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Efficient Estimation of Population Mean of a Sensitive Variable Under ORRT Models Using Two-Phase Sampling

Estimación eficiente de la media poblacional de una variable sensible según modelos ORRT utilizando un muestreo de dos fases

Sunil Kumar[1,a], Chanda Rani[1,b], Sanam Preet Kour[2,c], Housila P. Singh[3,d], Rakesh Chib[1,e]

[1]Department of Statistics, University of Jammu, J&K, India
[2]Department of Clinical Research, Sharda School of Allied Health Sciences, Sharda University, Uttar Pradesh, India
[3]School of Studies in Statistics, Vikram University Ujjain, Madhya Pradesh, India

Abstract

During human survey, participants are asked highly personal questions about a sensitive variable. This paper focus on estimation of population mean of sensitive study variable in the existence of non-response and measurement error under Optional Randomized Response Technique model using two-phase sampling. The bias and mean squared error of the proposed and considered family of estimators have been derived up to first order of approximation. Further, the properties of the proposed estimator have been discussed and efficiency conditions have been derived. To demonstrate the theoretical findings, simulation study is conducted under various conditions using data set from hypothetical population and it is clear that our proposed family of estimator is always better than the other considered family of estimators.

***Key words*:** Study variable; Auxiliary variable(s); Non-response; Measurement error; Mean squared error (MSE); Percent relative efficiency; Optional Randomized Response Technique (ORRT).

[a]Ph.D. E-mail: sunilbhougal06@gmail.com
[b]Ph.D. E-mail: chandaheer65@gmail.com
[c]Ph.D. E-mail: sanam.kour@sharda.ac.in
[d]Ph.D. E-mail: hpsujn@gmail.com
[e]Ph.D. E-mail: jurakeshchib@gmail.com

Resumen

Durante una encuesta a personas, se les hacen preguntas muy personales a los participantes sobre una variable sensible. Este artículo se centra en la estimación de la media poblacional de la variable sensible del estudio en la existencia de falta de respuesta y error de medición bajo el modelo de técnica de respuesta aleatoria opcional utilizando un muestreo de dos fases. El sesgo y el error cuadrático medio de la familia de estimadores propuesta y considerada se han derivado hasta el primer orden de aproximación. Además, se han discutido las propiedades del estimador propuesto y se han derivado las condiciones de eficiencia. Para demostrar los hallazgos teóricos, se lleva a cabo un estudio de simulación en varias condiciones utilizando un conjunto de datos de una población hipotética y está claro que nuestra familia de estimadores propuesta es siempre mejor que la otra familia de estimadores considerada.

***Palabras clave*:** Variable de estudio; Variable(s) auxiliar(es); Falta de respuesta; Error de medición; Error cuadrático medio (EMC); Porcentaje de eficiencia relativa; Técnica de respuesta aleatoria opcional (ORRT).

1. Introduction

During a sensitive survey, there is a chance of receiving erroneous or incomplete data. Each survey question that causes a respondent to feel uncomfortable or ashamed to answer is considered as sensitive. For example, inquiries concerning illegal activities such as drug or alcohol abuse, criminal activity or drunkenness, as well as financial information like income, etc. It is typically challenging to ask someone a sensitive topic because they can be embarrassed to answer or might give you a socially acceptable response that covers up their situation when an interviewer asks about a sensitive topic, the social desirability bias (SDB) is evident.

To tackle the SDB, the Randomized Response Technique (RRT) was first established by Warner (1965) then Greenberg et al. (1971), expanded Warner's original RRT to include quantitative response scenarios. Moreover, Eichhorn & Hayre (1983) propose a multiplicative scrambling RRT model. In addition, Gupta et al. (2002) revised the multiplicative model of Eichhorn & Hayre (1983) and implemented an ORRT model, allowing researchers to ascertain the sensitivity level in addition to the variables(s) mean. Estimating the relatively small population mean of a sensitive variable has become the topic of studies by Diana & Perri (2011), Gupta et al. (2012), Gupta et al. (2014), Gupta et al. (2018), Zhang et al. (2018), Waseem et al. (2021), Tiwari & Pandey (2022), and others.

Non-response and measurement error in human-related surveys are real problems that almost arise in all surveys. It occurs when interviewers neglect to follow up with non-respondents or when respondents refuse to participate in the survey. In order to address this problem, Hansen & Hurwitz (1946) were the first to develop a plan for selecting a subsample from non-respondents and obtaining data by a costly technique like a personal interview and postal interview questions. Additional study has been conducted on measurement error using auxiliary data by other researchers, such as Allen et al. (2003), Kumar et al. (2015) and others. Furthermore, Khalil et al. (2021) have suggested the mean estimation of the sensitive variable in the presence of measurement error if the survey question is sensitive. Also, Azeem (2014), Azeem & Hanif (2017), Singh & Sharma (2015), Kumar et al. (2023), Kumar & Kour (2022) etc, studied the problem of non-response and measurement error when both occurs at the same time and in case when the survey question is sensitive in nature then Zhang et al. (2021), Kumar & Kour (2022), Kumar et al. (2023), Azeem et al. (2024) studied the mean estimation of sensitive variable in presence non-response and measurement error simultaneously under ORRT model.

In a situation, especially treatment response times, can be sensitive and potentially impact patient outcomes or treatment decisions, its essential to handle this data with care. So, to estimate the population mean of sensitive variable under above situation, we suggest a family of estimators under the simultaneous presence of non-response and measurement error via ORRT model. Section 2 discusses the notations and ORRT model with some existing estimators under two-phase sampling scheme. Section 3 addressed the proposed family of estimators and their properties. In Section 4, we obtain the efficiency conditions of the estimators. To validate the theoretical findings, a simulation study is carried out in section 5 followed by conclusion in Section 6.

2. Notations and ORRT Model under Two-Phase Sampling

Suppose that $\omega = \omega_1, \omega_2, \omega_3, \ldots, \omega_N$ be a finite population of size N then in the first phase, a random sample of size n' is selected from the population ω and in the second phase a sub sample of size n is taken from n' under two-phase sampling. Let Y be the sensitive study variable with mean $\bar{Y}$ and variance S_y^2 and (X_1, X_2) be two non-sensitive auxiliary variables with mean $(\bar{X}_1, \bar{X}_2)$ and variances $(S_{x_1}^2, S_{x_2}^2)$, respectively. Also, T and S be two scrambling variables with mean $(\bar{T}, \bar{S})$ and variances S_t^2 and S_s^2. Let P state the probability that respondents find the question sensitive. If the respondent believes that the question to be sensitive, they are instructed to submit a scrambled response for the sensitive variable (Y) and a correct response in all other cases.

Motivated by Gupta et al. (2002), Zhang et al. (2021) and Kumar & Kour (2022), the ORRT model is given by

$$Z = \begin{cases} Y & \text{with probability } (1-P) \\ TY + S & \text{with probability } P. \end{cases} \tag{1}$$

Therefore, the mean and variance of Z are given as

$$E(Z) = E(Y)$$

and

$$Var(Z) = S_y^2 + S_s^2 P + S_t^2 (S_y^2 + \bar{Y}^2) P.$$

The randomized linear model is given by

$$Z = (TY + S)W + Y(1 - W),$$

where $W \sim Bernoulli(P)$ with $E(W)$=P, $Var(W)$=$P(1-P)$ and $E(W^2)$=P. And the mean and variance of Randomized linear model is $E_R(Z) = (\bar{T}P + 1 - P)Y + \bar{S}P$ and $Var_R(Z) = (Y^2 S_t^2 + S_s^2)P$.

Therefore, the ORRT model outperforms the non-optional RRT model significantly when the variance increases as the probability increases.

Let us take a transformation of the randomized response as $\bar{y}_i$, whose expectation under the randomization mechanism is the true response y_i and is given as

$$\hat{y}_i^* = \frac{z_i - \bar{S}P}{\bar{T}P + 1 - P},$$

with $E(\hat{y}_i^*) = y_i$ and $Var(\hat{y}_i^*) = \frac{(y_i^2 S_t^2 + S_s^2)P}{(\bar{T}P+1-P)^2}$.

- Hansen & Hurwitz (1946) were the first to gather data by mail survey in first attempt and personal interview in second attempt by taking a sub-sample. The modified Hansen & Hurwitz (1946) estimator in the existence of non-response under ORRT model is given by

$$\hat{\zeta}_y^* = \hat{\bar{y}}^* = w_1 \bar{y}_1^* + w_2 \hat{\bar{y}}_{2s}^*,$$

where $\hat{\bar{y}}_{2s}^* = \frac{1}{n_s} \sum_{i=1}^{n_s} \hat{y}_i^*$, $w_1 = \frac{n_1}{n}$ and $w_2 = \frac{n_2}{n}$.
For auxiliary variables X_1 and X_2, the estimators are

$$\bar{x}_1^* = w_1 \bar{x}_{11}^* + w_2 \bar{x}_{12}^*$$

and

$$\bar{x}_2^* = w_1 \bar{x}_{21}^* + w_2 \bar{x}_{22}^*.$$

Thus, mean and variance of the $\hat{\zeta}_y^*, \bar{x}_1^*$ and $\bar{x}_2^*$ in presence of non-response are given as

$$E(\hat{\zeta}_y^*) = \bar{Y}, E(\bar{x}_1^*) = \bar{X}_1, E(\bar{x}_2^*) = \bar{X}_2$$

and

$$Var(\hat{\zeta}_y^*) = \lambda S_y^2 + \theta S_{y_{(2)}}^2 + G,$$
$$Var(\bar{x}_1^*) = \lambda S_{x_1}^2 + \theta S_{x_{1(2)}}^2,$$
$$Var(\bar{x}_2^*) = \lambda S_{x_2}^2 + \theta S_{x_{2(2)}}^2,$$

where $\lambda = \left(\frac{1-f}{n}\right)$, $\theta = \frac{W_2(k-1)}{n}$, $G = \frac{W_2 k}{n}\left[\frac{\left\{(S_{y_{(2)}}^2 + \bar{y}_{(2)}^2)S_t^2 + S_s^2\right\}P}{(\bar{T}P+1-P)^2}\right]$ and $f = \frac{n}{N}$.
For each of the $i^{th}(i = 1, 2, 3, \ldots, n)$ sample unit, let $(y_i, x_{1i}, x_{2i}, z_i)$ be the predicted value and $(Y_i, X_{1i}, X_{2i}, Z_i)$ be the actual value for the sensitive

study variable (Y), two auxiliary variables (X_1, X_2) and for the scrambled response variable Z, respectively. Additionally, the measurement error associated with the (Y, X_1, X_2, Z) are denoted by $U_i = (y_i - Y_i)$, $V_i = (x_{1i} - X_{1i})$, $W_i = (x_{2i} - X_{2i})$ and $P_i = (z_i - Z_i)$. It is presumed that these measurement errors are random and uncorrelated with mean zero and variances S_u^2, S_v^2, S_w^2, $S_{u_{(2)}}^2$, $S_{v_{(2)}}^2$, $S_{w_{(2)}}^2$ and S_p^2, respectively.

Furthermore, when measurement error is present, the variances of $\hat{\zeta}_y^{**}$, $\bar{x}_1^{**}$ and $\bar{x}_2^{**}$ are given as

$$Var(\hat{\zeta}_y^{**}) = \lambda(S_y^2 + S_u^2) + \theta(S_{y_{(2)}}^2 + S_p^2) + G, \tag{2}$$

$$Var(\bar{x}_1^{**}) = \lambda(S_{x_1}^2 + S_v^2) + \theta(S_{x_{1(2)}}^2 + S_{v_{(2)}}^2)$$

and

$$Var(\bar{x}_2^{**}) = \lambda(S_{x_2}^2 + S_w^2) + \theta(S_{x_{2(2)}}^2 + S_{w_{(2)}}^2).$$

- A ratio estimator corresponding to Gupta et al. (2014) is given by

$$\hat{\zeta}_{rg}^{**} = \frac{\hat{\bar{y}}^{**}}{\bar{x}_1^{**}}\bar{x}_1',$$

where $\bar{x}_1^{**}$ is a non-sensitive auxiliary variable in the presence of non-response and measurement error simultaneously.

Therefore, the MSE of $\hat{\zeta}_{rg}^{**}$ is given by

$$\begin{aligned} MSE(\hat{\zeta}_{rg}^{**}) =& \lambda(S_y^2 + L^2 S_{x_1}^2 - 2L\rho_{yx_1} S_y S_{x_1}) + \theta(S_{y_{(2)}}^2 + L^2 S_{x_{1(2)}}^2 \\ & - 2L\rho_{zx_{1(2)}} S_z S_{x_{1(2)}}) - \lambda'(L^2 S_{x_1}^2 - 2L\rho_{yx_1} S_y S_{x_1}) \\ & + \lambda(S_u^2 + L^2 S_v^2) + \theta(S_p^2 + L^2 S_v^2) + G, \end{aligned} \tag{3}$$

where $L = \frac{\bar{Y}}{\bar{X}_1}$, $\lambda' = \left(\frac{1}{n'} - \frac{1}{N'}\right)$, $\rho_{yx_1} = \frac{S_{yx_1}}{S_y S_{x_1}}$ be the correlation coefficient between Y and X_1.

Thus, by putting $S_u^2 = S_v^2 = S_p^2 = 0$ in (3), we can obtain the MSE of Gupta et al. (2014) estimator $(\hat{\zeta}_{rg}^{**})$ without measurement error.

- In two-phase sampling, the Zhang et al. (2021) mean estimate of sensitive variable under measurement error and non-response has been given by

$$\hat{\zeta}_{zq}^{**} = \left[\hat{\bar{y}}^{**} + K(\bar{x}_1' - \bar{x}_1^{**})\right]\left(\frac{\bar{D}}{\bar{d}}\right)^{\nu},$$

where $\bar{d} = \phi(\alpha\bar{x}_1^{**} + \beta) + (1 - \phi)(\alpha\bar{x}_1' + \beta)$, $\bar{D} = (\alpha\bar{x}_1' + \beta)$. Moreover, K and ν are well-chosen constants, α and β are thought to represent some known parameters of the auxiliary variable X_1, and ϕ seems to be an unknown constant whose value must be determined through optimal conditions.

Therefore, the MSE of $\hat{\zeta}_{zq}^{**}$ is given by

$$\begin{aligned} MSE(\hat{\zeta}_{zq}^{**}) =& \lambda\big[S_y^2 + (K+\phi\nu\tau)^2 S_{x_1}^2 - 2(K+\phi\nu\tau)\rho_{yx_1} S_y S_{x_1}\big] \\ &+ \theta\big[S_{y_2}^2 + (K+\phi\nu\tau)^2 S_{x_{1(2)}}^2 - 2(K+\phi\nu\tau)\rho_{zx_{1(2)}} S_z S_{x_{1(2)}}\big] \\ &- \lambda'\big[(K+\phi\nu\tau)^2 S_{x_1}^2 - 2(K+\phi\nu\tau)\rho_{yx_1} S_y S_{x_1}\big] \\ &+ \lambda\big[S_u^2 + (K+\phi\nu\tau)^2 S_v^2\big] + \theta\big[S_p^2 + (K+\phi\nu\tau)^2 S_v^2\big] + G, \end{aligned} \tag{4}$$

where $\tau = \dfrac{\alpha\bar{Y}}{\alpha\bar{X}_1 + \beta}$.

The optimum value of ϕ is given as

$$\hat{\phi}_{opt} = \frac{\begin{array}{r}\lambda[\rho_{yx_1} S_y S_{x_1} - K(S_{x_1}^2 + S_v^2)] + \theta[\rho_{zx_{1(2)}} S_z S_{x_{1(2)}} - K(S_{x_{1(2)}}^2 + S_v^2)] \\ -\lambda'[K S_{x_1}^2 - \rho_{yx_1} S_y S_{x_1}]\end{array}}{\nu\tau[\lambda(S_{x_1}^2 + S_v^2) + \theta(S_{x_{1(2)}}^2 + S_v^2) - \lambda' S_{x_1}^2]}. \tag{5}$$

Thus, the resulting minimum MSE of $\hat{\zeta}_{zq}^{**}$ is given as

$$\begin{aligned} MSE_{\min}(\hat{\zeta}_{zq}^{**}) =& \lambda(S_y^2 + H^2 S_{x_1}^2 - 2H\rho_{yx_1} S_y S_{x_1}) + \theta(S_{y_{1(2)}}^2 + H^2 S_{x_{1(2)}}^2 \\ &- 2H\rho_{zx_{1(2)}} S_z S_{x_{1(2)}}) - \lambda'(H^2 S_{x_1}^2 - 2H\rho_{yx_1} S_y S_{x_1}) \\ &+ \lambda(S_u^2 + H^2 S_v^2) + \theta(S_p^2 + H^2 S_v^2) + G, \end{aligned} \tag{6}$$

where $H = \frac{\lambda\rho_{yx_1} S_y S_{x_1} + \theta\rho_{zx_{1(2)}} S_z S_{x_{1(2)}} - \lambda'\rho_{yx_1} S_y S_{x_1}}{\lambda(S_{x_1}^2 + S_v^2) + \theta(S_{x_{1(2)}}^2 + S_v^2) - \lambda' S_{x_1}^2}$.

Therefore, by putting $S_u^2 = S_v^2 = S_p^2 = 0$ in (6), we can obtain the MSE of Zhang et al. (2021) estimator ($\hat{\zeta}_{rg}^{**}$) without measurement error.

- Kumar et al. (2023) estimator for the estimation of sensitive variable when population mean of $\bar{X}_1$ and $\bar{X}_2$ are not known under two-phase sampling which is given by

$$\hat{\zeta}_{ssq}^{**} = \big[\alpha\hat{\bar{y}}^{**} + \beta_{yx_1}^{**}(\bar{x}_1' - \bar{x}_1^{**}) + \beta_{yx_2}^{**}(\bar{x}_2' - \bar{x}_2^{**})\big]\left(\frac{\bar{x}_1'}{\bar{x}_1^{**}} + \frac{\bar{x}_1^{**}}{\bar{x}_1'}\right)\left(\frac{\bar{x}_2'}{\bar{x}_2^{**}} + \frac{\bar{x}_2^{**}}{\bar{x}_2'}\right), \tag{7}$$

where $\beta_{yx_1}^{**} = \frac{s_{yx_1}^{**}}{s_{x_1}^{**2}}$ is the estimate of the population regression coefficient $\beta_{yx_1} = \frac{S_{yx_1}^{**}}{S_{x_1}^{**2}}$, $\beta_{yx_2}^{**} = \frac{s_{yx_2}^{**}}{s_{x_2}^{**2}}$ is the estimate of the population regression coefficient $\beta_{yx_2} = \frac{S_{yx_2}^{**}}{S_{x_2}^{**2}}$ and α be a finite quantity.

Therefore, the MSE of $\hat{\zeta}_{ssq}^{**}$ is given as

$$\begin{aligned} MSE(\hat{\zeta}_{ssq}^{**}) =& \bar{Y}^2(4\alpha - 1)^2 + 16\alpha^2 A + 16\beta_{yx_1}^2 B^* + 16\beta_{yx_2}^2 C^* \\ &- 16\lambda'\beta_{yx_1}^2 S_{x_1}^2 - 16\lambda'\beta_{yx_2}^2 S_{x_2}^2 - 32\alpha\beta_{yx_1} D^* \\ &+ 32\lambda'\alpha\beta_{yx_1}\rho_{yx_1} S_y S_{x_1} + 32\beta_{yx_1}\beta_{yx_2} E^* \\ &+ 32\lambda'\alpha\beta_{yx_2}\rho_{yx_2} S_y S_{x_2} - 32\alpha\beta_{yx_2} F^* \\ &+ 32\lambda'\beta_{yx_1}\beta_{yx_2} S_{x_1} S_{x_2} + 16\alpha^2 G, \end{aligned} \tag{8}$$

where
$A = \left[\lambda(S_y^2+S_u^2)+\theta(S_{y_{(2)}}^2+S_p^2)\right]$, $B^* = \left[(\lambda-\lambda^{'})(S_{x_1}^2+S_v^2)+\theta(S_{x_{1(2)}}^2+S_{v_{(2)}}^2)\right]$,
$C^* = \left[(\lambda-\lambda^{'})(S_{x_2}^2+S_w^2)+\theta(S_{x_{2(2)}}^2+S_{w_{(2)}}^2)\right]$,
$D^* = \left[(\lambda-\lambda^{'})\rho_{yx_1}S_yS_{x_1}+\theta\rho_{yx_{1(2)}}S_{y_{(2)}}S_{x_{1(2)}}\right]$,
$E^* = \left[(\lambda-\lambda^{'})\rho_{x_1x_2}S_{x_1}S_{x_2}+\theta\rho_{x_1x_{2(2)}}S_{x_{1(2)}}S_{x_{2(2)}}\right]$,
$F^* = \left[(\lambda-\lambda^{'})\rho_{yx_2}S_yS_{x_2}+\theta\rho_{yx_{2(2)}}S_{y_{(2)}}S_{x_{2(2)}}\right]$.
The optimum value of α is given as

$$\hat{\alpha}_{opt}^{**} = \frac{\frac{1}{4}\bar{Y}^2+\beta_{yx_1}(D^*-\lambda'\rho_{yx_1}S_yS_{x_1})+\beta_{yx_2}(F^*\lambda'\rho_{yx_2}S_yS_{x_2})}{\bar{Y}^2+A+G}. \tag{9}$$

Thus, the resulting minimum MSE of $\hat{\zeta}_{ssq}^{**}$ is given as

$$\begin{aligned} MSE_{min}(\hat{\zeta}_{ssq}^{**}) =& \bar{Y}^2(4\hat{\alpha}_{opt}^{**}-1)^2+16\hat{\alpha}_{opt}^{**2}A+16\beta_{yx_1}^2B^*+16\beta_{yx_2}^2C^* \\ &-16\lambda'\beta_{yx_1}^2S_{x_1}^2-16\lambda'\beta_{yx_2}^2S_{x_2}^2-32\hat{\alpha}_{opt}^{**}\beta_{yx_1}D^* \\ &+32\lambda'\hat{\alpha}_{opt}^{**}\beta_{yx_1}\rho_{yx_1}S_yS_{x_1}+32\lambda'\hat{\alpha}_{opt}^{**}\beta_{yx_2}\rho_{yx_2}S_yS_{x_2} \\ &+32\beta_{yx_1}\beta_{yx_2}E^*+32\lambda'\beta_{yx_1}\beta_{yx_2}S_{x_1}S_{x_2} \\ &-32\hat{\alpha}_{opt}^{**}\beta_{yx_2}F^*+16\hat{\alpha}_{opt}^{**2}G. \end{aligned} \tag{10}$$

3. Proposed Family of Estimator

In a situation, where a company needs to respond to privacy breaches or security incidents. The response time i.e., the time it takes for the company's security team to detect and respond to a privacy breach, could be modeled using an exponential distribution. In such situations where we have sensitive exponential information, to estimate the population mean of sensitive variables, we propose a family of estimators for collecting information from more than one auxiliary variable(s) under non-response and measurement error simultaneously. Using ORRT model under which our study is sensitive in nature, can provide more accurate estimates of population mean. When the population mean of $\bar{X}_1$ and $\bar{X}_2$ are not known, two-phase sampling is used. Motivated by Zhang et al. (2021), Kumar & Kour (2022) and Kumar et al. (2023), the family of estimators of $\hat{\zeta}_d^{**}$ is defined by

$$\begin{aligned} \hat{\zeta}_d^{**} =& \hat{\bar{y}}^{**}\Bigg[w_0\left(\frac{\bar{x}_1'}{\bar{x}_1^{**}}\right)^{\alpha_1}\left(\frac{\bar{x}_2'}{\bar{x}_2^{**}}\right)^{\alpha_2}\exp\left\{\frac{\alpha_3(\bar{x}_1'-\bar{x}_1^{**})}{\bar{x}_1'+\bar{x}_1^{**}}\right\}\exp\left\{\frac{\alpha_4(\bar{x}_2'-\bar{x}_2^{**})}{\bar{x}_2'+\bar{x}_2^{**}}\right\} \\ &+w_1\left(\frac{\bar{x}_1^{**}}{\bar{x}_1'}\right)^{\delta_1}\left(\frac{\bar{x}_2^{**}}{\bar{x}_2'}\right)^{\delta_2}\exp\left\{\frac{\delta_3(\bar{x}_1^{**}-\bar{x}_1')}{\bar{x}_1'+\bar{x}_1^{**}}\right\}\exp\left\{\frac{\delta_4(\bar{x}_2^{**}-\bar{x}_2')}{\bar{x}_2'+\bar{x}_2^{**}}\right\}\Bigg], \end{aligned} \tag{11}$$

where $(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\delta_1,\delta_2,\delta_3,\delta_4)$ are real constants and (w_0,w_1) are constants to be determined such that the MSE of $\hat{\zeta}_d^{**}$ is minimum.

To obtain the bias and MSE of $\hat{\zeta}_d^{**}$, we have
$\hat{\bar{y}}^{**} = \bar{Y}(1+\hat{e}_0^{**})$, $\bar{x}_1^{**} = \bar{X}_1(1+e_1^{**})$, $\bar{x}_2^{**} = \bar{X}_2(1+e_2^{**})$, $\bar{x}_1' = \bar{X}_1(1+e_1')$, $\bar{x}_2' = \bar{X}_2(1+e_2')$ such that
$E(\hat{e}_0^{**}) = E(e_1^{**}) = E(e_2^{**}) = E(e_1') = E(e_2') = 0$ and
$E(\hat{e}_0^{**2}) = \frac{1}{\bar{Y}^2}(A+G)$, $E(e_1^{**2}) = \frac{B^*}{\bar{X}_1^2}$, $E(e_2^{**2}) = \frac{C^*}{\bar{X}_2^2}$, $E(\hat{e}_0^{**}e_1^{**}) = \frac{D^*}{\bar{Y}\bar{X}_1}$,
$E(\hat{e}_0^{**}e_2^{**}) = \frac{F^*}{\bar{Y}\bar{X}_2}$, $E(e_1^{**}e_2^{**}) = \frac{E^*}{\bar{X}_1\bar{X}_2}$, $E(e_1'^2) = \lambda' \frac{(S_{x_1}^2+S_v^2)}{\bar{X}_1^2}$,
$E(e_2'^2) = \lambda' \frac{(S_{x_2}^2+S_w^2)}{\bar{X}_2^2}$, $E(\hat{e}_0^{**}e_1') = \lambda' \frac{\rho_{yx_1}S_yS_{x_1}}{\bar{Y}\bar{X}_1}$, $E(\hat{e}_0^{**}e_2') = \lambda' \frac{\rho_{yx_2}S_yS_{x_2}}{\bar{Y}\bar{X}_2}$,
$E(e_1^{**}e_2') = \lambda' \frac{\rho_{x_1x_2}S_{x_1}S_{x_2}}{\bar{X}_1\bar{X}_2}$, $E(e_2^{**}e_2') = \lambda' \frac{S_{x_2}^2 S_w^2}{\bar{X}_2^2}$.
We express $\hat{\zeta}_d^{**}$ in terms of $\hat{e}_0^{**}, e_1^{**}, e_2^{**}, e_1'$ and e_2' as

$$\begin{aligned}\hat{\zeta}_d^{**} =& \bar{Y}(1+\hat{e}_0^{**})\Bigg[w_0(1+e_1^{**})^{-\alpha_1}(1+e_1')^{\alpha_1}(1+e_2^{**})^{-\alpha_2}(1+e_2')^{\alpha_2}\\ &\exp\left\{\frac{-\alpha_3 d_1}{2}\left(1+\frac{d_1^*}{2}\right)^{-1}\right\}\exp\left\{\frac{-\alpha_4 d_2}{2}\left(1+\frac{d_2^*}{2}\right)^{-1}\right\}\\ &+ w_1(1+e_1^{**})^{\delta_1}(1+e_2^{**})^{\delta_2}(1+e_1')^{-\delta_1}(1+e_2')^{-\delta_2}\\ &\exp\left\{\frac{\delta_3 d_1}{2}\left(1+\frac{d_1^*}{2}\right)^{-1}\right\}\exp\left\{\frac{\delta_4 d_2}{2}\left(1+\frac{d_2^*}{2}\right)^{-1}\right\}\Bigg],\end{aligned} \tag{12}$$

where $d_1 = (e_1^{**} - e_1')$, $d_2 = (e_2^{**} - e_2')$, $d_1^* = (e_1^{**} + e_1')$ and $d_2^* = (e_2^{**} + e_2')$.

Expanding the right hand side of (12), multiplying out and neglecting terms of $e's$ having power greater than two and subtracting $\bar{Y}$ from both sides, we have

$$\begin{aligned}(\hat{\zeta}_d^{**} - \bar{Y}) =& \bar{Y}\Bigg[w_0\big\{1+\hat{e}_0^{**} - \theta_1(d_1+\hat{e}_0^{**}d_1) - \theta_2(d_2+\hat{e}_0^{**}d_2) + \frac{1}{2}(d_1^2\theta_1^2\\ &+ d_2^2\theta_2^2 + 2\theta_1\theta_2 d_1 d_2 + \theta_1 d_1 d_1^* + \theta_2 d_2 d_2^*)\big\} + w_1\big\{1+\hat{e}_0^{**}\\ &+ \phi_1(d_1+\hat{e}_0^{**}d_1) + \phi_2(d_2+\hat{e}_0^{**}d_2) + \frac{1}{2}(d_1^2\phi_1^2 + d_2^2\phi_2^2\\ &+ 2\phi_1\phi_2 d_1 d_2 - \phi_1 d_1 d_1^* - \phi_2 d_2 d_2^*)\big\} - 1\Bigg],\end{aligned} \tag{13}$$

where $\theta_1 = \left(\alpha_1 + \frac{1}{2}\alpha_3\right)$, $\theta_2 = \left(\alpha_2 + \frac{1}{2}\alpha_4\right)$, $\phi_1 = \left(\delta_1 + \frac{1}{2}\delta_3\right)$ and $\phi_2 = \left(\delta_2 + \frac{1}{2}\delta_4\right)$.

Taking expectation on both sides of (13), we get the bias of $\hat{\zeta}_d^{**}$ to the first degree of approximation as

$$\begin{aligned}Bias(\hat{\zeta}_d^{**}) =& \bar{Y}\Bigg[w_0\Bigg\{1 - \theta_1\frac{D^*}{\bar{Y}\bar{X}_1} - \theta_2\frac{F^*}{\bar{Y}\bar{X}_2} + \frac{1}{2}\Bigg(\theta_1(\theta_1+1)\frac{B^*}{\bar{X}_1^2} + \theta_2(\theta_2\\ &+1)\frac{C^*}{\bar{X}_2^2} + 2\theta_1\theta_2\frac{E^*}{\bar{X}_1\bar{X}_2}\Bigg)\Bigg\} + w_1\Bigg\{1 + \phi_1\frac{D^*}{\bar{Y}\bar{X}_1} + \phi_2\frac{F^*}{\bar{Y}\bar{X}_2}\\ &+ \frac{1}{2}\Bigg(\phi_1(\phi_1-1)\frac{B^*}{\bar{X}_1^2} + \phi_2(\phi_2-1)\frac{C^*}{\bar{X}_2^2} + 2\phi_1\phi_2\frac{E^*}{\bar{X}_1\bar{X}_2}\Bigg)\Bigg\} - 1\Bigg].\end{aligned} \tag{14}$$

Squaring both sides of (13) and neglecting terms of $e's$ having power greater than two, we have

$$
\begin{aligned}
(\hat{\zeta}_d^{**}-\bar{Y})^2 =& \bar{Y}^2\big[1+w_0^2\big\{1+2\hat{e}_0^{**}-2\theta_1 d_1-2\theta_2 d_2+\hat{e}_0^{**2}-4\theta_1\hat{e}_0^{**}d_1\\
&-4\theta_2\hat{e}_0^{**}d_2+4\theta_1\theta_2 d_1 d_2+2\theta_1^2 d_1^2+\theta_1 d_1 d_1^*+2\theta_2^2 d_2^2+\theta_2 d_2 d_2^*\big\}\\
&+w_1^2\big\{1+2\hat{e}_0^{**}+2\phi_1 d_1+2\phi_2 d_2+4\phi_1\hat{e}_0^{**}d_1+4\phi_2\hat{e}_0^{**}d_2\\
&+\hat{e}_0^{**2}+4\phi_1\phi_2 d_1 d_2+2\phi_1^2 d_1^2+2\phi_2^2 d_2^2-\phi_1 d_1 d_1^*-\phi_2 d_2 d_2^*\big\}\\
&+2w_0 w_1\big\{1+2\hat{e}_0^{**}+(\phi_1-\theta_1)(d_1+2\hat{e}_0^{**}d_1)\\
&+(\phi_2-\theta_2)(d_2+2\hat{e}_0^{**}d_2)+(\phi_1-\theta_1)(\phi_2-\theta_2)d_1 d_2\\
&+\frac{1}{2}d_1^2(\phi_1-\theta_1)^2-\frac{1}{2}(\phi_1-\theta_1)d_1 d_1^*+\frac{1}{2}d_2^2(\phi_2-\theta_2)^2\\
&-\frac{1}{2}(\phi_2-\theta_2)d_2 d_2^*+\hat{e}_0^{**2}\big\}-2w_0\big\{1+\hat{e}_0^{**}-\theta_1(d_1+\hat{e}_0^{**}d_1)\\
&-\theta_2(d_2+\hat{e}_0^{**}d_2)+\frac{1}{2}\big(d_1^2\theta_1^2+d_2^2\theta_2^2+2\theta_1\theta_2 d_1 d_2+\theta_1 d_1 d_1^*\\
&+\theta_2 d_2 d_2^*\big)\big\}-2w_1\big\{1+\hat{e}_0^{**}+\phi_1(d_1+\hat{e}_0^{**}d_1)+\phi_2(d_2+\hat{e}_0^{**}d_2)\\
&+\frac{1}{2}\big(d_1^2\phi_1^2+d_2^2\phi_2^2+2\phi_1\phi_2 d_1 d_2-\phi_1 d_1 d_1^*-\phi_2 d_2 d_2^*\big)\big\}\big].
\end{aligned}
\tag{15}
$$

Taking expectation on both sides of (15), we get the MSE of $\hat{\zeta}_d^{**}$ to the first degree of approximation as

$$
MSE(\hat{\zeta}_d^{**})=\bar{Y}^2\big[1+w_0^2 A_0^*+w_1^2 A_1^*+2w_0 w_1 A_2^*-2w_0 A_3^*-2w_1 A_4^*\big], \tag{16}
$$

where

$$A_0^* = \Big[1+\tfrac{1}{\bar{Y}^2}(A+G)-4\theta_1\tfrac{D^*}{\bar{Y}\bar{X}_1}-4\theta_2\tfrac{F^*}{\bar{Y}\bar{X}_2}+4\theta_1\theta_2\tfrac{E^*}{\bar{X}_1\bar{X}_2}+\theta_1(2\theta_1+1)\tfrac{B^*}{\bar{X}_1^2}+\theta_2(2\theta_2+1)\tfrac{C^*}{\bar{X}_2^2}\Big],$$

$$A_1^* = \Big[1+\tfrac{1}{\bar{Y}^2}(A+G)+4\phi_1\tfrac{D^*}{\bar{Y}\bar{X}_1}+4\phi_2\tfrac{F^*}{\bar{Y}\bar{X}_2}+4\phi_1\phi_2\tfrac{E^*}{\bar{X}_1\bar{X}_2}+\phi_1(2\phi_1-1)\tfrac{B^*}{\bar{X}_1^2}+\phi_2(2\phi_2-1)\tfrac{C^*}{\bar{X}_2^2}\Big],$$

$$A_2^* = \Big[1+\tfrac{1}{\bar{Y}^2}(A+G)+2(\phi_1-\theta_1)\tfrac{D^*}{\bar{Y}\bar{X}_1}+2(\phi_2-\theta_2)\tfrac{F^*}{\bar{Y}\bar{X}_2}+(\phi_1-\theta_1)(\phi_2-\theta_2)\tfrac{E^*}{\bar{X}_1\bar{X}_2}+\tfrac{(\phi_1-\theta_1)(\phi_1-\theta_1-1)}{2}\tfrac{B^*}{\bar{X}_1^2}+\tfrac{(\phi_2-\theta_2)(\phi_2-\theta_2-1)}{2}\tfrac{C^*}{\bar{X}_2^2}\Big],$$

$$A_3^* = \Big[1-\theta_1\tfrac{D^*}{\bar{Y}\bar{X}_1}-\theta_2\tfrac{F^*}{\bar{Y}\bar{X}_2}+\theta_1\theta_2\tfrac{E^*}{\bar{X}_1\bar{X}_2}+\tfrac{\theta_1(\theta_1+1)}{2}\tfrac{B^*}{\bar{X}_1^2}+\tfrac{\theta_2(\theta_2+1)}{2}\tfrac{C^*}{\bar{X}_2^2}\Big],$$

$$A_4^* = \Big[1+\phi_1\tfrac{D^*}{\bar{Y}\bar{X}_1}+\phi_2\tfrac{F^*}{\bar{Y}\bar{X}_2}+\phi_1\phi_2\tfrac{E^*}{\bar{X}_1\bar{X}_2}+\tfrac{\phi_1(\phi_1-1)}{2}\tfrac{B^*}{\bar{X}_1^2}+\tfrac{\phi_2(\phi_2-1)}{2}\tfrac{C^*}{\bar{X}_2^2}\Big].$$

Setting $\frac{\partial MSE(\hat{\zeta}_d^{**})}{\partial(w_i)} = 0$, $i = 0, 1$; we have

$$\begin{bmatrix} A_0^* & A_2^* \\ A_2^* & A_1^* \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} A_3^* \\ A_4^* \end{bmatrix}. \tag{17}$$

Solving (17) we get the optimum values of (w_0, w_1) as

$$\begin{aligned} w_{0(opt)} &= \frac{(A_1^* A_3^* - A_2^* A_4^*)}{(A_0^* A_1^* - A_2^{*2})} \\ \text{and } w_{1(opt)} &= \frac{(A_0^* A_4^* - A_2^* A_3^*)}{(A_0^* A_1^* - A_2^{*2})}. \end{aligned} \tag{18}$$

Thus, the resulting minimum MSE of $\hat{\zeta}_d^{**}$ is given by

$$MSE_{\min}(\hat{\zeta}_d^{**}) = \bar{Y}^2 \left[1 - \frac{\left(A_1^* A_3^{*2} - 2A_2^* A_3^* A_4^* + A_0^* A_4^{*2}\right)}{(A_0^* A_1^* - A_2^{*2})} \right], \tag{19}$$

which is true when $0 < \frac{(A_1^* A_3^{*2} - 2A_2^* A_3^* A_4^* + A_0^* A_4^{*2})}{(A_0^* A_1^* - A_2^{*2})} < 1$ and $(A_0^* A_1^* - A_2^{*2}) > 0$.

Theorem 1. *The MSE of $\hat{\zeta}_d^{**}$ is greater than or equal to the minimum MSE of $\hat{\zeta}_d^{**}$ i.e.*

$$MSE(\hat{\zeta}_d^{**}) \geq MSE_{\min}(\hat{\zeta}_d^{**}) = \bar{Y}^2 \left[1 - \frac{(A_1^* A_3^{*2} - 2A_2^* A_3^* A_4^* + A_0^* A_4^{*2})}{(A_0^* A_1^* - A_2^{*2})} \right],$$

with equality holdings if $w_0 = w_{0(opt)}$ and $w_1 = w_{1(opt)}$, where $w_{0(opt)}$ and $w_{1(opt)}$ are given by (18).

Thus, the members of the proposed family of estimator $\hat{\zeta}_d^{**}$ under two-phase sampling with population mean $\bar{Y}$ can be obtain by putting the suitable values of $(w_0, w_1, \alpha_i, \delta_i; i = 1, 2, 3, 4)$ in (11) and are given in Appendix.

4. Efficiency Conditions

From (2), (3), (6), (10) and (19), efficiency conditions are obtained as

1. $MSE_{\min}(\hat{\zeta}_d^{**}) < Var(\hat{\zeta}_y^{**})$, if

$$\left[\bar{Y}^2 \left\{ 1 - \frac{(A_1^* A_3^{*2} - 2A_2^* A_3^* A_4^* + A_0^* A_4^{*2})}{(A_0^* A_1^* - A_2^{*2})} \right\} - \{ \lambda(S_y^2 + S_u^2) \right.$$
$$\left. + \theta(S_{y_{(2)}}^2 + S_p^2) + G \} \right] < 0.$$

2. $MSE_{\min}(\hat{\zeta}_d^{**}) < MSE(\hat{\zeta}_{rg}^{**})$, if

$$\begin{aligned}&\Bigg[\bar{Y}^2\Bigg\{1-\frac{(A_1^*A_3^{*2}-2A_2^*A_3^*A_4^*+A_0^*A_4^{*2})}{(A_0^*A_1^*-A_2^{*2})}\Bigg\}-\big\{\lambda(S_y^2+L^2S_{x_1}^2\\&-2L\rho_{yx_1}S_yS_{x_1})+\theta(S_{y_{(2)}}^2+L^2S_{x_{1(2)}}^2-2L\rho_{zx_{1(2)}}S_zS_{x_{1(2)}})\\&-\lambda'(L^2S_{x_1}^2-2L\rho_{yx_1}S_yS_{x_1})+\lambda(S_u^2+L^2S_v^2)\\&+\theta(S_p^2+L^2S_v^2)+G\big\}\Bigg]<0.\end{aligned}$$

3. $MSE_{\min}(\hat{\zeta}_d^{**}) < MSE_{\min}(\hat{\zeta}_{zq}^{**})$, if

$$\begin{aligned}&\Bigg[\bar{Y}^2\Bigg\{1-\frac{(A_1^*A_3^{*2}-2A_2^*A_3^*A_4^*+A_0^*A_4^{*2})}{(A_0^*A_1^*-A_2^{*2})}\Bigg\}-\big\{\lambda(S_y^2+H^2S_{x_1}^2\\&-2H\rho_{yx_1}S_yS_{x_1})+\theta(S_{y_{1(2)}}^2+H^2S_{x_{1(2)}}^2-2H\rho_{zx_{1(2)}}S_zS_{x_{1(2)}})\\&-\lambda'(H^2S_{x_1}^2-2H\rho_{yx_1}S_yS_{x_1})+\lambda(S_u^2+H^2S_v^2)\\&+\theta(S_p^2+H^2S_v^2)+G\big\}\Bigg]<0.\end{aligned}$$

4. $MSE_{\min}(\hat{\zeta}_d^{**}) < MSE_{\min}(\hat{\zeta}_{ssq}^{**})$, if

$$\begin{aligned}&\Bigg[\bar{Y}^2\Bigg\{1-\frac{(A_1^*A_3^{*2}-2A_2^*A_3^*A_4^*+A_0^*A_4^{*2})}{(A_0^*A_1^*-A_2^{*2})}\Bigg\}-\big\{\bar{Y}^2(4\hat{\alpha}_{opt}^{**}-1)^2\\&+16\hat{\alpha}_{opt}^{**2}A+16\beta_{yx_1}^2B^*+16\beta_{yx_2}^2C^*-16\lambda'\beta_{yx_1}^2S_{x_1}^2-16\lambda'\beta_{yx_2}^2S_{x_2}^2\\&-32\hat{\alpha}_{opt}^{**}\beta_{yx_1}D^*+32\lambda'\hat{\alpha}_{opt}^{**}\beta_{yx_1}\rho_{yx_1}S_yS_{x_1}+32\lambda'\hat{\alpha}_{opt}^{**}\beta_{yx_2}\rho_{yx_2}S_yS_{x_2}\\&+32\beta_{yx_1}\beta_{yx_2}E^*+32\lambda'\beta_{yx_1}\beta_{yx_2}S_{x_1}S_{x_2}-32\hat{\alpha}_{opt}^{**}\beta_{yx_2}F^*+16\hat{\alpha}_{opt}^{**2}G\big\}\Bigg]<0.\end{aligned}$$

If the above conditions (1-4) are satisfied, we conclude that our proposed class of estimator $(\hat{\zeta}_d^{**})$ is better than the considered estimators $(\hat{\zeta}_y^{**}), (\hat{\zeta}_{rg}^{**}), (\hat{\zeta}_{zq}^{**}), (\hat{\zeta}_{ssq}^{**})$. Further to validate the theoretical findings, we perform a simulation study in the next section.

5. Simulation Study

Our study compares the performance of the proposed ratio-cum-product and exponential type estimator to the other estimators in the class using R software.

Using the normal distribution, an artificial population of size $N = 5000$ has been generated. From N, a sample of size $n' = 2000$, is taken and from n', a sample of size $n = 1400$ is selected using two-phase sampling. As a result, during the first phase, only $n_1(1200)$ units provides response to the survey questions, while $n_2 = n - n_1$ do not. In the second stage, we change $k = 2, 3, 4, 5$ and $P = 0.25$ to 0.85, in order to gather another sample $n_s = \frac{n_2}{k}$; $(k > 1)$ from the non-respondent group. Thus, the study variable Y is generated from a normal distribution and is defined as $Y = aX_1 + aX_2 + N(0, 1)$, where $X_1 = N(0.5, 1)$, $X_2 = N(0.5, 1)$ and $a = 0.2$. The scrambling variables T and S are also generated from a normal distribution with mean $(1, 0)$, respectively and variance is 0.5 for both the scrambling variables. In both phases, the measurement error of $\bar{X}_1$ and $\bar{X}_2$ follow a normal distribution with mean zero. In the first phase, the measurement error of Y and Z follow a normal distribution with mean zero and variance approximately equal to one.

Therefore, the percent relative efficiency(PRE) of the proposed class of estimator with respect to considered estimators is obtained by using the equation given below

$$PRE = \left(\frac{MSE(\hat{\zeta}_y^{**})}{MSE(\hat{\zeta}_j^{**})} \right) * 100, \tag{20}$$

where $MSE(\hat{\zeta}_j^{**}); j = y, rg, zq, ssq, d.$

Table 1 illustrate the comparison of the MSE's and PRE's of the proposed and considered class of estimator(s) for different values of k and different sensitivity level (P) in the existence of non-response and measurement error simultaneously under two-phase sampling using ORRT model.

- From Table 1, with the increase in k and P, the MSE's of the estimator(s) under ORRT increases but it is also proofs that our proposed estimator$(\hat{\zeta}_d^{**})$ has lesser MSE among all other considered estimators $(\hat{\zeta}_y^{**}), (\hat{\zeta}_{rg}^{**}), (\hat{\zeta}_{zq}^{**}), (\hat{\zeta}_{ssq}^{**})$.
- Also, we seen that PRE's of the proposed estimator$(\hat{\zeta}_d^{**})$ are highest among all the considered estimators $(\hat{\zeta}_y^{**}), (\hat{\zeta}_{rg}^{**}), (\hat{\zeta}_{zq}^{**}), (\hat{\zeta}_{ssq}^{**})$ and increases for different values of k and P.

Thus, based on the above explanation, we conclude that our proposed class of estimator is more trustworthy and efficient than the other estimators under consideration.

TABLE 1: MSE and PRE of the proposed and other considered family of estimators.

Estimator	***P***	**MSE**				**PRE**			
		k				k			
		2	3	4	5	2	3	4	5
$\hat{\zeta}_y^{**}$	0.25	0.002884	0.004191	0.004794	0.005939	100.000	100.000	100.000	100.000
	0.45	0.003017	0.004513	0.004794	0.005939	100.000	100.000	100.000	100.000
	0.65	0.003217	0.004960	0.005798	0.006820	100.000	100.000	100.000	100.000
	0.85	0.003488	0.005539	0.00656	0.007568	100.000	100.000	100.000	100.000
$\hat{\zeta}_{rg}^{**}$	0.25	0.002781	0.003931	0.005272	0.006216	96.4207	96.3173	95.5596	99.6898
	0.45	0.002930	0.004171	0.005541	0.006548	97.4677	97.5977	96.0447	100.3374
	0.65	0.003134	0.004498	0.005916	0.007024	98.7832	99.3546	97.1367	101.6934
	0.85	0.003392	0.004917	0.006403	0.007656	100.2582	101.4114	98.6764	103.5435
$\hat{\zeta}_{zq}^{**}$	0.25	0.002679	0.003786	0.005033	0.006197	100.0007	100.0064	100.093	100.004
	0.45	0.002856	0.004070	0.005318	0.006570	100.0006	100.0024	100.0755	100.0005
	0.65	0.003096	0.004469	0.005743	0.007143	100.0005	100.0004	100.0592	100.0005
	0.85	0.003401	0.004986	0.006315	0.007927	100.0005	100.0011	100.045	100.0005
$\hat{\zeta}_{ssq}^{**}$	0.25	0.002718	0.003799	0.004304	0.005218	106.0902	110.2923	111.3986	113.8199
	0.45	0.002838	0.004073	0.004634	0.005479	106.3231	110.7996	112.5109	114.6162
	0.65	0.003013	0.004441	0.005082	0.005879	106.7704	111.9748	114.0921	116.0073
	0.85	0.003247	0.004904	0.005644	0.006413	107.4420	112.9473	116.1554	118.0071
$\hat{\zeta}_d^{**}$	0.25	0.002647	0.003710	0.004156	0.004988	108.9491	112.9722	115.3424	119.0641
	0.45	0.002760	0.003962	0.004477	0.005233	109.3084	113.8884	116.4414	120.0108
	0.65	0.002929	0.004307	0.004913	0.005612	109.8499	115.1427	118.008	121.5157
	0.85	0.003154	0.004744	0.005460	0.006123	110.5851	116.7518	120.0646	123.6033

6. Conclusion

In a scenario where dealing with a sensitive variable, such as personal income or medical treatment response time, exponential data can be encountered in various ways. So, to handle such situation where collection of data on sensitive variable is required has always been a challenge for researchers. The biggest issues encountered by the researcher during data collection on sensitive study variable are non-response and measurement error, simultaneously. Motivated by Hansen and Hurwitz's method, we propose a generalized family of estimator for estimating the population mean in the presence of non-response and measurement error under ORRT using two-phase sampling. Different estimators can be obtained from proposed estimator by considering different values of constant used. The properties of the proposed estimator has been examined. The performance of the proposed class of estimator$(\hat{\zeta}_d^{**})$ with other considered estimators $(\hat{\zeta}_y^{**}), (\hat{\zeta}_{rg}^{**}), (\hat{\zeta}_{zq}^{**})$ and $(\hat{\zeta}_{ssq}^{**})$ are also studied and efficiency conditions have been obtained. A simulation study is carried out to verify the theoretical findings to assess the effectiveness and performance of the proposed class of estimator. From the results given in Section 4 and Section 5, it is clear that the proposed class of estimator is more reliable than the other considered estimator(s). Therefore, we recommend researchers and practitioners to employ the proposed class of estimator for efficiently estimate the mean of the study variable which is sensitive in nature.

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Appendix

Members of the Proposed Family of Estimator

In this section, by taking different values of constants in (11), we obtained ratio-product-exponential type estimators which are sensitive in nature.

1. When $w_1 = 0$, we have

$$\hat{\zeta}_{d1}^{**} = \hat{\bar{y}}^{**}\left[w_0\left(\frac{\bar{x}_1'}{\bar{x}_1^{**}}\right)^{\alpha_1}\left(\frac{\bar{x}_2'}{\bar{x}_2^{**}}\right)^{\alpha_2}\exp\left\{\frac{\alpha_3(\bar{x}_1' - \bar{x}_1^{**})}{\bar{x}_1' + \bar{x}_1^{**}}\right\}\exp\left\{\frac{\alpha_4(\bar{x}_2' - \bar{x}_2^{**})}{\bar{x}_2' + \bar{x}_2^{**}}\right\}\right].$$

2. When $(\alpha_3 = \alpha_4 = 0)$ and $w_1 = 0$, we have

$$\hat{\zeta}_{d2}^{**} = \hat{\bar{y}}^{**}\left[w_0\left(\frac{\bar{x}_1'}{\bar{x}_1^{**}}\right)^{\alpha_1}\left(\frac{\bar{x}_2'}{\bar{x}_2^{**}}\right)^{\alpha_2}\right].$$

3. When $(\alpha_1 = \alpha_2 = 0)$ and $w_1 = 0$, we have

$$\hat{\zeta}_{d3}^{**} = \hat{\bar{y}}^{**}\left[w_0\exp\left\{\frac{\alpha_3(\bar{x}_1' - \bar{x}_1^{**})}{\bar{x}_1' + \bar{x}_1^{**}}\right\}\exp\left\{\frac{\alpha_4(\bar{x}_2' - \bar{x}_2^{**})}{\bar{x}_2' + \bar{x}_2^{**}}\right\}\right].$$

4. When $(\alpha_3 = \alpha_4 = 0)$ and $(\delta_3 = \delta_4 = 0)$, we have

$$\hat{\zeta}_{d4}^{**} = \hat{\bar{y}}^{**}\left[w_0\left(\frac{\bar{x}_1'}{\bar{x}_1^{**}}\right)^{\alpha_1}\left(\frac{\bar{x}_2'}{\bar{x}_2^{**}}\right)^{\alpha_2} + w_1\left(\frac{\bar{x}_1^{**}}{\bar{x}_1'}\right)^{\delta_1}\left(\frac{\bar{x}_2^{**}}{\bar{x}_2'}\right)^{\delta_2}\right].$$

5. When $(\alpha_1 = \alpha_2 = 0)$ and $(\delta_1 = \delta_2 = 0)$, we have

$$\hat{\zeta}_{d5}^{**} = \hat{\bar{y}}^{**}\left[w_0\exp\left\{\frac{\alpha_3(\bar{x}_1' - \bar{x}_1^{**})}{\bar{x}_1' + \bar{x}_1^{**}}\right\}\exp\left\{\frac{\alpha_4(\bar{x}_2' - \bar{x}_2^{**})}{\bar{x}_2' + \bar{x}_2^{**}}\right\} + w_1\exp\left\{\frac{\delta_3(\bar{x}_1^{**} - \bar{x}_1')}{\bar{x}'_1 + \bar{x}_1^{**}}\right\}\exp\left\{\frac{\delta_4(\bar{x}_2^{**} - \bar{x}_2')}{\bar{x}_2' + \bar{x}_2^{**}}\right\}\right].$$

6. When $(\alpha_2 = \alpha_3 = 0)$ and $(\delta_2 = \delta_3 = 0)$, we have

$$\begin{aligned}\hat{\zeta}^{**}_{d6} = \hat{\bar{y}}^{**}\Bigg[&w_0\left(\frac{\bar{x}'_1}{\bar{x}^{**}_1}\right)^{\alpha_1}\exp\left\{\frac{\alpha_4(\bar{x}'_2 - \bar{x}^{**}_2)}{\bar{x}'_2 + \bar{x}^{**}_2}\right\}\\ &+w_1\left(\frac{\bar{x}^{**}_1}{\bar{x}'_1}\right)^{\delta_1}\exp\left\{\frac{\delta_4(\bar{x}^{**}_2 - \bar{x}'_2)}{\bar{x}'_2 + \bar{x}^{**}_2}\right\}\Bigg].\end{aligned}$$

7. When $(\alpha_2 = \alpha_3 = \alpha_4 = 0)$ and $(\delta_1 = \delta_3 = \delta_4 = 0)$, we have

$$\hat{\zeta}^{**}_{d7} = \hat{\bar{y}}^{**}\left[w_0\left(\frac{\bar{x}'_1}{\bar{x}^{**}_1}\right)^{\alpha_1} + w_1\left(\frac{\bar{x}^{**}_2}{\bar{x}'_2}\right)^{\delta_2}\right].$$

8. When $(\alpha_2 = \alpha_3 = \alpha_4 = 0)$ and $(\delta_1 = \delta_2 = \delta_3 = 0)$, we have

$$\hat{\zeta}^{**}_{d8} = \hat{\bar{y}}^{**}\left[w_0\left(\frac{\bar{x}'_1}{\bar{x}^{**}_1}\right)^{\alpha_1} + w_1\exp\left\{\frac{\delta_4(\bar{x}^{**}_2 - \bar{x}'_2)}{\bar{x}'_2 + \bar{x}^{**}_2}\right\}\right].$$

9. When $(\alpha_1 = \alpha_2 = \alpha_4 = 0)$ and $(\delta_1 = \delta_3 = \delta_4 = 0)$, we have

$$\hat{\zeta}^{**}_{d9} = \hat{\bar{y}}^{**}\left[w_0\exp\left\{\frac{\alpha_3(\bar{x}'_1 - \bar{x}^{**}_1)}{\bar{x}'_1 + \bar{x}^{**}_1}\right\} + w_1\left(\frac{\bar{x}^{**}_2}{\bar{x}'_2}\right)^{\delta_2}\right].$$

10. When $(\alpha_1 = \alpha_2 = \alpha_4 = 0)$ and $(\delta_1 = \delta_2 = \delta_3 = 0)$, we have

$$\hat{\zeta}^{**}_{d10} = \hat{\bar{y}}^{**}\left[w_0\exp\left\{\frac{\alpha_3(\bar{x}'_1 - \bar{x}^{**}_1)}{\bar{x}'_1 + \bar{x}^{**}_1}\right\} + w_1\exp\left\{\frac{\delta_4(\bar{x}^{**}_2 - \bar{x}'_2)}{\bar{x}'_2 + \bar{x}^{**}_2}\right\}\right].$$

11. When $(\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0)$, we have

$$\begin{aligned}\hat{\zeta}^{**}_{d11} = \hat{\bar{y}}^{**}\Bigg[&w_0 + w_1\left(\frac{\bar{x}^{**}_1}{\bar{x}'_1}\right)^{\delta_1}\left(\frac{\bar{x}^{**}_2}{\bar{x}'_2}\right)^{\delta_2}\exp\left\{\frac{\delta_3(\bar{x}^{**}_1 - \bar{x}'_1)}{\bar{x}'_1 + \bar{x}^{**}_1}\right\}\\ &\exp\left\{\frac{\delta_4(\bar{x}^{**}_2 - \bar{x}'_2)}{\bar{x}'_2 + \bar{x}^{**}_2}\right\}\Bigg].\end{aligned}$$

12. When $(\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0)$ and $(\delta_3 = \delta_4 = 0)$, we have

$$\hat{\zeta}^{**}_{d12} = \hat{\bar{y}}^{**}\left[w_0 + w_1\left(\frac{\bar{x}^{**}_1}{\bar{x}'_1}\right)^{\delta_1}\left(\frac{\bar{x}^{**}_2}{\bar{x}'_2}\right)^{\delta_2}\right].$$

13. When $(\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0)$ and $(\delta_1 = \delta_2 = 0)$, we have

$$\hat{\zeta}^{**}_{d13} = \hat{\bar{y}}^{**}\left[w_0 + w_1\exp\left\{\frac{\delta_3(\bar{x}^{**}_1 - \bar{x}'_1)}{\bar{x}'_1 + \bar{x}^{**}_1}\right\}\exp\left\{\frac{\delta_4(\bar{x}^{**}_2 - \bar{x}'_2)}{\bar{x}'_2 + \bar{x}^{**}_2}\right\}\right].$$

14. When $(\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0)$ and $(\delta_2 = \delta_3 = 0)$, we have

$$\hat{\zeta}^{**}_{d14} = \hat{\bar{y}}^{**}\left[w_0 + w_1\left(\frac{\bar{x}^{**}_1}{\bar{x}'_1}\right)^{\delta_1}\exp\left\{\frac{\delta_4(\bar{x}^{**}_2 - \bar{x}'_2)}{\bar{x}'_2 + \bar{x}^{**}_2}\right\}\right].$$

15. When ($\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$) and ($\delta_1 = \delta_4 = 0$), we have

$$\hat{\zeta}_{d15}^{**} = \hat{\bar{y}}^{**}\left[w_0 + w_1\left(\frac{\bar{x}_2^{**}}{\bar{x}_2^{'}}\right)^{\delta_2} \exp\left\{\frac{\delta_3(\bar{x}_1^{**} - \bar{x}_1^{'})}{\bar{x}_1^{'} + \bar{x}_1^{**}}\right\}\right].$$

Hence by putting the suitable values of $(w_0, w_1, \alpha_i, \delta_i; i = 1, 2, 3, 4)$, we can easily get the biases and MSE's of the members of the suggested family of estimator from (13) and (15).