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Logarithmic Imputation Methods Under Correlated Measurement Errors

Métodos de imputación logarítmica bajo errores de medida correlacionados

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Abstract

In sample survey, missing data is a common issue. Various imputation techniques have been developed to handle the missing data issue. But, a miniscule work has been done to handle missing data issue in the presence of measurement errors (ME) and correlated measurement errors (CME). This manuscript proposes a few logarithmic imputation techniques and the accompanying point estimators to address the missing data issue when the data are affected by CME. The mean square error (MSE) of the proposed imputation methods is reported to the first order approximation. The dominance conditions of the proposed imputation methods over the coeval imputation methods are obtained. Afterward, a simulation study using an artificially drawn population and a real data application are carried out to support the theoretical findings.

***Key words*:** Correlated measurement errors; Imputation; Missing data; Mean squre error.

Resumen

En las encuestas por muestreo, la falta de datos es un problema común. Se han desarrollado varias técnicas de imputación para abordar el problema de la falta de datos. Sin embargo, se ha realizado un trabajo minúsculo para abordar el problema de la falta de datos en presencia de errores de medición (ME) y errores de medición correlacionados (CME). Este manuscrito propone algunas técnicas de imputación logarítmica y los estimadores puntuales

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que las acompañan para abordar el problema de la falta de datos cuando los datos se ven afectados por CME. El error cuadrático medio (MSE) de los métodos de imputación propuestos se informa a la aproximación de primer orden. Se obtienen las condiciones de dominancia de los métodos de imputación propuestos sobre los métodos de imputación coetáneos. Luego, se lleva a cabo un estudio de simulación utilizando una población dibujada artificialmente y una aplicación de datos reales para respaldar los hallazgos teóricos.

***Palabras clave*:** Error de medición correlacionado; Error cuadrático medio; Datos faltantes; Imputación.

1. Introduction

In sample survey, missing data is a common problem which reduces the quality of data and ultimately results incorrect conclusions. Therefore, it is essential to cautiously handle the problem of missing data. Imputation is an established technique that is efficiently used to tackle the situation of missing data. Three cardinal mechanisms–observed at random (OAR), missing at random (MAR), and parameter distribution were first described by Rubin (1976). The OAR mechanism arises when, provided the observed and unobserved data, the probability of the observed missingness pattern does not depend on the observed data values for every conceivable value of the missing data. The MAR mechanism takes place when, given the observed and unobserved data, the probability of the observed missingness pattern does not rely on the values of the unobserved data. The missing completely at random (MCAR) is the combination of OAR and MAR. The processes of MAR and MCAR were also separated by Heitjan & Basu (1996). Various prominent authors developed a number of imputation methods over the last few decades namely, mean, regression, ratio, cold deck, hot deck and other imputation methods. The study of Singh & Horn (2000), Singh & Deo (2003), Rueda & Gonzalez (2004), Ahmed et al. (2006), Toutenburg et al. (2008), Singh (2009), Prasad (2018), Shahzad & Hanif (2019), Bhushan & Pandey (2021), Audu & Singh (2021), Prasad (2021), Shahzad et al. (2022), Anas et al. (2022), Audu et al. (2023), Bhushan & Kumar (2023), Bhushan, Kumar, Pandey & Singh (2023), Bhushan & Kumar (2023), and Bhushan et al. (2024) are recommended to the interested readers. To prevent the circumstance of missing data, these authors took into account the MCAR mechanism and offered several imputation techniques and the related point estimators based on auxiliary data. This study also considers MCAR mechanism to impute the missing data.

Measurement or observational errors are the another source of non-sampling error that arises when the observed value of the sample units deviates from the real value. Faulty machinery, hiring people with little expertise, or other circumstances might cause ME. Shalabh (1997) measured the effect of ME on the population mean using classical ratio and product estimators. Manisha & Singh (2001) considered the regression type estimator in the presence of ME under simple random sampling (SRS). Sahoo et al. (2006) computed the effect of ME using ratio and regression estimators. Gregoire & Salas (2008) measured the performance of the ratio estimator of population total in the presence of ME. Singh & Karpe (2008) considered the ratio and product estimators in case of ME. Singh (2009) developed a class of population variance estimators under ME. Singh & Karpe (2010*b*) proposed the mean, product and ratio estimators under SRS. In stratified random sampling, Singh & Karpe (2010*a*) recommended the separate and combined ratio and product estimators when data were tainted by ME. The computation of the population variance under ME was addressed by Diana & Giordan (2012). The regression and ratio type calibration estimators for the population mean in stratified random sampling were developed by Vishwakarma et al. (2020). Singh et al. (2022) computed the effect of MEs on some efficient variant of the product and ratio estimators of mean utilizing auxiliary information. Singh et al. (2022) computed the effect of MEs by using auxiliary information under systematic sampling. Vishwakarma & Singh (2022) computed the effect of MEs on ranked set sampling estimators of the population mean. Bhushan, Kumar, Shukla, Bakr, Tashkandy & Hossain (2023) suggested some novel logarithmic type estimators in the presence of MEs.

Bhushan, Kumar & Shukla (2023*a*) investigated some classes of the robust estimators in the presence of CMEs. Bhushan, Kumar & Shukla (2023*b*) assessed the impact of CMEs using logarithmic-type estimators. Bhushan et al. (2024) evaluated the performance of novel logarithmic estimators under CMEs. Kumar et al. (2023) measured the impact of CMEs on some efficient classes of estimators. Kumar et al. (2024) investigated some robust imputation methods in the presence of CMEs. To resolve the issue of missing data in the presence of CME, some new logarithmic type imputation methods (LIM) have been introduced in this work. The corresponding point estimators have been obtained to compute the population mean and to analyze the effect of CME.

The subsequent section provides the formulation of the model and a recap of the conventional imputation methods (CIM) in the case of CME. Some novel LIM and their properties under CME have been proposed in Section 3 and the theoretical conditions are obtained under which the LIM surpass the CIM. To support the theoretical results and to study the impact of CME, an extensive simulation study based on artificially generated data is carried out in Section 4, and a real data application of the proposed LIM is provided in Section 5. The study is concluded in Section 6.

2. CIMs and Their Properties

Consider a finite population $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_N)$ of size N from which a sample of size n is drawn using simple random sampling without replacement to compute the population mean μ_Y of study variable Y. Let r be the responding units out of n sampled units. Let S_r and $\bar{S}_r$, respectively, stand for the set of responding units and non-responding units. The amount y_i is retrieved for every unit, $i \in S_r$, but the units $i \in \bar{S}_r$, have missing values and require imputation in order to complete the creation of the sample data set. Let the imputation be performed consisting of the additional auxiliary information, Z, so $Z_i > 0$, for all $i \in s$ provided that the data $Z_s = \{Z_i; i \in s\}$ are available.

Further, we take the situation where data values may be measured with ME. Let the observed values be represented by (y_i, z_i); $i = 1, 2, \ldots, n$ and the true values be denoted by (Y_i, Z_i). Let the observed values be expressed in additive forms as $y_i = Y_i + U_i$ and $z_i = Z_i + V_i$. Assuming that μ_Y and μ_Z are the population means, σ_Y^2 and σ_Z^2 are the population variances, C_Y and C_Z are the population coefficients of variation of variables Y and Z, respectively, and ρ_{YZ} is the coefficient of correlation between variables Y and Z. The ME U_i and V_i are also unobservable with means 0, variances σ_U^2 and σ_V^2, respectively and correlation coefficient ρ_{UV}. In the presence of ME, $s_z^2 = (n-1)^{-1}\sum_{i=1}^{n}(z_i-\bar{z})^2$ and $s_y^2 = (n-1)^{-1}\sum_{i=1}^{n}(y_i-\bar{y})^2$ are not unbiased estimators of the population variances σ_Z^2 and σ_Y^2, respectively. Thus, the expected values of s_z^2 and s_y^2 in the case of ME are given by $E(s_z^2) = \sigma_Z^2 + \sigma_V^2$ and $E(s_y^2) = \sigma_Y^2 + \sigma_U^2$, respectively.

When determining the characteristics of the imputation methods, we take into account the notations described as follows: $\bar{y}_r = \mu_Y(1 + \epsilon_0)$, $\bar{z}_r = \mu_Z(1 + \epsilon_1)$ and $\bar{z}_n = \mu_Z(1 + \epsilon_2)$ where, $\epsilon_k,\ k = 0, 1, 2$ are error terms such that $E(\epsilon_k) = 0,\ k = 0, 1, 2$, $E(\epsilon_0^2) = f_r\aleph_y C_Y^2$, $E(\epsilon_1^2) = f_r\aleph_z C_Z^2$, $E(\epsilon_2^2) = f_n\aleph_z C_Z^2$, $E(\epsilon_0\epsilon_1) = f_r\aleph_{yz}C_Y C_Z$, $E(\epsilon_0\epsilon_2) = f_n\aleph_{yz}C_Y C_Z$, $E(\epsilon_1\epsilon_2) = f_n\aleph_z C_Z^2$, where $f_n = n^{-1}-N^{-1}$, $f_r = r^{-1}-N^{-1}$, $f_{rn} = r^{-1} - n^{-1}$, $C_Y = S_Y/\mu_Y$ and $C_Z = S_Z/\mu_Z$. Also, $\aleph_y = (\sigma_Y^2 + \sigma_U^2)/\sigma_Y^2$, $\aleph_z = (\sigma_Z^2 + \sigma_V^2)/\sigma_Z^2$ and $\aleph_{yz} = (\rho_{yz}\sigma_Y\sigma_Z + \rho_{uv}\sigma_U\sigma_V)/\sigma_Y\sigma_Z$.

2.1. Mean Imputation Method

Consider the notations of Lee et al. (1994), and for the imputation of unit value, when the data are ridden with ME and the i^{th} sample unit is absent and requires imputation, then the procedure of imputation is provided as

$$y_{.i_m} = \begin{cases} y_i & \text{for } i \in S_r \\ \bar{y}_r & \text{for } i \in \bar{S}_r \end{cases}$$

The point estimator is given below as

$$\zeta_m = \bar{y}_r$$

The mean square error (MSE) of estimator ζ_m is given below as

$$MSE(\zeta_m) = f_r\aleph_y C_Y^2 \tag{1}$$

Taking the additional information on auxiliary variable into account, the methods of imputation are divided into the following strategies:
Strategy I: When μ_Z is known and $\bar{z}_n$ is utilized.
Strategy II: When μ_Z is known and $\bar{z}_r$ is utilized.
Strategy III: When μ_Z is not known and $\bar{z}_n$, $\bar{z}_r$ are utilized.

2.2. Ratio Imputation Methods

The conventional ratio imputation methods under CME are given below as.
Strategy I

$$y_{.i_{r_1}} = \begin{cases} y_i & \text{for } i \in S_r \\ \frac{1}{n-r}\left[n\bar{y}_r\left(\frac{\mu_Z}{\bar{z}_n}\right) - r\bar{y}_r\right] & \text{for } i \in \bar{S}_r \end{cases}$$

Strategy II

$$y_{.i_{r_2}} = \begin{cases} y_i & \text{for } i \in S_r \\ \frac{1}{n-r}\left[n\bar{y}_r\left(\frac{\mu_Z}{\bar{z}_r}\right) - r\bar{y}_r\right] & \text{for } i \in \bar{S}_r \end{cases}$$

Strategy III

$$y_{.i_{r_3}} = \begin{cases} y_i & \text{for } i \in S_r \\ \frac{1}{n-r}\left[n\bar{y}_r\left(\frac{\bar{z}_n}{\bar{z}_r}\right) - r\bar{y}_r\right] & \text{for } i \in \bar{S}_r \end{cases}$$

Under the strategies I, II and III, the point estimators are given below as

$$\zeta_{r_1} = \bar{y}_r\left(\frac{\mu_Z}{\bar{z}_n}\right)$$
$$\zeta_{r_2} = \bar{y}_r\left(\frac{\mu_Z}{\bar{z}_r}\right)$$
$$\zeta_{r_3} = \bar{y}_r\left(\frac{\bar{z}_n}{\bar{z}_r}\right)$$

The MSE equations of these point estimators are given below as

$$MSE(\zeta_{r_1}) = \mu_Y^2(f_r\aleph_y C_Y^2 + f_n\aleph_z C_Z^2 - 2f_n\aleph_{yz}C_Y C_Z) \quad (2)$$
$$MSE(\zeta_{r_2}) = \mu_Y^2(f_r\aleph_y C_Y^2 + f_r\aleph_z C_Z^2 - 2f_r\aleph_{yz}C_Y C_Z) \quad (3)$$
$$MSE(\zeta_{r_3}) = \mu_Y^2(f_r\aleph_y C_Y^2 + f_{rn}\aleph_z C_Z^2 - 2f_{rn}\aleph_{yz}C_Y C_Z) \quad (4)$$

2.3. Regression Imputation Methods

We define the classical regression imputation methods in the presence of CME as.
Strategy I

$$y_{.i_{l_1}} = \begin{cases} y_i & \text{for } i \in S_r \\ \bar{y}_r + \frac{n\beta_1}{n-r}(\mu_Z - \bar{z}_n) & \text{for } i \in \bar{S}_r \end{cases}$$

Strategy II

$$y_{.i_{l_2}} = \begin{cases} y_i & \text{for } i \in S_r \\ \bar{y}_r + \frac{n\beta_2}{n-r}(\mu_Z - \bar{z}_r) & \text{for } i \in \bar{S}_r \end{cases}$$

Strategy III

$$y_{.i_{l_3}} = \begin{cases} y_i & \text{for } i \in S_r \\ \bar{y}_r + \frac{n\beta_3}{n-r}(\bar{z}_n - \bar{z}_r) & \text{for } i \in \bar{S}_r \end{cases}$$

The point estimators are given as

$$\zeta_{l_1} = \bar{y}_r + \beta_1(\mu_Z - \bar{z}_n)$$
$$\zeta_{l_2} = \bar{y}_r + \beta_2(\mu_Z - \bar{z}_r)$$
$$\zeta_{l_3} = \bar{y}_r + \beta_3(\bar{z}_n - \bar{z}_r)$$

where β_1, β_2 and β_3 are regression coefficients of y on z.

The MSE equations of the above point estimators at optimum value of $\beta_{i(opt)} = \mu_Y \aleph_{yz} C_Y / \mu_Z \aleph_z C_Z$ are given by

$$\min .MSE(\zeta_{l_1}) = \mu_Y^2 C_Y^2 \left(f_r \aleph_y - f_n \frac{\aleph_{yz}^2}{\aleph_z} \right) \tag{5}$$

$$\min .MSE(\zeta_{l_2}) = f_r \mu_Y^2 C_Y^2 \left(\aleph_y - \frac{\aleph_{yz}^2}{\aleph_z} \right) \tag{6}$$

$$\min .MSE(\zeta_{l_3}) = \mu_Y^2 C_Y^2 \left(f_r \aleph_y - f_{rn} \frac{\aleph_{yz}^2}{\aleph_z} \right) \tag{7}$$

3. Proposed Imputation Methods

Logarithms are found to be fruitful because they demonstrate numbers on the realistic scale that people can easily comprehend. Additionally, because logarithms treat multiplication as a series of steps, they may depict events whose magnitudes might differ enormously, such as the pH scale used to measure acidity. Such frequently varying magnitudes can be measured efficiently in a single scale using logarithmic scale graphs. In logarithmic scale graphs, the exponential changes are shown by a straight line and make them easier to draw conclusion. The utilization of logarithms in real life can be found in Richter scale for measuring earthquakes, decibels for measuring sound and pH scale for measuring acidity. It can also be utilized in exponential decay and growth namely, radioactive decay in radiocarbon dating interest rate, invasive species, Covid-19 pandemic, bacterial growth and compound rate, etc. Apart from these benefits of logarithms, the motivation of using the logarithmic imputation methods is that the logarithmic imputations provide efficient alternative imputations for the estimation of population mean in the presence of missing data. Therefore, to deal with the issue of missing data in the presence of CMEs, it is essential to explore and study the utilization of LIM for the estimation of population mean. Following Bhushan & Kumar (2022) and Bhushan & Kumar (2023), we construct the following LIM in the presence of CME as

Strategy I

$$y_{.i_{a_1}} = \begin{cases} y_i & \text{for } i \in S_r \\ \frac{1}{n-r}\left[n\psi_1\bar{y}_r\left\{1+\log\left(\frac{\bar{z}_n}{\mu_Z}\right)\right\}^{\delta_1} - r\bar{y}_r\right] & \text{for } i \in \bar{S}_r \end{cases}$$

$$y_{.i_{a_4}} = \begin{cases} y_i & \text{for } i \in S_r \\ \frac{1}{n-r}\left[n\psi_4\bar{y}_r\left\{1+\delta_4\log\left(\frac{\bar{z}_n}{\mu_Z}\right)\right\} - r\bar{y}_r\right] & \text{for } i \in \bar{S}_r \end{cases}$$

Design II

$$y_{.i_{a_2}} = \begin{cases} y_i & \text{for } i \in S_r \\ \frac{1}{n-r}\left[n\psi_2\bar{y}_r\left\{1+\log\left(\frac{\bar{z}_r}{\mu_Z}\right)\right\}^{\delta_2} - r\bar{y}_r\right] & \text{for } i \in \bar{S}_r \end{cases}$$

$$y_{.i_{a_5}} = \begin{cases} y_i & \text{for } i \in S_r \\ \frac{1}{n-r}\left[n\psi_5\bar{y}_r\left\{1+\delta_5\log\left(\frac{\bar{z}_r}{\mu_Z}\right)\right\} - r\bar{y}_r\right] & \text{for } i \in \bar{S}_r \end{cases}$$

Design III

$$y_{.i_{a_3}} = \begin{cases} y_i & \text{for } i \in S_r \\ \frac{1}{n-r}\left[n\psi_3\bar{y}_r\left\{1+\log\left(\frac{\bar{z}_r}{\bar{z}_n}\right)\right\}^{\delta_3} - r\bar{y}_r\right] & \text{for } i \in \bar{S}_r \end{cases}$$

$$y_{.i_{a_6}} = \begin{cases} y_i & \text{for } i \in S_r \\ \frac{1}{n-r}\left[n\psi_6\bar{y}_r\left\{1+\delta_6\log\left(\frac{\bar{z}_r}{\bar{z}_n}\right)\right\} - r\bar{y}_r\right] & \text{for } i \in \bar{S}_r \end{cases}$$

The point estimators are given by

$$\zeta_{a_1} = \psi_1\bar{y}_r\left[1+\log\left(\frac{\bar{z}_n}{\mu_Z}\right)\right]^{\delta_1}$$

$$\zeta_{a_2} = \psi_2\bar{y}_r\left[1+\log\left(\frac{\bar{z}_r}{\mu_Z}\right)\right]^{\delta_2}$$

$$\zeta_{a_3} = \psi_3\bar{y}_r\left[1+\log\left(\frac{\bar{z}_r}{\bar{z}_n}\right)\right]^{\delta_3}$$

$$\zeta_{a_4} = \psi_4\bar{y}_r\left[1+\delta_4\log\left(\frac{\bar{z}_n}{\mu_Z}\right)\right]$$

$$\zeta_{a_5} = \psi_5\bar{y}_r\left[1+\delta_5\log\left(\frac{\bar{z}_r}{\mu_Z}\right)\right]$$

$$\zeta_{a_6} = \psi_6\bar{y}_r\left[1+\delta_6\log\left(\frac{\bar{z}_r}{\bar{z}_n}\right)\right]$$

where ψ_i and δ_i, $i = 1, 2, \ldots, 6$ are appropriately selected scalars.

Theorem 1. *The first order approximated MSE and minimum MSE of the consequent point estimators ζ_{a_i}, $i = 1, 2, \ldots, 6$ of the proposed imputation methods are given by*

$$MSE(\zeta_{a_i}) \approx \mu_Y^2(1+\psi_i^2 L_i - 2\psi_i M_i) \tag{8}$$

$$min.MSE(\zeta_{a_i}) \approx \mu_Y^2\left(1-\frac{M_i^2}{L_i}\right) \tag{9}$$

where $L_1 = 1 + f_r\aleph_y C_Y^2 + 2\delta_1(\delta_1-1)f_n\aleph_z C_Z^2 + 4\delta_1 f_n\aleph_{yz}C_Z C_Y$; $M_1 = 1 + \{\delta_1(\delta_1-2)/2\}f_n\aleph_z C_Z^2 + \delta_1 f_n\aleph_{yz}C_Z C_Y$; $L_2 = 1 + f_r\aleph_y C_Y^2 + 2\delta_2(\delta_2-1)f_r\aleph_z C_Z^2 + 4\delta_2 f_r\aleph_{yz}C_Z C_Y$; $M_2 = 1 + \{\delta_2(\delta_2-2)/2\}f_r\aleph_z C_Z^2 + \delta_2 f_r\aleph_{yz}C_Z C_Y$, $L_3 = 1 + f_r\aleph_y C_Y^2 + 2\delta_3(\delta_3-1)f_{rn}\aleph_z C_Z^2 + 4\delta_3 f_{rn}\aleph_{yz}C_Z C_Y$; $M_3 = 1 + \{\delta_3(\delta_3-2)/2\}f_{rn}\aleph_z C_Z^2 + \delta_3 f_{rn}\aleph_{yz}C_Z C_Y$; $L_4 = 1 + f_r\aleph_y C_Y^2 + \delta_4(\delta_4-1)f_n C_Z^2 + 4\delta_4 f_n\aleph_{yz}C_Y C_Z$; $M_4 = 1 - (\delta_4/2)f_n C_Z^2 + \delta_4 f_n\aleph_{yz}C_Y C_Z$; $L_5 = 1 + f_r\aleph_y C_Y^2 + \delta_5(\delta_5-1)f_r C_Z^2 + 4\delta_5 f_r\aleph_{yz}C_Y C_Z$; $M_5 = 1 - (\delta_5/2)f_r C_Z^2 + \delta_5 f_r\aleph_{yz}C_Y C_Z$; $L_6 = 1 + f_r\aleph_y C_Y^2 + \delta_6(\delta_6-1)f_{rn}C_Z^2 + 4\delta_6 f_{rn}\aleph_{yz}C_Y C_Z$; *and* $M_6 = 1 - (\delta_6/2)f_{rn}C_Z^2 + \delta_6 f_{rn}\aleph_{yz}C_Y C_Z$.

Proof. Consider the first proposed point estimator ζ_{a_1} as

$$\zeta_{a_1} = \psi_1\bar{y}_r\left[1+\log\left(\frac{\bar{z}_n}{\mu_Z}\right)\right]^{\delta_1}$$

Employing the notations defined in Section 2, the proposed estimator ζ_{a_1} can be expressed as

$$\zeta_{a_1} = \psi_1\mu_Y(1+\epsilon_0)\left[1+\log\left\{\frac{\mu_Z(1+\epsilon_2)}{\mu_Z}\right\}\right]^{\delta_1} \tag{10}$$

$$= \psi_1\mu_Y(1+\epsilon_0)\left(1+\epsilon_2-\frac{\epsilon_2^2}{2}\right)^{\delta_1} \tag{11}$$

Using Taylor series expansion in the right hand side, excluding the error terms having power more than two, we get

$$\zeta_{a_1} \approx \psi_1\mu_Y(1+\epsilon_0)\left[1+\delta_1\left(\epsilon_2-\frac{\epsilon_2^2}{2}\right)+\frac{\delta_1(\delta_1-1)}{2}\epsilon_2^2\right] \tag{12}$$

$$\approx \mu_Y\psi_1\left(1+\epsilon_0+\delta_1\epsilon_2+\delta_1\epsilon_0\epsilon_2-\delta_1\epsilon_2^2+\frac{\delta_1^2}{2}\epsilon_2^2\right) \tag{13}$$

Subtracting μ_Y both the sides of (13), we get

$$\zeta_{a_1}-\mu_Y \approx \mu_Y\left\{\psi_1\left(1+\epsilon_0+\delta_1\epsilon_1+\delta_1\epsilon_0\epsilon_1-\delta_1\epsilon_1^2+\frac{\delta_1^2}{2}\epsilon_1^2\right)-1\right\} \tag{14}$$

Squaring and taking expectation on both side of (14), we obtain the MSE of the point estimator ζ_{a_1} up to first order approximation as

$$MSE(\zeta_{a_1}) \approx \mu_Y^2\left[\begin{array}{l} 1+\psi_1^2\left\{1+f_r\aleph_y C_Y^2+2\delta_1(\delta_1-1)f_n\aleph_z C_Z^2+4\delta_1 f_n\aleph_{yz}C_Z C_Y\right\} \\ -2\psi_1\left\{1+\frac{\delta_1(\delta_1-2)}{2}f_n\aleph_z C_Z^2+\delta_1 f_n\aleph_{yz}C_Y C_Z\right\} \end{array}\right] \tag{15}$$

The MSE of the estimator ζ_{a_1} of (15) can further be written as

$$MSE(\zeta_{a_1}) \approx \mu_Y^2(1+\psi_1^2 L_1 - 2\psi_1 M_1) \tag{16}$$

where $L_1 = 1 + f_r \aleph_y C_Y^2 + 2\delta_1(\delta_1 - 1) f_n \aleph_z C_Z^2 + 4\delta_1 f_n \aleph_{yz} C_Z C_Y$ and $M_1 = 1 + \{\delta_1(\delta_1 - 2)/2\} f_n \aleph_z C_Z^2 + \delta_1 f_n \aleph_{yz} C_Z C_Y$.

Partially differentiating (16) with respect to ψ_1 and equating to zero, we get

$$\frac{\partial MSE(\zeta_{a_1})}{\partial \psi_1} = 0$$
$$\mu_Y^2(2\psi_1 L_1 - 2M_1) = 0$$
$$\psi_{1(opt)} = \frac{M_1}{L_1}$$

Putting the value of $\psi_{1(opt)}$ in (16), we get the minimum MSE of the estimator ζ_{a_1} as

$$\min .MSE(\zeta_{a_1}) \approx \mu_Y^2 \left(1 - \frac{M_1^2}{L_1}\right) \tag{17}$$

The derivations of the MSE expressions of other suggested point estimators can be deduced on the same lines. In general, we can write

$$MSE(\zeta_{a_i}) \approx \mu_Y^2(1 + \psi_i^2 L_i - 2\psi_i M_i),\ i = 1, 2, \ldots, 6 \tag{18}$$

Partially differentiating (18) with respect to ψ_i and equating to zero, we get

$$\frac{\partial MSE(\zeta_{a_i})}{\partial \psi_i} = 0$$
$$\mu_Y^2(2\psi_i L_i - 2M_i) = 0$$
$$\psi_{i(opt)} = \frac{M_i}{L_i}$$

Putting the value of $\psi_{i(opt)}$ in (18), we get the minimum MSE of the estimator ζ_{a_i} as

$$\min .MSE(\zeta_{a_i}) \approx \mu_Y^2 \left(1 - \frac{M_i^2}{L_i}\right) \tag{19}$$

Note that the simultaneous optimization of ψ_i and $\delta_i,\ i = 1, 2, \ldots, 6$ of the MSE is not possible. Hence, we consider $\delta_i = \delta_{i(opt)}$ such that $\psi_i = 1$ and use it inside $\psi_i = \psi_{i(opt)}$ to obtain (19).

The optimum value of $\delta_i,\ i = 1, 2, \ldots, 6$ is given below as

$$\delta_{i(opt)} = -\aleph_{yz} \frac{C_Y}{\aleph_z C_Z}$$

which can be obtained by putting $\psi_i = 1$ in the respective point estimators and minimizing their MSE expressions for the scalars $\delta_i,\ i = 1, 2, \ldots, 6$, respectively. □

Further, we compare the MSE of the proposed LIM with the MSE of the CIM under strategies I, II and III. The proposed LIM dominate the CIM under the following conditions.

Condition 1.
The proposed LIM $y_{.i_{a_i}},\ i = 1, 2, \ldots, 6$ perform superior than the usual mean imputation method $y_{.i_m}$ under strategies I, II and III, if

$$MSE(\zeta_{a_i}) < MSE(\zeta_m) \implies \frac{M_i^2}{L_i} > 1 - f_r \aleph_y C_Y^2$$

Condition 2.
(i). The proposed LIM $y_{.i_{a_i}},\ i = 1, 4$ performs superior than the ratio imputation method $y_{.i_{r_1}}$ under design I, if

$$MSE(\zeta_{a_i}) < MSE(\zeta_{r_1}) \implies \frac{M_i^2}{L_i} > 1 - f_r \aleph_y C_Y^2 - f_n \aleph_z C_Z^2 - 2 f_n \aleph_{yz} C_Z C_Y$$

(ii). The proposed LIM $y_{.i_{a_i}},\ i = 2, 5$ performs superior than the ratio imputation method $y_{.i_{r_2}}$ under design II, if

$$MSE(\zeta_{a_i}) < MSE(\zeta_{r_2}) \implies \frac{M_i^2}{L_i} > 1 - f_r \aleph_y C_Y^2 + f_r \aleph_z C_Z^2 - 2 f_r \aleph_{yz} C_Z C_Y$$

(iii). The proposed LIM $y_{.i_{a_i}},\ i = 3, 6$ performs superior than the ratio imputation method $y_{.i_{r_3}}$ under design III, if

$$MSE(\zeta_{a_i}) < MSE(\zeta_{r_3}) \implies \frac{M_i^2}{L_i} > \begin{pmatrix} 1 - f_n \aleph_y C_Y^2 - f_{rn} \aleph_y C_Y^2 + f_{rn} \aleph_z C_Z^2 \\ -2 f_{rn} \aleph_{yz} C_Z C_Y \end{pmatrix}$$

Condition 3.
(i). The proposed LIM $y_{.i_{a_i}},\ i = 1, 4$ performs superior than the regression imputation method $y_{.i_{lr_1}}$ under design I, if

$$MSE(\zeta_{a_i}) < MSE(\zeta_{l_1}) \implies \frac{M_i^2}{L_i} > 1 - f_r \aleph_y C_Y^2 + f_n \aleph_y C_Y^2 \aleph_{yz}^2$$

(ii). The proposed LIM $y_{.i_{a_i}},\ i = 2, 5$ performs superior than the regression imputation method $y_{.i_{l_2}}$ under design II, if

$$MSE(\zeta_{a_i}) < MSE(\zeta_{l_2}) \implies \frac{M_i^2}{L_i} > 1 - f_r \aleph_y C_Y^2 (1 - \aleph_{yz}^2)$$

(iii). The proposed LIM $y_{.i_{a_i}},\ i = 3, 6$ performs superior than the regression imputation method $y_{.i_{l_3}}$ under design III, if

$$MSE(\zeta_{a_i}) < MSE(\zeta_{l_3}) \implies \frac{M_i^2}{L_i} > 1 - f_n \aleph_y C_Y^2 - f_{rn} \aleph_y C_Y^2 (1 - \aleph_{yz}^2)$$

4. Simulation Study

We conducted a simulation study to evaluate the performance of the LIM with the CIM, employing an artificially generated normal population $N(\underline{\mu}, \Sigma)$ of size $N = 800$ such that mean vector $\underline{\mu} = \begin{pmatrix} \mu_Y \\ \mu_Z \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 0 \\ 0 \end{pmatrix}$, covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_Y^2 & \rho_{YZ}\sigma_Z\sigma_Y & 0 & 0 \\ \rho_{YZ}\sigma_Z\sigma_Y & \sigma_Z^2 & 0 & 0 \\ 0 & 0 & \sigma_U^2 & \rho_{UV}\sigma_U\sigma_V \\ 0 & 0 & \rho_{UV}\sigma_U\sigma_V & \sigma_V^2 \end{pmatrix},$$

$\rho_{YZ} = 0.3,\ 0.6,\ 0.9,\ \rho_{UV} = -0.9,\ -0.6,\ 0,\ 0.6,\ 0.9$ and different values of σ_Y, σ_Z, σ_U, σ_V which are given in the header of each Table. We have taken different values of response rates such as $r = 250,\ 260,\ 270,\ 280$ to study the superiority of the point estimators. We take 20,000 replications to tabulate the percent relative efficiency (PRE) of the point estimators. The simulation steps are defined below.

Step 1. From a population of size N, select a random sample s of size n.

Step 2. Each time, select $(n - r)$ sample units at random from the sample s.

Step 3. Apply suggested imputation techniques examined for quantified samples to impute the selected units.

Step 4. Tabulate the required statistics.

Step 5. Repeat the above steps 20,000 times. The formula utilized to tabulate the PRE is given below.

$$PRE = \frac{MSE(\zeta_m)}{MSE(\zeta_i)} \times 100$$

where
$$MSE(\zeta_i) = \frac{1}{20,000} \sum_{i=1}^{20,000} (\zeta_i - \mu_Y)^2$$

such that $\zeta_i = \zeta_m,\ \zeta_{r_i},\ \zeta_{l_i},\ i = 1, 2, 3$ and $\zeta_{a_i},\ i = 1, 2, \ldots, 6$. Table 1 to Table 8 provide an overview of the outcomes of the simulation study reported in terms of PRE. The results reported in Table 1 to Table 4 are obtained utilizing different valuations of σ_Y^2, σ_Z^2, σ_U^2, σ_V^2 and response rate r such as 250, 260, 270 and 280, whereas the results reported in Table 5 to Table 8 are obtained utilizing different valuations of σ_Y^2, σ_Z^2, σ_U^2, σ_V^2 and level of ME such as 15, 20, 25 and 30. After profoundly observing the outcomes of the simulation study, which are described in Table 1 to Table 8, we draw the following interpretation:

(1). From Table 1 based on $\sigma_Y^2 = 27$, $\sigma_Z^2 = 24$, $\sigma_U^2 = 17$ and $\sigma_V^2 = 19$, it has been observed that: (a). when the response rate is $r = 250$ then the

PRE of the ratio estimator ζ_{r_1} under design I increases with the successive increase in the values of correlation coefficient ρ_{YZ} from 0.3 to 0.9. The PRE also increases with the successive increase in the magnitude and direction of ρ_{UV} from -0.9 to 0.9. This tendency in the PRE values of ζ_{r_1} can be also observed from strategies II and III. (b). The similar tendency as observed in the change in PRE of ζ_{r_1} discussed in previous point can be also observed from the PRE values of the conventional point estimators ζ_{l_i}, $i = 1, 2, 3$ of CIM $y_{.i_{l_i}}$ and proposed point estimators ζ_{a_i}, $i = 1, 2, \ldots, 6$ of LIM $y_{.i_{a_i}}$. (c). The similar tendency in the PRE of the mean estimator ζ_m, conventional regression and ratio estimators ζ_{l_i}, ζ_{r_i}, $i = 1, 2, 3$ and proposed estimators ζ_{a_i}, $i = 1, 2, \ldots, 6$ under strategies I, II, III can be observed for the other response rates $r = 260$, $r = 270$ and $r = 280$. (d). The PRE of the point estimators based on CIM and LIM increases with the successive incremental change in the values of response rate r from 250 to 280. (e). Moreover, the point estimators ζ_{a_i}, $i = 1, 2, \ldots, 6$ based on LIM dominate the point estimators such as ζ_m, ζ_r and ζ_{l_i}, $i = 1, 2, 3$ based on CIM for different values of response rate r under strategies I, II and III.

(2). The interpretation as drawn in points (1) for the results of Table 1 can also be drawn from the results revealed in Tables 2-4 based on different valuations of σ_Y^2, σ_Z^2, σ_U^2 and σ_V^2.

(3). From Table 5 based on $\sigma_Y^2 = 27$, $\sigma_Z^2 = 24$, it has been noticed that: (a). when the level of ME is 15% then the PRE of the ratio estimator ζ_{r_1} under design I increases with the successive increase in the values of correlation coefficient ρ_{YZ} from 0.3 to 0.9. The PRE also increases with the successive increase in the magnitude and direction of ρ_{UV} from -0.9 to 0.9. This pattern in the PRE values of ζ_{r_1} can be also observed from strategies II and III. (b). The similar pattern as observed in the change in PRE of ζ_{r_1} discussed in previous point can be also observed from the PRE values of the point estimators ζ_{l_i}, $i = 1, 2, 3$ based on CIM $y_{.i_{l_i}}$ and point estimators ζ_{a_i}, $i = 1, 2, \ldots, 6$ based on LIM $y_{.i_{a_i}}$. (c). The similar pattern in the PRE of the point estimators ζ_r, ζ_{l_i}, $i = 1, 2, 3$ based on CIM $y_{.i_{r_i}}$, $y_{.i_{l_i}}$ and point estimators ζ_{a_i}, $i = 1, 2, \ldots, 6$ based on LIM $y_{.i_{a_i}}$ can be noticed for different levels of ME such as 20%, 25% and 30%. (d). The PRE of the point estimators based on CIM and LIM increases with the successive incremental change in the level of ME from 10% to 40%. (e). Moreover, the point estimators ζ_{a_i}, $i = 1, 2, \ldots, 6$ based on LIM surpass the point estimators such as ζ_r and ζ_{l_i}, $i = 1, 2, 3$ based on CIM for different levels of ME under strategies I, II and III.

(4). The interpretation as discussed in points (3) for the results of Table 5 can also be drawn from the results of Tables 6-8 based on different valuations of σ_Y^2, σ_Z^2, σ_U^2 and σ_V^2.

5. Real Data Application

This section provides an example of applying the suggested imputation techniques to a real population. The real population is obtained from Sarndal et al. (2003, pp. 652-659), which is based on 284 municipalities in Sweden. In general, a municipality consists of a town and its surroundings. The municipalities differ significantly in terms of size and other characteristics. In this study, Y_i is considered as the actual number of seats in 1982 (S82) in municipal council, and y_i is considered as the measured number of seats in 1982 (S82) in municipal council, whereas X_i is taken as the true number of conservative seats in 1982 (CS82) in municipal council, and x_i is taken as the measured number of conservative seats in 1982 (CS82) in municipal council. The needed descriptive statistics are as follows: $N = 284$, $n = 60$, $r = 54$, $\mu_Y = 46.0704$, $\mu_X = 9.095$, $\sigma_y^2 = 158.5533$, $\sigma_x^2 = 24.3690$, $\sigma_u^2 = 67.4993$, $\sigma_v^2 = 1.4220$, $\rho_{xy} = 0.6878$, and $\rho_{uv} = -0.0155$. The graphical representation of the data is shown by the density plot presented in Figure 1. Figure 1 illustrates that the variable X has a right-skewed density.

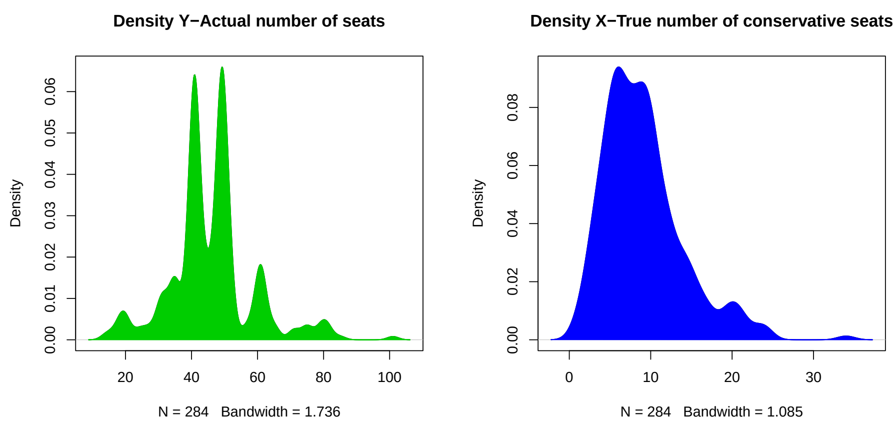

Figure 1: The density plots of the seats data.

For the above data, the PRE of the suggested estimators is determined using the following expression:

$$PRE = \frac{MSE(\zeta_m)}{MSE(\zeta_i)} \times 100. \tag{20}$$

The results are shown in Table 9, based on the existing imputation techniques and the proposed imputation techniques.

Based on Table 9, it can be seen that the performance of the suggested estimators is better than the existing estimators for all strategies considered in this study.

Table 1: PRE of different point estimators when $\sigma_Z^2 = 24$, $\sigma_Y^2 = 27$, $\sigma_U^2 = 17$ and $\sigma_V^2 = 19$.

			Strategy I				Strategy II				Strategy III			
r	ρ_{YZ}	ρ_{UV}	ζ_{r_1}	ζ_{l_1}	ζ_{a_1}	ζ_{a_4}	ζ_{r_2}	ζ_{l_2}	ζ_{a_2}	ζ_{a_5}	ζ_{r_3}	ζ_{l_3}	ζ_{a_3}	ζ_{a_6}
250	0.3	-0.9	72.45	103.63	105.23	105.31	66.58	104.85	106.34	106.44	89.15	101.13	102.97	102.99
		-0.6	74.79	100.62	101.73	101.75	69.21	100.82	101.90	101.93	90.26	100.20	101.35	101.36
		0	89.15	102.58	103.91	103.78	86.16	103.43	104.82	104.65	96.25	100.81	102.03	101.99
		0.6	114.14	117.67	119.21	118.71	119.55	124.72	126.41	125.71	104.13	105.05	106.32	106.18
		0.9	130.12	131.55	134.01	132.89	144.00	146.32	149.05	147.43	108.00	108.31	110.39	110.09
	0.6	-0.9	77.79	100.41	101.57	101.56	72.63	100.55	101.70	101.69	91.63	100.13	101.29	101.29
		-0.6	84.85	101.37	102.62	102.54	80.93	101.81	103.10	103.00	94.60	100.43	101.61	101.59
		0	106.48	112.61	114.03	113.66	108.73	117.35	118.90	118.37	101.99	103.72	104.91	104.80
		0.6	139.67	140.56	142.18	141.28	159.98	161.53	163.42	162.08	110.00	110.17	111.43	111.20
		0.9	164.37	169.02	170.72	169.43	207.00	216.93	219.08	217.07	114.33	115.03	116.30	115.99
	0.9	-0.9	86.84	102.67	103.88	103.75	83.33	103.55	104.81	104.65	95.38	100.84	101.94	101.90
		-0.6	97.89	107.22	108.56	108.31	97.24	109.75	111.19	110.85	99.32	102.20	103.37	103.30
		0	128.11	129.58	131.10	130.41	140.77	143.13	144.87	143.85	107.55	107.88	109.07	108.89
		0.6	180.30	188.04	189.73	188.28	242.65	261.78	264.08	261.82	116.62	117.62	118.83	118.49
		0.9	230.32	262.46	264.32	262.47	395.06	546.63	550.48	550.64	122.11	124.70	125.86	125.44
260	0.3	-0.9	71.41	103.81	105.30	105.38	66.72	104.80	106.20	106.30	91.03	100.91	102.66	102.68
		-0.6	73.67	100.66	101.70	101.72	69.19	100.82	101.85	101.87	91.91	100.16	101.25	101.26
		0	88.58	102.73	103.99	103.87	86.15	103.43	104.73	104.57	96.92	100.66	101.80	101.77
		0.6	115.17	118.96	120.43	119.93	119.64	124.78	126.38	125.72	103.35	104.08	105.26	105.16
		0.9	132.56	134.12	136.48	135.34	144.11	146.41	148.98	147.46	106.44	106.68	108.62	108.39
	0.6	-0.9	76.73	100.44	101.52	101.52	72.58	100.55	101.63	101.63	93.05	100.11	101.20	101.20
		-0.6	84.04	101.43	102.62	102.54	80.87	101.79	103.01	102.91	95.53	100.35	101.45	101.43
		0	106.84	113.43	114.79	114.41	108.67	117.31	118.77	118.28	101.60	103.00	104.10	104.02
		0.6	143.10	144.08	145.64	144.72	160.08	161.61	163.39	162.12	108.01	108.14	109.30	109.13
		0.9	170.99	176.33	177.99	176.67	207.20	217.13	219.14	217.26	111.38	111.93	113.10	112.87
	0.9	-0.9	86.06	102.80	103.94	103.82	83.20	103.51	104.69	104.54	96.16	100.67	101.70	101.67
		-0.6	97.65	107.62	108.90	108.65	97.09	109.67	111.02	110.71	99.41	101.77	102.86	102.80
		0	130.21	131.84	133.29	132.59	140.67	143.04	144.68	143.72	106.06	106.32	107.42	107.28
		0.6	189.40	198.44	200.10	198.61	242.84	261.90	264.06	261.93	113.15	113.91	115.02	114.77
		0.9	249.84	290.46	292.36	290.53	395.85	546.82	550.45	550.57	117.32	119.25	120.32	120.01
270	0.3	-0.9	70.14	104.09	105.47	105.54	66.61	104.85	106.17	106.26	92.96	100.70	102.37	102.38
		-0.6	72.58	100.69	101.67	101.69	69.21	100.81	101.78	101.81	93.71	100.12	101.16	101.16
		0	88.00	102.89	104.09	103.96	86.17	103.42	104.65	104.50	97.63	100.50	101.57	101.55
		0.6	116.09	120.19	121.61	121.1	119.51	124.67	126.18	125.55	102.53	103.08	104.18	104.10
		0.9	135.00	136.67	138.93	137.78	143.96	146.20	148.62	147.18	104.83	105.01	106.81	106.64
	0.6	-0.9	75.75	100.46	101.48	101.48	72.62	100.54	101.56	101.56	94.62	100.08	101.11	101.11
		-0.6	83.31	101.52	102.65	102.57	80.91	101.80	102.94	102.85	96.56	100.27	101.30	101.29
		0	107.32	114.34	115.64	115.26	108.74	117.33	118.71	118.24	101.23	102.28	103.31	103.25
		0.6	146.67	147.76	149.28	148.34	159.94	161.47	163.15	161.96	106.00	106.10	107.17	107.05
		0.9	178.15	184.28	185.91	184.55	206.90	216.75	218.65	216.88	108.46	108.85	109.94	109.77
	0.9	-0.9	85.45	102.99	104.08	103.96	83.29	103.54	104.66	104.51	97.06	100.52	101.48	101.46
		-0.6	97.62	108.15	109.38	109.13	97.21	109.74	111.02	110.72	99.57	101.36	102.37	102.33
		0	132.58	134.34	135.75	135.03	140.73	143.07	144.61	143.71	104.57	104.76	105.78	105.68
		0.6	199.66	210.26	211.92	210.38	242.66	261.53	263.58	261.57	109.74	110.28	111.31	111.13
		0.9	273.55	326.39	328.39	326.62	395.62	546.24	549.67	549.76	112.71	114.07	115.06	114.84
280	0.3	-0.9	68.97	104.31	105.60	105.68	66.61	104.83	106.08	106.16	95.11	100.47	102.07	102.08
		-0.6	71.50	100.71	101.63	101.66	69.25	100.79	101.71	101.73	95.64	100.08	101.06	101.07
		0	87.45	103.07	104.21	104.09	86.21	103.43	104.60	104.45	98.39	100.34	101.34	101.33
		0.6	117.28	121.65	123.02	122.51	119.64	124.74	126.17	125.58	101.71	102.08	103.10	103.05
		0.9	137.89	139.70	141.89	140.71	144.13	146.34	148.63	147.27	103.24	103.36	105.04	104.93
	0.6	-0.9	74.76	100.47	101.44	101.43	72.66	100.52	101.49	101.49	96.28	100.05	101.03	101.03
		-0.6	82.57	101.61	102.68	102.6	80.96	101.80	102.88	102.80	97.64	100.18	101.16	101.15
		0	107.84	115.33	116.59	116.20	108.82	117.39	118.69	118.25	100.84	101.54	102.50	102.46
		0.6	150.81	151.99	153.47	152.51	160.10	161.59	163.18	162.04	104.00	104.07	105.07	104.99
		0.9	186.67	193.77	195.39	193.98	207.19	217.03	218.84	217.15	105.60	105.85	106.87	106.76
	0.9	-0.9	84.80	103.16	104.20	104.08	83.36	103.53	104.59	104.45	97.99	100.35	101.26	101.24
		-0.6	97.55	108.65	109.83	109.57	97.28	109.73	110.94	110.66	99.71	100.92	101.87	101.84
		0	135.24	137.13	138.51	137.78	140.91	143.21	144.67	143.82	103.07	103.19	104.14	104.08
		0.6	211.82	224.62	226.29	224.69	242.85	261.93	263.87	261.96	106.42	106.77	107.73	107.62
		0.9	303.64	375.03	377.19	375.61	395.74	546.95	550.20	550.31	108.30	109.15	110.07	109.93

TABLE 2: PRE of different point estimators when $\sigma_Z^2 = 31$, $\sigma_Y^2 = 33$, $\sigma_U^2 = 21$ and $\sigma_V^2 = 23$.

			Strategy I				Strategy II				Strategy III			
r	ρ_{YZ}	ρ_{UV}	ζ_{r_1}	ζ_{l_1}	ζ_{a_1}	ζ_{a_4}	ζ_{r_2}	ζ_{l_2}	ζ_{a_2}	ζ_{a_5}	ζ_{r_3}	ζ_{l_3}	ζ_{a_3}	ζ_{a_6}
250	0.3	-0.9	72.63	103.36	105.46	105.56	66.78	104.48	106.46	106.58	89.24	101.05	103.45	103.48
		-0.6	74.89	100.56	101.94	101.96	69.33	100.74	102.10	102.13	90.31	100.18	101.61	101.62
		0	89.14	102.66	104.31	104.15	86.14	103.55	105.26	105.05	96.25	100.84	102.35	102.30
		0.6	113.82	117.61	119.51	118.90	119.09	124.63	126.73	125.86	104.04	105.03	106.61	106.44
		0.9	129.54	131.21	134.40	132.97	143.07	145.77	149.29	147.23	107.87	108.24	110.94	110.55
	0.6	-0.9	78.01	100.41	101.83	101.82	72.89	100.55	101.97	101.96	91.73	100.13	101.55	101.55
		-0.6	85.06	101.51	103.06	102.95	81.17	102.01	103.60	103.46	94.68	100.48	101.93	101.89
		0	106.49	112.89	114.62	114.16	108.75	117.75	119.64	118.99	101.99	103.79	105.24	105.11
		0.6	139.73	140.77	142.75	141.65	160.09	161.89	164.21	162.55	110.01	110.21	111.75	111.47
		0.9	164.22	168.74	170.83	169.25	206.69	216.33	218.96	216.51	114.30	114.99	116.54	116.17
	0.9	-0.9	88.74	102.99	104.54	104.37	85.65	103.98	105.60	105.38	96.10	100.94	102.35	102.3
		-0.6	98.54	107.78	109.43	109.11	98.08	110.54	112.30	111.86	99.53	102.37	103.79	103.69
		0	129.10	130.59	132.45	131.58	142.35	144.76	146.91	145.62	107.77	108.10	109.55	109.33
		0.6	181.88	189.38	191.46	189.67	246.45	265.24	268.11	265.29	116.83	117.79	119.26	118.85
		0.9	226.38	263.18	265.42	263.20	380.10	550.77	555.17	556.38	121.75	124.75	126.22	125.7
260	0.3	-0.9	71.60	103.52	105.49	105.58	66.93	104.43	106.29	106.41	91.11	100.84	103.13	103.15
		-0.6	73.78	100.60	101.89	101.92	69.31	100.75	102.02	102.05	91.96	100.15	101.49	101.50
		0	88.57	102.82	104.38	104.22	86.14	103.54	105.15	104.95	96.92	100.68	102.09	102.05
		0.6	114.83	118.89	120.71	120.09	119.19	124.69	126.66	125.84	103.28	104.07	105.53	105.40
		0.9	131.94	133.74	136.80	135.35	143.19	145.86	149.18	147.23	106.34	106.62	109.14	108.84
	0.6	-0.9	76.96	100.44	101.77	101.76	72.83	100.55	101.88	101.87	93.14	100.11	101.44	101.44
		-0.6	84.26	101.59	103.05	102.94	81.12	101.99	103.48	103.35	95.60	100.39	101.74	101.71
		0	106.86	113.73	115.38	114.91	108.69	117.71	119.49	118.87	101.60	103.06	104.40	104.30
		0.6	143.17	144.30	146.21	145.09	160.19	161.96	164.14	162.58	108.02	108.17	109.59	109.38
		0.9	170.81	176.00	178.04	176.42	206.87	216.51	218.99	216.68	111.36	111.89	113.33	113.05
	0.9	-0.9	88.06	103.14	104.60	104.44	85.54	103.94	105.46	105.25	96.77	100.75	102.07	102.03
		-0.6	98.34	108.22	109.79	109.47	97.93	110.46	112.11	111.70	99.58	101.91	103.22	103.15
		0	131.32	132.95	134.74	133.86	142.28	144.68	146.69	145.49	106.24	106.50	107.83	107.66
		0.6	191.31	200.04	202.09	200.24	246.77	265.39	268.09	265.43	113.31	114.04	115.39	115.08
		0.9	244.89	291.43	293.71	291.54	380.64	551.14	555.29	556.39	117.05	119.29	120.64	120.25
270	0.3	-0.9	70.34	103.78	105.60	105.69	66.81	104.48	106.23	106.34	93.03	100.65	102.82	102.84
		-0.6	72.70	100.63	101.84	101.87	69.33	100.74	101.94	101.97	93.74	100.11	101.39	101.39
		0	87.99	102.98	104.47	104.31	86.16	103.53	105.05	104.87	97.63	100.52	101.84	101.81
		0.6	115.73	120.12	121.87	121.25	119.06	124.58	126.43	125.67	102.48	103.07	104.43	104.33
		0.9	134.32	136.25	139.18	137.71	143.05	145.64	148.77	146.93	104.76	104.97	107.30	107.09
	0.6	-0.9	75.99	100.46	101.72	101.71	72.88	100.54	101.80	101.79	94.68	100.08	101.34	101.34
		-0.6	83.54	101.69	103.07	102.97	81.16	101.99	103.40	103.28	96.62	100.30	101.56	101.55
		0	107.34	114.66	116.25	115.78	108.76	117.73	119.41	118.83	101.23	102.33	103.57	103.5
		0.6	146.75	148.01	149.87	148.72	160.05	161.82	163.88	162.41	106.00	106.12	107.44	107.29
		0.9	177.95	183.90	185.90	184.23	206.58	216.14	218.47	216.29	108.45	108.83	110.16	109.96
	0.9	-0.9	87.52	103.35	104.75	104.58	85.62	103.97	105.40	105.21	97.53	100.58	101.81	101.78
		-0.6	98.35	108.80	110.30	109.98	98.06	110.53	112.10	111.71	99.70	101.46	102.69	102.64
		0	133.77	135.55	137.29	136.39	142.32	144.70	146.60	145.46	104.70	104.89	106.13	106.01
		0.6	201.84	212.16	214.21	212.30	246.47	265.01	267.55	265.05	109.85	110.37	111.62	111.41
		0.9	267.24	327.67	330.03	328.00	380.31	550.46	554.38	555.40	112.52	114.09	115.35	115.08
280	0.3	-0.9	69.17	103.98	105.68	105.77	66.82	104.46	106.11	106.21	95.15	100.44	102.52	102.53
		-0.6	71.62	100.65	101.79	101.82	69.37	100.72	101.86	101.89	95.67	100.07	101.29	101.29
		0	87.44	103.17	104.59	104.43	86.20	103.54	104.99	104.81	98.38	100.35	101.59	101.57
		0.6	116.90	121.58	123.27	122.63	119.20	124.65	126.41	125.69	101.68	102.07	103.33	103.28
		0.9	137.15	139.25	142.08	140.58	143.23	145.79	148.75	147.01	103.19	103.33	105.51	105.38
	0.6	-0.9	75.00	100.47	101.66	101.65	72.92	100.52	101.72	101.71	96.33	100.05	101.25	101.25
		-0.6	82.81	101.79	103.11	103.00	81.21	101.99	103.33	103.22	97.68	100.20	101.40	101.38
		0	107.86	115.67	117.21	116.73	108.84	117.78	119.38	118.83	100.84	101.57	102.73	102.69
		0.6	150.90	152.27	154.09	152.9	160.22	161.94	163.89	162.50	104.01	104.08	105.31	105.21
		0.9	186.45	193.34	195.32	193.60	206.89	216.43	218.64	216.57	105.60	105.84	107.08	106.95
	0.9	-0.9	86.95	103.54	104.88	104.71	85.67	103.96	105.32	105.14	98.31	100.39	101.55	101.53
		-0.6	98.29	109.34	110.78	110.45	98.10	110.52	112.01	111.63	99.80	100.99	102.14	102.10
		0	136.55	138.48	140.17	139.25	142.50	144.85	146.65	145.57	103.16	103.28	104.43	104.35
		0.6	214.47	226.91	228.98	226.99	246.75	265.40	267.82	265.44	106.50	106.83	107.99	107.86
		0.9	295.55	376.78	379.31	377.61	380.61	551.10	554.82	555.79	108.18	109.16	110.34	110.17

TABLE 3: PRE of different point estimators when $\sigma_Z^2 = 35$, $\sigma_Y^2 = 40$, $\sigma_U^2 = 20$ and $\sigma_V^2 = 15$.

			Strategy I				Strategy II				Strategy III			
r	ρ_{YZ}	ρ_{UV}	ζ_{r_1}	ζ_{l_1}	ζ_{a_1}	ζ_{a_4}	ζ_{r_2}	ζ_{l_2}	ζ_{a_2}	ζ_{a_5}	ζ_{r_3}	ζ_{l_3}	ζ_{a_3}	ζ_{a_6}
250	0.3	-0.9	72.95	100.67	101.97	101.99	67.14	100.89	102.16	102.19	89.39	100.21	101.56	101.57
		-0.6	77.68	100.39	101.79	101.77	72.50	100.52	101.93	101.90	91.58	100.13	101.50	101.50
		0	90.19	104.39	105.95	105.73	87.44	105.88	107.53	107.23	96.64	101.36	102.75	102.68
		0.6	109.81	115.71	117.50	116.96	113.36	121.83	123.81	123.04	102.94	104.54	106.03	105.87
		0.9	121.25	124.71	126.59	125.83	130.10	135.42	137.54	136.44	105.94	106.77	108.28	108.07
	0.6	-0.9	89.11	101.90	103.62	103.50	86.10	102.52	104.30	104.13	96.24	100.60	102.21	102.18
		-0.6	95.49	104.52	106.32	106.09	94.14	106.06	107.94	107.63	98.51	101.40	103.03	102.96
		0	114.70	117.46	119.36	118.76	120.36	124.42	126.49	125.64	104.28	104.99	106.59	106.42
		0.6	139.74	141.32	143.38	142.24	160.10	162.85	165.227	163.52	110.01	110.32	112.00	111.70
		0.9	155.89	161.40	163.51	162.02	189.84	200.87	203.38	201.14	112.96	113.86	115.54	115.17
	0.9	-0.9	104.80	110.12	111.94	111.54	106.44	113.80	115.76	115.21	101.49	103.03	104.62	104.50
		-0.6	114.85	117.49	119.36	118.77	120.58	124.45	126.51	125.67	104.32	105.00	106.57	106.41
		0	142.90	143.98	145.98	144.82	165.65	167.57	169.90	168.16	110.63	110.83	112.41	112.12
		0.6	184.80	201.13	203.22	201.31	253.63	297.37	300.18	297.56	117.21	119.18	120.74	120.29
		0.9	215.08	264.10	266.21	264.14	340.46	556.13	559.69	563.48	120.66	124.82	126.39	125.84
260	0.3	-0.9	71.72	100.72	101.93	101.96	67.05	100.90	102.09	102.12	91.15	100.18	101.45	101.45
		-0.6	76.59	100.42	101.73	101.71	72.41	100.52	101.84	101.81	93.00	100.10	101.39	101.39
		0	89.64	104.63	106.11	105.89	87.41	105.84	107.39	107.11	97.23	101.10	102.39	102.34
		0.6	110.47	116.81	118.53	117.98	113.39	121.85	123.71	122.99	102.39	103.67	105.04	104.93
		0.9	122.85	126.59	128.40	127.63	130.18	135.46	137.46	136.42	104.80	105.45	106.84	106.69
	0.6	-0.9	88.49	102.00	103.62	103.5	86.05	102.50	104.17	104.01	96.90	100.48	101.99	101.96
		-0.6	95.19	104.78	106.48	106.25	94.08	106.03	107.80	107.50	98.77	101.14	102.65	102.60
		0	115.69	118.65	120.46	119.86	120.34	124.36	126.31	125.52	103.45	104.03	105.50	105.37
		0.6	143.16	144.91	146.89	145.73	160.18	162.91	165.14	163.54	108.02	108.26	109.81	109.58
		0.9	161.31	167.57	169.61	168.09	189.97	201.01	203.37	201.26	110.32	111.02	112.57	112.30
	0.9	-0.9	105.01	110.72	112.45	112.05	106.32	113.71	115.55	115.04	101.19	102.44	103.91	103.82
		-0.6	115.77	118.63	120.41	119.82	120.44	124.33	126.26	125.47	103.47	104.02	105.47	105.35
		0	146.61	147.77	149.69	148.51	165.61	167.45	169.64	168.01	108.49	108.64	110.10	109.88
		0.6	194.73	213.99	216.04	214.09	253.95	297.45	300.10	297.63	113.60	115.09	116.54	116.20
		0.9	230.98	292.61	294.71	292.78	340.90	556.40	559.76	563.27	116.22	119.34	120.80	120.38
270	0.3	-0.9	70.61	100.75	101.89	101.92	67.10	100.89	102.01	102.04	93.11	100.13	101.34	101.34
		-0.6	75.61	100.43	101.67	101.65	72.47	100.51	101.76	101.73	94.58	100.08	101.29	101.29
		0	89.23	104.94	106.35	106.13	87.56	105.87	107.33	107.07	97.90	100.84	102.05	102.01
		0.6	111.12	117.87	119.52	118.96	113.36	121.74	123.49	122.81	101.81	102.77	104.04	103.96
		0.9	124.44	128.44	130.18	129.40	130.09	135.27	137.16	136.18	103.62	104.10	105.39	105.28
	0.6	-0.9	87.93	102.12	103.67	103.54	86.09	102.51	104.08	103.93	97.62	100.37	101.79	101.76
		-0.6	94.96	105.08	106.70	106.47	94.12	106.03	107.71	107.43	99.07	100.87	102.29	102.25
		0	116.79	119.98	121.71	121.10	120.38	124.40	126.24	125.48	102.62	103.05	104.42	104.33
		0.6	146.73	148.66	150.58	149.40	160.02	162.74	164.84	163.33	106.00	106.18	107.62	107.46
		0.9	167.10	174.19	176.18	174.62	189.73	200.66	202.89	200.90	107.69	108.19	109.64	109.44
	0.9	-0.9	105.39	111.46	113.12	112.72	106.41	113.78	115.51	115.02	100.92	101.86	103.23	103.17
		-0.6	116.94	120.01	121.73	121.13	120.58	124.44	126.26	125.52	102.64	103.06	104.41	104.32
		0	150.71	152.00	153.87	152.67	165.64	167.49	169.55	168.02	106.36	106.48	107.83	107.67
		0.6	205.96	229.00	231.03	229.05	253.77	297.14	299.64	297.30	110.07	111.13	112.48	112.24
		0.9	249.89	329.20	331.33	329.69	340.67	555.59	558.76	562.02	111.94	114.13	115.49	115.20
280	0.3	-0.9	69.53	100.77	101.84	101.87	67.19	100.86	101.92	101.95	95.23	100.09	101.23	101.24
		-0.6	74.66	100.45	101.63	101.61	72.56	100.50	101.69	101.66	96.27	100.05	101.20	101.20
		0	88.73	105.24	106.59	106.37	87.60	105.87	107.26	107.01	98.57	100.57	101.70	101.68
		0.6	111.96	119.18	120.77	120.21	113.51	121.85	123.51	122.86	101.24	101.87	103.06	103.01
		0.9	126.36	130.69	132.39	131.59	130.28	135.44	137.23	136.30	102.44	102.76	103.96	103.89
	0.6	-0.9	87.39	102.25	103.72	103.60	86.15	102.51	104.00	103.86	98.38	100.25	101.58	101.57
		-0.6	94.75	105.40	106.95	106.72	94.19	106.06	107.64	107.38	99.37	100.59	101.92	101.90
		0	118.00	121.41	123.09	122.47	120.48	124.46	126.20	125.49	101.77	102.06	103.34	103.28
		0.6	150.91	153.03	154.90	153.69	160.24	162.91	164.90	163.47	104.01	104.12	105.47	105.37
		0.9	173.96	182.10	184.05	182.44	190.02	200.95	203.07	201.18	105.11	105.43	106.78	106.66
	0.9	-0.9	105.78	112.21	113.81	113.40	106.48	113.79	115.44	114.98	100.63	101.26	102.54	102.50
		-0.6	118.14	121.41	123.07	122.46	120.64	124.46	126.19	125.48	101.79	102.06	103.32	103.26
		0	155.35	156.81	158.63	157.39	165.84	167.69	169.65	168.19	104.24	104.32	105.58	105.48
		0.6	219.32	247.39	249.43	247.41	253.95	297.50	299.87	297.65	106.63	107.31	108.57	108.42
		0.9	273.38	379.03	381.25	380.17	340.94	556.49	559.49	562.62	107.81	109.19	110.46	110.28

Table 4: PRE of different point estimators when $\sigma_Z^2 = 50$, $\sigma_Y^2 = 45$, $\sigma_U^2 = 30$ and $\sigma_V^2 = 25$.

			Strategy I				Strategy II				Strategy III			
r	ρ_{YZ}	ρ_{UV}	ζ_{r_1}	ζ_{l_1}	ζ_{a_1}	ζ_{a_4}	ζ_{r_2}	ζ_{l_2}	ζ_{a_2}	ζ_{a_5}	ζ_{r_3}	ζ_{l_3}	ζ_{a_3}	ζ_{a_6}
250	0.3	-0.9	72.65	101.54	103.43	103.50	66.81	102.04	103.86	103.95	89.25	100.49	102.54	102.57
		-0.6	77.71	100.36	102.43	102.44	72.53	100.47	102.54	102.55	91.59	100.11	102.21	102.22
		0	94.64	103.85	106.51	106.21	93.04	105.14	107.92	107.52	98.22	101.20	103.64	103.55
		0.6	113.08	116.84	119.54	118.71	118.01	123.49	126.44	125.28	103.84	104.83	107.11	106.88
		0.9	126.20	128.14	130.96	129.75	137.76	140.82	143.99	142.23	107.12	107.56	109.87	109.54
	0.6	-0.9	82.02	100.73	102.82	102.75	77.55	100.97	103.09	102.99	93.44	100.23	102.25	102.23
		-0.6	88.79	102.60	104.82	104.61	85.72	103.46	105.77	105.49	96.12	100.82	102.86	102.8
		0	109.20	114.56	116.97	116.27	112.51	120.15	122.80	121.82	102.77	104.24	106.25	106.05
		0.6	139.88	141.22	143.96	142.43	160.34	162.69	165.88	163.59	110.04	110.30	112.45	112.06
		0.9	160.69	165.29	168.14	166.04	199.42	208.94	212.47	209.24	113.75	114.47	116.63	116.13
	0.9	-0.9	94.66	105.52	107.74	107.4	93.07	107.42	109.77	109.31	98.23	101.70	103.66	103.56
		-0.6	104.62	111.45	113.76	113.19	106.20	115.68	118.20	117.40	101.43	103.40	105.35	105.19
		0	134.92	135.79	138.35	137.03	151.89	153.35	156.34	154.37	109.03	109.21	111.20	110.86
		0.6	184.53	193.20	196.03	193.54	252.96	275.31	279.25	275.38	117.18	118.25	120.25	119.68
		0.9	225.29	262.96	265.98	262.99	376.05	549.53	555.37	557.41	121.65	124.74	126.75	126.03
260	0.3	-0.9	71.50	101.65	103.41	103.48	66.81	102.06	103.77	103.86	91.06	100.40	102.35	102.36
		-0.6	76.70	100.38	102.33	102.34	72.54	100.48	102.41	102.42	93.04	100.09	102.07	102.07
		0	94.33	104.06	106.57	106.28	93.03	105.11	107.71	107.33	98.54	100.97	103.24	103.17
		0.6	114.03	118.04	120.61	119.78	118.10	123.52	126.30	125.20	103.12	103.91	106.02	105.84
		0.9	128.27	130.36	133.06	131.84	137.86	140.89	143.86	142.21	105.74	106.08	108.22	107.97
	0.6	-0.9	81.11	100.76	102.73	102.66	77.51	100.95	102.95	102.85	94.58	100.19	102.08	102.06
		-0.6	88.16	102.73	104.84	104.63	85.67	103.43	105.60	105.34	96.80	100.66	102.57	102.52
		0	109.78	115.52	117.82	117.12	112.49	120.11	122.59	121.67	102.24	103.42	105.28	105.13
		0.6	143.32	144.79	147.43	145.87	160.42	162.73	165.73	163.58	108.04	108.24	110.22	109.93
		0.9	166.75	172.00	174.77	172.63	199.54	209.06	212.37	209.34	110.93	111.49	113.48	113.11
	0.9	-0.9	94.26	105.82	107.92	107.58	92.95	107.36	109.56	109.13	98.52	101.37	103.19	103.11
		-0.6	104.80	112.13	114.34	113.76	106.05	115.59	117.94	117.2	101.14	102.74	104.54	104.42
		0	137.73	138.66	141.14	139.8	151.83	153.25	156.06	154.21	107.23	107.37	109.20	108.94
		0.6	194.40	204.54	207.33	204.77	253.25	275.41	279.12	275.48	113.58	114.39	116.22	115.80
		0.9	243.48	291.12	294.17	291.27	376.44	549.75	555.25	557.10	116.97	119.27	121.13	120.60
270	0.3	-0.9	70.36	101.73	103.38	103.45	66.84	102.04	103.66	103.74	93.03	100.30	102.15	102.17
		-0.6	75.70	100.40	102.23	102.24	72.57	100.47	102.30	102.30	94.60	100.07	101.94	101.94
		0	94.00	104.32	106.71	106.41	93.01	105.13	107.58	107.22	98.88	100.74	102.86	102.81
		0.6	114.87	119.20	121.67	120.83	117.99	123.41	126.03	125	102.36	102.95	104.91	104.78
		0.9	130.28	132.55	135.15	133.91	137.69	140.69	143.50	141.94	104.31	104.56	106.55	106.38
	0.6	-0.9	80.28	100.81	102.67	102.6	77.56	100.96	102.84	102.75	95.82	100.14	101.92	101.91
		-0.6	87.61	102.90	104.90	104.70	85.72	103.44	105.49	105.24	97.55	100.50	102.29	102.26
		0	110.44	116.59	118.81	118.10	112.53	120.14	122.48	121.61	101.71	102.60	104.32	104.22
		0.6	146.91	148.54	151.10	149.52	160.28	162.58	165.40	163.37	106.02	106.17	108.01	107.80
		0.9	173.31	179.29	182.00	179.80	199.28	208.71	211.83	208.97	108.13	108.53	110.38	110.12
	0.9	-0.9	94.03	106.22	108.22	107.88	93.04	107.40	109.49	109.07	98.88	101.05	102.75	102.70
		-0.6	105.20	113.00	115.12	114.54	106.18	115.68	117.91	117.20	100.89	102.09	103.77	103.68
		0	140.84	141.87	144.27	142.90	151.86	153.28	155.94	154.19	105.44	105.54	107.23	107.05
		0.6	205.55	217.58	220.37	217.73	253.03	275.09	278.58	275.15	110.05	110.63	112.32	112.03
		0.9	265.49	327.22	330.39	327.69	376.17	548.98	554.17	555.87	112.46	114.08	115.80	115.43
280	0.3	-0.9	69.22	101.80	103.35	103.41	66.87	102.01	103.53	103.61	95.16	100.20	101.96	101.97
		-0.6	74.70	100.40	102.14	102.15	72.60	100.45	102.18	102.19	96.27	100.05	101.81	101.82
		0	93.77	104.58	106.87	106.56	93.10	105.14	107.46	107.12	99.25	100.50	102.49	102.46
		0.6	115.99	120.60	122.98	122.13	118.15	123.50	125.98	125.00	101.60	101.99	103.82	103.74
		0.9	132.76	135.18	137.71	136.44	137.92	140.84	143.50	142.03	102.90	103.07	104.91	104.80
	0.6	-0.9	79.43	100.85	102.62	102.54	77.60	100.95	102.73	102.65	97.12	100.10	101.77	101.77
		-0.6	87.04	103.08	104.99	104.78	85.77	103.45	105.39	105.15	98.33	100.34	102.02	102.00
		0	111.19	117.76	119.89	119.17	112.63	120.19	122.42	121.59	101.16	101.75	103.36	103.30
		0.6	151.12	152.90	155.40	153.76	160.50	162.73	165.42	163.49	104.02	104.12	105.83	105.70
		0.9	181.12	188.00	190.68	188.40	199.63	209.02	211.98	209.27	105.39	105.65	107.38	107.21
	0.9	-0.9	93.76	106.60	108.52	108.18	93.09	107.41	109.38	108.99	99.24	100.71	102.31	102.27
		-0.6	105.56	113.85	115.89	115.30	106.23	115.68	117.79	117.12	100.61	101.41	102.98	102.92
		0	144.36	145.48	147.83	146.41	152.07	153.45	155.97	154.31	103.64	103.70	105.28	105.16
		0.6	218.87	233.43	236.24	233.50	253.27	275.42	278.74	275.49	106.62	106.99	108.57	108.38
		0.9	293.37	376.25	379.63	377.42	376.59	549.84	554.76	556.40	108.15	109.16	110.76	110.53

Table 5: PRE of different point estimators when $\sigma_Z^2 = 24$, $\sigma_Y^2 = 27$.

			Strategy I				Strategy II				Strategy III			
% of ME	ρ_{YZ}	ρ_{UV}	ζ_{r_1}	ζ_{l_1}	ζ_{a_1}	ζ_{a_4}	ζ_{r_2}	ζ_{l_2}	ζ_{a_2}	ζ_{a_5}	ζ_{r_3}	ζ_{l_3}	ζ_{a_3}	ζ_{a_6}
15	0.3	-0.9	91.16	102.36	103.36	103.28	88.65	103.14	104.17	104.06	96.99	100.74	101.68	101.65
		-0.6	93.82	103.41	104.43	104.32	92.01	104.54	105.61	105.47	97.94	101.06	102.01	101.97
		0	99.98	106.55	107.61	107.44	99.97	108.83	109.95	109.72	99.99	102.01	102.96	102.91
		0.6	106.28	110.56	111.66	111.42	108.46	114.43	115.60	115.27	101.93	103.15	104.11	104.04
		0.9	109.79	113.13	114.24	113.96	113.34	118.09	119.28	118.89	102.94	103.86	104.82	104.74
	0.6	-0.9	113.25	115.77	116.86	116.55	118.27	121.93	123.11	122.67	103.89	104.56	105.49	105.40
		-0.6	117.44	119.18	120.28	119.91	124.38	126.97	128.17	127.66	104.99	105.43	106.36	106.26
		0	127.47	128.39	129.52	129.03	139.75	141.22	142.48	141.78	107.41	107.62	108.55	108.42
		0.6	137.97	139.49	140.63	140.02	157.05	159.66	160.97	160.06	109.66	109.96	110.90	110.74
		0.9	144.06	146.58	147.73	147.04	167.71	172.25	173.58	172.56	110.85	111.32	112.26	112.08
	0.9	-0.9	151.53	152.90	154.00	153.28	181.44	184.07	185.40	184.30	112.21	112.45	113.28	113.11
		-0.6	160.01	162.67	163.78	162.98	198.03	203.48	204.87	203.62	113.64	114.06	114.89	114.70
		0	180.95	192.59	193.74	192.73	244.22	273.68	275.21	273.72	116.71	118.18	119.03	118.79
		0.6	208.87	234.46	235.65	234.47	320.53	411.52	413.46	412.41	120.02	122.48	123.30	123.02
		0.9	224.19	264.27	265.48	264.28	372.05	557.11	559.46	560.47	121.55	124.83	125.65	125.36
20	0.3	-0.9	88.35	101.43	102.45	102.39	85.17	101.90	102.94	102.87	95.95	100.45	101.42	101.40
		-0.6	91.57	102.46	103.51	103.42	89.17	103.28	104.36	104.24	97.14	100.77	101.75	101.72
		0	99.19	106.07	107.17	107	98.94	108.17	109.33	109.10	99.74	101.87	102.85	102.80
		0.6	107.27	111.23	112.37	112.11	109.82	115.37	116.59	116.23	102.22	103.34	104.34	104.26
		0.9	111.84	114.73	115.89	115.57	116.24	120.40	121.66	121.21	103.51	104.28	105.28	105.19
	0.6	-0.9	107.86	111.72	112.83	112.56	110.64	116.07	117.26	116.9	102.39	103.47	104.44	104.36
		-0.6	112.75	115.33	116.46	116.14	117.55	121.28	122.51	122.05	103.76	104.44	105.41	105.32
		0	124.81	125.80	126.97	126.49	135.58	137.12	138.41	137.73	106.79	107.02	108.00	107.87
		0.6	137.98	139.48	140.67	140.03	157.07	159.64	161.00	160.06	109.66	109.96	110.93	110.77
		0.9	145.95	148.78	149.99	149.25	171.12	176.31	177.70	176.60	111.20	111.72	112.69	112.50
	0.9	-0.9	138.97	139.61	140.73	140.12	158.76	159.88	161.18	160.27	109.86	109.99	110.85	110.69
		-0.6	148.14	149.15	150.28	149.57	175.12	176.98	178.35	177.26	111.61	111.79	112.65	112.47
		0	172.15	179.75	180.93	179.96	223.82	241.35	242.87	241.39	115.49	116.55	117.43	117.20
		0.6	205.02	227.49	228.72	227.52	308.79	384.25	386.19	384.87	119.61	121.85	122.71	122.43
		0.9	223.35	264.19	265.45	264.21	369.02	556.67	559.07	560.20	121.47	124.82	125.68	125.38
25	0.3	-0.9	85.90	100.84	101.88	101.84	82.19	101.12	102.17	102.11	95.01	100.27	101.26	101.25
		-0.6	89.59	101.75	102.83	102.75	86.70	102.33	103.43	103.33	96.41	100.55	101.56	101.54
		0	98.48	105.65	106.79	106.62	98.01	107.59	108.79	108.57	99.51	101.74	102.77	102.72
		0.6	108.20	111.87	113.06	112.78	111.11	116.28	117.56	117.17	102.48	103.51	104.55	104.47
		0.9	113.84	116.32	117.53	117.18	119.12	122.73	124.04	123.54	104.05	104.70	105.74	105.64
	0.6	-0.9	102.35	108.59	109.68	109.47	103.13	111.66	112.83	112.53	100.74	102.60	103.56	103.49
		-0.6	107.81	112.22	113.34	113.06	110.58	116.78	117.99	117.60	102.37	103.61	104.57	104.49
		0	122.46	123.58	124.79	124.33	131.94	133.67	135.00	134.34	106.23	106.50	107.51	107.39
		0.6	139.86	141.08	142.30	141.63	160.31	162.44	163.85	162.84	110.04	110.28	111.25	111.08
		0.9	149.82	152.84	154.08	153.28	178.23	183.93	185.40	184.19	111.91	112.44	113.42	113.22
	0.9	-0.9	129.09	130.20	131.33	130.81	142.34	144.12	145.42	144.65	107.77	108.02	108.91	108.78
		-0.6	138.64	139.29	140.44	139.81	158.20	159.32	160.67	159.73	109.79	109.92	110.82	110.66
		0	164.75	169.73	170.95	170.02	207.81	218.48	220.01	218.57	114.39	115.14	116.06	115.83
		0.6	201.59	221.49	222.77	221.53	298.68	362.36	364.33	362.81	119.23	121.29	122.18	121.90
		0.9	223.26	264.18	265.50	264.2	368.69	556.63	559.13	560.31	121.46	124.82	125.72	125.40
30	0.3	-0.9	83.75	100.51	101.56	101.54	79.61	100.68	101.74	101.71	94.15	100.16	101.19	101.18
		-0.6	87.83	101.23	102.32	102.27	84.53	101.63	102.75	102.67	95.75	100.39	101.43	101.41
		0	97.83	105.28	106.46	106.29	97.16	107.09	108.32	108.1	99.30	101.63	102.69	102.64
		0.6	108.95	112.54	113.77	113.47	112.16	117.24	118.57	118.15	102.70	103.70	104.77	104.68
		0.9	115.83	117.88	119.15	118.75	122.01	125.03	126.41	125.86	104.57	105.10	106.19	106.08
	0.6	-0.9	98.28	106.23	107.34	107.16	97.74	108.39	109.57	109.32	99.44	101.91	102.90	102.85
		-0.6	104.11	109.74	110.88	110.64	105.49	113.27	114.49	114.15	101.28	102.92	103.92	103.84
		0	120.35	121.66	122.91	122.46	128.74	130.73	132.09	131.46	105.72	106.04	107.09	106.97
		0.6	139.59	140.93	142.20	141.50	159.85	162.17	163.63	162.59	109.98	110.25	111.27	111.09
		0.9	151.78	154.98	156.27	155.42	181.93	188.07	189.61	188.31	112.26	112.81	113.82	113.61
	0.9	-0.9	121.13	123.28	124.42	123.98	129.91	133.20	134.49	133.84	105.91	106.43	107.36	107.24
		-0.6	131.74	132.43	133.62	133.05	146.63	147.76	149.13	148.28	108.35	108.50	109.46	109.31
		0	158.45	161.71	162.96	162.06	194.91	201.49	203.03	201.65	113.38	113.91	114.87	114.65
		0.6	195.76	215.62	216.94	215.68	282.26	342.26	344.17	342.61	118.56	120.71	121.66	121.37
		0.9	224.49	264.30	265.67	264.31	373.12	557.29	559.94	561.05	121.57	124.83	125.75	125.42

TABLE 6: PRE of different point estimators when $\sigma_Z^2 = 31$, $\sigma_Y^2 = 33$.

			Strategy I				Strategy II				Strategy III			
% of ME	ρ_{YZ}	ρ_{UV}	ζ_{r_1}	ζ_{l_1}	ζ_{a_1}	ζ_{a_4}	ζ_{r_2}	ζ_{l_2}	ζ_{a_2}	ζ_{a_5}	ζ_{r_3}	ζ_{l_3}	ζ_{a_3}	ζ_{a_6}
15	0.3	-0.9	90.70	102.37	103.62	103.52	88.08	103.15	104.45	104.31	96.82	100.75	101.91	101.88
		-0.6	93.37	103.42	104.69	104.56	91.43	104.56	105.89	105.71	97.78	101.07	102.24	102.20
		0	99.58	106.58	107.90	107.69	99.44	108.87	110.26	109.97	99.86	102.01	103.20	103.13
		0.6	105.93	110.61	111.98	111.68	107.98	114.50	115.96	115.54	101.82	103.17	104.36	104.27
		0.9	109.47	113.18	114.56	114.21	112.88	118.16	119.65	119.16	102.85	103.87	105.07	104.97
	0.6	-0.9	112.86	115.79	117.13	116.74	117.70	121.95	123.41	122.85	103.78	104.56	105.71	105.59
		-0.6	117.09	119.18	120.54	120.09	123.86	126.97	128.46	127.82	104.90	105.43	106.58	106.45
		0	127.31	128.42	129.81	129.21	139.49	141.26	142.82	141.95	107.37	107.62	108.77	108.61
		0.6	138.04	139.54	140.95	140.19	157.17	159.75	161.37	160.24	109.67	109.97	111.12	110.92
		0.9	144.26	146.62	148.05	147.19	168.05	172.33	173.99	172.71	110.89	111.33	112.48	112.26
	0.9	-0.9	151.20	152.90	154.27	153.37	180.82	184.05	185.71	184.34	112.15	112.45	113.51	113.28
		-0.6	159.55	162.71	164.10	163.10	197.11	203.56	205.28	203.73	113.56	114.07	115.12	114.88
		0	183.10	192.76	194.19	192.93	249.42	274.12	276.08	274.17	116.99	118.20	119.23	118.94
		0.6	206.92	234.27	235.74	234.29	314.55	410.75	413.09	411.92	119.81	122.46	123.50	123.16
		0.9	221.71	264.12	265.62	264.14	363.13	556.27	559.06	560.68	121.31	124.82	125.86	125.49
20	0.3	-0.9	87.87	101.44	102.71	102.63	84.59	101.91	103.21	103.11	95.77	100.46	101.66	101.64
		-0.6	91.10	102.47	103.78	103.67	88.58	103.29	104.64	104.49	96.97	100.78	101.99	101.96
		0	98.79	106.10	107.47	107.26	98.40	108.21	109.65	109.37	99.61	101.87	103.10	103.04
		0.6	106.93	111.28	112.70	112.38	109.36	115.44	116.97	116.52	102.12	103.35	104.60	104.50
		0.9	111.52	114.78	116.22	115.83	115.79	120.47	122.04	121.48	103.42	104.30	105.54	105.43
	0.6	-0.9	107.38	111.73	113.10	112.77	109.98	116.08	117.56	117.10	102.25	103.48	104.66	104.57
		-0.6	112.33	115.33	116.72	116.32	116.95	121.28	122.79	122.23	103.64	104.44	105.63	105.52
		0	124.60	125.82	127.26	126.68	135.25	137.16	138.76	137.91	106.74	107.03	108.22	108.07
		0.6	138.05	139.53	141.00	140.21	157.20	159.73	161.42	160.25	109.67	109.97	111.17	110.96
		0.9	146.17	148.82	150.31	149.39	171.52	176.37	178.11	176.73	111.25	111.73	112.92	112.69
	0.9	-0.9	138.89	139.61	141.01	140.25	158.62	159.88	161.51	160.37	109.84	109.99	111.09	110.89
		-0.6	147.95	149.19	150.62	149.73	174.76	177.07	178.77	177.42	111.57	111.80	112.89	112.67
		0	173.74	179.89	181.36	180.16	227.40	241.68	243.61	241.73	115.72	116.57	117.64	117.35
		0.6	203.23	227.29	228.82	227.33	303.48	383.51	385.86	384.34	119.41	121.83	122.92	122.57
		0.9	220.91	264.12	265.67	264.14	360.32	556.24	559.10	560.89	121.23	124.82	125.91	125.52
25	0.3	-0.9	85.41	100.84	102.13	102.08	81.60	101.12	102.43	102.36	94.82	100.27	101.51	101.49
		-0.6	89.10	101.76	103.09	103.00	86.10	102.34	103.71	103.59	96.23	100.56	101.81	101.78
		0	98.07	105.68	107.09	106.89	97.46	107.63	109.12	108.84	99.37	101.75	103.03	102.96
		0.6	107.87	111.92	113.41	113.05	110.66	116.36	117.95	117.46	102.39	103.53	104.82	104.72
		0.9	113.62	116.36	117.87	117.43	118.79	122.78	124.43	123.80	103.99	104.71	106.01	105.88
	0.6	-0.9	101.70	108.61	109.95	109.68	102.25	111.69	113.12	112.75	100.54	102.60	103.77	103.69
		-0.6	107.25	112.26	113.63	113.28	109.79	116.84	118.33	117.84	102.21	103.62	104.79	104.69
		0	122.21	123.60	125.09	124.52	131.56	133.70	135.35	134.53	106.17	106.51	107.75	107.59
		0.6	139.86	141.08	142.58	141.74	160.30	162.42	164.17	162.93	110.03	110.27	111.46	111.25
		0.9	150.33	152.89	154.42	153.43	179.18	184.04	185.87	184.36	112.00	112.45	113.64	113.39
	0.9	-0.9	129.17	130.20	131.63	130.97	142.47	144.13	145.75	144.80	107.79	108.02	109.16	108.98
		-0.6	138.63	139.34	140.79	140.00	158.18	159.40	161.10	159.92	109.79	109.93	111.07	110.87
		0	165.93	169.85	171.36	170.21	210.30	218.75	220.67	218.86	114.57	115.15	116.27	116.00
		0.6	199.93	221.29	222.88	221.34	293.92	361.65	364.03	362.25	119.04	121.27	122.40	122.04
		0.9	221.97	264.14	265.77	264.16	364.06	556.37	559.40	561.13	121.33	124.82	125.95	125.55
30	0.3	-0.9	83.26	100.51	101.81	101.79	79.02	100.68	101.99	101.95	93.95	100.16	101.44	101.43
		-0.6	87.33	101.23	102.59	102.52	83.93	101.63	103.03	102.93	95.56	100.39	101.69	101.66
		0	97.41	105.30	106.77	106.56	96.61	107.12	108.66	108.38	99.16	101.64	102.96	102.90
		0.6	108.76	112.54	114.09	113.71	111.89	117.24	118.91	118.38	102.64	103.70	105.04	104.93
		0.9	115.48	117.95	119.53	119.03	121.51	125.14	126.86	126.16	104.48	105.12	106.46	106.32
	0.6	-0.9	97.58	106.25	107.61	107.39	96.82	108.42	109.86	109.55	99.21	101.92	103.12	103.06
		-0.6	103.49	109.78	111.18	110.87	104.66	113.32	114.83	114.41	101.09	102.93	104.14	104.06
		0	120.07	121.68	123.22	122.66	128.31	130.75	132.45	131.66	105.65	106.05	107.33	107.18
		0.6	139.89	141.08	142.64	141.78	160.36	162.44	164.25	162.96	110.04	110.28	111.51	111.29
		0.9	151.93	155.00	156.60	155.54	182.20	188.10	190.01	188.40	112.28	112.81	114.06	113.79
	0.9	-0.9	121.32	123.28	124.73	124.17	130.20	133.20	134.83	134.02	105.96	106.43	107.61	107.46
		-0.6	131.56	132.49	133.96	133.25	146.34	147.86	149.55	148.51	108.31	108.52	109.67	109.49
		0	159.32	161.81	163.37	162.25	196.65	201.71	203.64	201.91	113.53	113.93	115.09	114.82
		0.6	197.90	215.65	217.30	215.73	288.19	342.35	344.80	342.75	118.81	120.72	121.87	121.51
		0.9	220.79	264.13	265.81	264.16	359.90	556.33	559.41	561.37	121.22	124.82	126.00	125.58

Table 7: PRE of different point estimators when $\sigma_Z^2 = 35$, $\sigma_Y^2 = 40$.

% of ME	ρ_{YZ}	ρ_{UV}	Strategy I				Strategy II				Strategy III			
			ζ_{r_1}	ζ_{l_1}	ζ_{a_1}	ζ_{a_4}	ζ_{r_2}	ζ_{l_2}	ζ_{a_2}	ζ_{a_5}	ζ_{r_3}	ζ_{l_3}	ζ_{a_3}	ζ_{a_6}
15	0.3	-0.9	84.04	102.42	103.59	103.47	79.96	103.22	104.44	104.28	94.27	100.76	101.82	101.79
		-0.6	86.88	103.49	104.68	104.54	83.38	104.66	105.92	105.72	95.39	101.09	102.16	102.12
		0	93.52	106.68	107.93	107.70	91.62	109.01	110.35	110.04	97.83	102.04	103.13	103.06
		0.6	100.47	110.74	112.05	111.73	100.62	114.68	116.11	115.66	100.15	103.20	104.3	104.21
		0.9	104.37	113.33	114.67	114.29	105.85	118.38	119.85	119.33	101.36	103.91	105.02	104.91
	0.6	-0.9	113.23	115.77	117.36	116.90	118.24	121.92	123.64	123.00	103.88	104.56	105.93	105.79
		-0.6	117.38	119.17	120.78	120.25	124.29	126.97	128.72	127.97	104.97	105.43	106.80	106.65
		0	127.28	128.38	130.02	129.32	139.46	141.21	143.03	142.02	107.36	107.61	108.99	108.79
		0.6	137.64	139.47	141.14	140.25	156.50	159.64	161.52	160.22	109.59	109.96	111.33	111.10
		0.9	143.65	146.56	148.24	147.24	166.98	172.23	174.15	172.68	110.77	111.32	112.69	112.43
	0.9	-0.9	150.61	152.85	154.45	153.41	179.72	183.96	185.86	184.29	112.05	112.44	113.70	113.43
		-0.6	158.60	162.59	164.21	163.04	195.21	203.31	205.28	203.51	113.41	114.05	115.30	115.01
		0	181.40	192.82	194.46	193.01	245.30	274.28	276.49	274.34	116.77	118.21	119.42	119.08
		0.6	203.80	234.34	236.02	234.36	305.17	411.05	413.61	412.56	119.47	122.47	123.70	123.30
		0.9	217.65	264.12	265.82	264.15	349.05	556.26	559.22	561.82	120.91	124.82	126.05	125.62
20	0.3	-0.9	81.09	101.47	102.64	102.56	76.46	101.95	103.16	103.05	93.05	100.47	101.56	101.53
		-0.6	84.50	102.53	103.74	103.62	80.51	103.36	104.63	104.47	94.46	100.80	101.90	101.87
		0	92.68	106.19	107.49	107.26	90.56	108.34	109.73	109.42	97.54	101.90	103.03	102.96
		0.6	101.59	111.40	112.77	112.43	102.10	115.61	117.12	116.63	100.50	103.38	104.53	104.43
		0.9	106.65	114.94	116.35	115.92	108.97	120.71	122.27	121.67	102.04	104.34	105.49	105.37
	0.6	-0.9	107.88	111.72	113.34	112.96	110.67	116.07	117.81	117.28	102.39	103.47	104.89	104.78
		-0.6	112.74	115.33	116.98	116.52	117.53	121.28	123.07	122.41	103.75	104.44	105.87	105.73
		0	124.66	125.79	127.49	126.81	135.34	137.11	138.98	138.00	106.76	107.02	108.45	108.26
		0.6	137.65	139.46	141.19	140.27	156.51	159.62	161.58	160.22	109.59	109.96	111.39	111.14
		0.9	145.52	148.77	150.52	149.45	170.33	176.29	178.30	176.71	111.12	111.72	113.14	112.87
	0.9	-0.9	138.71	139.57	141.21	140.32	158.32	159.80	161.69	160.37	109.81	109.98	111.28	111.05
		-0.6	147.43	149.08	150.74	149.71	173.80	176.85	178.81	177.26	111.48	111.78	113.08	112.82
		0	172.46	179.95	181.65	180.25	224.52	241.81	244.00	241.86	115.53	116.57	117.84	117.51
		0.6	200.34	227.39	229.13	227.42	295.08	383.86	386.45	384.94	119.09	121.84	123.13	122.72
		0.9	216.84	264.07	265.83	264.11	346.33	555.99	559.03	561.85	120.84	124.82	126.10	125.65
25	0.3	-0.9	78.54	100.86	102.04	101.98	73.50	101.14	102.35	102.27	91.96	100.27	101.40	101.38
		-0.6	82.42	101.80	103.04	102.94	78.03	102.39	103.67	103.54	93.61	100.57	101.71	101.68
		0	91.92	105.76	107.10	106.88	89.61	107.75	109.18	108.87	97.26	101.77	102.94	102.88
		0.6	102.64	112.03	113.48	113.10	103.52	116.52	118.10	117.57	100.83	103.56	104.76	104.65
		0.9	109.05	116.51	118.00	117.52	112.30	123.01	124.67	123.99	102.73	104.75	105.96	105.82
	0.6	-0.9	102.31	108.58	110.17	109.86	103.07	111.65	113.35	112.92	100.73	102.60	103.99	103.90
		-0.6	107.74	112.20	113.83	113.43	110.47	116.76	118.52	117.96	102.35	103.61	105.00	104.89
		0	122.33	123.57	125.33	124.67	131.75	133.66	135.59	134.64	106.20	106.50	107.98	107.80
		0.6	139.61	141.09	142.85	141.88	159.87	162.44	164.47	163.03	109.98	110.28	111.69	111.44
		0.9	149.68	152.86	154.65	153.49	177.97	183.98	186.09	184.35	111.88	112.44	113.86	113.57
	0.9	-0.9	129.28	130.16	131.84	131.07	142.64	144.06	145.96	144.85	107.81	108.01	109.37	109.16
		-0.6	138.41	139.23	140.93	140.01	157.80	159.22	161.18	159.82	109.75	109.91	111.26	111.03
		0	164.97	169.90	171.65	170.31	208.27	218.85	221.05	218.98	114.42	115.16	116.48	116.16
		0.6	197.23	221.40	223.21	221.45	286.33	362.04	364.67	362.82	118.73	121.28	122.62	122.2
		0.9	217.90	264.15	265.99	264.18	349.93	556.42	559.65	562.42	120.94	124.82	126.15	125.68
30	0.3	-0.9	76.33	100.51	101.70	101.67	70.96	100.68	101.88	101.84	90.97	100.16	101.32	101.31
		-0.6	80.59	101.26	102.52	102.44	75.87	101.67	102.97	102.86	92.84	100.40	101.58	101.56
		0	91.23	105.38	106.77	106.54	88.74	107.23	108.70	108.40	97.02	101.66	102.87	102.81
		0.6	103.64	112.65	114.15	113.75	104.86	117.40	119.05	118.48	101.14	103.73	104.97	104.86
		0.9	111.26	118.09	119.65	119.12	115.42	125.35	127.10	126.34	103.35	105.16	106.41	106.26
	0.6	-0.9	98.25	106.23	107.84	107.58	97.70	108.39	110.09	109.74	99.43	101.91	103.35	103.27
		-0.6	104.05	109.73	111.39	111.03	105.42	113.25	115.03	114.54	101.26	102.92	104.36	104.26
		0	120.25	121.66	123.47	122.82	128.58	130.72	132.70	131.78	105.70	106.04	107.58	107.40
		0.6	139.65	141.10	142.93	141.92	159.94	162.47	164.57	163.07	109.99	110.28	111.75	111.49
		0.9	151.43	154.97	156.84	155.61	181.26	188.06	190.27	188.41	112.19	112.81	114.28	113.97
	0.9	-0.9	121.62	123.24	124.95	124.30	130.66	133.15	135.05	134.11	106.03	106.42	107.83	107.65
		-0.6	131.76	132.56	134.28	133.46	146.67	147.98	149.95	148.73	108.36	108.53	109.91	109.69
		0	158.61	161.86	163.66	162.37	195.22	201.80	204.01	202.03	113.41	113.93	115.31	115.00
		0.6	195.42	215.65	217.54	215.74	281.32	342.36	345.08	342.87	118.52	120.72	122.08	121.66
		0.9	218.02	264.17	266.08	264.20	350.33	556.52	559.88	562.75	120.95	124.82	126.21	125.72

TABLE 8: PRE of different point estimators when $\sigma_Z^2 = 50$, $\sigma_Y^2 = 45$.

			Strategy I				Strategy II				Strategy III			
% of ME	ρ_{YZ}	ρ_{UV}	ζ_{r_1}	ζ_{l_1}	ζ_{a_1}	ζ_{a_4}	ζ_{r_2}	ζ_{l_2}	ζ_{a_2}	ζ_{a_5}	ζ_{r_3}	ζ_{l_3}	ζ_{a_3}	ζ_{a_6}
15	0.3	-0.9	89.33	102.40	104.24	104.08	86.38	103.19	105.09	104.89	96.32	100.76	102.46	102.42
		-0.6	92.01	103.45	105.32	105.12	89.71	104.60	106.56	106.29	97.30	101.08	102.80	102.74
		0	84.00	100.44	103.00	102.98	79.91	100.58	103.15	103.12	94.25	100.14	102.70	102.69
		0.6	88.20	100.99	103.69	103.59	84.99	101.32	104.05	103.92	95.90	100.32	102.93	102.90
		0.9	90.37	101.60	104.34	104.19	87.67	102.12	104.92	104.72	96.70	100.51	103.13	103.09
	0.6	-0.9	111.51	115.82	117.73	117.16	115.77	121.99	124.08	123.28	103.42	104.57	106.18	106.02
		-0.6	115.88	119.18	121.12	120.46	122.08	126.98	129.11	128.18	104.59	105.43	107.04	106.86
		0	126.62	128.48	130.47	129.60	138.41	141.37	143.61	142.34	107.21	107.64	109.24	109.01
		0.6	138.06	139.67	141.72	140.61	157.21	159.99	162.35	160.7	109.68	110.00	111.62	111.33
		0.9	144.65	146.74	148.81	147.56	168.76	172.54	174.98	173.10	110.96	111.35	112.97	112.66
	0.9	-0.9	111.51	115.82	117.73	117.16	115.77	121.99	124.08	123.28	103.42	104.57	106.18	106.02
		-0.6	115.88	119.18	121.12	120.46	122.08	126.98	129.11	128.18	104.59	105.43	107.04	106.86
		0	126.62	128.48	130.47	129.6	138.41	141.37	143.61	142.34	107.21	107.64	109.24	109.01
		0.6	138.06	139.67	141.72	140.61	157.21	159.99	162.35	160.70	109.68	110.00	111.62	111.33
		0.9	144.65	146.74	148.81	147.56	168.76	172.54	174.98	173.10	110.96	111.35	112.97	112.66
20	0.3	-0.9	86.46	101.45	103.32	103.21	82.87	101.93	103.84	103.69	95.23	100.46	102.23	102.19
		-0.6	89.70	102.49	104.41	104.24	86.84	103.31	105.30	105.08	96.46	100.78	102.56	102.51
		0	97.40	106.16	108.17	107.85	96.59	108.30	110.42	109.99	99.15	101.89	103.68	103.59
		0.6	105.9	111.41	113.51	113.02	107.94	115.63	117.89	117.21	101.82	103.39	105.22	105.07
		0.9	110.53	114.89	117.03	116.43	114.38	120.64	122.96	122.12	103.14	104.33	106.16	105.99
	0.6	-0.9	105.84	111.75	113.69	113.21	107.85	116.11	118.21	117.55	101.80	103.48	105.15	105.01
		-0.6	110.92	115.32	117.3	116.72	114.94	121.26	123.43	122.62	103.25	104.44	106.11	105.95
		0	123.75	125.88	127.93	127.08	133.93	137.25	139.55	138.32	106.54	107.04	108.71	108.48
		0.6	138.10	139.67	141.80	140.65	157.27	159.99	162.45	160.73	109.68	110.00	111.69	111.39
		0.9	146.64	148.90	151.06	149.72	172.37	176.53	179.09	177.07	111.33	111.74	113.43	113.09
	0.9	-0.9	138.63	139.72	141.70	140.60	158.18	160.06	162.41	160.76	109.79	110.01	111.51	111.23
		-0.6	148.57	149.49	151.51	150.23	175.91	177.61	180.09	178.11	111.68	111.85	113.34	113.03
		0	176.57	179.79	181.91	180.17	233.87	241.42	244.34	241.51	116.11	116.55	118.01	117.61
		0.6	210.16	227.05	229.34	227.11	324.59	382.60	386.41	383.54	120.15	121.81	123.30	122.81
		0.9	231.26	264.21	266.59	264.23	398.74	556.80	561.76	562.51	122.19	124.83	126.32	125.78
25	0.3	-0.9	83.97	100.85	102.74	102.66	79.88	101.12	103.05	102.95	94.24	100.27	102.09	102.07
		-0.6	87.67	101.77	103.73	103.59	84.34	102.35	104.37	104.19	95.69	100.56	102.40	102.36
		0	96.63	105.73	107.81	107.50	95.60	107.71	109.90	109.47	98.90	101.77	103.62	103.53
		0.6	106.87	112.06	114.25	113.72	109.27	116.55	118.91	118.17	102.10	103.57	105.47	105.31
		0.9	112.72	116.50	118.74	118.07	117.51	122.99	125.43	124.49	103.75	104.75	106.66	106.47
	0.6	-0.9	99.63	108.66	110.54	110.15	99.51	111.76	113.77	113.24	99.88	102.62	104.23	104.12
		-0.6	105.42	112.38	114.30	113.80	107.28	117.02	119.11	118.42	101.67	103.65	105.27	105.13
		0	121.21	123.65	125.77	124.94	130.04	133.78	136.14	134.96	105.93	106.52	108.25	108.03
		0.6	139.47	141.02	143.17	141.97	159.64	162.34	164.87	163.06	109.96	110.26	111.91	111.61
		0.9	150.85	152.94	155.15	153.71	180.17	184.14	186.83	184.61	112.09	112.46	114.12	113.77
	0.9	-0.9	128.23	130.29	132.29	131.36	140.96	144.28	146.59	145.22	107.58	108.04	109.59	109.35
		-0.6	138.49	139.61	141.66	140.53	157.94	159.88	162.31	160.60	109.76	109.99	111.54	111.25
		0	167.85	169.76	171.93	170.27	214.41	218.54	221.41	218.73	114.86	115.14	116.65	116.27
		0.6	206.29	221.00	223.37	221.10	312.63	360.66	364.47	361.32	119.74	121.24	122.79	122.29
		0.9	232.26	264.12	266.60	264.14	402.68	556.23	561.48	562.02	122.28	124.82	126.37	125.81
30	0.3	-0.9	81.80	100.51	102.42	102.38	77.30	100.67	102.60	102.55	93.35	100.16	102.03	102.02
		-0.6	85.88	101.24	103.24	103.13	82.16	101.64	103.69	103.54	95.00	100.39	102.29	102.26
		0	95.93	105.35	107.49	107.19	94.70	107.19	109.44	109.03	98.66	101.65	103.58	103.48
		0.6	107.78	112.68	114.97	114.39	110.53	117.45	119.91	119.11	102.36	103.74	105.71	105.55
		0.9	114.58	118.07	120.40	119.66	120.19	125.31	127.86	126.82	104.25	105.15	107.13	106.92
	0.6	-0.9	95.39	106.29	108.19	107.87	94.01	108.48	110.50	110.06	98.48	101.93	103.59	103.50
		-0.6	101.54	109.90	111.85	111.41	102.04	113.49	115.60	114.99	100.49	102.97	104.64	104.51
		0	118.94	121.73	123.91	123.10	126.62	130.82	133.25	132.10	105.37	106.06	107.85	107.63
		0.6	139.48	141.01	143.24	141.99	159.65	162.31	164.94	163.06	109.96	110.26	111.97	111.66
		0.9	152.73	155.12	157.43	155.89	183.73	188.34	191.16	188.79	112.42	112.83	114.57	114.19
	0.9	-0.9	119.90	123.36	125.38	124.58	128.06	133.33	135.62	134.47	105.61	106.45	108.06	107.84
		-0.6	130.29	132.32	134.38	133.37	144.27	147.59	149.99	148.50	108.04	108.48	110.05	109.79
		0	160.53	161.73	163.95	162.35	199.09	201.53	204.39	201.85	113.72	113.91	115.48	115.11
		0.6	204.05	215.64	218.08	215.76	305.91	342.30	346.21	342.72	119.50	120.71	122.29	121.79
		0.9	231.16	264.25	266.81	264.27	398.34	556.99	562.34	563.18	122.18	124.83	126.45	125.86

TABLE 9: PRE of the estimators for real population.

Strategy I				Strategy II				Strategy III			
ζ_{r_1}	ζ_{l_1}	ζ_{a_1}	ζ_{a_4}	ζ_{r_2}	ζ_{l_2}	ζ_{a_2}	ζ_{a_5}	ζ_{r_3}	ζ_{l_3}	ζ_{a_3}	ζ_{a_6}
43.76	151.77	152.48	151.99	40.55	163.72	164.62	163.95	84.67	105.04	105.20	105.17

6. Conclusion

To tackle the issue of missing data in the presence of ME, some prominent studies have been conducted under SRS which are useful to estimate the characteristics of population but, a miniscule work has been done to deal with the problem of missing data under CME. In this study, we have proposed some LIM for the estimation of population mean under CME and evaluated the impact of CME at different levels. The MSE of the point estimators of the LIM is deduced to the first order of approximation. The comparative study is carried out to determine the efficiency conditions under which the LIM surpass the CIM. We have executed a simulation study to support the theoretical results and to compute the impact of the CME over the efficiency of the point estimators based on LIM. The simulation results are revealed in Tables 1-8 by PRE. The results of Table 1 to Table 4 exhibit that the PRE of the point estimators $\zeta_{a_i},\ i = 1, 2, \ldots, 6$ based on the reasonably chosen values of σ_Y^2, σ_Z^2, σ_U^2 and σ_V^2 varies as the value of ρ_{YZ} varies from 0.3 to 0.9 which is also depend on the sign and value of ρ_{UV}. The same pattern in the PRE of the LIM can be observed from Table 5 to Table 8 which are based on different percentages of ME. The PRE of the LIM are found to be rather different under uncorrelated ME and CME. Furthermore, the point estimators based on LIM surpass the point estimators based on CIM for reasonably chosen values of σ_Y^2, σ_Z^2, σ_U^2 and σ_V^2, ρ_{YZ}, ρ_{UV} and percentages of ME. The PRE of the point estimator ζ_{a_1} repress the PRE of the point estimator ζ_{a_2}. Therefore, the proposed LIM are strongly advised to the survey experts in real-world circumstances to deal with the issue of uncorrelated ME and CME.

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