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# Estimation of Inverse Pareto Distribution under Unified Hybrid Censoring

## Estimación de la distribución inversa de Pareto bajo censura híbrida unificada

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**Abstract**

In this paper, some estimators of the unknown parameter, reliability function and hazard rate function of Inverse Pareto distribution under Unified Hybrid Censoring were derived. The maximum likelihood method, Bayes and E-Bayes method were used for estimating the parameter, reliability function and hazard rate function of the Inverse Pareto Distribution. Approximate confidence intervals(confidence interval and credible interval) were also derived. Comparisons were made in sense of mean squared error and asymptotic relative efficiency through Monte Carlo simulation. Finally, the proposed methods can be understood through illustrating the results of the real data analysis.

***Key words*:** Inverse Pareto distribution; Unified Hybrid Censoring; Confidence interval; Credible interval; Monte Carlo simulation.

**Resumen**

En este artículo, algunos estimadores del parámetro desconocido, la función de confiabilidad y la función de tasa de riesgo de Se obtuvo la distribución inversa de Pareto bajo la censura híbrida unificada. Se utilizó el método de máxima verosimilitud, el método de Bayes y el de E-Bayes para estimar la parámetro, función de confiabilidad y función de tasa de riesgo de la distribución inversa de Pareto. Aproximado También se derivaron intervalos de confianza (intervalo de confianza e intervalo de credibilidad). Las comparaciones se realizaron en sentido de error cuadrático medio y eficiencia relativa asintótica mediante Monte Carlo. simulación. Finalmente, los métodos propuestos pueden entenderse ilustrando los resultados del análisis de datos reales.

***Palabras clave*:** Distribución inversa de Pareto; Censura híbrida unificada; Intervalo de confianza; Intervalo de credibilidad; Simulación de Monte Carlo.

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# 1. Introduction

The Pareto distribution is widely used in modeling real-world phenomena that exhibit a power law relationship, such as income and wealth distributions, city sizes, and earthquake magnitudes. However, in some cases, the Pareto distribution may not be an appropriate model, as it assumes that the minimum possible value is zero, which may not hold in all contexts. To address this limitation, the inverse Pareto distribution has been proposed as an alternative model that can accommodate non-zero minimum values.

Inverse distributions are special cases of the class of ratio distributions, in which the numerator random variable has a degenerate distribution. If the random variable (r.v.) Y has Pareto distribution, then the r.v. $X = \frac{1}{Y}$ has an inverse Pareto distribution (IPD) with parameter $\theta$ with cumulative distribution function(cdf)

$$F(x;\theta) = \left(\frac{x}{1+x}\right)^{\theta}; x > 0, \theta > 0 \tag{1}$$

and probability density function (pdf)

$$f(x;\theta) = \frac{\theta x^{\theta-1}}{(1+x)^{1+\theta}}; x > 0, \theta > 0 \tag{2}$$

Also the Reliability and Hazard rate functions of IPD are given by

$$R(t;\theta) = P(T > t) = 1 - F(t) = 1 - \left(\frac{t}{1+t}\right)^{\theta}; t > 0, \theta > 0 \tag{3}$$

$$H(t;\theta) = \frac{f(t)}{R(t)} = \frac{\theta t^{\theta-1}}{(1+t)^{1+\theta}\left[1 - \left(\frac{t}{1+t}\right)^{\theta}\right]}; t > 0, \theta > 0 \tag{4}$$

The inverse Pareto distribution has been studied in various fields, including finance, economics, engineering, and environmental science, among others. It has also been used to model a wide range of phenomena, such as the size of oil reserves, the value of portfolios, and the concentration of pollutants in water bodies. Despite its practical relevance and theoretical interest, the inverse Pareto distribution has received relatively little attention in the literature, particularly in comparison to the Pareto distribution.

Inverse Pareto distribution was used to describe demand for the pay-as-bid auctions model in Holmberg (2009). Naghettini et al. (1996) estimated the upper tail of floodpeak frequency distribution by the inverse Pareto distribution. For other applications, see Wildani et al. (2014). Guo & Gui (2018) studied IPD lifetime model based on stress-strength reliability in case of both classical and Bayesian approaches. Kumar & Kumar (2020) discussed IPD model based on randomly censored data. The application of IPD in extreme events is studied by Dankunprasert et al. (2021). Kumar et al. (2021) discussed different estimation methods of the parameter and reliability characteristics of the inverse Pareto distribution from both classical and Bayesian approaches.

Censoring occurs when the exact value of a variable of interest is unknown, and only partial information is available, such as an interval or a threshold value. In some cases, the censoring mechanism may be a mixture of different types, such as right, left, and interval censoring. Unified hybrid censoring is a method that can handle a mixture of different censoring mechanisms in a single model, providing a more flexible and efficient approach for analyzing censored data. This technique is becoming increasingly important in various fields, including engineering, biomedical research, and reliability analysis. Examples of applications include the analysis of failure time data in medical studies, the estimation of survival probabilities in ecological studies, and the modeling of financial data with censoring.

In the Type-I censoring scheme, the test is terminated at fixed $T \in (0, \infty)$ and uses the observed samples until that $T$. In the Type-II censoring scheme, the test is terminated when samples are observed until the predetermined observation number ($r$). These were combined by Childs et al. (2003) and are called the hybrid censoring scheme that also has two types. Chandrasekar et al. (2004) combined these two types as the generalized Type-I hybrid censoring scheme and generalized Type-II hybrid censoring scheme. A mixture of generalized Type-I and Type-II hybrid censoring scheme is known as the unified hybrid censoring scheme combined by Balakrishnan et al. (2008). They proposed exact likelihood inference based on the unified hybrid censoring scheme from the exponential distribution. Panahi & Sayyareh (2016) studied the estimation and prediction for the Burr Type XII distribution based on the unified hybrid censoring scheme. Ateya (2017) discussed the estimations under the inverse Weibull distribution based on the unified hybrid censoring scheme (UHCS). Kaushik (2019) proposed a progressive interval Type-I censored life test plan for the Rayleigh distribution. Jeon & Kang (2021) Estimated the Rayleigh Distribution under Unified Hybrid Censoring. Kumar et al. (2024) studied parameter and reliability characteristics of Inverse Pareto Distribution under unified hybrid censoring scheme using classical and Bayesian technique.

Consider a life testing experiment in which n identical unites are placed on a life test. Let the lifespan of experimental objects be expressed by $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ taken from Inverse Pareto distribution with pdf given by equation (2). Assuming $T_1$ is the predetermined experiment end time and $T_2$ is the extended experiment end time, let k be the object number that must be observed. Let $d_i$ $(i = 1, 2)$ be the number of objects that have been observed until $T_i$ $(i = 1, 2)$ and $r$ be the predetermined observation number. In other words, it is presumptive that $k < r < n$. If the $k_{th}$ failure occurs before time $T_1$, terminate the experiment at $\min(\max(X_{r:n}, T_1), T_2)$. If the $k_{th}$ failure occurs between $T_1$ and $T_2$, terminate the experiment at min $(X_{r:n}, T_2)$ and if the $k_{th}$ failure occurs after time $T_2$, then terminate the experiment at $X_{k:n}$. Under this censoring scheme, one can guarantee that the experiment would be completed at most in time $T_2$ with at least $k$ failures and if not, we can guarantee exactly $k$ failures. Thus, we have the following six cases:
Case I: $0 < X_{k:n} < X_{r:n} < T_1 < T_2$ in which case we terminate at $T_1$,
Case II: $0 < X_{k:n} < T_1 < X_{r:n} < T_2$ in which case we terminate at $X_{r:n}$,
Case III: $0 < X_{k:n} < T_1 < T_2 < X_{r:n}$ in which case we terminate at $T_2$,
Case IV: $0 < T_1 < X_{k:n} < X_{r:n} < T_2$ in which case we terminate at $X_{r:n}$,

Case V: $0 < T_1 < X_{k:n} < T_2 < X_{r:n}$ in which case we terminate at $T_2$,
Case VI: $0 < T_1 < T_2 < X_{k:n} < X_{r:n}$ in which case we terminate at $X_{k:n}$.

These six cases make up the unified hybrid censoring system, depending on the end point and observed number.

In this research paper, we aim to contribute to the literature on the inverse Pareto distribution by providing a comprehensive overview of its properties and applications under UHCS. Specifically, we will review the existing literature on the inverse Pareto distribution, including its mathematical properties, and estimation methods. We will also examine its empirical performance in various contexts and compare it to other alternative models. Additionally, we aim to provide a comprehensive review of unified hybrid censoring, focusing on its properties and applications in various fields.

To our best knowledge no study has been done to estimate Inverse Pareto distribution under Unified Hybrid Censoring. The paper is carried out as follows: In Section 2 the likelihood function is obtained. MlEs for the unknown paremeter, reliability function and hazard rate function under UHCS is obtained in Section 3 followed by approximate confidence interval in Section 4. In Section 5 and 6 Bayes and E-Bayes estimated and Credible intervals are computed. Section 7 depicts the simulation study conducted for performance evaluation along with real data analysis. References used for literature review are given in the last section.

## 2. Maximum Likelihood Estimation

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n taken from the Inverse Pareto distribution. Then the likelihood function for the six cases of the UHCS are as follows:

$$L(x;\theta) = \frac{n!}{(n-m)!}\left[\prod_{i=1}^{m} f(x_i)\right][1-F(C)]^{n-m} \tag{5}$$

$$(m, C) = \begin{cases} (d_1, T_1), & \text{for Case I} \\ (r, x_r), & \text{for Case II and Case IV} \\ (d_2, T_2), & \text{for Case III and Case V} \\ (k, x_k), & \text{for Case VI} \end{cases}$$

where m indicates the number of total failures in experiment up to time C(end time point of experiment) and $d_1$ and $d_2$ indicate the number of failures that occur before time points $T_1$ and $T_2$ respectively, where $d = d_1 = d_2$ for Case I.

From (1),(2) and (5) we have,

$$L(x,\theta) = \frac{n!}{(n-m)!}\prod_{i=1}^{m}\frac{\theta x_i^{\theta-1}}{(1+x_i)^{\theta+1}}\left[1-\left(\frac{C}{1+C}\right)^{\theta}\right]^{n-m}$$

$$L(x,\theta) = K\theta^m\prod_{i=1}^{m}\frac{x_i^{\theta-1}}{(1+x_i)^{\theta+1}}\left[1-\left(\frac{C}{1+C}\right)^{\theta}\right]^{n-m} \tag{6}$$

where

$$K = \frac{n!}{(n-m)!}$$

The log likelihood function for equation (6) is given as

$$\begin{aligned}
\ln L(x,\theta) &= \ln(K) + m\ln(\theta) + \sum_{i=1}^{m} \ln\left[\frac{x_i^{\theta-1}}{(1+x_i)^{\theta+1}}\right] + (n-m)\ln\left[1-\left(\frac{C}{1+C}\right)^{\theta}\right] \\
&= \ln(K) + m\ln(\theta) + \sum_{i=1}^{m} \ln(x_i^{\theta-1}) - \sum_{i=1}^{m} \ln(1+x_i)^{\theta+1} \\
&\quad + (n-m)\ln\left[1-\left(\frac{C}{1+C}\right)^{\theta}\right] \\
&= \ln(K) + m\ln(\theta) + (\theta-1)\sum_{i=1}^{m} \ln(x_i) - (\theta+1)\sum_{i=1}^{m} \ln(1+x_i) \\
&\quad + (n-m)\ln\left[1-\left(\frac{C}{1+C}\right)^{\theta}\right]
\end{aligned}$$

Taking derivation on both sides with respect to $\theta$, we get

$$\begin{aligned}
\frac{\partial \ln L}{\partial \theta} &= \frac{m}{\theta} + \sum_{i=1}^{m} \ln(x_i) - \sum_{i=1}^{m} \ln(1+x_i) \\
&\quad - (n-m)\left[1-\left(\frac{C}{1+C}\right)^{\theta}\right]^{-1}\left(\frac{C}{1+C}\right)^{\theta} \ln\left(\frac{C}{1+C}\right)
\end{aligned} \tag{7}$$

Equating equation (7) to zero, we obtain the likelihood equations for the parameter $\theta$ as follow

$$\frac{\partial \ln L}{\partial \theta} = \frac{m}{\theta} + \sum_{i=1}^{m} \ln\left(\frac{x_i}{1+x_i}\right) - (n-m)\left[R^{-\theta} - 1\right]^{-1} \ln(R) = 0 \tag{8}$$

where $R = \ln\left(\frac{C}{1+C}\right)$.

The MLE of $\theta$ can be found by solving the above likelihood equation. As the suggested estimator cannot be expressed in closed form, we can use a suitable numerical technique to obtain the estimator. Moreover, we can obtain the MLE's of $R(t)$ and $H(t)$ after replacing $\theta$ by their MLE as following

$$\hat{R}(t) = 1 - \left(\frac{t}{1+t}\right)^{\hat{\theta}}; t > 0 \tag{9}$$

$$\hat{H}(t) = \frac{\hat{\theta} t^{\hat{\theta}-1}}{(1+t)^{1+\hat{\theta}}\left[1-\left(\frac{t}{1+t}\right)^{\hat{\theta}}\right]}; t > 0 \tag{10}$$

# 3. Approximate Confidence Interval

To construct the $100(1-\alpha)\%$ confidence interval for $\theta$, we can use the asymptotic normality of the MLE with $Var(\hat{\theta}_{MLE})$ estimated from the inverse of the Fisher information. Due to its complicated form we derive the asymptotic variance by using the observed Fisher information

$$\begin{aligned}
I(\theta) &\simeq -\frac{\partial^2 \ln L}{\partial \theta^2} \\
&= \frac{m}{\theta^2} + (n-m)\big[R^{-\theta} - 1\big]^{-2} R^{-\theta} (\ln R)^2 \\
&= \frac{m}{\theta^2} + (n-m)\big[R^{-\theta} + R^{\theta} - 2\big]^{-1} (\ln R)^2
\end{aligned}$$

Under some mild regularity conditions, $\hat{\theta_M}$ is approximately normal with mean $\sigma$ and variance $Var(\hat{\theta}_{MLE}) = \frac{1}{I(\hat{\theta}_M)}$.

Hence we can obtain the approximate CI based on the MLE for the parameter $\theta$ as

$$\left(\hat{\theta}_M \pm z_{\alpha/2}\sqrt{\hat{Var}(\hat{\theta}_{MLE})}\right)$$

where $z_{\alpha/2}$ denotes the percentile of the standard normal distribution with right tail probability $\alpha/2$.

Moreover to construct the Approximate confidence interval(ACI) of the $R(t)$ and $H(t)$, which they are the functions in the parameter $\theta$, we need to find the variances of them. In order to find the approximate estimates of the variances of $\hat{R}(t)$ and $\hat{H}(t)$ we use the delta method. The delta method is a general approach for computing confidence intervals for functions of MLEs. According to this method, the variance of $\hat{R}(t)$ and $\hat{H}(t)$ can be approximated, respectively by

$$\hat{\sigma}^2_{R(t)} = [\Delta\hat{R}(t)]^T[\hat{V}][\Delta\hat{R}(t)] \quad \text{and} \quad \hat{\sigma}^2_{H(t)} = [\Delta\hat{H}(t)]^T[\hat{V}][\Delta\hat{H}(t)],$$

where $\Delta\hat{R}(t)$ and $\Delta\hat{H}(t)$ are the gradient of $\hat{R}(t)$ and $\hat{H}(t)$, respectively, with respect to $\theta$ given by

$$\begin{aligned}
\Delta\hat{R}(t) =& \frac{\partial \hat{R}(t)}{\partial \theta} = \frac{\partial}{\partial \theta}\left(1 - \left(\frac{t}{1+t}\right)^{\theta}\right) = -\left(\frac{t}{1+t}\right)^{\theta} \ln\left(\frac{t}{1+t}\right) \\
\Delta\hat{H}(t) =& \frac{\partial \hat{H}(t)}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\frac{\theta t^{\theta-1}}{(1+t)^{1+\theta}\left[1 - \left(\frac{t}{1+t}\right)^{\theta}\right]}\right) \\
=& t^{\theta-1}(1+t)^{-\theta-1}\frac{\left[\ln\left(\frac{t}{1+t}\right)^{\theta} - \left(\frac{t}{1+t}\right)^{\theta} + 1\right]}{\left[1 - \left(\frac{t}{1+t}\right)^{\theta}\right]^2}
\end{aligned}$$

and $\hat{V} = \frac{1}{I(\theta)}$. Then, $100(1-\alpha)\%$ intervals for $S(t)$ and $H(t)$ become

$$\left(\hat{R}(t) \pm z_{\alpha/2}\sqrt{\hat{\sigma}^2_{R(t)}}\right) \quad \text{and} \quad \left(\hat{H}(t) \pm z_{\alpha/2}\sqrt{\hat{\sigma}^2_{H(t)}}\right).$$

# 4. Bayes and E-Bayesian Estimation

In this section, we obtain Bayesian estimates of the unknown parameter $\theta$, $S(t)$ and $H(t)$ of $IPD(\theta)$ under squared error loss function based on unified hybrid censoring scheme.

Let the prior distribution of $\theta$ be $Gamma(a,b)$ where a and b are the hyper-parameters such that $a, b > 0$ i.e.

$$g(\theta) = \frac{e^{-b\theta}\theta^{a-1}b^a}{\Gamma a} \tag{11}$$

The posterior distribution is obtained by combining equation (6) and (11) as follows

$$\pi(\theta|data) = \frac{L(data;\theta) * g(\theta)}{\int_0^\infty L(data;\theta) * g(\theta)d\theta}$$

$$\pi(\theta|x) = \frac{K\theta^m \prod_{i=1}^m \frac{x_i^{\theta-1}}{(1+x_i)^{\theta+1}} \left[1 - \left(\frac{C}{1+C}\right)^\theta\right]^{n-m} \frac{e^{-b\theta}\theta^{a-1}b^a}{\Gamma a}}{\int_0^\infty K\theta^m \prod_{i=1}^m \frac{x_i^{\theta-1}}{(1+x_i)^{\theta+1}} \left[1 - \left(\frac{C}{1+C}\right)^\theta\right]^{n-m} \frac{e^{-b\theta}\theta^{a-1}b^a}{\Gamma a} d\theta}$$

$$= \frac{\theta^{m+a-1} \prod_{i=1}^m \frac{1}{x_i(1+x_i)} \left(\frac{x_i}{1+x_i}\right)^\theta \left[1 - R^\theta\right]^{n-m} e^{-b\theta}}{\int_0^\infty \theta^{m+a-1} \prod_{i=1}^m \frac{1}{x_i(1+x_i)} \left(\frac{x_i}{1+x_i}\right)^\theta \left[1 - R^\theta\right]^{n-m} e^{-b\theta} d\theta}$$

$$= \frac{\theta^{m+a-1} \prod_{i=1}^m \frac{1}{x_i(1+x_i)} e^{\theta \sum_{i=1}^m \ln\left(\frac{x_i}{1+x_i}\right)} \left[1 - R^\theta\right]^{n-m} e^{-b\theta}}{\int_0^\infty \theta^{m+a-1} \prod_{i=1}^m \frac{1}{x_i(1+x_i)} e^{\theta \sum_{i=1}^m \ln\left(\frac{x_i}{1+x_i}\right)} \left[1 - R^\theta\right]^{n-m} e^{-b\theta} d\theta}$$

$$\pi(\theta|x) \propto \theta^{m+a-1} e^{-\theta\left[b + \sum_{i=1}^m \ln\left(1 + \frac{1}{x_i}\right)\right]} \left[1 - R^\theta\right]^{n-m} \tag{12}$$

The Bayes estimator of $\theta$ using the SELF is

$$\hat{\theta_B} \propto \theta^{m+a} e^{-\theta\left[b + \sum_{i=1}^m \ln\left(1 + \frac{1}{x_i}\right)\right]} \left[1 - R^\theta\right]^{n-m} \tag{13}$$

Here, three different prior distributions of hyper-parameters are investigated in this section to see how they affect the E-Bayesian estimates of $\theta$. We select the hyper-parameters a and $b$ to prove that $g(\theta)$ is $a$ decreasing function of $\theta$. The first derivative of $g(\theta)$ regarding $\theta$ is as follows:

$$\frac{\partial g(\theta)}{\partial \theta} \propto \theta e^{b\theta}[(\theta - 1) - b\theta] \tag{14}$$

Thus, for $0 < a < 1$ and $b > 0$, the prior PDF $g(\theta)$ is a decreasing function of $\theta$. Suppose that $a$ and $b$, are independent with bivariate PDF given by

$$p(a,b) = p(a)p(b) \tag{15}$$

the EB estimates of the parameter $\theta$ are expectation of the Bayesian estimate of $\theta$ can be obtained as follows:

$$\hat{\theta}_{EB} = E[\hat{\theta}_B|x] = \int_A \hat{\theta}_B(a,b)p(a,b)dadb, \tag{16}$$

According to three various prior PDF of the hyper-parameters $a$ and $b$, the EB estimates of the parameter $\theta$ can be derived. As a result, prior distributions chosen to show how different prior distributions affect the estimation of the EB of $\theta$. We suggest the following prior PDFs

$$\begin{aligned} p_1(a,b) &= \frac{1}{c}, \quad 0<a<1, 0<b<c,\\ p_2(a,b) &= \frac{2b}{c^2}, \quad 0<a<1, 0<b<c,\\ p_3(a,b) &= \frac{2(c-b)}{c^2}, \quad 0<a<1, 0<b<c, \end{aligned} \tag{17}$$

The EB estimate of $\theta$ under the SELF based on $p_1(a,b)$, $p_2(a,b)$, and $p_3(a,b)$ are computed from (15), as follows

$$\hat{\theta}_{EB1} \propto \int_0^1 \int_0^c \frac{1}{c}\theta^{m+a} e^{-\theta\left[b+\sum_{i=1}^m \ln\left(1+\frac{1}{x_i}\right)\right]} \left[1-R^\theta\right]^{n-m} dbda \tag{18}$$

$$\hat{\theta}_{EB2} \propto \int_0^1 \int_0^c \frac{2b}{c^2}\theta^{m+a} e^{-\theta\left[b+\sum_{i=1}^m \ln\left(1+\frac{1}{x_i}\right)\right]} \left[1-R^\theta\right]^{n-m} dbda \tag{19}$$

$$\hat{\theta}_{EB3} \propto \int_0^1 \int_0^c \frac{2(c-b)}{c^2}\theta^{m+a} e^{-\theta\left[b+\sum_{i=1}^m \ln\left(1+\frac{1}{x_i}\right)\right]} \left[1-R^\theta\right]^{n-m} dbda \tag{20}$$

It is evident that it is not possible to compute (13) and (18)-(20) analytically because it is very difficult to get explicit forms for the marginal posterior distribution. So we use MCMC method to approximate these.

## 5. Credibility Interval

For a specified value of $\tau$, we define the $100(1-\tau)\%$ Credibility interval $(L_\theta, U_\theta)$ for $\theta$ by

$$\int_{L_\theta}^{U_\theta} \pi(\theta|x)d\theta = 1-\tau$$

where $\pi(\theta|x)$ is the marginal distribution of $\theta$. In many cases it will be very difficult to obtain the marginal pdf. So, Metropolis Hastings algorithms have been used to generate $(\theta^1),(\theta^2)\cdots(\theta^N)$ from $\pi(\theta \mid x)$. Using these generated values of $\theta$, simple formulas have been obtained to compute the credibility intervals for $\theta$ in the following form:

$$\frac{1}{N}\sum_{i=1}^N \int_{L_\theta}^\infty \pi(\theta^i \mid x)d\theta = 1-\frac{\tau}{2}, \quad \frac{1}{N}\sum_{i=1}^N \int_{U_\theta}^\infty \pi(\theta^i \mid x)d\theta = 1-\frac{\tau}{2}.$$

# 6. MCMC Method

The posterior distribution of $\theta$ under Bayes and E-bayes technique cannot be reduced analytically to well known distributions and therefore it is impossible to sample directly by standard methods. So, to generate random numbers from this distribution, we use Metropolis-Hastings algorithm in order to obtain estimates of $\theta$ and functions of $\theta$ such as $R(t)$ and $H(t)$ and the credible intervals.

Algorithm of Metropolis-Hastings(MH) method:

1. Take some initial guess of $\theta$ say $\theta^{(0)}$, $M =$ burn-in.
2. Set $j = 1$.
3. Using Metropolis-Hastings, generate $\theta^{(j)}$ from the posterior distribution $\pi(\theta \mid x)$ with $N(\theta^{(j-1)}, \sigma^2)$ as proposal distribution where $\sigma^2$ is the variance of $\theta$ obtained using variance-covariance matrix.
   (i) Calculate the acceptance probability
   $$\phi = \min\left[1, \frac{\pi(\theta|x)}{\pi(\theta^{j-1}|x)}\right] \tag{21}$$
   (ii) Generate $u$ from a $Uniform(0,1)$ distribution.
   (iii) If $u \leq \phi$, accept the proposal and set $\theta^i = \theta^{i-1}$
4. Compute $R(t)$ and $H(t)$ as
   $$R(t)^{(j)} = 1 - \left(\frac{t}{1+t}\right)^{\theta^{(j)}}$$
   $$H(t)^{(j)} = \frac{\theta^{(j)} t^{\theta^{(j)}-1}}{(1+t)^{1+\theta^{(j)}}\left[1 - \left(\frac{t}{1+t}\right)^{\theta^{(j)}}\right]}$$
5. Set $j = j + 1$.
6. Repeat steps 3-5 $N$ times and obtain $\theta^{(j)}$, $R(t)^{(j)}$ and $H(t)^{(j)}$, $j = M + 1, \ldots, N$.

   The Bayes estimate of $\lambda = [\theta, R(t), H(t)]$ under SELF is given by
   $$\lambda = \frac{1}{N-M}\sum_{k=M+1}^{N}\lambda^{(k)}$$
   To compute the credible intervals, sort the samples of the parameters generated by MH algorithm in increasing order and then compute the $100(1-\gamma)\%$ credible intervals of $\theta$, $R(t)$ and $H(t)$ as
   $$\left[\lambda^{(N-M)(\frac{\gamma}{2})}, \lambda^{(N-M)(1-\frac{\gamma}{2})}\right] \tag{22}$$

# 7. Illustration Example and Simulation Results

## 7.1. Illustration Example

In this section, we use the following real data presented in Sana et al.(2023) which is the vinyl chloride data obtained from clean upgrading, monitoring wells in mg/L; this data set was used by Bhaumik et al. (2009). The data is 5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2.

For this data set, the value of K-S test statistic is 0.144 with $p$-value 0.472 which favors the null hypothesis that the data follows the Inverse Pareto Distribution. Figure 1 and Figure 2 indicate that the Inverse Pareto distribution provides a satisfactory fit. According to UHCS, we consider the six cases as following:
Case I ($r = 20, k = 15, T1 = 1.5, T2 = 4.5$), Case II ($r = 25, k = 20, T1 = 1.5, T2 = 4.5$), Case III ($r = 25, k = 23, T1 = 1.5, T2 = 4.5$), Case IV ($r = 30, k = 20, T1 = 1.5, T2 = 4.5$), Case V ($r = 30, k = 25, T1 = 1.5, T2 = 4.5$), and Case VI ($r = 30, k = 28, T1 = 1.5, T2 = 4.5$). For the Bayesian inference, the prior parameters are chosen $(a, b) = (1.5, 1.18)$.

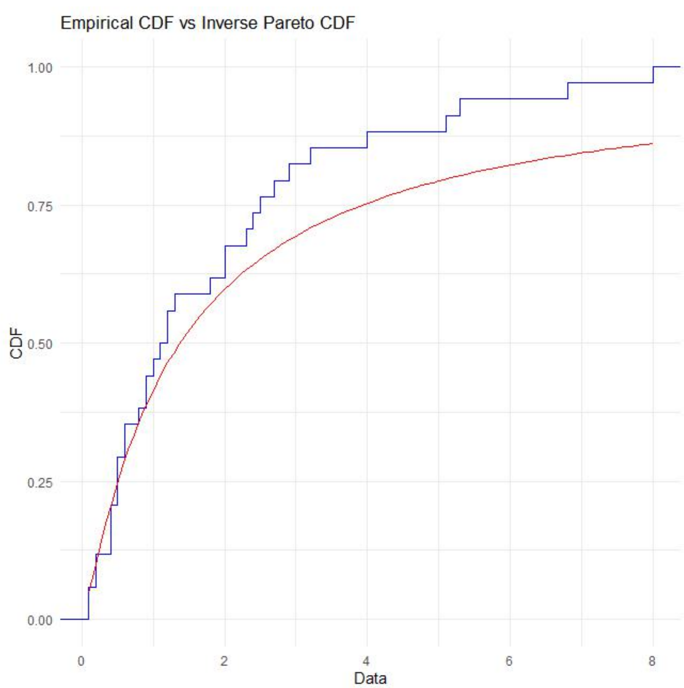


FIGURE 1: Empirical and Fitted CDF.

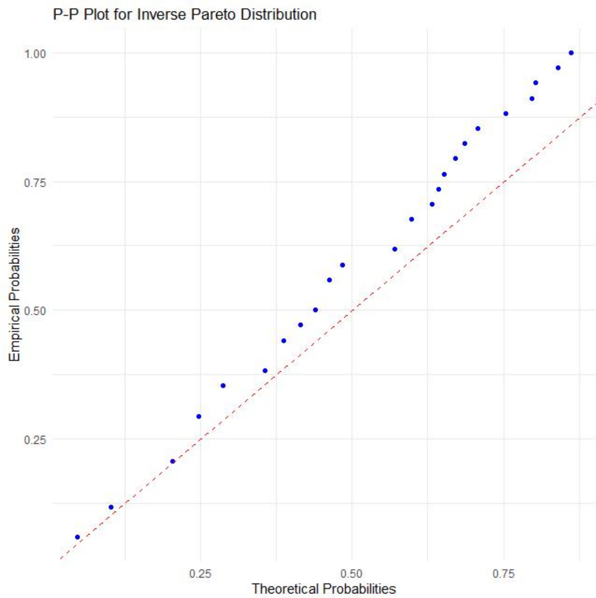


FIGURE 2: pp plot.

TABLE 1: Estimates of $\theta$, Reliability and Hazard rate function for the example.

| Case | $n$ | $r$ | $k$ | $T_1$ | $T_2$ | $\hat{\theta}_M$ | $\hat{\theta}_M$ | $\hat{\theta}_{EB1}$ | $\hat{\theta}_{EB2}$ | $\hat{\theta}_{EB3}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 34 | | | 1.5 | 4.5 | 1.2251 | 1.2186 | 1.2324 | 1.2165 | 1.2214 |
| 2 | | | | | | 1.2233 | 1.2159 | 1.2298 | 1.2146 | 1.2193 |
| 3 | | | | | | 1.2231 | 1.2245 | 1.2298 | 1.2144 | 1.2192 |
| 4 | | | | | | 1.2186 | 1.2292 | 1.2254 | 1.2127 | 1.2135 |
| 5 | | | | | | 1.2403 | 1.2477 | 1.2436 | 1.2250 | 1.2349 |
| 6 | | | | | | 1.2375 | 1.2369 | 1.2422 | 1.2226 | 1.2338 |
| Case | $n$ | $r$ | $k$ | $T_1$ | $T_2$ | $\hat{R}_M$ | $\hat{R}_M$ | $\hat{R}_{EB1}$ | $\hat{R}_{EB2}$ | $\hat{R}_{EB3}$ |
| 1 | 15 | | | 1.5 | 4.5 | 0.7397 | 0.7313 | 0.7348 | 0.7302 | 0.7316 |
| 2 | | | | | | 0.7391 | 0.7303 | 0.7340 | 0.7297 | 0.7310 |
| 3 | | | | | | 0.7382 | 0.7331 | 0.7340 | 0.7296 | 0.7310 |
| 4 | | | | | | 0.7378 | 0.7344 | 0.7329 | 0.7291 | 0.7294 |
| 5 | | | | | | 0.7440 | 0.7388 | 0.7378 | 0.7328 | 0.7519 |
| 6 | | | | | | 0.7432 | 0.7367 | 0.7375 | 0.7321 | 0.7348 |
| Case | $n$ | $r$ | $k$ | $T_1$ | $T_2$ | $\hat{H}_M$ | $\hat{H}_M$ | $\hat{H}_{EB1}$ | $\hat{H}_{EB2}$ | $\hat{H}_{EB3}$ |
| 1 | 15 | | | 1.5 | 4.5 | 0.5747 | 0.5815 | 0.5765 | 0.5826 | 0.5808 |
| 2 | | | | | | 0.5755 | 0.5827 | 0.5775 | 0.5833 | 0.5816 |
| 3 | | | | | | 0.5751 | 0.5792 | 0.5737 | 0.5834 | 0.5816 |
| 4 | | | | | | 0.5773 | 0.5774 | 0.5791 | 0.5841 | 0.5837 |
| 5 | | | | | | 0.5689 | 0.5709 | 0.5724 | 0.5792 | 0.5758 |
| 6 | | | | | | 0.5700 | 0.5744 | 0.5729 | 0.5801 | 0.5762 |

### 7.2. Simulation Results

To compare the performance of all previously proposed estimates for the parameter, reliability and hazard rate function, we simulate the estimates, their MSEs and also the Asymptotic relative efficiencies (ARE) between two estimators. Following are the steps:

1. For Bayesian estimation, for the given set of prior parameters $a$ and $b$, we generate $\theta$ using equation (13).
2. Similarly for E-Bayesian estimation, we generate $\theta$ for a given set of prior parameters $a$, $b$ and $c$ using equation (17).
3. Making use of $\theta$ obtained in Step 1 and Step 2, we generate a sample of size $n$ of upper ordered values from IPD.
4. For different values of $r$, $k$, $T_1$ and $T_2$, the MLE's have been computed as explained in Section 2.
5. For the same values of $r$, $k$, $T_1$ and $T_2$, the Bayesian and E-Bayesian estimates of $\theta$ based on SELF using MCMC method.
6. The above steps (2-5) have been repeated $N$ times.
7. If $\hat{\xi}_j$ is an estimate of $\xi$, based on sample $j$, $j = 1, 2, \ldots, N$, then the average estimate $\bar{\hat{\xi}}$ and the MSE over the $N$ samples have been given, respectively, by $\bar{\hat{\xi}} = \frac{1}{N}\sum_{j=1}^{N}\hat{\xi}_j$ and $MSE(\hat{\xi}_j) = \frac{1}{N}\sum_{j=1}^{N}(\hat{\xi}_j - \xi)^2$.
8. Using Step 7, the quantities MSE($\hat{\theta}$), MSE($\hat{R}$) and MSE($\hat{h}$) have been computed.
9. The approximate confidence and credible intervals have been computed for different values of $r$, $k$, $T_1$ and $T_2$.
10. The results have been summarized in Tables 2 to 7.

The average lengths of the CrIs is obtained by 10,000 sampling from the posterior distribution. The UHCS samples from Inverse Pareto distribution are generated for sample $n = 25$,50 and 75. Using these samples MSEs of all estimators are calculated for $\theta = 0.8$ and 1.75. For reliability function, we set $t = 0.5$. The prior parameters for the Bayesian inference are chosen $(a, b) = (0.4, 0.5)$ when value of true parameter $\theta = 0.8$ and $(a, b) = (0.7, 0.4)$ when value of true parameter $\theta = 1.75$. For E-Bayesian inference the prior parameters are chosen $(a, b, c) = (0.5, 1.4, 1.75)$.

The ARE is helpful for comparing the performance of the Bayes and E-Bayes estimator with the MLE. The ARE is defined as follows:

$$ARE(\hat{\xi}^*, \hat{\xi}) = \frac{MSE(\hat{\xi})}{MSE(\hat{\xi}^*)} \tag{23}$$

where the $MSE(\hat{x}i)$ is the mean square error of the parameter $\hat{\xi}$. If $ARE(\hat{\xi}^*, \hat{\xi}) > 1$, $MSE(\hat{\theta}^*)$ is more efficient than $MSE(\hat{\xi})$. Figure 3 depicts the plot of the MCMC output and histograms of $\theta$ under Bayesian and all three E-Bayesian estimation techniques.

**Under Gamma prior**

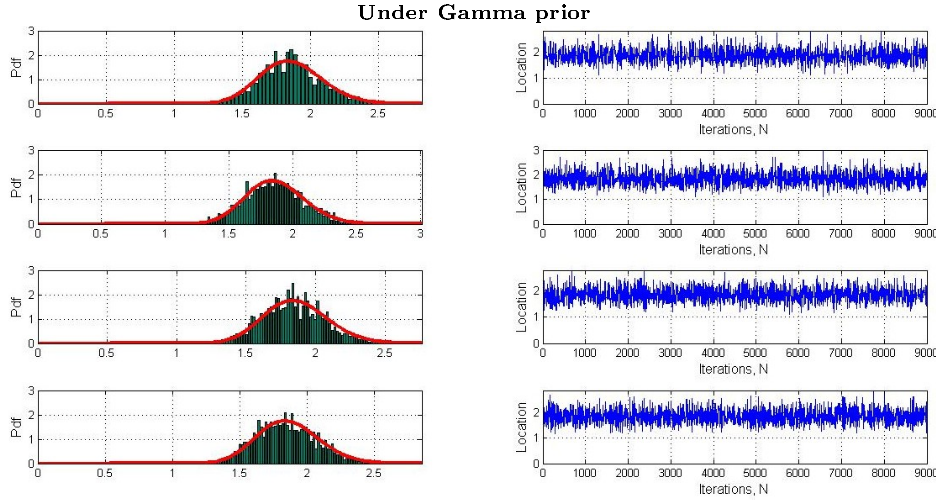


FIGURE 3: Histogram and Simulation number of $\theta(= 1.75)$ by MCMC method using Bayesian(top) and E-Bayesian methods (last 3).

Table 2: Estimates of $\theta$ (MSE) and Asymptotic relative efficiency (ARE) of the estimators.

| ($\theta$,$T_1$,$T_2$) | $(n, r, k)$ | $\hat{\theta}_M$ | $\hat{\theta}_B$ | $\hat{\theta}_{EB1}$ | $\hat{\theta}_{EB2}$ | $\hat{\theta}_{EB3}$ | ARE(B,M) | ARE(EB,M) | ARE(EB,B) |
|---|---|---|---|---|---|---|---|---|---|
| (0.8,0.8,1.6) | (25,20,15) | 0.8612(0.848) | 0.8544(0.794) | 0.8518(0.781) | 0.8524(0.769) | 0.8528(0.764) | 1.1216 | 1.0993 | 1.0293 |
| | (25,20,17) | 0.8503(0.764) | 0.8492(0.726) | 0.8450(0.669) | 0.8477(0.684) | 0.8479(0.691) | 1.0523 | 1.1213 | 1.0655 |
| | (50,40,34) | 0.8491(0.620) | 0.8478(0.587) | 0.8441(0.518) | 0.8387(0.561) | 0.8364(0.551) | 1.0562 | 1.1411 | 1.0803 |
| | (50,40,37) | 0.8380(0.598) | 0.8377(0.516) | 0.8309(0.473) | 0.8287(0.487) | 0.8291(0.502) | 1.1589 | 1.2270 | 1.0588 |
| | (75,65,58) | 0.8371(0.456) | 0.8348(0.393) | 0.8279(0.322) | 0.8238(0.337) | 0.8243(0.328) | 1.1603 | 1.372 | 1.1825 |
| | (75,65,63) | 0.8222(0.357) | 0.8220(0.321) | 0.8197(0.313) | 0.8188(0.320) | 0.8153(0.315) | 1.1121 | 1.1297 | 1.0158 |
| (0.8,1.2,1.9) | (25,20,15) | 1.8095(0.353) | 1.8052(0.304) | 1.7876(0.271) | 1.7838(0.259) | 1.7820(0.223) | 1.1611 | 1.4063 | 1.2111 |
| | (25,20,17) | 1.7866(0.286) | 1.7859(0.285) | 1.7802(0.186) | 1.7668(0.192) | 1.7710(0.124) | 1.0035 | 1.7125 | 1.7065 |
| | (50,40,34) | 1.7863(0.248) | 1.7809(0.241) | 1.7727(0.179) | 1.7717(0.183) | 1.7649(0.175) | 1.0290 | 1.3854 | 1.3463 |
| | (50,40,37) | 1.7719(0.246) | 1.7690(0.235) | 1.7676(0.151) | 1.7558(0.146) | 1.7565(0.165) | 1.0468 | 1.5974 | 1.5259 |
| | (75,65,58) | 1.7629(0.229) | 1.7622(0.223) | 1.7615(0.153) | 1.7620(0.149) | 1.7617(0.150) | 1.0269 | 1.5199 | 1.4801 |
| | (75,65,63) | 1.7615(0.211) | 1.7590(0.209) | 1.7586(0.145) | 1.7592(0.138) | 1.7582(0.135) | 1.0095 | 1.5143 | 1.500 |
| (1.75,0.8,1.6) | (25,20,15) | 0.8544(0.791) | 0.8466(0.700) | 0.8449(0.662) | 0.8441(0.659) | 0.8458(0.668) | 1.1285 | 1.1915 | 1.0558 |
| | (25,20,17) | 0.8527(0.751) | 0.8427(0.673) | 0.8425(0.547) | 0.8402(0.525) | 0.8419(0.543) | 1.1158 | 1.3950 | 1.2501 |
| | (50,40,34) | 0.8316(0.660) | 0.8283(0.619) | 0.8271(0.455) | 0.8253(0.426) | 0.8260(0.450) | 1.0662 | 1.4876 | 1.3952 |
| | (50,40,37) | 0.8303(0.573) | 0.8279(0.551) | 0.8265(0.358) | 0.8245(0.370) | 0.8251(0.355) | 1.0399 | 1.5872 | 1.5263 |
| | (75,65,58) | 0.8256(0.413) | 0.8229(0.352) | 0.8204(0.329) | 0.8188(0.311) | 0.8175(0.307) | 1.1732 | 1.3083 | 1.1151 |
| | (75,65,63) | 0.8155(0.335) | 0.8135(0.320) | 0.8073(0.279) | 0.8082(0.262) | 0.08087(0.288) | 1.0468 | 1.2123 | 1.1580 |
| (1.75,1.2,1.9) | (25,20,15) | 1.7897(0.344) | 1.7843(0.313) | 1.7822(0.267) | 1.7838(0.291) | 1.7809(0.263) | 1.0990 | 1.2570 | 1.1437 |
| | (25,20,17) | 1.7797(0.278) | 1.7776(0.270) | 1.7637(0.182) | 1.7655(0.207) | 1.7690(0.191) | 1.0296 | 1.4379 | 1.3965 |
| | (50,40,34) | 1.7712(0.218) | 1.7681(0.206) | 1.7629(0.153) | 1.7611(0.159) | 1.7601(0.147) | 1.0296 | 1.4248 | 1.3464 |
| | (50,40,37) | 1.7645(0.196) | 1.7629(0.188) | 1.7577(0.132) | 1.7574(0.148) | 1.7535(0.138) | 1.0425 | 1.4067 | 1.3493 |
| | (75,65,58) | 1.7571(0.155) | 1.7531(0.147) | 1.7549(0.120) | 1.7551(0.133) | 1.7524(0.121) | 1.0544 | 1.2433 | 1.1792 |
| | (75,65,63) | 1.7555(0.147) | 1.7517(0.131) | 1.7509(0.111) | 1.7511(0.125) | 1.7514(0.113) | 1.1221 | 1.2636 | 1.1261 |

Table 3: Estimates of $\mathbf{R}$(MSE) and Asymptotic relative efficiency (ARE) of the estimators.

| ($\theta$,$T_1$,$T_2$) | ($n, r, k$) | $\hat{\theta}_M$ | $\hat{\theta}_B$ | $\hat{\theta}_{EB1}$ | $\hat{\theta}_{EB2}$ | $\hat{\theta}_{EB3}$ | ARE(B,M) | ARE(EB,M) | ARE(EB,B) |
|---|---|---|---|---|---|---|---|---|---|
| (0.8,0.8,1.6) | (25,20,15) | 0.6298(0.717) | 0.6224(0.716) | 0.6132(0.710) | 0.6129(0.711) | 0.6121(0.708) | 1.0013 | 1.0103 | 1.0089 |
| | (25,20,17) | 0.6256(0.713) | 0.6180(0.705) | 0.6118(0.701) | 0.6093(0.699) | 0.6088(0.702) | 1.0113 | 1.0176 | 1.0061 |
| | (50,40,34) | 0.6191(0.604) | 0.6147(0.593) | 0.6106(0.590) | 0.6074(0.587) | 0.6063(0.590) | 1.0185 | 1.0254 | 1.0067 |
| | (50,40,37) | 0.6172(0.596) | 0.6116(0.586) | 0.6091(0.579) | 0.6051(0.573) | 0.6048(0.578) | 1.0170 | 1.0333 | 1.0161 |
| | (75,65,58) | 0.6129(0.481) | 0.6090(0.477) | 0.6083(0.465) | 0.6036(0.460) | 0.6029(0.456) | 1.0083 | 1.0449 | 1.0362 |
| | (75,65,63) | 0.6086(0.469) | 0.6079(0.462) | 0.6061(0.454) | 0.5999(0.449) | 0.6014(0.442) | 1.0151 | 1.0461 | 1.0304 |
| (0.8,1.2,1.9) | (25,20,15) | 0.8721(0.230) | 0.8718(0.218) | 0.8693(0.205) | 0.8686(0.196) | 0.8691(0.191) | 1.0550 | 1.1655 | 1.1047 |
| | (25,20,17) | 0.8698(0.222) | 0.8689(0.209) | 0.8671(0.189) | 0.8674(0.176) | 0.8685(0.170) | 1.0622 | 1.2448 | 1.1719 |
| | (50,40,34) | 0.8681(0.201) | 0.8643(0.184) | 0.8639(0.171) | 0.8658(0.167) | 0.8669(0.169) | 1.0923 | 1.1893 | 1.0887 |
| | (50,40,37) | 0.8625(0.185) | 0.8630(0.178) | 0.8619(0.160) | 0.8619(0.154) | 0.8619(0.159) | 1.0393 | 1.1734 | 1.1290 |
| | (75,65,58) | 0.8610(0.152) | 0.8604(0.140) | 0.8595(0.131) | 0.8600(0.122) | 0.8597(0.138) | 1.0857 | 1.1662 | 1.0741 |
| | (75,65,63) | 0.8589(0.147) | 0.8580(0.133) | 0.8583(0.123) | 0.8585(0.117) | 0.8579(0.129) | 1.1052 | 1.1951 | 1.0813 |
| (1.75,0.8,1.6) | (25,20,15) | 0.6243(0.714) | 0.6287(0.710) | 0.6116(0.692) | 0.6120(0.688) | 0.6078(0.687) | 1.0056 | 1.0362 | 1.0304 |
| | (25,20,17) | 0.6168(0.705) | 0.6145(0.695) | 0.6078(0.649) | 0.6054(0.636) | 0.6033(0.650) | 1.0143 | 1.0930 | 1.0775 |
| | (50,40,34) | 0.6150(0.598) | 0.6118(0.591) | 0.6031(0.575) | 0.6047(0.543) | 0.6018(0.547) | 1.0118 | 1.0774 | 1.0648 |
| | (50,40,37) | 0.6106(0.590) | 0.6089(0.577) | 0.5991(0.552) | 0.6012(0.559) | 0.5987(0.565) | 1.0225 | 1.0560 | 1.0328 |
| | (75,65,58) | 0.5998(0.465) | 0.5975(0.458) | 0.5936(0.434) | 0.5989(0.439) | 0.5966(0.441) | 1.0152 | 1.0616 | 1.0456 |
| | (75,65,63) | 0.5974(0.450) | 0.5968(0.440) | 0.5917(0.422) | 0.5963(0.426) | 0.5938(0.433) | 1.0227 | 1.0538 | 1.0304 |
| (1.75,1.2,1.9) | (25,20,15) | 0.8741(0.200) | 0.8724(0.193) | 0.8693(0.187) | 0.8677(0.180) | 0.8676(0.176) | 1.0362 | 1.1049 | 1.0662 |
| | (25,20,17) | 0.8720(0.189) | 0.8704(0.185) | 0.8680(0.177) | 0.8675(0.171) | 0.8654(0.166) | 1.0216 | 1.1031 | 1.0797 |
| | (50,40,34) | 0.8686(0.167) | 0.8675(0.159) | 0.8633(0.151) | 0.8621(0.147) | 0.8639(0.143) | 1.0503 | 1.1360 | 1.0816 |
| | (50,40,37) | 0.8655(0.158) | 0.8651(0.153) | 0.8619(0.143) | 0.8615(0.140) | 0.8627(0.138) | 1.0326 | 1.1259 | 1.0902 |
| | (75,65,58) | 0.8630(0.149) | 0.8618(0.141) | 0.8602(0.130) | 0.8598(0.131) | 0.8611(0.128) | 1.0567 | 1.1491 | 1.0874 |
| | (75,65,63) | 0.8610(0.130) | 0.8591(0.125) | 0.8582(0.117) | 0.8577(0.124) | 0.8593(0.119) | 1.0400 | 1.0833 | 1.0416 |

Table 4: Estimates of $\mathbf{H}$(MSE) and Asymptotic relative efficiency (ARE) of the estimators.

| $(\theta,T_1,T_2)$ | $(n,r,k)$ | $\hat{\theta}_M$ | $\hat{\theta}_B$ | $\hat{\theta}_{EB1}$ | $\hat{\theta}_{EB2}$ | $\hat{\theta}_{EB3}$ | ARE(B,M) | ARE(EB,M) | ARE(EB,B) |
|---|---|---|---|---|---|---|---|---|---|
| (0.8,0.8,1.6) | (25,20,15) | 0.7149(0.761) | 0.7155(0.739) | 0.7238(0.690) | 0.7232(0.696) | 0.7276(0.692) | 1.0297 | 1.0986 | 1.0669 |
| | (25,20,17) | 0.7181(0.729) | 0.7206(0.710) | 0.7279(0.677) | 0.7271(0.683) | 0.7290(0.679) | 1.0267 | 1.0725 | 1.0446 |
| | (50,40,34) | 0.7244(0.625) | 0.7263(0.618) | 0.7314(0.586) | 0.7312(0.590) | 0.7361(0.583) | 1.0113 | 1.0659 | 1.0540 |
| | (50,40,37) | 0.7290(0.603) | 0.7301(0.591) | 0.7366(0.553) | 0.7386(0.558) | 0.7396(0.551) | 1.0203 | 1.0884 | 1.0667 |
| | (75,65,58) | 0.7311(0.520) | 0.7337(0.478) | 0.7394(0.417) | 0.7401(0.422) | 0.7412(0.415) | 1.0878 | 1.2440 | 1.1435 |
| | (75,65,63) | 0.7401(0.507) | 0.7410(0.453) | 0.7417(0.411) | 0.7433(0.416) | 0.7458(0.408) | 1.1192 | 1.2315 | 1.1004 |
| (0.8,1.2,1.9) | (25,20,15) | 0.3621(0.354) | 0.3660(0.344) | 0.3679(0.280) | 0.3711(0.291) | 0.3702(0.282) | 1.0290 | 1.2450 | 1.2098 |
| | (25,20,17) | 0.3652(0.279) | 0.3705(0.254) | 0.3713(0.240) | 0.3754(0.255) | 0.3741(0.251) | 1.0984 | 1.1220 | 1.0214 |
| | (50,40,34) | 0.3720(0.184) | 0.3782(0.171) | 0.3796(0.150) | 0.3817(0.158) | 0.3801(0.155) | 1.0760 | 1.1922 | 1.1080 |
| | (50,40,37) | 0.3786(0.165) | 0.3805(0.161) | 0.3833(0.136) | 0.3891(0.127) | 0.3883(0.126) | 1.0248 | 1.2725 | 1.2417 |
| | (75,65,58) | 0.3829(0.140) | 0.3898(0.132) | 0.3929(0.129) | 0.3943(0.120) | 0.3962(0.119) | 1.0606 | 1.1413 | 1.0761 |
| | (75,65,63) | 0.3895(0.131) | 0.3931(0.124) | 0.3955(0.119) | 0.3963(0.112) | 0.3988(0.108) | 1.0564 | 1.1592 | 1.0973 |
| (1.75,0.8,1.6) | (25,20,15) | 0.7124(0.758) | 0.7136(0.738) | 0.7220(0.683) | 0.7262(0.685) | 0.7235(0.687) | 1.0271 | 1.1065 | 1.0773 |
| | (25,20,17) | 0.7170(0.717) | 0.7185(0.705) | 0.7247(0.672) | 0.7285(0.677) | 0.7255(0.670) | 1.0170 | 1.0653 | 1.0475 |
| | (50,40,34) | 0.7205(0.620) | 0.7218(0.615) | 0.7298(0.582) | 0.7309(0.587) | 0.7317(0.580) | 1.0081 | 1.0274 | 1.0548 |
| | (50,40,37) | 0.7271(0.599) | 0.7280(0.590) | 0.7325(0.551) | 0.7356(0.556) | 0.7368(0.554) | 1.0152 | 1.0818 | 1.0656 |
| | (75,65,58) | 0.7388(0.518) | 0.7399(0.475) | 0.7438(0.416) | 0.7479(0.420) | 0.7488(0.419) | 1.0905 | 1.2382 | 1.1354 |
| | (75,65,63) | 0.7441(0.504) | 0.7431(0.447) | 0.7509(0.398) | 0.7516(0.407) | 0.7524(0.403) | 1.1275 | 1.2516 | 1.1101 |
| (1.75,1.2,1.9) | (25,20,15) | 0.3649(0.348) | 0.3662(0.327) | 0.3712(0.275) | 0.3736(0.283) | 0.3725(0.280) | 1.0642 | 1.2458 | 1.1706 |
| | (25,20,17) | 0.3681(0.257) | 0.3694(0.251) | 0.3757(0.238) | 0.3773(0.256) | 0.3740(0.249) | 1.0239 | 1.0377 | 1.0134 |
| | (50,40,34) | 0.3722(0.170) | 0.3769(0.164) | 0.3846(0.146) | 0.3859(0.158) | 0.3866(0.140) | 1.0365 | 1.1486 | 1.1081 |
| | (50,40,37) | 0.3802(0.166) | 0.3811(0.150) | 0.3863(0.139) | 0.3879(0.125) | 0.3871(0.122) | 1.1066 | 1.2902 | 1.1658 |
| | (75,65,58) | 0.3863(0.137) | 0.3883(0.128) | 0.3888(0.122) | 0.3916(0.121) | 0.3896(0.118) | 1.0703 | 1.1385 | 1.0637 |
| | (75,65,63) | 0.3916(0.130) | 0.3938(0.123) | 0.3961(0.119) | 0.3972(0.110) | 0.3969(0.111) | 1.0569 | 1.1470 | 1.0853 |

TABLE 5: 95 % Confidence and Credible Intervals of the estimates of $\theta$.

| $(\theta,T_1,T_2)$ | $(n,r,k)$ | $\hat{\theta}_M$ | | $\hat{\theta}_B$ | | $\hat{\theta}_{EB1}$ | | $\hat{\theta}_{EB2}$ | | $\hat{\theta}_{EB3}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | L | U | L | U | L | U | L | U | L | U |
| (0.8,0.8,1.6) | (25,20,15) | 0.5002 | 1.2823 | 0.5682 | 1.3667 | 0.5467 | 1.2934 | 0.5402 | 1.2929 | 0.5314 | 1.2884 |
| | (25,20,17) | 0.5162 | 1.1820 | 0.5521 | 1.3152 | 0.5470 | 1.2035 | 0.5455 | 1.1975 | 0.5413 | 1.1976 |
| | (50,40,34) | 0.6677 | 1.1340 | 0.6329 | 1.1197 | 0.6610 | 1.1476 | 0.6492 | 1.1449 | 0.6302 | 1.1458 |
| | (50,40,37) | 0.6449 | 1.1246 | 0.6204 | 1.0964 | 0.6400 | 1.1046 | 0.6495 | 1.1135 | 0.6179 | 1.1272 |
| | (75,65,58) | 0.6984 | 1.1026 | 0.6906 | 1.0953 | 0.7162 | 1.1122 | 0.7044 | 1.1123 | 0.6978 | 1.1304 |
| | (75,65,63) | 0.7134 | 1.1308 | 0.7304 | 1.1369 | 0.7331 | 1.1374 | 0.7240 | 1.1498 | 0.7135 | 1.1450 |
| (0.8,1.2,1.9) | (25,20,15) | 0.9865 | 2.5365 | 1.0973 | 2.6016 | 1.0051 | 2.3506 | 0.9943 | 2.3812 | 1.0192 | 2.4100 |
| | (25,20,17) | 1.3005 | 2.3185 | 1.3336 | 2.3676 | 1.3080 | 2.3275 | 1.3151 | 2.3100 | 1.2998 | 2.3328 |
| | (50,40,34) | 1.2828 | 2.2899 | 1.3221 | 2.3192 | 1.2915 | 2.3000 | 1.2934 | 2.2858 | 1.2811 | 2.3140 |
| | (50,40,37) | 1.2451 | 2.2607 | 1.2844 | 2.3176 | 1.2801 | 2.2982 | 1.2985 | 2.2900 | 1.2850 | 2.3102 |
| | (75,65,58) | 1.4438 | 2.2999 | 1.4737 | 2.3381 | 1.4451 | 2.3381 | 1.4367 | 2.2980 | 1.4514 | 2.3078 |
| | (75,65,63) | 1.4556 | 2.3176 | 1.4885 | 2.3585 | 1.4537 | 2.3488 | 1.4593 | 2.3100 | 1.4774 | 2.3268 |
| (1.75,0.8,1.6) | (25,20,15) | 0.5022 | 1.2867 | 0.5706 | 1.3695 | 0.5483 | 1.3176 | 0.5447 | 1.2970 | 0.5368 | 1.2930 |
| | (25,20,17) | 0.6977 | 1.1003 | 0.6857 | 1.0876 | 0.7074 | 1.1079 | 0.7082 | 1.1080 | 0.6977 | 0.1029 |
| | (50,40,34) | 0.6029 | 1.0774 | 0.5984 | 1.0582 | 0.6209 | 1.0924 | 0.6127 | 1.0767 | 0.5951 | 1.0867 |
| | (50,40,37) | 0.6418 | 1.1495 | 0.6711 | 1.1541 | 0.6654 | 1.1821 | 0.6602 | 1.1502 | 0.6362 | 1.1813 |
| | (75,65,58) | 0.7141 | 1.1319 | 0.7307 | 1.1369 | 0.7341 | 1.1415 | 0.7240 | 1.1510 | 0.7135 | 1.1456 |
| | (75,65,63) | 0.7083 | 1.1228 | 0.7207 | 1.1351 | 0.7237 | 1.1248 | 0.7181 | 1.1413 | 0.7104 | 1.1435 |
| (1.75,1.2,1.9) | (25,20,15) | 0.9931 | 2.5493 | 1.0989 | 2.6041 | 1.0076 | 2.3592 | 1.0055 | 2.3925 | 1.0219 | 2.4225 |
| | (25,20,17) | 1.2806 | 2.2794 | 1.3195 | 2.3070 | 1.2909 | 2.2957 | 1.2850 | 2.2861 | 1.2795 | 2.2724 |
| | (50,40,34) | 1.2794 | 2.2758 | 1.3179 | 2.3065 | 1.2836 | 2.2860 | 1.2901 | 2.2705 | 1.2750 | 2.2862 |
| | (50,40,37) | 1.2846 | 2.2839 | 1.3195 | 2.3194 | 1.2900 | 2.2913 | 1.2960 | 2.2766 | 1.2834 | 2.2962 |
| | (75,65,58) | 1.4461 | 2.2978 | 1.4761 | 2.3381 | 1.4498 | 2.3307 | 1.4435 | 2.2880 | 1.4529 | 2.3062 |
| | (75,65,63) | 1.4332 | 2.2788 | 1.4644 | 2.2918 | 1.4304 | 2.3116 | 1.4298 | 2.2582 | 1.4407 | 2.2861 |

TABLE 6: 95 % Confidence and Credible Intervals of the estimates of $R$.

| $(\theta, T_1, T_2)$ | $(n, r, k)$ | $\hat{R}_M$ | | $\hat{R}_B$ | | $\hat{R}_{EB1}$ | | $\hat{R}_{EB2}$ | | $\hat{R}_{EB3}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | L | U | L | U | L | U | L | U | L | U |
| (0.8,0.8,1.6) | (25,20,15) | 0.4629 | 0.7857 | 0.4643 | 0.7771 | 0.4515 | 0.7585 | 0.4476 | 0.7583 | 0.4422 | 0.7571 |
| | (25,20,17) | 0.4627 | 0.7504 | 0.4621 | 0.7825 | 0.4515 | 0.7333 | 0.4508 | 0.7316 | 0.4482 | 0.7317 |
| | (50,40,34) | 0.5198 | 0.7123 | 0.5160 | 0.7202 | 0.5162 | 0.7165 | 0.5099 | 0.7157 | 0.4996 | 0.7160 |
| | (50,40,37) | 0.5076 | 0.7123 | 0.5160 | 0.7202 | 0.5162 | 0.7034 | 0.5081 | 0.7057 | 0.4927 | 0.7101 |
| | (75,65,58) | 0.5357 | 0.7021 | 0.5417 | 0.7084 | 0.5447 | 0.7053 | 0.5388 | 0.7053 | 0.5354 | 0.7111 |
| | (75,65,63) | 0.5536 | 0.7201 | 0.5519 | 0.7132 | 0.5531 | 0.7133 | 0.5486 | 0.7172 | 0.5433 | 0.7157 |
| (0.8,1.2,1.9) | (25,20,15) | 0.7326 | 0.9785 | 0.7004 | 0.9426 | 0.6685 | 0.9244 | 0.6645 | 0.9269 | 0.6736 | 0.9291 |
| | (25,20,17) | 0.7864 | 0.9396 | 0.7689 | 0.9258 | 0.7623 | 0.9224 | 0.7642 | 0.9209 | 0.7602 | 0.9229 |
| | (50,40,34) | 0.7941 | 0.9479 | 0.7785 | 0.9327 | 0.7706 | 0.9309 | 0.7711 | 0.9297 | 0.7679 | 0.9321 |
| | (50,40,37) | 0.7874 | 0.9396 | 0.7691 | 0.9258 | 0.7629 | 0.9225 | 0.7651 | 0.9209 | 0.7615 | 0.9226 |
| | (75,65,58) | 0.8119 | 0.9322 | 0.8022 | 0.9233 | 0.7955 | 0.9233 | 0.7936 | 0.9199 | 0.7969 | 0.9207 |
| | (75,65,63) | 0.8145 | 0.9337 | 0.8050 | 0.9250 | 0.7975 | 0.9242 | 0.7987 | 0.9209 | 0.8027 | 0.9224 |
| (1.75,0.8,1.6) | (25,20,15) | 0.4643 | 0.7870 | 0.4657 | 0.7778 | 0.4525 | 0.7648 | 0.4503 | 0.7595 | 0.4455 | 0.7585 |
| | (25,20,17) | 0.5353 | 0.7014 | 0.5391 | 0.7058 | 0.5403 | 0.7039 | 0.5407 | 0.7039 | 0.5354 | 0.7081 |
| | (50,40,34) | 0.4843 | 0.6938 | 0.4958 | 0.6991 | 0.4945 | 0.6988 | 0.4899 | 0.6936 | 0.4799 | 0.6969 |
| | (50,40,37) | 0.4923 | 0.6962 | 0.4920 | 0.6852 | 0.4890 | 0.6930 | 0.4863 | 0.6835 | 0.4737 | 0.6927 |
| | (75,65,58) | 0.5540 | 0.7204 | 0.5519 | 0.7132 | 0.5536 | 0.7146 | 0.5486 | 0.7176 | 0.5433 | 0.7160 |
| | (75,65,63) | 0.5410 | 0.7075 | 0.5369 | 0.7026 | 0.5384 | 0.6993 | 0.5357 | 0.7045 | 0.5318 | 0.7052 |
| (1.75,1.2,1.9) | (25,20,15) | 0.7350 | 0.9792 | 0.7009 | 0.9427 | 0.6694 | 0.9251 | 0.6686 | 0.9278 | 0.6746 | 0.9301 |
| | (25,20,17) | 0.7808 | 0.9361 | 0.7653 | 0.9207 | 0.7578 | 0.9197 | 0.7562 | 0.9188 | 0.7547 | 0.9176 |
| | (50,40,34) | 0.7804 | 0.9359 | 0.7649 | 0.9206 | 0.7559 | 0.9188 | 0.7576 | 0.9174 | 0.7535 | 0.9188 |
| | (50,40,37) | 0.7818 | 0.9364 | 0.7653 | 0.9218 | 0.7576 | 0.9193 | 0.7592 | 0.9180 | 0.7558 | 0.9197 |
| | (75,65,58) | 0.8122 | 0.9319 | 0.8023 | 0.9233 | 0.7966 | 0.9227 | 0.7952 | 0.9199 | 0.7973 | 0.9206 |
| | (75,65,63) | 0.8093 | 0.9303 | 0.8002 | 0.9196 | 0.7922 | 0.9211 | 0.7921 | 0.9163 | 0.7945 | 0.9188 |

TABLE 7: 95 % Confidence and Credible Intervals of the estimates of $H$.

| $(\theta,T_1,T_2)$ | $(n,r,k)$ | $\hat{H}_M$ | | $\hat{H}_B$ | | $\hat{H}_{EB1}$ | | $\hat{H}_{EB2}$ | | $\hat{H}_{EB3}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | L | U | L | U | L | U | L | U | L | U |
| (0.8,0.8,1.6) | (25,20,15) | 0.5367 | 0.8931 | 0.5224 | 0.8732 | 0.5490 | 0.8854 | 0.5476 | 0.8889 | 0.5509 | 0.8936 |
| | (25,20,17) | 0.5794 | 0.8891 | 0.5239 | 0.8611 | 0.5832 | 0.8852 | 0.5855 | 0.8859 | 0.5854 | 0.8883 |
| | (50,40,34) | 0.6107 | 0.8224 | 0.6100 | 0.8334 | 0.6052 | 0.8258 | 0.6063 | 0.8313 | 0.6059 | 0.8405 |
| | (50,40,37) | 0.6145 | 0.8334 | 0.6198 | 0.8402 | 0.6216 | 0.8365 | 0.6190 | 0.8335 | 0.6134 | 0.8479 |
| | (75,65,58) | 0.6235 | 0.8069 | 0.6220 | 0.8062 | 0.6195 | 0.7981 | 0.6195 | 0.8040 | 0.6121 | 0.8073 |
| | (75,65,63) | 0.6072 | 0.7946 | 0.6095 | 0.7909 | 0.6093 | 0.7893 | 0.6043 | 0.7942 | 0.6062 | 0.7994 |
| (0.8,1.2,1.9) | (25,20,15) | 0.1763 | 0.6164 | 0.2108 | 0.6251 | 0.2560 | 0.6645 | 0.2503 | 0.6690 | 0.2449 | 0.6584 |
| | (25,20,17) | 0.2425 | 0.5233 | 0.2529 | 0.5342 | 0.2608 | 0.5435 | 0.2643 | 0.5410 | 0.2596 | 0.5463 |
| | (50,40,34) | 0.2465 | 0.5266 | 0.2496 | 0.5248 | 0.2542 | 0.5360 | 0.2568 | 0.5351 | 0.2515 | 0.5398 |
| | (50,40,37) | 0.2422 | 0.5217 | 0.2526 | 0.5337 | 0.2599 | 0.5428 | 0.2643 | 0.5397 | 0.2600 | 0.5447 |
| | (75,65,58) | 0.2523 | 0.4797 | 0.2587 | 0.4849 | 0.2587 | 0.4943 | 0.2667 | 0.4979 | 0.2647 | 0.4929 |
| | (75,65,63) | 0.2487 | 0.4756 | 0.2547 | 0.4804 | 0.2563 | 0.4919 | 0.2643 | 0.4902 | 0.2609 | 0.4836 |
| (1.75,0.8,1.6) | (25,20,15) | 0.5349 | 0.8919 | 0.5203 | 0.8726 | 0.5399 | 0.8845 | 0.5476 | 0.8863 | 0.5490 | 0.8907 |
| | (25,20,17) | 0.6244 | 0.8073 | 0.6252 | 0.8087 | 0.6213 | 0.8025 | 0.6208 | 0.8021 | 0.6160 | 0.8073 |
| | (50,40,34) | 0.6333 | 0.8545 | 0.6360 | 0.8520 | 0.6276 | 0.8463 | 0.6341 | 0.8506 | 0.6300 | 0.8593 |
| | (50,40,37) | 0.6395 | 0.8681 | 0.6435 | 0.8606 | 0.6337 | 0.8635 | 0.6462 | 0.8661 | 0.6340 | 0.8782 |
| | (75,65,58) | 0.6568 | 0.8442 | 0.6595 | 0.8909 | 0.6573 | 0.8384 | 0.6538 | 0.8442 | 0.6559 | 0.8994 |
| | (75,65,63) | 0.6605 | 0.8572 | 0.6602 | 0.8589 | 0.6644 | 0.8542 | 0.6577 | 0.8571 | 0.6569 | 0.8510 |
| (1.75,1.2,1.9) | (25,20,15) | 0.1739 | 0.6132 | 0.2107 | 0.6250 | 0.2546 | 0.6632 | 0.2481 | 0.6642 | 0.2424 | 0.6567 |
| | (25,20,17) | 0.2509 | 0.5313 | 0.2645 | 0.5394 | 0.2672 | 0.5499 | 0.2691 | 0.5520 | 0.2716 | 0.5542 |
| | (50,40,34) | 0.2517 | 0.5319 | 0.2650 | 0.5400 | 0.2691 | 0.5526 | 0.2722 | 0.5498 | 0.2691 | 0.5558 |
| | (50,40,37) | 0.2500 | 0.5298 | 0.2623 | 0.5394 | 0.2681 | 0.5502 | 0.2711 | 0.5480 | 0.2671 | 0.5524 |
| | (75,65,58) | 0.2529 | 0.4741 | 0.2487 | 0.4845 | 0.2599 | 0.4934 | 0.2667 | 0.4956 | 0.2651 | 0.4914 |
| | (75,65,63) | 0.2569 | 0.4836 | 0.2673 | 0.4875 | 0.2637 | 0.5000 | 0.2746 | 0.5003 | 0.2690 | 0.4961 |

## 8. Conclusion

In this paper, we derived Classical, Bayesian and E-Bayesian estimators of the the unknown parameter, reliability and hazard rate function of Inverse Pareto distribution when the data is collected under the unified hybrid censored data. The MLEs, AREs, CIs and CrIs are also discussed. The techniques are applied to real life data set.

The Bayes estimators perform better than the MLEs in most of the cases while the E-Bayesian estimators perform better than both the MLE and Bayes estimators in all the cases.

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