frac{1-G(x;\zeta)}{1-(1-\theta)G(x;\zeta)}\right]^{\theta}$. From equations (5) and (6), we can write

$$\begin{aligned} f(x)[F(x)]^r &= \frac{\left[-\log(1-y)\right]^{\delta-1} \theta^2 g(x;\zeta) \left[1-G(x;\zeta)\right]^{\theta-1}}{\Gamma(\delta) \left[1-(1-\theta)G(x;\zeta)\right]^{\theta+1}} \\ &\times \left[\frac{\gamma\left(\delta, -\log(1-y)\right)}{\Gamma(\delta)}\right]^r. \end{aligned}$$

Using the incomplete gamma function series representation by Gradshteyn & Ryzhik (2014)

$$\gamma(\delta, x) = \sum_{j=0}^{\infty} \frac{(-1)^j x^{j+\delta}}{(j+\delta)j!},$$

we get

$$\begin{aligned}\left[\frac{\gamma\left(\delta, -\log\left(1-y\right)\right)}{\Gamma(\delta)}\right]^r &= \left[\sum_{m=0}^{\infty} \frac{(-1)^m \left[-\log\left(1-y\right)\right]^{m+\delta}}{(m+\delta)m!\Gamma(\delta)}\right]^r \\ &= \sum_{m=0}^{\infty} \frac{d_{m+\delta,r}}{\left(\Gamma(\delta)\right)^r} \left[-\log\left(1-y\right)\right]^{m+\delta},\end{aligned}$$

where

$$d_{m+\delta,r} = \left((m+\delta)c_0\right)^{-1} \sum_{k=1}^{m+\delta} \left[k(r+1) - m - \delta\right] c_k d_{m+\delta-k,r},$$

and $d_0 = c_0^r$, we can write

$$f(x)[F(x)]^r = \sum_{m=0}^{\infty} \frac{(-1)^m d_{m+\delta,r}\theta^2 g(x;\zeta)\left[1-G(x;\zeta)\right]^{\theta-1}\left[-\log\left(1-y\right)\right]^{m+2\delta-1}}{\left(\Gamma(\delta)\right)^{r+1}\left[1-(1-\theta)G(x;\zeta)\right]^{\theta+1}}.$$

Considering

$$\left[-\log\left(1-y\right)\right]^{m+2\delta-1} = y^{m+2\delta-1} \sum_{n=0}^{\infty} \binom{m+2\delta-1}{n} y^n \left[\sum_{p=0}^{\infty} \frac{y^p}{p+2}\right]^n,$$

let $a_p = (p+2)^{-1}$, then $\left[\sum_{p=0}^{\infty} a_p y^p\right]^n = \sum_{p=0}^{\infty} b_{p,n} y^p$,
where $b_{p,n} = (pa_0)^{-1} \sum_{l=1}^{p} \left[l(p+1) - p\right] a_l b_{p-l,n}$ and $b_{0,n} = a_0^n$, we have

$$\begin{aligned}f(x)[F(x)]^r &= \sum_{m,n,p=0}^{\infty} \frac{(-1)^m d_{m+\delta,r} b_{p,n}\theta^2 g(x;\zeta)\left[1-G(x;\zeta)\right]^{\theta-1}}{\left(\Gamma(\delta)\right)^{r+1}\left[1-(1-\theta)G(x;\zeta)\right]^{\theta+1}} \\ &\times \left[1 - \left[\frac{1-G(x;\zeta)}{1-(1-\theta)G(x;\zeta)}\right]^\theta\right]^{p+n+m+2\delta-1}.\end{aligned}$$

Using the generalized binomial expansions:

$$\begin{aligned}\left[1 - \left[\frac{1-G(x;\zeta)}{1-(1-\theta)G(x;\zeta)}\right]^\theta\right]^{p+n+m+2\delta-1} &= \sum_{q=0}^{\infty} (-1)^q \binom{p+n+m+2\delta-1}{q} \\ &\times \frac{\left[1-G(x;\zeta)\right]^{\theta q}}{\left[1-(1-\theta)G(x;\zeta)\right]^{\theta q}},\end{aligned}$$

$$[1-G(x;\zeta)]^{\theta q+\theta-1} = \sum_{r=0}^{\infty}(-1)^r\binom{\theta q+\theta-1}{r}G^r(x;\zeta),$$

$$[1-(1-\theta)G(x;\zeta)]^{-(\theta q+\theta+1)} = \sum_{s=0}^{\infty}(-1)^s\binom{-(\theta q+\theta+1)}{s}(1-\theta)^s G^s(x;\zeta),$$

$$[1-(1-G(x;\zeta))]^{r+s} = \sum_{t=0}^{\infty}(-1)^t\binom{r+s}{t}[1-G(x;\zeta)]^t$$

and

$$[1-G(x;\zeta)]^t = \sum_{v=0}^{\infty}(-1)^v\binom{t}{v}G^v(x;\zeta),$$

we get

$$\begin{aligned} f(x)[F(x)]^r &= \sum_{m,n,p,q,r,s,t,v=0}^{\infty}\frac{(-1)^{m+q+r+s+t+v}\,d_{m+\delta,r}b_{p,n}\theta^2\,(1-\theta)^s}{(\Gamma(\delta))^{r+1}}\binom{m+2\delta-1}{n}\\ &\times \binom{p+n+m+2\delta-1}{q}\binom{\theta q+\theta-1}{r}\binom{-(\theta q+\theta+1)}{s}\binom{r+s}{t}\\ &\times \binom{t}{v}g(x;\zeta)G^v(x;\zeta)\\ &= \sum_{v=0}^{\infty}\tau_{v+1}h_{v+1}(x;\zeta), \end{aligned}$$

where $h_{v+1}(x;\zeta)=(v+1)g(x;\zeta)G^v(x;\zeta)$ is the exp-G density with power parameter $(v+1)$ and parameter vector ζ, and

$$\begin{aligned} \tau_{v+1} &= \sum_{m,n,p,q,r,s,t,v=0}^{\infty}\frac{(-1)^{m+q+r+s+t+v}\,d_{m+\delta,r}b_{p,n}\theta^2\,(1-\theta)^s}{(v+1)\,(\Gamma(\delta))^{r+1}}\binom{m+2\delta-1}{n}\\ &\times \binom{p+n+m+2\delta-1}{q}\binom{\theta q+\theta-1}{r}\binom{-(\theta q+\theta+1)}{s}\binom{r+s}{t}\binom{t}{v}. \end{aligned}$$

Appendix A.3. Derivation of Rényi entropy

Taking $y=1-\left[\frac{1-G(x;\zeta)}{1-(1-\theta)G(x;\zeta)}\right]^{\theta}$ $0<y<1$. Using the series presentation $-\log(1-y)=\sum_{i=0}^{\infty}\frac{y^{i+1}}{i+1}$, we have

$$[-\log(1-y)]^{v(\delta-1)} = y^{v(\delta-1)}\sum_{j=0}^{\infty}\binom{v(\delta-1)}{j}y^j\left[\sum_{k=0}^{\infty}\frac{y^k}{k+2}\right]^j.$$

Letting $a_k = (k+2)^{-1}$, $\sum_{k=0}^{\infty} a_k y^k = \sum_{k=0}^{\infty} b_{k,j} y^k$, where

$$b_{k,j} = (ka_0)^{-1} \sum_{l=0}^{k} \left[l(j+1) - k\right] a_l b_{k-1,j}$$

and $b_{0,j} = a_0^j$, we have

$$\begin{aligned} f^v(x) &= \sum_{j,k=0}^{\infty} \binom{v(\delta-1)}{j} b_{k,j} y^{k+j+v(\delta-1)} \frac{\theta^{2v} g^v(x;\zeta) \left[1-G(x;\zeta)\right]^{v(\theta-1)}}{(\Gamma(\delta))^v \left[1-(1-\theta)G(x;\zeta)\right]^{v(\theta+1)}} \\ &= \sum_{j,k=0}^{\infty} \binom{v(\delta-1)}{j} b_{k,j} \left(1 - \left[\frac{1-G(x;\zeta)}{1-(1-\theta)G(x;\zeta)}\right]^{\theta}\right)^{k+j+v(\delta-1)} \\ &\times \frac{\theta^{2v} g^v(x;\zeta) \left[1-G(x;\zeta)\right]^{v(\theta-1)}}{(\Gamma(\delta))^v \left[1-(1-\theta)G(x;\zeta)\right]^{v(\theta+1)}}. \end{aligned}$$

Now, considering the generalized binomial expansions

$$\left(1 - \left[\frac{1-G(x;\zeta)}{1-(1-\theta)G(x;\zeta)}\right]^{\theta}\right)^{k+j+v(\delta-1)} = \sum_{l=0}^{\infty} (-1)^l \binom{k+j+\delta-1}{l} \frac{\left[1-G(x;\zeta)\right]^{\theta l}}{\left[1-(1-\theta)G(x;\zeta)\right]^{\theta l}},$$

$$\left[1-G(x;\zeta)\right]^{\theta l + v(\theta-1)} = \sum_{m=0}^{\infty} (-1)^m \binom{\theta l + v(\theta-1)}{m} G^m(x;\zeta),$$

$$\left[1-(1-\theta)G(x;\zeta)\right]^{-(\theta l + v(\theta+1))} = \sum_{n=0}^{\infty} (-1)^n \binom{-(\theta l + v(\theta+1))}{n} (1-\theta)^n G^n(x;\zeta),$$

$$\left[1-(1-G(x;\zeta))\right]^{m+n} = \sum_{p=0}^{\infty} (-1)^p \binom{m+n}{p} \left[1-G(x;\zeta)\right]^p,$$

$$\left[1-G(x;\zeta)\right]^p = \sum_{q=0}^{\infty} (-1)^q \binom{p}{q} G^q(x;\zeta),$$

we obtain

$$\begin{aligned} f^v(x) &= \sum_{j,k,l,m,n,p,q=0}^{\infty} \frac{(-1)^{l+m+n+p+q}}{(\Gamma(\delta))^v} \binom{v(\delta-1)}{j} \binom{k+j+\delta-1}{l} \binom{\theta l + v(\theta-1)}{m} \\ &\times \binom{-(\theta l + v(\theta+1))}{n} \binom{m+n}{p} \binom{p}{q} (1-\theta)^n \theta^{2v} g^v(x;\zeta) G^q(x;\zeta). \end{aligned}$$

Now, we can write Rényi entropy of the ZBTIHT-G family of distributions as

$$\begin{aligned} H_R(v) &= \frac{1}{1-v}\log\left[\int_0^\infty f^v(x)dx\right] \\ &= \frac{1}{1-v}\log\left[\sum_{q=0}^\infty U_q\left(\int_0^\infty g^v(x;\zeta)G^q(x;\zeta)dx\right)\right], \end{aligned}$$

where

$$\begin{aligned} U_q &= \sum_{j,k,l,m,n,p=0}^\infty \frac{(-1)^{l+m+n+p+q}}{(\Gamma(\delta))^v}\binom{v(\delta-1)}{j}\binom{k+j+\delta-1}{l}\binom{\theta l+v(\theta-1)}{m} \\ &\times \binom{-(\theta l+v(\theta+1))}{n}\binom{m+n}{p}\binom{p}{q}(1-\theta)^n\theta^{2v}. \end{aligned}$$

Note that, $\int_0^\infty g^v(x;\zeta)G^p(x;\zeta)dx$ can also be obtained numerically.

Appendix A.4. Pdfs used in Applications

$$f_{APW}(x;\alpha,\lambda,\beta) = \frac{\log(\alpha)}{\alpha-1}\lambda\beta x^{1-e^{-\lambda x^\beta}}x^{\beta-1}e^{-\lambda x^\beta}$$

for $\alpha,\lambda,\beta>0$, and $x>0$,

$$f_{TIHT-W}(x;\alpha,\theta,\gamma) = \frac{\alpha\theta^2\gamma x^{\alpha-1}e^{(-\theta\gamma x^\alpha)}}{\left[1-(1-\theta)\left(1-e^{(-\theta\gamma x^\alpha)}\right)\right]^{\theta+1}}$$

for $\alpha,\theta,\gamma>0$ and $x>0$,

$$f_{HTBPT-W}(x;\alpha,\gamma,\beta) = \alpha\gamma x^{\alpha-1}e^{(-\gamma x^\alpha)}\left[\beta-(\log\beta)\,\beta^{e^{(-\gamma x^\alpha)}}\right]$$

for $\alpha,\gamma,\beta>0$, and $x>0$,

$$f_{HLG-W}(x;w,\lambda,\delta) = \frac{2w\lambda\gamma x^{\lambda-1}\left(1+\gamma x^\lambda\right)^{w-1}\exp\left(1-\left(1+\gamma x^\lambda\right)^w\right)}{\left[1+\exp\left(1-(1+\gamma x^\lambda)^w\right)\right]^2}$$

for $w,\lambda,\delta>0$ and $x>0$,

$$f_{ZBBXII}(x;a,\alpha,\beta) = \frac{\alpha\beta x^{\alpha-1}\left(1+x^\alpha\right)^{-\beta-1}\left[-\log\left(1+x^\alpha\right)^{-\beta}\right]^{a-1}}{\Gamma(a)}$$

for $a,\alpha,\beta>0$ and $x>0$, and

$$f_{TIHLW}(x;\lambda,\delta,\gamma) = 2\lambda\delta\gamma x^{\gamma-1}e^{-\lambda\delta x^\lambda}\left[1+e^{-\lambda\delta x^\lambda}\right]^{-2}$$

for $\alpha,\gamma,\beta>0$ and $x>0$.

Appendix A.5. Datasets Used for Applications

Growth Hormone Data

The data are: 2.15, 2.20, 2.55, 2.56, 2.63, 2.74, 2.81, 2.90, 3.05, 3.41, 3.43, 3.43, 3.84, 4.16, 4.18, 4.36, 4.42, 4.51, 4.60, 4.61, 4.75, 5.03, 5.10, 5.44, 5.90, 5.96, 6.77, 7.82, 8.00, 8.16, 8.21, 8.72, 10.40, 13.20, 13.70.

Endurance of Deep Groove Ball Bearings Data

The data are: 17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.40, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40.

Remission Times Data

The data are: 0.08, 4.98, 25.74, 3.7, 10.06, 2.69, 7.62, 1.26, 7.87, 4.4, 2.02, 21.73, 2.09, 6.97, 0.5, 5.17, 14.77, 4.18, 10.75, 2.83, 11.64, 5.85, 3.31, 2.07, 3.48, 9.02, 2.46, 7.28, 32.15, 5.34, 16.62, 4.33, 17.36, 8.26, 4.51, 3.36, 4.87, 13.29, 3.64, 9.74, 2.64, 7.59, 43.01, 5.49, 1.4, 11.98, 6.54, 6.93, 6.94, 0.4, 5.09, 14.76, 3.88, 10.66, 1.19, 7.66, 3.02, 19.13, 8.53, 8.65, 8.66, 2.26 ,7.26 ,26.31, 5.32, 15.96, 2.75, 11.25, 4.34, 1.76, 12.03, 12.63, 13.11, 3.57, 9.47, 0.81, 7.39, 36.66, 4.26, 17.14, 5.71, 3.25, 20.28, 22.69, 23.63, 5.06, 14.24, 2.62, 10.34, 1.05, 5.41, 79.05, 7.93, 4.5, 2.02, 0.2 ,7.09, 25.82, 3.82, 14.83, 2.69, 7.63, 1.35, 11.79, 6.25, 3.36, 2.23, 9.22, 0.51, 5.32, 34.26 ,4.23, 17.12, 2.87, 18.1, 8.37, 6.76 ,3.52, 13.8, 2.54, 7.32, 0.9, 5.41, 46.12, 5.62, 1.46, 12.02, 12.07.