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A New Technique to Generate Families of Continuous Distributions

Una nueva técnica para generar familias de distribuciones continuas

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Abstract

We introduce a novel technique for producing several families of distributions: the alpha-log-power transformed method. The novelty of our new approach lies in the fact that it adds one new shape parameter and was not derived from any established parent model. Some examples of the new family are presented. Also, some important statistical properties of the new family are studied. The maximum likelihood estimation approach is utilized to estimate the model parameters of the new family. To evaluate the performance of the estimators, Monte Carlo simulation is conducted using some arbitrary baseline distributions namely the Weibull, Burr-XII and Pareto distribution. Two real datasets are used to empirically show the potential significance and applicability of the alpha log power transformed Weibull. The alpha log power transformed Weibull is a very competitive model for characterizing observations in survival analysis.

***Key words*:** Alpha-Power transformation; Alpha-Power transformed-G; Goodness-of-fit statistics; Maximum Likelihood estimation; Power distribution.

Resumen

Introducimos una técnica novedosa para producir varias familias de distribuciones: el método transformado de potencia logarítmica alfa. La novedad de nuestro nuevo enfoque radica en el hecho de que agrega un nuevo parámetro de forma y no se deriva de ningún modelo principal establecido. Se presentan algunos ejemplos de la nueva familia. Además, se estudian algunas propiedades estadísticas importantes de la nueva familia. Se utiliza el enfoque de estimación de máxima verosimilitud para estimar los parámetros del

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modelo de la nueva familia. Para evaluar el desempeño de los estimadores, se realiza una simulación de Monte Carlo utilizando algunas distribuciones de referencia arbitrarias, a saber, la distribución de Weibull, Burr-XII y Pareto. Se utilizan dos conjuntos de datos reales para mostrar empíricamente la importancia potencial y la aplicabilidad de Weibull transformado en potencia logarítmica alfa. El poder logarítmico alfa transformado de Weibull es un modelo muy competitivo para caracterizar observaciones en el análisis de supervivencia.

***Palabras clave*:** Alfa-Poder Transformado-G; Distribución de poder; Estadísticas de bondad de ajuste; Estimación de máxima verosimilitud; Transformación de poder alfa.

1. Introduction

Most fascinating data sets are typically neither described by nor predicted by standard distributions like the exponential, half-logistic, normal and Weibull distributions. To deal with this issue, attempts have recently been made to create new families of probability distributions that extend existing classical distributions and enable more adaptable modeling. Some recent techniques include the Marshall-Olkin-G (MO-G) by Marshall & Olkin (1997), exponentiated-G (exp-G) by Gupta et al. (1998), the transmuted-G (Tr-G) by Shaw & Buckley (2009), the T-X strategy by Alzaatreh et al. (2013), the exponentiated generalized-G (EGG) generator by Cordeiro et al. (2013), the alpha-power transformation (APT) technique by Mahdavi & Kundu (2017), the cubic rank transmuted-G (CRTr-G) by Granzotto et al. (2017), and the new flexible generalized family (NFGF) by Tahir et al. (2022).

During the recent decade, unlike in the aforementioned approaches, many new distributions in the literature seem to be developed from well-known parent distributions, for example, the Weibull-G family of distributions (Bourguignon et al., 2014), the beta generated family of distributions (Eugene et al., 2002; Zografos & Balakrishnan, 2009), the gamma generated family of distributions (Zografos & Balakrishnan, 2009), the flexible generalized XLindley distribution (Musekwa & Makubate, 2024), the type II half logistic Topp-Leone exponentiated generalized-G family of distributions (Keganne et al., 2023) and the odd-Topp–Leone–Gompertz-G family of distributions (Musekwa & Makubate, 2023) among others.

For an arbitrary parent cumulative distribution function (cdf) $G(x;\gamma)$ with γ being a vector of parameters, (Mahdavi & Kundu, 2017) defined the cdf of the alpha-power transformed-G (APTG) family of distributions by

$$F_{APT}(x) = \begin{cases} \frac{\alpha^{G(x;\gamma)}-1}{\alpha-1}, & \alpha > 0, \alpha \neq 1 \\ G(x;\gamma), & \alpha = 1, \end{cases} \tag{1}$$

where $x \in \Re$ such that $G(x;\gamma)$ is a well-defined cdf. In this article, we propose a similar generator to the APT called the Alpha-Log-Power Transformed-G (ALPTG) technique. Alongside the MO-G, Exp-G, Tr-G, T-X, EGG, APT, CRTr-G, and NFGF approaches, the ALPTG is not based on any well-known parent model. The method takes any arbitrary continuous cdf and adds one shape parameter to it.

2. Material and Methods

2.1. The Alpha-Log-Power Transformed-G

For any continuous baseline cdf, $G(x;\gamma)$, the cdf of the ALPTG family of distributions is given by;

$$F_{ALPTG}(x) = \alpha^{\log\left(\frac{1}{G(x;\gamma)}\right)}, \tag{2}$$

where $0 < \alpha < 1$, $x \in \Re$, and γ is the parameter vector. Clearly Equation (2) satisfies to be a cdf since

- $\forall\ x_1 \leq x_2$ then $F_{ALPTG}(x_1) \leq F_{ALPTG}(x_2)$,
- $\lim\limits_{x\to-\infty} F_{ALPTG}(x) = 0,$ and
- $\lim\limits_{x\to\infty} F_{ALPTG}(x) = 1.$

The probability density function (pdf) corresponding to (2) is given by

$$f_{ALPTG}(x) = \alpha^{\log\left(\frac{1}{G(x;\gamma)}\right)} \log(1/\alpha) g(x;\gamma)(G(x;\gamma))^{-1}, \tag{3}$$

where $g(x;\gamma)$ is the pdf of the continuous random variable X. Thus the associated hazard rate function (hrf) is defined by

$$h_{ALPTG}(x) = \alpha^{\log\left(\frac{1}{G(x;\gamma)}\right)} \frac{\log(1/\alpha)g(x;\gamma)}{G(x;\gamma)} \left(1 - \alpha^{\log\left(\frac{1}{G(x;\gamma)}\right)}\right)^{-1}. \tag{4}$$

For $0 < u < 1$, the quantile function of the NAPTG family of distributions is given by

$$Q(u) = G^{-1}\left\{\exp\left(-\left(\frac{\log(u)}{\log(1/\alpha)}\right)\right)\right\}. \tag{5}$$

As a result, the median of ALPTG distribution is given by

$$Q(0.5) = G^{-1}\left\{\exp\left(-\left(\frac{\log(0.5)}{\log(1/\alpha)}\right)\right)\right\}. \tag{6}$$

The pdf in (3) can be as expressed as

$$f_{ALPTG}(x) = \sum_{e=0}^{\infty} a_e^* g_e^*(x;\gamma) \tag{7}$$

where

$$a_e^* = \sum_{a,b,c=0}^{\infty} \frac{(\log(\alpha))^a \log(1/\alpha)(-1)^e b_{c,b}}{a!e} \binom{a}{b} \binom{a+b+c}{e}, \tag{8}$$

is a constant and

$$g_e^*(x;\gamma) = eg(x;\gamma)G^{e-1}(x;\gamma), \tag{9}$$

is an exp-G pdf with power parameter e (please see Appendix A for more details).

Lemma 1. *Let the random variable X be described by ALPTG given in (3). Then, the pdf of the random variable $Z = G(X)$ is given by*

$$f(z) = \log(1/\alpha) z^{-1} \alpha^{\log\left(\frac{1}{z}\right)}, \quad 0 < z < 1. \tag{10}$$

***Proof*.** We know that;

$$\begin{aligned} Z = G(X) \quad &\Rightarrow \quad X = G^{-1}(Z) \\ \frac{\partial Z}{\partial x} = g(x) \quad &\Rightarrow \quad \frac{\partial x}{\partial Z} = \frac{1}{g(x)}, \end{aligned}$$

now

$$\begin{aligned} f(z) &= \frac{\partial F_{ALPTG}(x)}{\partial z} = \frac{\partial F_{ALPTG}(x)}{\partial x} \times \frac{\partial x}{\partial z} \\ &= \log(1/\alpha) z^{-1} \alpha^{\log\left(\frac{1}{z}\right)}. \end{aligned}$$

□

2.2. The r^{th} Moment and r^{th} Incomplete Moment

The r^{th} moment and the r^{th} incomplete moment of the ALPTG can be easily obtained using Equation (10) and are given by

$$E(X^r) = \int_0^{\infty} x^r f_{ALPTG}(x) dx = \int_0^1 [G^{-1}(z)]^r f(z) dz = E_{f(z)}[G^{-1}(z)]^r, \tag{11}$$

and

$$\ell_{r,a}(x) = \int_0^a x^r f_{ALPTG}(x) dx = \int_0^{G^{-1}(a)} [G^{-1}(z)]^r f(z) dz = E_z[G^{-1}(z)]^r, \tag{12}$$

respectively, where $0 < a < \infty$, and $0 < G^{-1}(a) < 1$.

2.3. Probability Weighted Moment and Moment Generating Function

The Probability Weighted Moment (PWM) is the expectation of a function of a random variable given that the mean of the variable exists. The $(i, j)^{th}$ PWM of a random variable Y, say $(\vartheta_{i,j})$ of the ALPTG is given by

$$\begin{aligned}\vartheta_{i,j} &= E(X^i F^j(X)) = \int_0^\infty x^i F^j_{ALPTG}(x) f_{ALPTG}(x)dx \\ &= \sum_{e=0}^{\infty}(j+1)^a a_e^* \int_0^\infty x^i g_e^*(x;\gamma)dx = \sum_{e=0}^{\infty}(j+1)^a a_e^* E[X^i_{e-1}].\end{aligned} \tag{13}$$

The derivation of expression (13) is provided in Appendix B. The moment generating function (mgf) of the ALPTG family of distributions can be derived from Equation (7) and is given by

$$M_X(t) = \int_0^1 \exp[tG^{-1}(z)]dz, \tag{14}$$

for $-\lambda < t < \lambda$, where $\lambda > 0$.

2.3.1. Residual and Reverse Residual Life Function

If $n > 1$ is an integer value and $x > t$, then the r^{th} moment of residual life is given by;

$$k_r^*(t) = \frac{1}{\bar{F}(x)} \sum_{e=0}^{\infty} a_e^* \sum_{s=0}^{\infty} \binom{n}{s} (-t)^s \int_t^\infty x^{n-s} g_e^*(x;\gamma)dx, \tag{15}$$

where $\bar{F}(x)$ is the survival function.

Suppose $n > 1$ is an integer and $y > t$, then the r^{th} moment of reverse residual life is given by;

$$p_r^*(t) = \frac{1}{F(x)} \sum_{e=0}^{\infty} a_e^* \sum_{s=0}^{\infty} \binom{n}{s} (t)^{n-s} (-1)^s \int_0^t x^s g_e^*(x;\gamma)dx. \tag{16}$$

2.4. Order Statistics

Let $X_1, X_2, \ldots, X_m$ denote an independent and identically distributed random sample of size m from a pdf defined in (3). Furthermore, let Z_1 be the minimum of these X_k, Z_2 the next X_k in order of magnitude, …, Z_m the largest of X_k. That is $Z_1 \leq Z_2 \leq Z_3 \leq \cdots \leq Z_m$ represent $X_1, X_2, \ldots, X_m$ when the latter are arranged in ascending order of magnitude. We call Z_k, $k = 1, 2, 3, \ldots, m$, the k^{th} order statistic of the random sample $X_1, X_2, \ldots, X_m$. The joint pdf of $Z_1, Z_2, Z_3, \ldots, Z_m$ is

$$g(z_1, z_2, z_3, \ldots, z_m) = m! f(z_1) f(z_2) f(z_3) \cdots f(z_m).$$

The marginal pdf of Z_k is given by

$$f_{k:m}(z) = \frac{m!}{(k-1)!(m-k)!} f(z) G^{k-1}(z)[1-G(z)]^{m-k}. \tag{17}$$

As a result, the pdf of the k^{th} order statistic of the ALPTG family of distributions can be written as

$$\begin{aligned} f_{k:m}(z) &= \frac{m!}{(k-1)!(m-k)!} \sum_{l=0}^{\infty} (-1)^l \binom{m-k}{l} f_{ALPTG}(z) F_{ALPTG}^{k+l-1}(z) \\ &= \frac{m!}{(k-1)!(m-k)!} \sum_{l,e=0}^{\infty} (-1)^l \binom{m-k}{l} (k+l)^a a_e^* g_e^*(z;\gamma). \end{aligned} \tag{18}$$

The proof of Equation (18) can be found in Appendix C.

2.5. Rényi Entropy

The Rényi entropy (Rényi, 1961) is an extension of Shannon entropy (Shannon, 1951). Rényi entropy is defined to be

$$I_R(\delta) = \frac{1}{1-\delta} \log\left(\int_0^{\infty} [f_{NAPTG}(x)]^{\delta} dx \right), \delta \neq 1, \delta > 0. \tag{19}$$

Such that the Rényi Entropy of the ALPTG family of distribution is given by

$$\begin{aligned} I_R(\delta) = \frac{1}{1-\delta} \log\Big(&\sum_{a=0}^{\infty} \frac{(\log(\alpha))^a \log(1/\alpha)^{\delta} \delta^a (-1)^e b_{c,b}}{a!} \binom{a}{b} \binom{a+b+c}{e} \\ &\times \int_0^{\infty} g^{\delta}(x;\gamma) G^{e-\delta}(x;\gamma) dx \Big). \end{aligned} \tag{20}$$

The derivation of expression (20) is provided in Appendix D.

2.6. Maximum Likelihood Estimation

Let $X \sim$ALPTG(α, γ) and $\Delta = (\alpha, \gamma)^T$ be the vector of model parameters. The log-likelihood function $\ell_n(\Delta) = \ell_n$ based on a random sample of size n from the ALPTG family of distributions is given by

$$\begin{aligned} \ell_n &= \sum_{i=1}^{n} \log(G(x_i;\gamma)) \log(\alpha) + n \log(\log(1/\alpha)) + \sum_{i=1}^{n} \log(g(x_i;\gamma)) \\ &- \sum_{i=1}^{n} \log(G(x_i;\gamma)). \end{aligned} \tag{21}$$

The score equations for Equation (21) are as follows,

$$\begin{aligned}
\frac{\partial \ell_n}{\partial \alpha} &= \sum_{i=1}^{n} \frac{\log(G(x_i;\gamma))}{\alpha} - \frac{n}{\alpha^2 \log(1/\alpha)}, \\
\frac{\partial \ell_n}{\partial \gamma} &= \sum_{i=1}^{n} \partial \left(\log(1/G(x_i;\gamma)) \log(\alpha)\right)/\partial\gamma + \sum_{i=1}^{n} \partial \left(\log(g(x_i;\gamma))\right)/\partial\gamma \\
&- \sum_{i=1}^{n} \partial \left(\log(G(x_i;\gamma))\right)/\partial\gamma.
\end{aligned}$$

The maximum likelihood estimates (MLEs) are obtained by equating the score equations to zero and solve for the parameters. Since the partial derivatives are non-linear systems of equations, iterative methods like the Newton-Raphson method are required to solve them (Musekwa & Makubate, 2023; Nyamajiwa et al., 2024). The observed Fisher information matrix assessed at MLEs can be used to generate confidence intervals for the model parameters.

3. Results and discussion

3.1. Some Special Cases

The ALPTG family of distributions special instances are discussed in this section. Examples of Weibull, Burr-XII and Pareto distributions for the baseline distribution function are given. The pdf and hrf plots are also shown. The first four ordinary moments, standard deviation (SD), variance ($Var(X)$), skewness (S) and kurtosis (K) for selected parameter values of $\mathbf{\Theta} = (\alpha, \theta, \beta)$ are provided.

3.2. The Alpha-Log-Power Transformed Weibull Distribution

Consider the case where the Weibull distribution is the baseline distribution with the cdf and pdf given by $G(x;\beta,\theta) = 1 - e^{-\beta x^\theta}$ and $g(x;\beta,\theta) = \beta\theta x^{\theta-1} e^{-\beta x^\theta}$, respectively, for $\beta, \theta > 0$. The alpha-log-power transformed-Weibull (ALPTW) distribution's cdf and pdf are provided by

$$F_{ALPTW}(x) = \alpha^{\log\left(\frac{1}{1-e^{-\beta x^\theta}}\right)}, \tag{22}$$

and

$$f_{ALPTW}(x) = \alpha^{\log\left(\frac{1}{1-e^{-\beta x^\theta}}\right)} \frac{\log(1/\alpha)\beta\theta x^{\theta-1} e^{-\beta x^\theta}}{1 - e^{-\beta x^\theta}}, \tag{23}$$

respectively. The quantile function of the ALPTW distribution is given by

$$Q(u) = \left[\frac{-\log\left\{1 - \exp\left(-\left(\frac{\log(u)}{\log(1/\alpha)}\right)\right)\right\}}{\beta}\right]^{\frac{1}{\theta}}, \tag{24}$$

with the corresponding median given by

$$Q(0.5) = \left[\frac{-\log\left\{1 - \exp\left(-\left(\frac{\log(0.5)}{\log(1/\alpha)}\right)\right)\right\}}{\beta} \right]^{\frac{1}{\theta}}. \tag{25}$$

Using Equation (11), the r^{th} moment of the ALPTW distribution is given by

$$E[G^{-1}(z)]^r \quad = \quad \int_0^1 \left[\frac{-\log(1-z)}{\beta}\right]^{r/\theta} \log(1/\alpha) z^{-1} \alpha^{\log\left(\frac{1}{z}\right)} dz. \tag{26}$$

Table 1 shows that, skewness and kurtosis is low for small parameter values of $\boldsymbol{\Theta}$. Skewness and kurtosis increases as we increase the values of θ and β.

TABLE 1: Table of moments for some selected parameter values of the ALPTW distribution

	(0.5,1.0,1.5)	(0.5,1.5,2.0)	(0.5,2.0,2.5)	(0.8,2.5,3.5)	(0.9,3.0,4.0)
$E(X)$	0.5202	0.4647	0.4730	0.2479	0.1648
$E(X^2)$	6475.0000	0.3573	0.5121	0.1182	0.0708
$E(X^3)$	1.2601	0.3641	0.2513	0.0727	0.0402
$E(X^4)$	3.3209	0.4508	0.2330	0.0522	0.0266
$E(X^5)$	11.0095	0.6480	0.2409	0.4170	0.0195
SD	80.4657	0.3760	0.5370	0.2382	0.2089
$Var(X)$	6474.7294	0.1414	0.2884	0.0567	0.0436
S	-0.0194	5.6051	-0.1801	7.9677	6.5280
K	0.0003	4.8575	3.5447	3.8397	4.9484

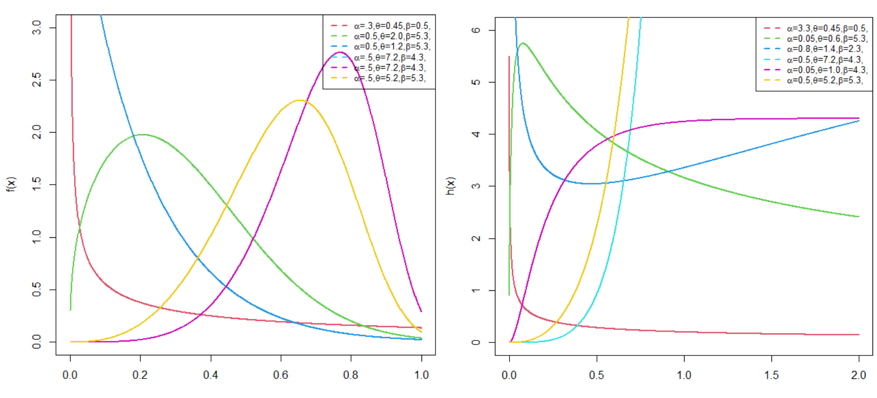

FIGURE 1: Plots of the pdf and hrf for the ALPTWdistribution

Figure 1 shows that the ALPTW distribution can handle left skewed, right-skewed and heavy right skewed data-sets. The model's hazard rate function exhibits increasing, decreasing, bathtub and upside down bathtub shapes.

3.3. The Alpha-Log-Power Transformed Burr-XII Distribution

We consider the burr-XII baseline distribution with the cdf and pdf given by $G(x;\beta,\theta) = 1-(1+x^{\beta})^{-\theta}$ and $g(x;\beta,\theta) = \beta\theta x^{\beta-1}(1+x^{\beta})^{-\theta-1}$, respectively, for $\beta,\theta > 0$. The alpha-log-power transformed-Burr-XII (ALPTBXII) distribution's cdf is provided by

$$F_{ALPTBXII}(x) = \alpha^{\log\left(\frac{1}{1-(1+x^{\beta})^{-\theta}}\right)}. \tag{27}$$

The corresponding pdf is given by

$$f_{ALPTBXII}(x) = \alpha^{\log\left(\frac{1}{1-(1+x^{\beta})^{-\theta}}\right)} \frac{\log(1/\alpha)(\beta\theta x^{\beta-1}(1+x^{\beta})^{-\theta-1})}{1-(1+x^{\beta})^{-\theta}}. \tag{28}$$

The quantile of the ALPTBXII distribution is given by

$$Q(u) = \left[\left\{1-\exp\left(-\left(\frac{\log(u)}{\log(1/\alpha)}\right)\right)\right\}^{\frac{-1}{\theta}} - 1\right]^{\frac{1}{\beta}}. \tag{29}$$

Now, the median of the ALPTBXII distribution is given by

$$Q(0.5) = \left[\left\{1-\exp\left(-\left(\frac{\log(0.5)}{\log(1/\alpha)}\right)\right)\right\}^{\frac{-1}{\theta}} - 1\right]^{\frac{1}{\beta}}. \tag{30}$$

Using Equation (11), the r^{th} moment of the ALPTBXII distribution is given by

$$E[G^{-1}(z)]^r = \int_0^1 [(1-z)^{\frac{-1}{\theta}} - 1]^{\frac{r}{\beta}} \log(1/\alpha) z^{-1} \alpha^{\log\left(\frac{1}{z}\right)} dz. \tag{31}$$

Table 2 shows that, skewness and kurtosis increases as we increases the values of θ and β for a constant value of α.

Table 2: Table of moments for some selected parameter values of the ALPTBXII distribution

	(0.9,2.0,7.0)	(0.9,2.0,7.5)	(0.9,2.5,8.5)	(0.7,3.0,9.0)	(0.7,1.5,7.0)
$E(X)$	0.3989	0.4160	0.4350	0.7029	0.7259
$E(X^2)$	0.2476	0.2606	0.2687	0.5326	0.5983
$E(X^3)$	0.1843	0.1935	0.1929	0.4257	0.5417
$E(X^4)$	0.1533	0.1596	0.1514	0.3549	0.5315
$E(X^5)$	6.1386	0.1422	0.1267	0.3062	0.5630
SD	0.2975	0.2958	0.2819	0.1963	0.2672
$Var(X)$	0.0885	0.0875	0.0795	0.0385	0.0714
S	7.8363	8.2853	9.8481	38.4609	15.3404
K	2.5117	2.3873	2.1391	2.9845	3.3639

It is shown in Figure 2 that the ALPTBXII can model symmetric data-sets together with left skewed, right skewed and heavily skewed data-sets. The model can also handle data-sets with decreasing, upside down bathtub and bathtub followed by upside down bathtub hrf.

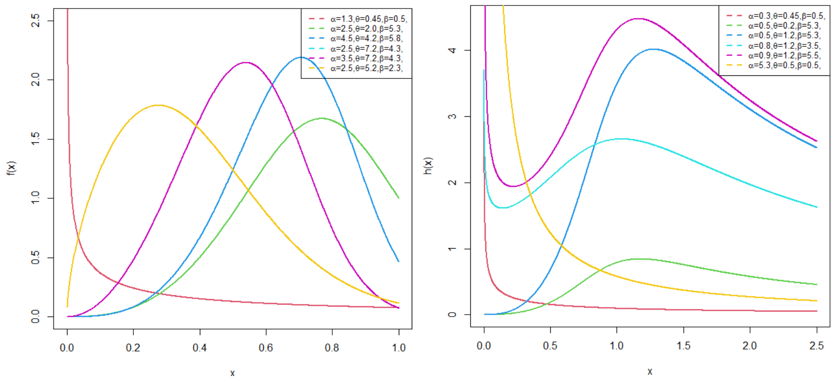

Figure 2: Plots of the pdf and hrf for the ALPTBXII distribution

3.4. The Alpha-Log-Power Transformed Pareto Distribution

We consider the Pareto baseline distribution with the cdf and pdf given by $G(x;\beta,\theta) = 1 - (\theta/x)^{\beta}$ and $g(x;\beta,\theta) = \beta\theta^{\beta}x^{-\beta-1}$, respectively, for $\beta,\theta > 0$. The alpha-log-power transformed-Pareto (ALPTP) distribution's cdf and pdf are provided by

$$F_{ALPTP}(x) = \alpha^{\log\left(\frac{1}{1-(\theta/x)^{\beta}}\right)}, \tag{32}$$

and

$$f_{ALPTP}(x) = \alpha^{\log\left(\frac{1}{1-(\theta/x)^{\beta}}\right)} \frac{\log(1/\alpha)\beta\theta^{\beta}x^{-\beta-1}}{1-(\theta/x)^{\beta}}, \tag{33}$$

respectively for $\alpha,\beta > 0$ and $\theta > 0$. The quantile of the ALPTP distribution is given by

$$Q(u) = \left[\frac{\theta}{\left(1 - \exp\left(-\left(\frac{\log(u)}{\log(1/\alpha)}\right)\right)\right)^{\frac{1}{\beta}}}\right]. \tag{34}$$

As such, we can calculate the median of the ALPTP distribution using the following equation

$$Q(0.5) = \left[\frac{\theta}{\left(1 - \exp\left(-\left(\frac{\log(0.5)}{\log(1/\alpha)}\right)\right)\right)^{\frac{1}{\beta}}}\right]. \tag{35}$$

Using Equation (11), the r^{th} moment of the ALPTP distribution is given by

$$E[G^{-1}(z)]^{r} = \int_0^1 \theta^{r}(1-z)^{\frac{-r}{\beta}} \log(1/\alpha) z^{-1} \alpha^{\log\left(\frac{1}{z}\right)} dz. \tag{36}$$

Table 3: Table of moments for some selected parameter values of the ALPTP distribution

	(0.9,2.0,7.0)	(0.9,2.0,7.5)	(0.9,2.5,7.5)	(0.8,2.5,8.5)	(0.5,1.5,7.5)
$E(X)$	2.0518	2.0479	2.5599	2.6032	1.6783
$E(X^2)$	4.2391	4.2186	6.5915	6.8298	2.8757
$E(X^3)$	8.8563	8.7686	17.1263	18.1198	5.0753
$E(X^4)$	18.8844	18.5105	45.1918	48.8914	9.3748
$E(X^5)$	42.1359	40.2983	122.9807	135.6617	18.7322
SD	0.1709	0.1572	0.1960	0.2305	0.2429
$Var(X)$	0.0292	0.0247	0.0384	0.0531	0.0590
S	-1765.5785	-2256.2221	-2708.1960	-1768.0960	-263.0168
K	124.5523	113.9693	111.5086	51.0095	29.2358

Based on results presented in Table 3, as we hold α constant, skewness and kurtosis increases as we increase the values of θ and β. For small values of α and θ, skewness and kurtosis decreases.

Figure 3 shows that the model can handle right skewed and heavily right skewed data-sets together with data-sets exhibiting upside down bathtub and decreasing hrf.

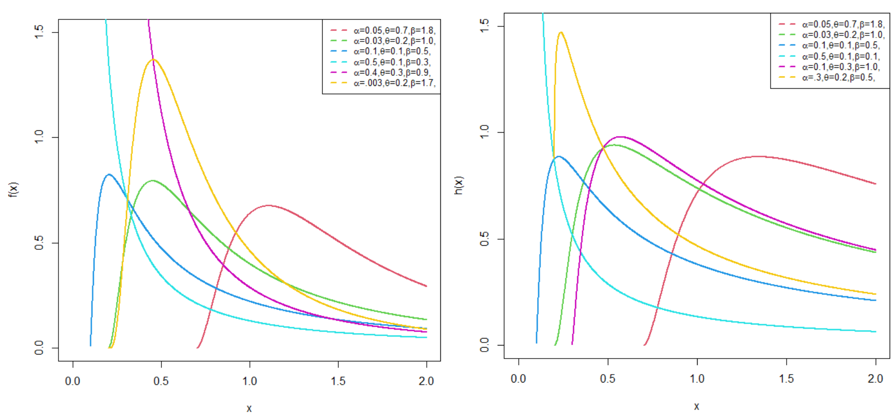

Figure 3: Plots of the pdf and hrf for the ALPTP distribution

4. Monte Carlo Simulation

Using a simulation study of 2000 replications each with a sample size of $n = 50, 100, 200, 400, 800$ and 1000, the MLEs of the ALPTW, ALPTBXII and ALPTP distributions, their Mean Square Error (MSE), as well as the average bias (ABias) are computed. Equations (24), (29) and (34) are used to produce simulated data from the ALPTW, ALPTBXII and ALPTP distributions respectively. The MLE, MSE and the ABias are calculated by

$$MLE(\tau) = \frac{1}{2000}\sum_{i}^{2000}\hat{\tau}_i,$$

$$MSE(\hat{\tau}) = \frac{1}{2000}\sum_{i=1}^{2000}(\hat{\tau}_i - \tau)^2,$$

and

$$ABias(\hat{\tau}) = \frac{1}{2000}\sum_{i=1}^{2000}(\hat{\tau}_i - \tau),$$

respectively, where $\tau = (\alpha, \beta, \theta)$, and $\hat{\tau}$ is an estimate of τ. Tables 4 to 9 show that, for all parameter combinations, the magnitude of both MSE and Abias decrease as the sample size n increases. Thus the MLEs are consistent.

4.1. Alpha Log Power Transformed Weibull Simulation

Table 4: Parameter estimation from the ALPTW results 1

		$(0.2, 0.8, 0.2)$			$(0.2, 1.1, 1.5)$		
Parameter	Sample Size	MLE	RMSE	ABias	MLE	RMSE	ABias
α	50	0.2213	0.1581	0.0213	0.2574	0.1848	0.0574
	100	0.2193	0.1314	0.0193	0.2290	0.1386	0.0290
	200	0.2061	0.1058	0.0061	0.1998	0.1024	-0.0002
	400	0.2074	0.0777	0.0074	0.1966	0.0773	-0.0034
	800	0.2013	0.0574	0.0013	0.1941	0.0561	-0.0059
	1000	0.2029	0.0507	0.0029	0.1932	0.0520	-0.0068
β	50	0.8926	0.2895	0.0926	1.3816	0.6205	0.2816
	100	0.8656	0.2209	0.0656	1.2454	0.3849	0.1454
	200	0.8277	0.1619	0.0277	1.1457	0.2208	0.0457
	400	0.8185	0.1131	0.0185	1.1240	0.1554	0.0240
	800	0.8066	0.0799	0.0066	1.1104	0.1079	0.0104
	1000	0.8069	0.0698	0.0069	1.1054	0.0979	0.0054
θ	50	0.2514	0.2104	0.0514	1.3623	0.5695	0.1377
	100	0.2307	0.1712	0.0307	1.4289	0.4389	0.0711
	200	0.2312	0.1431	0.0312	1.5110	0.3503	0.0110
	400	0.2136	0.0940	0.0136	1.5087	0.2539	0.0087
	800	0.2111	0.0676	0.0111	1.5152	0.1838	0.0152
	1000	0.2076	0.0583	0.0076	1.5151	0.1668	0.0151

Table 5: Parameter estimation from the ALPTW results 2

		(0.5, 1.5, 0.5)			(0.2, 1.5, 0.2)		
Parameter	Sample Size	Mean	RMSE	Bias	Mean	RMSE	A.Bias
α	50	0.4437	0.2051	0.0563	0.2159	0.1542	0.0159
	100	0.4682	0.1521	0.0318	0.2153	0.1295	0.0153
	200	0.4756	0.1087	0.0244	0.2005	0.1022	0.0005
	400	0.4874	0.0732	0.0126	0.1990	0.0744	-0.0010
	800	0.4891	0.0529	0.0109	0.1975	0.0558	-0.0025
	1000	0.4896	0.0482	0.0104	0.1967	0.0503	-0.0033
β	50	1.7044	0.7986	0.2044	1.6419	0.5210	0.1419
	100	1.6178	0.5014	0.1178	1.6031	0.4021	0.1031
	200	1.5507	0.3212	0.0507	1.5351	0.2865	0.0351
	400	1.5334	0.2188	0.0334	1.5163	0.1982	0.0163
	800	1.5132	0.1528	0.0132	1.5083	0.1444	0.0083
	1000	1.5082	0.1371	0.0082	1.5045	0.1277	0.0045
θ	50	0.6117	0.4543	0.1117	0.2448	0.2040	0.0448
	100	0.5507	0.3276	0.0507	0.2249	0.1636	0.0249
	200	0.5296	0.2190	0.0296	0.2268	0.1325	0.0268
	400	0.5090	0.1456	0.0090	0.2148	0.0908	0.0148
	800	0.5106	0.1064	0.0106	0.2088	0.0634	0.0088
	1000	0.5089	0.0949	0.0089	0.2079	0.0568	0.0079

4.2. Alpha Log Power Transformed Burr-XII Simulation

Table 6: Parameter estimation from the ALPTBXII results 1

		(0.3, 1.5, 1.5)			(0.7, 0.7, 2.5)		
Parameter	Sample Size	MLE	RMSE	ABias	MLE	RMSE	ABias
α	50	0.3333	0.2236	0.0333	0.5585	0.2573	-0.1415
	100	0.3204	0.2079	0.0204	0.5969	0.2045	-0.1031
	200	0.3012	0.1807	0.0012	0.6289	0.1608	-0.0711
	400	0.2985	0.1480	-0.0015	0.6598	0.1036	-0.0402
	800	0.2961	0.1149	-0.0039	0.6785	0.0692	-0.0215
	1000	0.2935	0.1058	-0.0065	0.6806	0.0632	-0.0194
β	50	1.4480	0.7953	-0.0520	1.1298	0.8844	0.4298
	100	1.5028	0.7575	0.0028	1.0055	0.6862	0.3055
	200	1.5547	0.6733	0.0547	0.9047	0.5281	0.2047
	400	1.5362	0.5364	0.0362	0.8085	0.3366	0.1085
	800	1.5264	0.4041	0.0264	0.7564	0.2310	0.0564
	1000	1.5295	0.3723	0.0295	0.7466	0.2089	0.0466
θ	50	2.0703	1.4002	0.5703	2.4572	1.3492	-0.0428
	100	1.8686	1.0203	0.3686	2.4242	1.0737	-0.0758
	200	1.6989	0.7286	0.1989	2.4531	0.8941	-0.0469
	400	1.6156	0.5112	0.1156	2.4685	0.6839	-0.0315
	800	1.5615	0.3512	0.0615	2.4955	0.5012	-0.0045
	1000	1.5428	0.3102	0.0428	2.4888	0.4563	-0.0112

Table 7: Parameter estimation from the ALPTBXII results 2

		(0.2, 0.2, 1.1)			(0.7, 0.7, 1.5)		
Parameter	Sample Size	Mean	RMSE	Bias	Mean	RMSE	A.Bias
α	50	0.1715	0.0851	-0.0285	0.5554	0.2650	-0.1446
	100	0.1787	0.0642	-0.0213	0.5766	0.2290	-0.1234
	200	0.1834	0.0459	-0.0166	0.6178	0.1745	-0.0822
	400	0.1823	0.0351	-0.0177	0.6400	0.1209	-0.0600
	800	0.1854	0.0256	-0.0146	0.6606	0.0843	-0.0394
	1000	0.1861	0.0219	-0.0139	0.6570	0.0819	-0.0430
β	50	0.2556	0.2220	0.0556	1.1457	0.9142	0.4457
	100	0.2316	0.1340	0.0316	1.0816	0.7793	0.3816
	200	0.2207	0.0793	0.0207	0.9441	0.5750	0.2441
	400	0.2257	0.0593	0.0257	0.8749	0.3911	0.1749
	800	0.2192	0.0407	0.0192	0.8187	0.2789	0.1187
	1000	0.2179	0.0336	0.0179	0.8271	0.2659	0.1271
θ	50	2.0625	3.1171	0.9625	1.4901	0.8367	-0.0099
	100	1.4060	1.4533	0.3060	1.4248	0.6963	-0.0752
	200	1.1680	0.3946	0.0680	1.4601	0.5758	-0.0399
	400	1.0724	0.2099	-0.0276	1.4220	0.4379	-0.0780
	800	1.0788	0.1603	-0.0212	1.4422	0.3370	-0.0578
	1000	1.0631	0.1254	-0.0369	1.4050	0.2954	-0.0950

4.3. Alpha Log Power Transformed Pareto Simulation

Table 8: Parameter estimation from the ALPTP results 1

		(0.2, 0.2, 1.5)			(0.7, 1.5, 2.5)		
Parameter	Sample Size	MLE	RMSE	ABias	MLE	RMSE	ABias
α	50	0.1794	0.0809	-0.0206	0.6064	0.2076	-0.0936
	100	0.1837	0.0628	-0.0163	0.6371	0.1540	-0.0629
	200	0.1900	0.0439	-0.0100	0.6576	0.1080	-0.0424
	400	0.1917	0.0316	-0.0083	0.6779	0.0675	-0.0221
	800	0.1932	0.0225	-0.0068	0.6871	0.0476	-0.0129
	1000	0.1940	0.0199	-0.0060	0.6866	0.0446	-0.0134
β	50	0.2308	0.1923	0.0308	1.8173	0.9049	0.3173
	100	0.2230	0.1318	0.0230	1.7063	0.6783	0.2063
	200	0.2088	0.0732	0.0088	1.6324	0.4840	0.1324
	400	0.2095	0.0518	0.0095	1.5571	0.3134	0.0571
	800	0.2063	0.0352	0.0063	1.5353	0.2363	0.0353
	1000	0.2050	0.0316	0.0050	1.5335	0.2151	0.0335
θ	50	3.1229	4.7044	1.6229	2.6559	1.2250	0.1559
	100	1.9734	2.1187	0.4734	2.5542	0.8644	0.0542
	200	1.6398	0.5679	0.1398	2.5070	0.6808	0.0070
	400	1.5359	0.3205	0.0359	2.5082	0.4663	0.0082
	800	1.5150	0.2091	0.0150	2.5098	0.3394	0.0098
	1000	1.5133	0.1896	0.0133	2.4949	0.3190	-0.0051

Table 9: Parameter estimation from the ALPTP results 2

		$(0.5, 0.5, 1.0)$			$(0.7, 1.5, 1.5)$		
Parameter	Sample Size	Mean	RMSE	Bias	Mean	RMSE	A.Bias
α	50	0.4240	0.2201	-0.0760	0.5955	0.2163	-0.1045
	100	0.4279	0.1926	-0.0721	0.6241	0.1647	-0.0759
	200	0.4413	0.1582	-0.0587	0.6439	0.1181	-0.0561
	400	0.4611	0.1134	-0.0389	0.6588	0.0769	-0.0412
	800	0.4774	0.0717	-0.0226	0.6649	0.0578	-0.0351
	1000	0.4803	0.0666	-0.0197	0.6633	0.0540	-0.0367
β	50	0.7174	0.6614	0.2174	1.8547	0.9236	0.3547
	100	0.6980	0.5742	0.1980	1.7609	0.7057	0.2609
	200	0.6459	0.4575	0.1459	1.6983	0.5186	0.1983
	400	0.5830	0.3068	0.0830	1.6419	0.3364	0.1419
	800	0.5394	0.1528	0.0394	1.6420	0.2698	0.1420
	1000	0.5316	0.1446	0.0316	1.6534	0.2436	0.1534
θ	50	1.1378	0.6004	0.1378	1.5648	0.7541	0.0648
	100	1.0415	0.4414	0.0415	1.5024	0.5426	0.0024
	200	0.9980	0.3148	-0.0020	1.4681	0.4279	-0.0319
	400	0.9882	0.2244	-0.0118	1.4347	0.2778	-0.0653
	800	0.9953	0.1582	-0.0047	1.4231	0.2044	-0.0769
	1000	0.9953	0.1487	-0.0047	1.4078	0.1867	-0.0922

5. Applications

We used rainfall (Ali et al., 2021) and Canada Covid-19 (Liu et al., 2021) datasets to illustrate the utility of the ALPTW distribution. We analyse both datasets using the ALPTW distribution and six other competitive distributions. The goodness-of-fit statistics (Chen & Balakrishnan, 1995) the ALPTW and non-nested comparative models are presented. Model performance was assessed using these GoF statistics, namely, $-2loglikelihood$ ($-2logL$), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Cramér-Von Mises (W^*), Anderson-Darling (A^*) and the Kolmogorov-Smirnov (K-S) statistic. The model with the smallest values of the presented goodness-of-fit statistics is identified as the best fitting model. We used R software to estimate the model parameters via the *nlm* function. Model parameter estimates (standard errors in parenthesis) and goodness-of-fit-statistics for the selected data sets are presented in Tables 10 and 11. To show how well our model fits the observed data sets, Figures 4, 5, 7 and 8 present plots of the fitted densities, the histogram of the data, probability plots, Kaplain Meier (KM), empirical cumulative distribution function (ECDF) and total-time-on-test (TTT) plots (Cleveland & McGill, 1984). The ALPTW is compared with the Weibull distribution (Rinne, 2008), Burr-XII distribution (Zimmer et al., 1998), generalised Rayleigh distribution (GRD) (Alamatsaz et al., 2016), Transmuted Inverse Rayleigh distribution (TIRD) (Ahmad et al., 2014), Dagum distribution (Dey et al., 2017) and the Gambel distribution (Cooray, 2010).

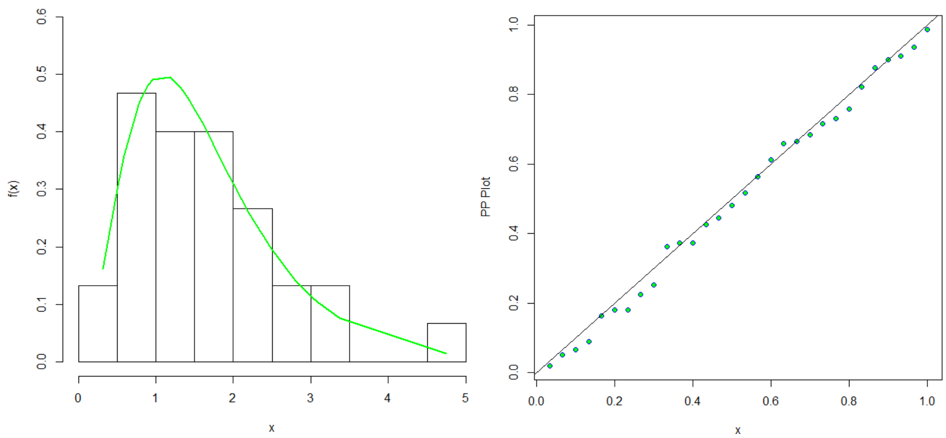

FIGURE 4: Fitted density and the probability plots of the ALPTW distribution for Rainfall data.

5.1. Rainfall Dataset

The first data set consists of thirty observations for the rainfall (in inches) of March in Minneapolis/St Paul (Ali et al., 2021).

The results in Table 10 shows clearly that the ALPTW model provides the best fit in fitting the rainfall data set for it has the lowest goodness-of-fit statistic values. The estimated pdf and probability plot are given in Figure 4, KM curves and ECDF plots are in Figure 5, also the estimated hrf and TTT plots are provided in Figure 6. These Figures clearly show that the ALPTW fits the rainfall data closely.

TABLE 10: Parameter estimates and goodness-of-fit statistics for various models fitted for the Rainfall data set

	Estimates			Statistics						
Model	α	β	θ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	K-S
ALPTW	0.048434	1.052742	1.062657	76.1771	82.1771	83.1002	86.3807	0.0143	0.1046	0.0621
	(0.185153)	(1.050686)	(0.600673)							
Weibull		0.3154	1.8088	77.2865	82.2865	83.7310	86.4889	0.0219	0.1693	0.0689
		(0.0906)	(0.2491)							
BurrXII		3.2554	0.5769	80.5160	84.5160	84.9604	87.3184	0.0696	0.4299	0.1378
		(0.6454)	(0.1371)							
GRD	0.9018	0.4973		77.6567	82.6567	83.1012	86.4591	0.0265	0.2009	0.0770
	(0.2147)	(0.0620)								
TIRD	0.0000		0.8587	88.2730	92.2730	92.7174	95.0754	0.1629	0.9880	0.2396
	(0.4016)		(0.2136)							
Dagum	11.0885	0.5493	3.6468	77.2324	83.2324	84.1555	87.4360	0.0179	0.1323	0.0641
	(21.8645)	(0.3814)	(1.3834)							
Gambel		0.7336	1.2352	77.3839	82.3839	83.8284	86.1863	0.0165	0.1314	0.0666
		(0.1079)	(0.1407)							

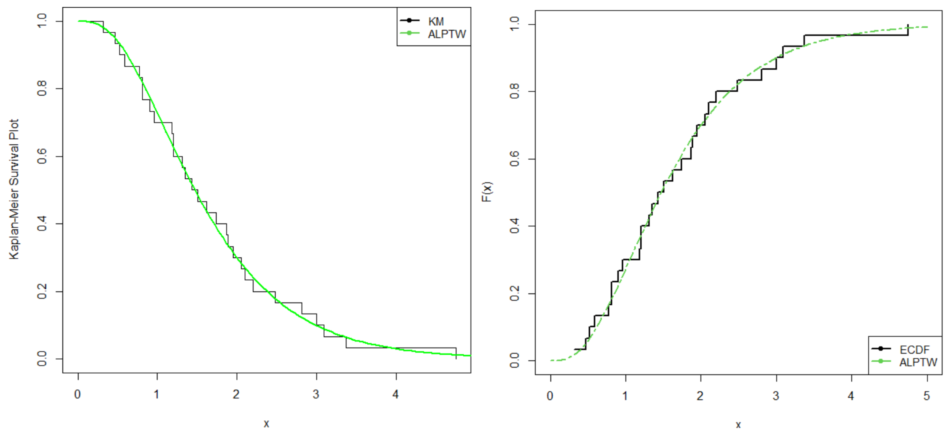

FIGURE 5: KM survival and ECDF plots of the ALPTW distribution for Rainfall data.

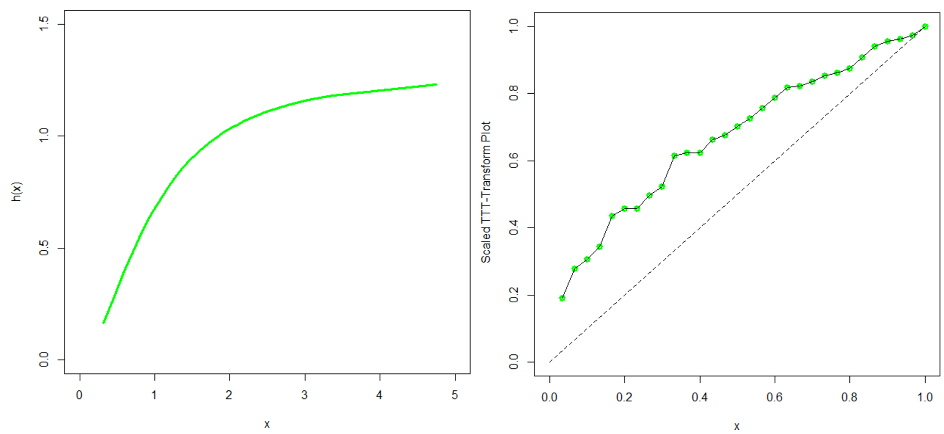

FIGURE 6: Estimated hrf and TTT plots of the ALPTW distribution for Rainfall data.

5.2. Canada Covid-19 Dataset

The second data set represents the mortality rate of the Covid-19 patients in Canada (see Liu et al., 2021).

The results in Table 11 demonstrates clearly that the ALPTW model provides the best fit in fitting the Canada Covid-19 data set for it has the lowest goodness-of-fit statistic values.The estimated pdf and probability plot are given in Figure 7, KM curves and ECDF plots are in Figure 8, also the estimated hrf and TTT plots are provided in Figure 9. These Figures clearly shows that our model fits the Canada Covid-19 data well.

Table 11: Parameter estimates and goodness-of-fit statistics for various models fitted for the Canada Covid-19 data set

	Estimates			Statistics						
Model	α	β	θ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	K-S
ALPTW	0.0002	0.4394	1.5112	96.1872	102.1872	102.9372	106.9378	0.0957	0.5597	0.1053
		(0.0026)	(0.5031)	(0.6043)						
Weibull		0.0138	3.3136	102.9485	106.9485	107.3122	110.1156	0.1729	0.9918	0.1499
		(0.0079)	(0.3788)							
BurrXII		34.6350	0.0252	164.2753	168.2753	168.6389	171.4423	0.1900	1.1445	0.4255
		(3.0561×10^{-6})	(4.2002×10^{-3})							
GRD	3.9991	0.4225		96.7606	102.7606	103.1242	107.9276	0.0999	0.5742	0.1205
	(1.1777)	(0.0351)								
Dagum	7.8890×10^{-4}	3.8097×10^{3}	1	157.6796	161.6796	162.0433	164.8467	0.1288	0.7749	0.4281
	(1.3150×10^{-4})	(2.7204×10^{-11})	(-)							
Gambel		0.8181	2.8368	96.6916	102.6917	103.0553	109.8587	0.1054	0.6283	0.1195
		(0.1011)	(0.1439)							

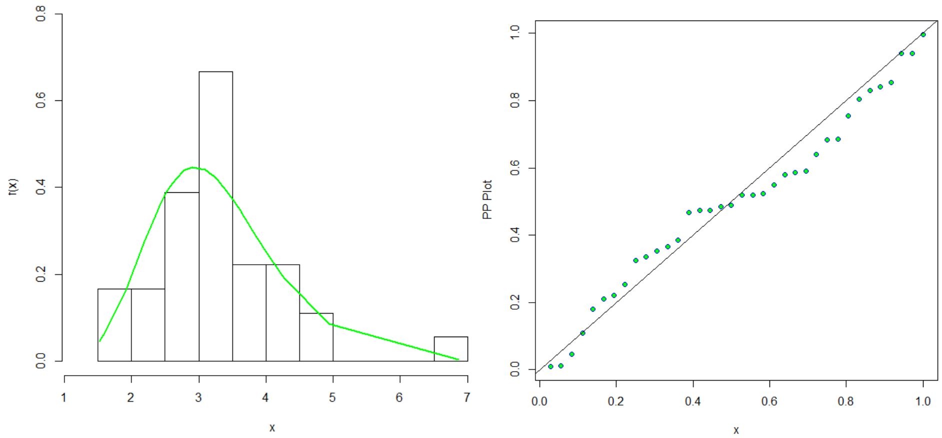

FIGURE 7: Fitted density and the probability plots of the ALPTW distribution for Canada Covid-19 data

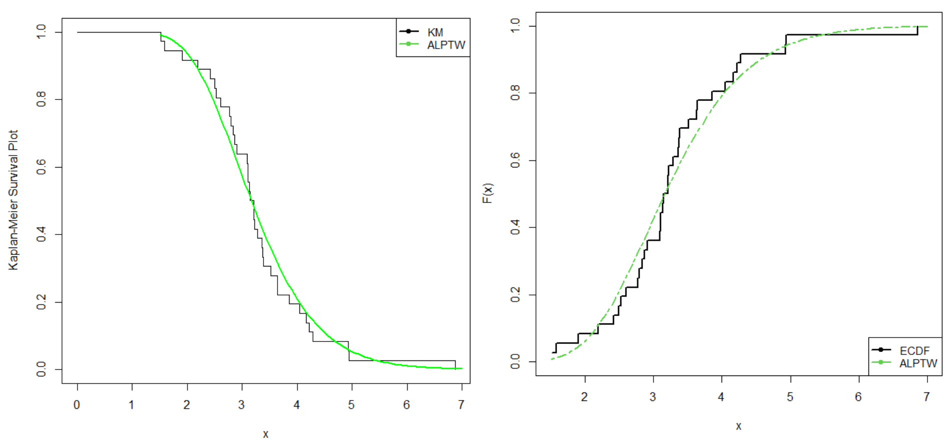

FIGURE 8: KM survival and ECDF plots of the ALPTW distribution for Canada Covid-19 data

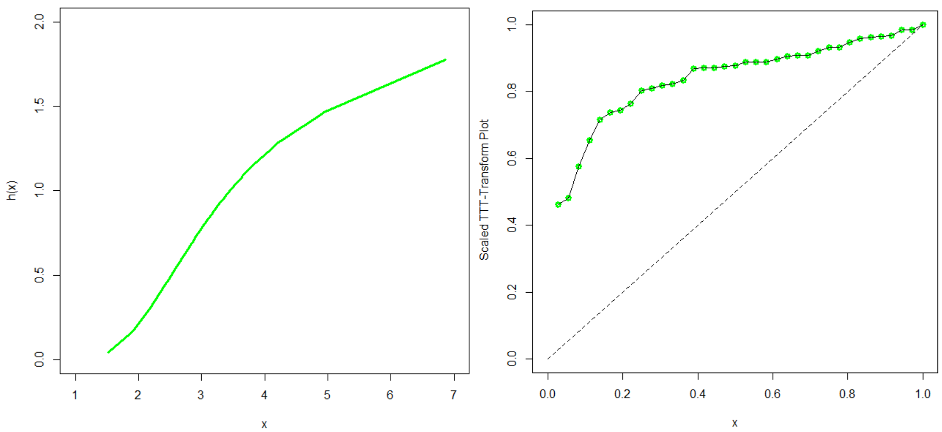

Figure 9: Estimated hrf and TTT plots of the ALPTW distribution for Canada Covid-19 data

6. Concluding Remarks

A new family of distributions called the alpha-log-power transformed-G was introduced. This new family is not based based on any well-known parent cumulative distribution. The generator takes any random continuous cdf and adds one shape parameter. Expansion of density, distribution of order statistics, R'enyi entropy, moments, quantile, generating functions, residual life function, and reverse residual life function are some of the structural components of the family that are studied. The proposed family of distributions can handle data with non-monotonic hazard rate functions as well as profoundly skewed data. The maximum likelihood estimation method was used to estimate the model parameters. A Monte-Carlo simulation exercise was performed using the alpha log power transformed Weibull distribution to investigate the consistency of the maximum likelihood estimates. The alpha log power transformed Weibull and other well-known classical models were used to model the Canada Covid-19 and Rainfall data sets. The observed patterns were more precisely captured by the alpha log power transformed Weibull. Our research suggests that the alpha log power transformed Weibull is a better option than the comparative models presented in this article for modelling the aforementioned datasets. Finally, depending on the choice of the baseline cdf, the quantile function of the new family of distributions can be expressed in a closed form.

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Appendix A. Linear Representation of the pdf

Using the following series expansion on Equation (3),

$$\alpha^{\eta} = \sum_{a=0}^{\infty} \frac{(\log(\alpha))^a}{a!} \eta^a,$$

we have

$$\alpha^{\log(1/G(x;\gamma))} = \alpha^{-\log(G(x;\gamma)} = \sum_{a=0}^{\infty} \frac{(\log(\alpha))^a}{a!} [-\log(G(x;\gamma))]^a.$$

Now, using the expansion,

$$[-\log(1-\bar{G}(x;\gamma))]^a = \bar{G}^a(x;\gamma) \sum_{b=0}^{\infty} \binom{a}{b} \bar{G}^b(x;\gamma) \left[\sum_{c=0}^{\infty} \frac{\bar{G}^c(x;\gamma)}{c+2}\right]^b,$$

and applying the results on a power series raised to a positive integer with $a_c = (c+2)^{-1}$, (see Gradshteyn & Ryzhik, 2014) we have,

$$\left[\sum_{c=0}^{\infty} a_c \bar{G}^c(x;\gamma)\right]^b = \sum_{c=0}^{\infty} b_{c,b} \bar{G}^c(x;\gamma),$$

where $b_{c,b} = (ca_0)^{-1} \sum_{d=0}^{\infty} [b(l+d)-c]_{a_l b_{c-l,b}}$ and $b_{0,b} = a_0^b$, we have

$$\begin{aligned}
[-\log(G(x;\gamma))]^a &= \bar{G}^a(x;\gamma) \sum_{b,c=0}^{\infty} \binom{a}{b} \bar{G}^{(b+c)}(x;\gamma) b_{c,b} \\
(1-G(x;\gamma))^{a+b+c} &= \sum_{e=0}^{\infty} (-1)^e \binom{a+b+c}{e} G^e(x;\gamma).
\end{aligned}$$

Therefore, Equation (3) can be expressed as

$$\begin{aligned}
f_{ALPTG}(x) &= \sum_{a,b,c,e=0}^{\infty} \frac{(\log(\alpha))^a \log(1/\alpha)(-1)^e b_{c,b}}{a!} \binom{a}{b} \binom{a+b+c}{e} \\
&\times \quad g(x;\gamma) G^{e-1}(x;\gamma).
\end{aligned}$$

Appendix B. Probability Weighted Moment

The $(i,j)^{th}$ PWM of a random variable Y, say $(\vartheta_{i,j})$ is given by

$$\vartheta_{i,j} = E(X^i F^j(X)) = \int_0^{\infty} x^i F_{NAPTG}^j(x) f_{NAPTG}(x) dx$$

Now, making use of Equations (2) and (3), and also using the following series expansion on the above equation,

$$\alpha^{\eta}=\sum_{a=0}^{\infty}\frac{(\log(\alpha))^{a}}{a!}\eta^{a},$$

we have

$$\alpha^{(k+1)\log(1/G(z;\gamma))}=\alpha^{-(k+1)\log(G(z;\gamma)}=\sum_{a=0}^{\infty}\frac{(\log(\alpha))^{a}}{a!}(k+1)^{a}[-\log(G(z;\gamma))]^{a}.$$

Now, using the expansion,

$$[-\log(1-\bar{G}(z;\gamma))]^{a}=\bar{G}^{a}(z;\gamma)\sum_{b=0}^{\infty}\binom{a}{b}\bar{G}^{b}(z;\gamma)\left[\sum_{c=0}^{\infty}\frac{\bar{G}^{c}(z;\gamma)}{c+2}\right]^{b},$$

and applying the results on a power series raised to a positive integer with $a_c=(c+2)^{-1}$, (see Gradshteyn & Ryzhik, (2014); Nyamajiwa et al., (2024)) we have,

$$\left[\sum_{c=0}^{\infty}a_{c}\bar{G}^{c}(z;\gamma)\right]^{b}=\sum_{c=0}^{\infty}b_{c,b}\bar{G}^{c}(z;\gamma),$$

where $b_{c,b}=(ca_0)^{-1}\sum_{d=0}^{\infty}[b(l+d)-c]_{a_l b_{c-l,b}}$ and $b_{0,b}=a_0^b$, we have

$$\begin{aligned}[-\log(G(z;\gamma))]^{a} &= \bar{G}^{a}(z;\gamma)\sum_{b,c=0}^{\infty}\binom{a}{b}\bar{G}^{(b+c)}(z;\gamma)b_{c,b}\\ (1-G(x;\gamma))^{a+b+c} &= \sum_{e=0}^{\infty}(-1)^{e}\binom{a+b+c}{e}G^{e}(z;\gamma).\end{aligned}$$

Therefore, the PWM of the ALPTG family of distributions can be expressed as

$$\begin{aligned}\vartheta_{i,j} &= \sum_{a,b,c,e=0}^{\infty}\frac{(\log(\alpha))^{a}\log(1/\alpha)(-1)^{e}(k+1)^{a}b_{c,b}}{a!}\\ &\times \binom{a}{b}\binom{a+b+c}{e}g(z;\gamma)G^{e-1}(z;\gamma).\end{aligned}$$

Appendix C. Order Statistics

The pdf of the k^{th} order statistic of the ALPTG family of distributions can be written as

$$f_{k:m}(z) = \frac{m!}{(k-1)!(m-k)!}\sum_{l=0}^{\infty}(-1)^{l}\binom{m-k}{l}f_{ALPTG}(z)F_{ALPTG}^{k+l-1}(z).$$

Now, making use of Equations (2) and (3), and also using the following series expansion on the above equation,

$$\alpha^{\eta} = \sum_{a=0}^{\infty} \frac{(\log(\alpha))^{a}}{a!}\eta^{a},$$

we have

$$\alpha^{(k+l)\log(1/G(z;\gamma))} = \alpha^{-(k+l)\log(G(z;\gamma)} = \sum_{a=0}^{\infty} \frac{(\log(\alpha))^{a}}{a!}(k+l)^{a}[-\log(G(z;\gamma))]^{a}.$$

Now, using the expansion,

$$[-\log(1-\bar{G}(z;\gamma))]^{a} = \bar{G}^{a}(z;\gamma)\sum_{b=0}^{\infty}\binom{a}{b}\bar{G}^{b}(z;\gamma)\left[\sum_{c=0}^{\infty}\frac{\bar{G}^{c}(z;\gamma)}{c+2}\right]^{b},$$

and applying the results on a power series raised to a positive integer with $a_c = (c+2)^{-1}$, (see Gradshteyn & Ryzhik, 2014; Nyamajiwa et al., 2024) we have,

$$\left[\sum_{c=0}^{\infty} a_c \bar{G}^{c}(z;\gamma)\right]^{b} = \sum_{c=0}^{\infty} b_{c,b}\bar{G}^{c}(z;\gamma),$$

where $b_{c,b} = (ca_0)^{-1}\sum_{d=0}^{\infty}[b(l+d)-c]_{a_l b_{c-l,b}}$ and $b_{0,b} = a_0^b$, we have

$$\begin{aligned}
[-\log(G(z;\gamma))]^{a} &= \bar{G}^{a}(z;\gamma)\sum_{b,c=0}^{\infty}\binom{a}{b}\bar{G}^{(b+c)}(z;\gamma)b_{c,b} \\
(1-G(x;\gamma))^{a+b+c} &= \sum_{e=0}^{\infty}(-1)^{e}\binom{a+b+c}{e}G^{e}(z;\gamma).
\end{aligned}$$

Therefore, the k^{th} order statistic of the ALPTG family of distributions can be expressed as

$$\begin{aligned}
f_{k:m}(z) &= \frac{m!}{(k-1)!(m-k)!}\sum_{l,a,b,c,e=0}^{\infty}(-1)^{l}\binom{m-k}{l}\binom{a}{b}\binom{a+b+c}{e} \\
&\times \frac{(\log(\alpha))^{a}\log(1/\alpha)(-1)^{e}(k+l)^{a}b_{c,b}}{a!}g(z;\gamma)G^{e-1}(z;\gamma).
\end{aligned}$$

Appendix D. Rényi Entropy

From the ALPTG pdf, $[f_{ALPTG}(x)]^{\delta}$, can be expressed as

$$f^{\delta}_{ALPTG}(x) = \alpha^{\delta\log\left(\frac{1}{G(x;\gamma)}\right)}\log(1/\alpha)^{\delta}g^{\delta}(x;\gamma)(G(x;\gamma))^{-\delta},$$

Now, using the expansions mentioned earlier, we have the following;

$$\alpha^{-\delta\log(G(x;\gamma))} = \sum_{a=0}^{\infty}\frac{(\log(\alpha))^{a}\delta^{a}}{a!}[-\log(G(x;\gamma)]^{a}$$

$$[-\log(G(x;\gamma)]^a = \bar{G}^a(x;\gamma) \sum_{b,c=0}^{\infty} (-1) \binom{a}{b} b_{c,b} \bar{G}^{b+c}(x;\gamma)$$

$$[1 - G(x;\gamma)]^{a+b+c} = \sum_{e=0}^{\infty} \binom{a+b+c}{e} (-1)^e G^e(x;\gamma)$$

Now, the Rényi Entropy of the ALPTG family of distribution is given by

$$\begin{aligned} I_R(\delta) &= \frac{1}{1-\delta} \log \Big(\sum_{a=0}^{\infty} \frac{(\log(\alpha))^a \log(1/\alpha)^\delta \delta^a (-1)^e b_{c,b}}{a!} \binom{a}{b} \binom{a+b+c}{e} \\ &\times \int_0^\infty g^\delta(x;\gamma) G^{e-\delta}(x;\gamma) dx \Big). \end{aligned}$$