Revista Colombiana de Estadística - Theoretical Statistics
June 2025, volume 48, issue 2, pp. 93–114
http://doi.org/10.15446/rce.v48n2.114887

Efficient Parameter Estimation for Claim-Time Behaviour in Insurance Portfolios: MCMC Simulation Analysis of MLE and MAP Techniques

Estimación eficiente de los parámetros del comportamiento siniestral de las carteras de seguros: Análisis de simulación MCMC de las técnicas MLE y MAP.

Christian Akrong Hesse[1,a], Emmanuel Dodzi Kpeglo[1,b], Dominic Buer Boyetey[2,c], Albert Ayi Ashiagbor[3,d]

[1]Faculty of Accounting and Finance, Department of Economics and Actuarial Science, University of Professional Studies, Accra, Ghana

[2]School of Sustainable Development, Department of Built Environment, University of Environment and Sustainable Development, Somanya, Ghana

[3]Faculty of Economic and Management Sciences, Department of Business Management, University of Pretoria, Pretoria, South Africa

Abstract

The study investigated the dynamics of "commencement-to-event-time" behaviour in life insurance portfolios, employing Maximum Likelihood Estimation (MLE) and Maximum A Posteriori (MAP) with the Markov Chain Monte Carlo (MCMC) simulation technique. Focusing on the Lognormal and Exponential distributions for their efficacy in modelling time-to-occurrence data, the research simulated 120 observations from both distributions and estimated parameters using the first 80 ordered samples. Remarkably, estimates for lognormal parameters obtained through MLE and MAP_MCMC were highly similar, with errors well within 10% of the actual values, highlighting the accuracy of both methods. The study also explored the robustness of the MAP_MCMC technique to various prior distributions, demonstrating its effectiveness across different priors, including Exponential, Normal, Gamma, Pareto, and Weibull prior distributions. In the case of the exponential distribution, both MLE and MAP_MCMC techniques performed exceptionally well, providing estimates within 5% of the true value, with MAP _MCMC exhibiting remarkable precision, just 1% off the true value.

[a]Ph.D. E-mail: christian.hesse@upsamail.edu.gh
[b]Ph.D. E-mail: emmanuel.kpeglo@upsamail.edu.gh
[c]Ph.D. E-mail: dbboyetey@uesd.edu.gh
[d]Ph.D. E-mail: albert.ashiagbor@up.ac.za

Real-life data fitted to the Gamma distribution showed that MLE and MAP _MCMC methods, using censored data, closely approximated benchmark estimates from the method of moments. The MAP_MCMC approach slightly outperformed the MLE.

***Key words*:** Bayesian Inference; Maximum Likelihood Estimation; Maximum a-Posteriori; Markov Chain Monte Carlo Simulation.

Resumen

El estudio investigó la dinámica del comportamiento "inicio-acontecimiento-tiempo" en las carteras de seguros de vida, empleando la Estimación de Máxima Verosimilitud (MLE) y la Máxima A Posteriori (MAP) con la técnica de simulación Markov Chain Monte Carlo (MCMC). Centrándose en las distribuciones Lognormal y Exponencial por su eficacia en la modelización de datos de tiempo de ocurrencia, la investigación simuló 120 observaciones de ambas distribuciones y estimó los parámetros utilizando las 80 primeras muestras ordenadas. Sorprendentemente, las estimaciones de los parámetros lognormales obtenidas mediante MLE y MAP_MCMC fueron muy similares, con errores muy inferiores al 10% de los valores reales, lo que pone de relieve la precisión de ambos métodos. El estudio también exploró la robustez de la técnica MAP_MCMC a varias distribuciones a priori, demostrando su eficacia a través de diferentes distribuciones a priori, incluyendo Exponencial, Normal, Gamma, Pareto y Weibull. En el caso de la distribución exponencial, tanto las técnicas MLE como MAP_MCMC obtuvieron resultados excepcionales, proporcionando estimaciones dentro del 5% del valor real, con MAP_MCMC mostrando una precisión notable, sólo un 1% por debajo del valor real. Los datos reales ajustados a la distribución Gamma mostraron que los métodos MLE y MAP_MCMC, utilizando datos censurados, se aproximaron mucho a las estimaciones de referencia del método de los momentos. El método MAP_MCMC superó ligeramente al MLE.

***Palabras clave*:** Estimación de máxima verosimilitud; Inferencia bayesiana; Maximum a posteriori; Simulación Monte Carlo con cadenas de Markov.

1. Introduction

Life is inherently uncertain, with unforeseen events like accidents, illnesses, or property damage posing significant financial challenges. Insurance serves as a collective risk-pooling mechanism, where individuals and businesses contribute to a shared fund to provide financial protection against losses (Cronk & Aktipis, 2021). This collective approach alleviates the financial burden on individuals and businesses, preventing potential financial ruin and highlighting the pivotal role of insurance in contemporary society (Liedtke, 2007; van der Heide, 2023).

According to Zakaria et al. (2016), life insurance primarily aims to mitigate the financial risks associated with the death or disability of an insured person, with secondary considerations for potential investment returns. Under an insurance contract, the insured pays a predetermined premium, and in the event of death or disability, the insurer compensates the designated beneficiary with a predefined claim amount. In this context, a claim represents compensation for the risk of loss (Ekberg, 2015; Kochenburger & Salve, 2023; Yohandoko et al., 2023), and these claims are submitted either at the time of maturity or in the event of death or disability. It is worth studying the time between the commencement of the life insurance policy and the time of the occurrence of the event (commencement-to-event-time). It stands as a vital financial imperative for insurance companies to fulfil their obligation to their clients as well as avoid institutional bankruptcy (Riaman et al., 2023). Therefore, understanding the "commencement-to-event-time", frequency, severity, and complexity of claims is essential for insurance pricing and forecasting future claims (Omari et al., 2018).

The characteristics of claims, including "commencement-to-event-time", size of claim, frequency of claim, and portfolio totals, were studied using various distribution models. Notably, Bahnemann (2015) found that discrete standard distributions are suitable for modelling claims' occurrences within a portfolio, while Omari et al. (2018) suggest the lognormal distribution for claims severity. Even though the Pareto distribution has seen extensive usage in claims data modelling (Clemente et al., 2023; Feng, 2023; Gilenko & Mironova, 2017), Moumeesri et al. (2020) profess heavy-tailed distributions like Gamma and Lognormal to be more accurate in modelling claim severity.

Earlier scholars, such as Kaplan & Meier (1958), Cox (1972), and Kleinbaum & Klein (1996), applied survival analysis techniques, including Kaplan-Meier estimates and the Cox proportional hazards model, in an attempt to fit the time between the occurrences of insurance claims. In the quest to find insightful approaches towards the estimation of claim-time patterns, other scholarly works (Ramani et al., 2023; Zhou, 2024; Abdulkadir & Fernando, 2024) explored machine learning methods together with other notable distributions such as the Marshall-Olkin extended Weibull distribution. The survival analysis techniques and the machine learning approaches prove distinct in their strengths and limitations in estimating different claim types (Arik et al., 2023).

In the recent works of Zaçaj et al. (2022), bootstrap methods were used to predict the distribution of future claims development. The approach involved the application of the maximum likelihood parameter estimation method to specify the probability distribution that best fits the data among a family of predefined distributions. It came to light that the Gamma distribution better describes the claim development data. In a related study, Cousineau & Helie (2013) proposed an improved alternative to the regular Maximum Likelihood Estimation (MLE) technique. They found that the Maximum A Posteriori (MAP) estimation technique yielded accurate estimates for the parameter estimation of the Weibull distribution for some simulated data. According to Edwards et al. (1963), the MAP is an extension of the Bayesian estimation (BE) technique that returns the posterior distribution of the parameters given the data. The sole use of the BE is difficult to implement and slow to operate for parameter estimation. Therefore, in its operationalization, some numerical estimations or the use of Markov Chain Monte Carlo (MCMC) techniques may be required to estimate the parameters. This approach is scantly used in "commencement-to-event-time" modelling.

Louzada & Ramos (2018) support the applicability of the maximum a posteriori estimator for the Gamma distribution parameters. They argued in a simulation study to compare different estimation procedures that the MAP approach performs better than the existing closed-form estimators and also produces highly efficient estimates for both parameters, even for small sample sizes. Similarly, empirical studies by Bolstad (2007) and Jaroengeratikun et al. (2012) found that the Bayesian methods with prior distributions, such as Gamma, outperformed some other heavy-tailed and skewed distributions when assessing time-to-claim data. In this instance, lognormal distribution was selected as the best distribution to model the time-to-claim data. These statistical approaches are important because they help actuaries assess the coverage probabilities of any chosen probability distribution and its expected length of claim time.

On the specific subject of "commencement-to-event-time," Landriault et al. (2014) propose a distribution that is contingent on the time of incurrence. While studies on claim severity, frequency, and aggregate claims abound, little is known in the literature about "commencement-to-event-time" modelling at the portfolio level.

Consequently, this study proceeds with the objective of applying the MAP estimation approach to model "commencement-to-event-time" behaviour in an insurance portfolio. The study relied on a censored approach by focusing on the first "commencement-to-event-time" r $(r < n)$ instead of using the traditional complete-data methods. This was necessary for an efficient parameter estimation of the underlying probability density function governing "commencement-to-event-time".

In addition, using simulated data, we compared the Bayesian MAP-MCMC estimation to the MLE with sensitivity analysis of different prior distributions. These theories were applied to real-world data from a life insurance portfolio of 180 insured individuals. The data recorded the time (in days) from each policy's commencement to the event's occurrence (death).

In the subsequent sections of this study, we focused on the methods, providing a detailed illustration of the estimation of the lognormal distribution parameters using MCMC-based approaches. Similar estimation methods are provided for the exponential and the Gamma "commencement-to-event-time" random variables. We continued with results and discussion sections and ended the study with conclusion remarks.

2. Materials and Methods

Consider a scenario where an insurance portfolio comprises n life insurance policies, and one-time death benefits are disbursed to the beneficiary if the insured person passes away during the policy term. The focus of interest is modelling "commencement-to-event-time" within this portfolio. Let the underlying "commencement-to-event-time" be denoted as $X_{(1)}, \dots, X_{(n)}$, where $X_{(i)} \leq X_{(i+1)}$, $i = 1, \dots, n-1$. The distribution function of the "commencement-to-event-time" variable X is represented by $F_X(x)$, and its probability density function (pdf) is denoted as $f_X(x)$.

In this study, we develop a model based on the first r, "commencement-to-event-time," within the portfolio, where $r \leq n$. The number of claims is considered fixed, while the "commencement-to-event-time" is treated as a random variable. Although there are other random variables, such as claim severity and claim frequency, our focus in this study centres on the "commencement-to-event-time" random variable.

This work compares two popular parameter estimation methods for time-to-occurrence data: Maximum Likelihood Estimation (MLE) and Maximum A Posteriori (MAP). Both methods utilize Markov Chain Monte Carlo (MCMC) algorithms to efficiently sample from complex, high-dimensional probability distributions encountered in practice, allowing for robust inferences.

We focus on the versatile Lognormal and Exponential distributions due to their effectiveness in modelling "commencement-to-event-time" data (Kundu et al., 2005). The Lognormal excels at scenarios where claim processes involve multiple independent factors and exhibit right-skewness (more early claims). Its link to the normal distribution through logarithmic transformation further strengthens its applicability for complex claim dynamics (Zuanetti et al., 2006).

Conversely, the Exponential distribution thrives when claim rates remain constant and the "memoryless" property applies. This renders it a suitable choice for portfolios with consistent claim patterns (Ndwandwe et al., 2024).

2.1. Lognormal "Commencement-to-Event-Time" Random Variable

When the random variable X has a lognormal distribution with parameters μ and σ, where $-\infty < \mu < \infty$ and $\sigma > 0$, its density function is given by

$$f_X(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left(\frac{\log x - \mu}{\sigma}\right)^2\right\}, \quad x > 0. \tag{1}$$

The distribution function can be obtained by integrating the density function as follows:

$$F_X(x) = \int_0^x \frac{1}{y\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left(\frac{\log y - \mu}{\sigma}\right)^2\right\} dy,$$

and the substitution $z = \log y$ yields

$$F_X(x) = \int_{-\infty}^{\log x} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left(\frac{z - \mu}{\sigma}\right)^2\right\} dz.$$

As the integrand is the $N\left(\mu, \sigma^2\right)$ density function,

$$F_X(x) = \Phi\left(\frac{\log x - \mu}{\sigma}\right), \tag{2}$$

where Φ is the distribution function of the standard normal. Thus, probabilities under a lognormal distribution can be calculated from the standard normal distribution function. We use the notation $LN(\mu, \sigma)$ to denote a lognormal distribution with parameters μ and σ. From the preceding argument, it follows that if $X \sim LN(\mu, \sigma)$, then $\log X \sim N(\mu, \sigma^2)$. In the next section, we illustrate the estimation of the lognormal distribution parameters using MCMC-based approaches. We employ both MLE and MAP techniques implemented through MCMC algorithm.

2.1.1. MLE_MCMC Approach

The likelihood function L of the first r-order statistics, $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(r)}$, is given by Ofosu & Hesse (2011)

$$\begin{aligned} L(\mu, \sigma) &= f_{X_{(1)},\ldots,X_{(r)}}(x_1, \ldots, x_r) \\ &= \tfrac{n!}{(n-r)!}\left[1 - F_X(x_r)\right]^{n-r} \prod_{i=1}^{r} f_X(x_i) \\ &= \tfrac{n!}{(n-r)!}\left[1 - \Phi\left(\tfrac{\log x_r - \mu}{\sigma}\right)\right]^{n-r} \prod_{i=1}^{r} \left\{\tfrac{1}{x_i \sigma \sqrt{2\pi}} \exp\left\{-\tfrac{1}{2}\left(\tfrac{\log x_i - \mu}{\sigma}\right)^2\right\}\right\} \\ &= k\left[1 - \Phi\left(\tfrac{\log x_r - \mu}{\sigma}\right)\right]^{n-r} \left(\tfrac{1}{\sigma}\right)^r \exp\left\{-\tfrac{1}{2}\sum_{i=1}^{r}\left(\tfrac{\log x_i - \mu}{\sigma}\right)^2\right\}, \end{aligned} \tag{3}$$

where k is independent of the parameters μ and σ. Determining the MLE estimates can be challenging, as finding a straightforward solution to the likelihood equations is not always feasible. Fortunately, diverse tools exist for such situations within the realm of the MLE. Prominent approaches include: Iterative methods, the Expectation-Maximization (EM) algorithm, Gradient Descent, Quasi-Newton methods, Monte Carlo methods, Profile Likelihood, Bootstrapping, and Numerical Optimization (Nocedal & Wright, 1999; Dempster et al., 1977; Gilks et al., 1995; Press, 1992).

Further, in this method, we employ MCMC sampling to generate samples from the likelihood function, a technique we will refer to as MLE_MCMC. This approach effectively circumvents the challenges of solving the likelihood equations directly and yields parameter estimates that maximize the likelihood function given the sample data. The core steps of MLE_MCMC are:

1. Simulate a large sample from the bivariate likelihood function in Equation (3) using MCMC techniques.

2. Identify the mode of this simulated sample, which represents the MLE_MCMC point estimate of the parameter vector $\theta = (\mu, \sigma)$. That is,

$$\theta_{\text{MLE}} = \arg\max\left\{k\left[1 - \Phi\left(\tfrac{\log x_r - \mu}{\sigma}\right)\right]^{n-r} \left(\tfrac{1}{\sigma}\right)^r \exp\left\{-\tfrac{1}{2}\sum_{i=1}^{r}\left(\tfrac{\log x_i - \mu}{\sigma}\right)^2\right\}\right\} \tag{4}$$

The following algorithm is the description for the multivariate Metropolis-Hastings procedure (Hesse et al., 2016):

1. Set $t = 1$
2. Generate an initial value for $\beta \sim U(u_1, u_2)$.
3. Repeat

   $t = t + 1$

   Do a MH step on α,

   Generate a proposal $\theta^* \sim N(\theta, \sigma^2)$;

   Evaluate the acceptance probability $a = \min\left[1, \frac{L(\theta*|\mathrm{x})}{L(\theta|\mathrm{x})}\right]$;

   Generate a u from a $Uniform(0, 1)$ distribution

   If $u \leq a$, accept the proposal and set $\theta = \theta^*$
4. Until $< t = T$.

2.1.2. MAP_MCMC Approach

Maximum A Posteriori (MAP) estimation is the Bayesian counterpart to Maximum Likelihood Estimation (MLE), incorporating additional information through the prior distribution. Now, the joint pdf of $X_{(1)}, \ldots, X_{(r)}$ and $\theta = (\alpha, \beta)$ is given by

$$g(x_1, \ldots, x_r, \theta) = f_{X_{(1)},\ldots,X_{(r)}}\left(x_1, \ldots, x_r | \, \theta\right) \pi(\theta),$$

where $\pi(\theta)$, is the prior distribution of the parameter vector Θ. We assume α and β are independent and exponentially distributed with means a and b, respectively. Thus,

$$\pi(\theta) = \tfrac{1}{ab} e^{-(\mu/a + \sigma/b)}, \;\; \alpha > 0, \beta > 0.$$

$$g(x_1, \ldots, x_r, \theta) = k\left[1 - \Phi\left(\tfrac{\log x_r - \mu}{\sigma}\right)\right]^{n-r} \left(\tfrac{1}{\sigma}\right)^r \exp\left\{-\tfrac{1}{2}\sum_{i=1}^{r}\left(\tfrac{\log x_i - \mu}{\sigma}\right)^2 - \left(\tfrac{\mu}{a} + \tfrac{\sigma}{b}\right)\right\} \quad (5)$$

Thus, the marginal pdf of $X_{(1)}, \ldots, X_{(r)}$ is

$$g_{X_{(1)},\ldots,X_{(r)}}\left(x_1, \ldots, x_r\right) = \int_{\Theta} g(x_1, \ldots, x_r, \theta) d\theta =$$
$$k\int_{\Theta}\left[1 - \Phi\left(\tfrac{\log x_r - \mu}{\sigma}\right)\right]^{n-r} \left(\tfrac{1}{\sigma}\right)^r \exp\left\{-\tfrac{1}{2}\sum_{i=1}^{r}\left(\tfrac{\log x_i - \mu}{\sigma}\right)^2 - \left(\tfrac{\mu}{a} + \tfrac{\sigma}{b}\right)\right\} d\mu d\sigma,$$

which is independent of μ and σ. The conditional pdf of Θ given $X_{(1)}, \ldots, X_{(r)}$ is therefore given by

$$
\begin{aligned}
\pi\left(\theta \mid x_1, \ldots, x_r\right) &= \frac{g(x_1, \ldots, x_r, \theta)}{g_{X_{(1)}, \ldots, X_{(r)}}\left(x_1, \ldots, x_r\right)} \\
&= K\left[1-\Phi\left(\tfrac{\log x_r-\mu}{\sigma}\right)\right]^{n-r}\left(\tfrac{1}{\sigma}\right)^r \exp\left\{-\tfrac{1}{2}\sum_{i=1}^{r}\left(\tfrac{\log x_i-\mu}{\sigma}\right)^2-\left(\tfrac{\mu}{a}+\tfrac{\sigma}{b}\right)\right\} \qquad (6)
\end{aligned}
$$

where K is independent of μ and σ. The typical approach in Bayesian estimation is to employ the posterior mean, $E(\Theta \mid x_1, \ldots, x_r)$, as a point estimate for θ (Hesse et al., 2016). The Maximum A Posteriori (MAP) estimator of θ is the value that maximizes the posterior distribution. Similar to the MLE_MCMC, we utilize the MCMC sampling approach to draw samples from the posterior distribution. This specific method of estimation, denoted as MAP_MCMC for the purpose of this study, identifies the mode of the posterior distribution, representing the point estimate for the parameter vector θ. Thus,

$$
\begin{aligned}
&\hat{\theta}_{\mathrm{MAP}} = \\
&\arg\max\left\{K\left[1-\Phi\left(\tfrac{\log x_r-\mu}{\sigma}\right)\right]^{n-r}\left(\tfrac{1}{\sigma}\right)^r \exp\left\{-\tfrac{1}{2}\sum_{i=1}^{r}\left(\tfrac{\log x_i-\mu}{\sigma}\right)^2-\left(\tfrac{\mu}{a}+\tfrac{\sigma}{b}\right)\right\}\right\}. \qquad (7)
\end{aligned}
$$

2.2. Exponential "Commencement-to-Event-Time" Random Variable

Suppose the "commencement-to-event-time" random variable X follow the exponential distribution with pdf

$$
f_X(x) = \tfrac{1}{\lambda}e^{-x/\lambda}, \quad x \geq 0, \lambda > 0. \qquad (8)
$$

and the distribution function, that is, $P(X \leq x)$, is

$$
F_X(x) = 1 - e^{-x/\lambda}, \quad x \geq 0, \lambda > 0.
$$

2.2.1. Maximum Likelihood Estimation

The joint density function of the first r-ordered observations $X_{(1)}, \ldots, X_{(r)}, X_{(i)} \leq X_{(i+1)}$, $i = 1, \ldots, n$, is given by

$$
\begin{aligned}
L &= f_{X_{(1)}, \ldots, X_{(r)}}\left(x_1, \ldots, x_r \mid \lambda\right) \\
&= \tfrac{n!}{(n-r)!}\left[1-F_X(x_r)\right]^{n-r}\prod_{i=1}^{r} f_X(x_i) \\
&= \tfrac{n!}{(n-r)!}\left[\exp\left(-x_r/\lambda\right)\right]^{n-r}\prod_{t=1}^{r}\tfrac{1}{\theta}\exp\left(-x_i/\lambda\right), \qquad (9) \\
&= \tfrac{n!}{(n-r)!\lambda^r}\exp\left\{-\tfrac{1}{\lambda}\left[\sum_{i=1}^{r} x_i + (n-r)x_r\right]\right\}, \quad 0 \leq x_1 \leq .. \leq x_r.
\end{aligned}
$$

$$\ln L = \ln\left(\tfrac{n!}{(n-r)!}\right) - r\ln\lambda - \tfrac{1}{\lambda}\left[\sum_{i=1}^{r} x_i + (n-r)x_r\right]$$
$$\tfrac{\partial \ln L}{\partial \lambda} = -\tfrac{r}{\lambda} + \tfrac{1}{\lambda^2}\left[\sum_{i=1}^{r} x_i + (n-r)x_r\right]$$

Hence, the maximum likelihood estimator of θ is

$$\hat{\lambda} = \tfrac{\sum_{i=1}^{r} X_{(i)} + (n-r)X_{(r)}}{r}. \tag{10}$$

It can be shown that $\hat{\lambda}$ is an unbiased estimator of λ and mean-squared error $MSE_{\text{MLE}}(\hat{\lambda}) = \frac{\lambda^2}{r}$(Mann et al., 1974).

2.2.2. MAP_MCMC Approach

We assume θ has the exponential distribution with mean a. Then, the posterior distribution can be written in the form

$$\pi\left(\lambda | \, x_1, \ldots, x_r\right) = K \cdot \tfrac{1}{\lambda^r} \exp\left\{-\tfrac{1}{\lambda}\left[\sum_{i=1}^{r} x_i + (n-r)x_r + \tfrac{\lambda^2}{a}\right]\right\}. \tag{11}$$

Since the moments of the posterior distribution cannot easily be obtained, we resort to the MCMC sampling technique to get samples from the posterior distribution. MAP finds the mode of the posterior distribution which represents the point estimate of the parameter λ. Thus, the MAP estimator of λ is the value of λ that maximizes the posterior distribution. That is,

$$\hat{\lambda}_{\text{MAP}} = \arg\max\left\{K \cdot \tfrac{1}{\lambda^r} \exp\left[-\tfrac{1}{\lambda}\left[\sum_{i=1}^{r} x_i + (n-r)x_r - \tfrac{\lambda^2}{a}\right]\right]\right\}. \tag{12}$$

2.3. Gamma "Commencement-to-Event-Time" Random Variable

The gamma distribution is often employed to model time-to-failure random variables in life testing when the failure rate is not constant. This distribution is particularly suitable when the failure rate follows a bathtub-shaped curve, exhibiting both an initial phase of decreasing failure rates (infant mortality) and a later phase of increasing failure rates (wear-out). The gamma distribution allows for flexibility in capturing diverse failure rate behaviours and is well-suited for scenarios where the hazard function varies over time (Eric et al., 2021). The continuous random variable T, is said to have the gamma distribution with parameters $\alpha > 0$ and $\beta > 0$ if its pdf is given by

$$f_T(t) = \tfrac{\beta^\alpha t^{\alpha-1} e^{-\beta t}}{\Gamma(\alpha)}, \quad t > 0 \tag{13}$$

2.3.1. Maximum Likelihood Estimation

It can be shown that the likelihood function L of the first r order statistics, $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(r)}$, is given by

$$L = \frac{n!}{(n-r)!} \left[\Gamma(\alpha) - \gamma(\alpha, \beta x_r)\right]^{n-r} \left(\frac{1}{\Gamma(\alpha)}\right)^n \beta^{r\alpha} \left(\prod_{i=1}^{r} x_i^{\alpha-1}\right) e^{-\beta \sum_{i=1}^{r} x_i}. \tag{14}$$

This function yields the following logarithmic likelihood equations:

$$\frac{\partial \ln L}{\partial \beta} = \frac{r\alpha}{\beta} - \sum_{i=1}^{r} \ln x_i + \frac{\beta^{\alpha-1}(n-r)x_r^{\alpha} e^{-\beta x_r}}{[\Gamma(\alpha) - \gamma(\alpha, \beta x_r)]} = 0. \tag{15}$$

$$\frac{\partial \ln L}{\partial \alpha} = \frac{(n-r)\left[\Gamma'(\alpha) - \Gamma'(\alpha, \beta x_r)\right]}{[\Gamma(\alpha) - \gamma(\alpha, \beta x_r)]} + \frac{n\Gamma'(\alpha)}{\Gamma(\alpha)} + r \ln \beta + \sum_{i=1}^{r} \ln x_i = 0. \tag{16}$$

Similarly, solving Equation (15) and Equation (16) is notably challenging. When a straightforward solution to the likelihood equations is elusive, various procedures are available for the MLE. Common methods encompass Iterative methods, the Expectation-Maximization (EM) algorithm, Gradient Descent, Quasi-Newton methods, Monte Carlo methods, Profile Likelihood, Bootstrapping, and Numerical Optimization (Nocedal & Wright, 1999; Dempster et al., 1977; Gilks et al., 1995; Press, 1992).

In cases where obtaining a solution to the log-likelihood equations proves difficult, we turn to MCMC sampling techniques to generate samples from the likelihood function. For the purposes of this study, the estimation technique is referred to as MLE_MCMC. The primary objective is to determine parameter estimates that maximize the likelihood function given the sample data. The MLE_MCMC approach identifies the mode of the simulated MCMC sample from the bivariate likelihood function in Equation (14), representing the point estimate of the parameter vector $\theta = (\alpha, \beta)$. That is,

$$\theta_{\text{MLE}} = \arg\max \left\{ k \left[\Gamma(\alpha) - \gamma(\alpha, \beta x_r)\right]^{n-r} \left(\frac{1}{\Gamma(\alpha)}\right)^n \beta^{r\alpha} \left(\prod_{i=1}^{r} x_i^{\alpha-1}\right) e^{-\beta \sum_{i=1}^{r} x_i} \right\}. \tag{17}$$

2.3.2. MAP_MCMC Approach

If we assume α and β are independent and exponentially distributed with means a and b, respectively, then it can be shown that the posterior distribution is

$$\pi\left(\theta \mid x_1, \ldots, x_r\right)$$
$$= K \left[\Gamma(\alpha) - \gamma(\alpha, \beta x_r)\right]^{n-r} \left(\frac{1}{\Gamma(\alpha)}\right)^n \beta^{r\alpha} \left(\prod_{i=1}^{r} x_i^{\alpha-1}\right) e^{-\left(\frac{\alpha}{a} + \frac{\beta}{b} + \beta \sum_{i=1}^{r} x_i\right)}. \tag{18}$$

The MAP estimator of θ is the value that maximizes the posterior distribution. Thus,

$$\hat{\theta}_{\text{MAP}} = \arg\max\{\Omega\} \tag{19}$$

where

$$\Omega = \left\{ K\left[\Gamma(\alpha) - \gamma(\alpha, \beta t_r)\right]^{n-r} \left(\tfrac{1}{\Gamma(\alpha)}\right)^n \beta^{r\alpha} \left(\prod_{i=1}^{r} t_i^{\alpha-1}\right) \; e^{-\left(\frac{\alpha}{a} + \frac{\beta}{b} + \beta \sum_{i=1}^{r} t_i\right)} \right\}.$$

3. Results

In this section, we present results of some simulated and real-life data for both the MAP_MCMC and the MLE estimation approaches, enabling efficient parameterization of the Lognormal, Exponential, and Gamma distributions.

3.1. Lognormal Distribution

MATLAB's 'lognquantile' function (MathWorks, 2020) was used to calculate quantiles for given probabilities simulated from the uniform distribution over the interval $(0, 1)$. Table 1 displays the first 80 out of 120 ordered data points simulated from the lognormal distribution with parameters $\mu = 3$ and $\sigma = 2$. These observations are assumed to represent the ordered "commencement-to-event-time" data of an insurance portfolio comprising 120 life insurance policies until the 80^{th} event-time.

Table 1: Ordered data simulated from the lognormal distribution with parameters $\mu = 3$ and $\sigma = 2$.

Simulated ordered data points									
0.063	0.067	0.286	0.327	0.376	0.382	0.656	0.866	0.868	1.038
1.121	1.177	1.408	1.484	1.545	1.614	1.687	1.828	1.990	2.181
2.208	2.270	2.336	3.468	3.870	3.896	4.809	4.987	5.015	5.085
5.526	5.780	6.114	6.133	6.153	6.535	7.176	7.277	8.872	9.013
9.115	9.191	9.695	9.821	10.012	10.667	10.753	11.147	12.056	12.090
12.655	12.803	13.091	13.147	13.174	13.271	13.277	13.482	13.766	13.777
16.200	16.365	16.739	16.923	17.142	20.491	21.608	21.734	22.659	23.491
24.629	26.271	26.941	27.058	27.197	27.587	28.210	29.548	32.526	38.423

3.1.1. MLE_MCMC Estimate

With $n = 120$, $r = 80$, and $x_r = 38.423$, as specified in Table 1, we implemented a Metropolis-Hastings algorithm to sample from the likelihood function in Equation (3). The MATLAB code for this component-wise Metropolis sampler is detailed in Listings A1 and A2 (see Appendix). Analyzing the mode of the resulting bivariate sample yielded maximum likelihood estimates for the lognormal distribution's parameters, μ and σ as; $\mu_{\text{MLE}} = 3.2215$ and $\sigma_{\text{MLE}} = 2.0824$, respectively.

The estimated parameters closely align with the true values ($\mu = 3$ and $\sigma = 2$), suggesting a highly accurate fit of the model to the data.

3.1.2. MAP_MCMC Estimate

To explore the Bayesian posterior distribution, we employed a Metropolis-Hastings algorithm to simulate a sample, leveraging the data from Table 1. We assumed independent exponential prior distributions for μ and σ with means $a = 6$ and $b = 4$, respectively. The MATLAB code for this posterior sampling process is analogous to Listings A1 and A2 in the appendix. MAP_MCMC estimates are: $\mu_{\text{MAP}} = 3.1726$ and $\sigma_{\text{MAP}} = 2.0281$.

These estimates precisely align with those obtained through direct sampling from the likelihood function, demonstrating robustness to prior assumptions. The results remain unchanged even with varying a and b (e.g., $a = 15$, $b = 10$).

To assess the MAP_MCMC estimator's sensitivity to prior assumptions, we performed repeated MCMC simulations with diverse prior distributions for μ and σ: Exponential (E), Normal (N), Gamma(G), Pareto(P), and Weibull (W). Table 2 summarizes the results. All estimated values of μ and σ landed within 10% of the true values, showcasing remarkable tolerance to variations in prior assumptions. Notably, the MAP_MCMC estimates for both parameters remained consistent across different priors, mirroring the results obtained with the MLE_MCMC approach.

Table 2: Comparison of Estimates (Lognormal).

			Bivariate Prior Distribution (MAP_MCMC)									
	MLE_MCMC		E		N		G		P		W	
Parameters	μ	σ	μ	σ	μ	σ	μ	σ	μ	σ	μ	σ
Actual	3	2	3	2	3	2	3	2	3	2	3	2
Estimate	3.22	2.08	3.17	2.03	3.15	2.11	3.15	2.12	3.14	2.10	3.14	2.10
Error %	7.40	4.10	5.80	1.40	4.90	5.50	5.10	5.80	4.80	5.20	4.60	5.00

3.2. Exponential Distribution

To generate "commencement-to-event-times" following an exponential distribution with a mean of 150, we leveraged the versatility of the gamma distribution. Specifically, we employed the MS Excel formula "$= GAMMA.INV(p, 1, 150)$" to calculate quantiles based on probabilities simulated from a uniform distribution over $(0, 1)$.

Recall that the exponential distribution emerges as a special case of the gamma distribution when the shape parameter α equals 1. Table 3 presents the initial 80 "commencement-to-event-times", extracted from a larger dataset of 120. These values mirror ordered "commencement-to-event-times" within an insurance portfolio encompassing 120 life insurance policies, observed up to the 80^{th} claim.

Figures 1 and 2 show the plot of the probability density functions of the log-normal distribution and exponential distribution with $\mu = 3$, $\sigma = 2$, and $\lambda = 150$.

TABLE 3: Ordered data simulated from the exponential distribution with mean 150.

Simulated ordered data points									
0.472	1.370	4.022	5.141	7.379	7.798	9.989	10.597	11.202	13.875
13.925	14.817	17.083	17.375	18.386	19.915	20.057	20.712	23.117	23.219
24.033	25.258	26.285	26.575	27.408	27.815	28.268	28.744	28.843	28.988
30.086	33.212	33.435	35.138	40.870	46.169	52.750	54.439	54.518	54.788
55.396	56.326	56.493	61.768	68.344	75.311	75.402	75.629	75.826	75.902
77.990	78.213	82.214	82.246	83.558	85.845	89.874	91.879	92.302	94.092
96.947	97.009	99.583	104.683	107.628	112.772	115.229	118.058	118.905	119.869
119.910	120.297	125.121	125.279	129.311	140.330	144.147	145.638	153.275	153.823

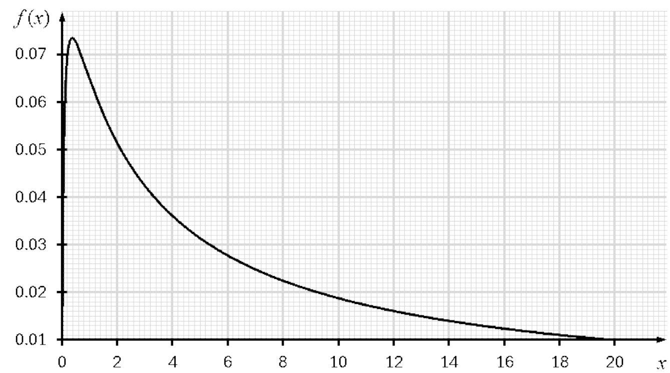

FIGURE 1: pdf of the lognormal distribution with $\mu = 3$ and $\sigma = 2$.

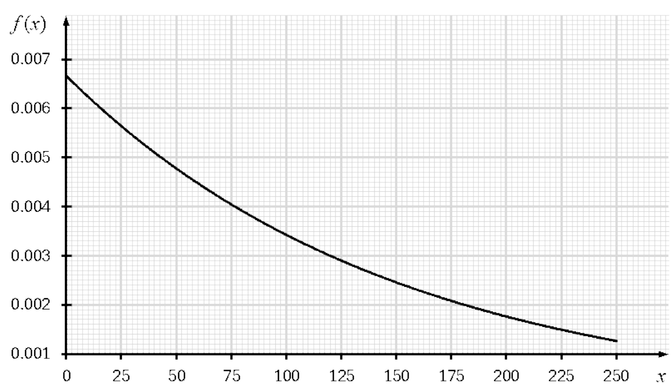

FIGURE 2: pdf of the exponential distribution with $\lambda = 150$.

3.2.1. MLE Estimate

From Equation (10) and Table 3, given $n = 120$, $r = 80$ and $x_r = 153.8225$, the maximum likelihood estimate of the parameter λ is $\hat{\lambda}_{\text{MLE}} = 138.793$.

3.2.2. MLE_MCMC Estimate

The MATLAB code for implementing the Metropolis-Hastings sampler for the likelihood function in Equation (9) is provided in Listings A3 and A4 of the appendix. The results show that the maximum likelihood estimate of the parameter λ is $\lambda_{\text{MLE}} = 155.533$. This estimation was based on the values $n = 120$, $r = 80$, and $x_r = 153.823$, from Table 3.

3.2.3. MAP_MCMC Estimate

Harnessing the data in Table 3, we employed the MAP_MCMC technique to estimate the value of λ. The Metropolis-Hastings algorithm was implemented analogously to Listings A3 and A4 in the Appendix, facilitating sampling from the posterior distribution outlined in Equation (12). This approach yielded a MAP_MCMC estimate of $\lambda_{\text{MAP}} = 148.732$, that closely aligned with the true value of $\lambda = 150$.

Table 4 reveals the parameter estimates and their absolute percentage differences from the true λ value (150). Both MCMC-based techniques, MLE_MCMC and MAP_MCMC, excel, with estimates hovering within 5% of the true value. While the MLE estimate falls within a reasonable 10% margin, it exhibits slightly lower precision compared to its MCMC counterpart. Notably, MAP_MCMC shines, with an estimate within 1% of the true value, highlighting the potential advantage of incorporating prior information through MCMC for tighter parameter estimation.

Table 4: Comparison of Estimates (Exponential).

	MLE	MLE_MCMC	MAP_MCMC
Estimate of λ	138.793	155.533	148.732
Error Percentage	7.5	3.7	0.8

3.3. Application to Real-Life Data

The theory was applied to real-world data from a life insurance portfolio of 180 insured individuals from Hollard Insurance Ghana. The data recorded the time (in days) from each policy's commencement to the occurrence of the event (death). Table 5 presents the frequency distribution, with 12 groups, of the number of days from the start of the policy until the event occurred for these 180 policyholders.

The data was fitted to the Normal, Exponential, Gamma, Weibull, and Lognormal distributions, and the test results are summarized in Table 6.

Table 5: Frequency Distribution of the Number of Days (Commencement to Occurrence of the Death).

Number of Days	Frequency	Percent	Valid Percent	Cumulative Percent
100 - 299	10	5.5	5.5	5.5
300 - 499	37	20.4	20.4	26.0
500 - 699	17	9.4	9.4	35.4
700 - 899	27	14.9	14.9	50.3
900 - 1099	17	9.4	9.4	59.7
1100 - 1299	21	11.6	11.6	71.3
1300 - 1499	17	9.4	9.4	80.7
1500 - 1699	12	6.6	6.6	87.3
1700 - 1899	4	2.2	2.2	89.5
1900 - 2099	14	7.7	7.7	97.2
2100 - 2299	2	1.1	1.1	98.3
2300 - 2499	3	1.7	1.7	100.0
Total	181	100.0	100.0	

Table 6: Summary of the Goodness of Fit Test.

Distribution	**Log-Likelihood**	**AIC**	**BIC**	**Best Fit?**
Normal	–443.12	890.24	896.64	No
Exponential	–471.91	945.82	949.01	No
Gamma	–428.26	860.52	866.91	**Yes**
Weibull	–428.49	860.98	867.38	Close Fit
Lognormal	–433.39	870.79	877.18	No

The Gamma distribution is the best fit for the data, as it has the lowest AIC and BIC values and the highest log-likelihood. It is closely followed by the Weibull distribution. Both distributions handle skewness well, but Gamma slightly outperforms the Weibull distribution.

The method of moment was used to estimate the parameters of the Gamma distribution based on the entire dataset, serving as a benchmark estimate for comparison. The method of moment estimates of α and β of the Gamma distribution, based on the complete data from 180 policies, were calculated as 3.2151 and 0.0032, respectively. Figure 3 shows the graph of the fitted gamma distribution using the estimated parameters.

From Equation (17) and given $n = 180$, $r = 100$, $x_r = 962$, the component-wise Metropolis sampler was used to determine maximum likelihood estimates for the α and β of the Gamma distribution using the first 100 ordered samples of the 180 policies, which are given by $\alpha_{\text{MLE}} = 3.3526$ and $\beta_{\text{MLE}} = 0.0029$, respectively. We may have noticed that the maximum likelihood estimates (using the first 100 ordered samples) are not significantly different from the method of moment estimates (using the complete data).

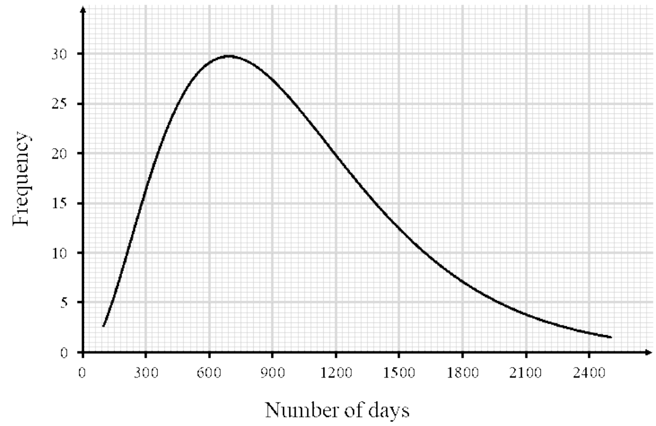

Figure 3: Graph of the fitted gamma.

Again, using the first 100 ordered samples from the 180 policies, the Metropolis-Hastings algorithm was employed to generate a sample from the Bayesian posterior distribution. It was assumed that α and β are independent and exponentially distributed with parameters $a = 3$ and $b = 0.02$, respectively. The results indicate that the MAP_MCMC estimates of α and β are $\alpha_{\text{MAP}} = 3.3561$ and $\beta_{\text{MAP}} = 0.0034$, precisely matching the estimates obtained by sampling directly from the likelihood function. Even when we vary the values of a and b, the findings remain consistent. Similar consistent parameter estimates were obtained (across the three estimation techniques above) for the Weibull distribution.

4. Concluding Remarks

The study explored the dynamics of "commencement-to-event-time" behaviour within life insurance portfolios by employing two parameter estimation techniques: Maximum Likelihood Estimation (MLE) and Maximum A Posteriori (MAP), utilizing the Markov Chain Monte Carlo (MCMC) simulation technique. Due to their effectiveness in modelling time-to-occurrence data, we focused on the versatile Lognormal, Exponential, and Gamma distributions.

We simulated 120 observations from both lognormal and exponential distributions by using the first 80 ordered samples. The estimates for the lognormal parameters (μ and σ) obtained through MLE and MAP_MCMC were remarkably similar, with errors within 10% of the actual values. This indicates the effectiveness of both methods in providing accurate estimations for the parameters of the lognormal distribution. Furthermore, the study explored the sensitivity of the MAP_MCMC technique to various prior distributions, demonstrating its robustness across different priors, including Exponential, Normal, Gamma, Pareto, and Weibull prior distributions.

The MLE and MAP_MCMC techniques demonstrated excellent performance regarding the exponential distribution, providing estimates within 5% of the true value. Although the MLE estimate remained within a reasonable 10% margin, it showed slightly lower precision than its MAP_MCMC counterpart. The exceptional performance of MAP_MCMC was particularly noteworthy, with an estimate within 1% of the true value. This underscores the potential advantage of incorporating prior distribution through MCMC to achieve more precise parameter estimation.

Further, we applied our methodology to real-life data, which was fitted to the Gamma distribution. The method of moment was used to estimate the parameters of the Gamma distribution based on the entire dataset, serving as a benchmark estimate for comparison. The result showed that based on censored data, the MLE and MAP_MCMC estimation techniques, produced estimates closer to the benchmark. It is important to note that our MAP_MCMC approach slightly outperformed the MLE approach.

This research pushes the boundaries of "commencement-to-event-time" modelling, opening doors to exciting future explorations. For example, within survival analysis, one may contemplate a statistical framework designed to analyse time-to-event data, which could apply to modelling the duration between instances of an insurance claim. Approaches such as the Cox proportional hazards model and Kaplan-Meier estimates offer insights into the analysis of "commencement-to-event-time". Another avenue worth exploring in modelling "commencement-to-event-time" involves renewal processes, which characterise the time intervals between recurrent events, similar to the arrival of claims in an insurance portfolio. Renewal theory and concepts, such as inter-arrival times and renewal intervals, provide a valuable framework for analysing and predicting "commencement-to-event-time" patterns.

[Received: February 2024 — Accepted: March 2025]

References

Abdulkadir, U. I. & Fernando, A. (2024), 'A deep learning model for insurance claims predictions.', *Journal of Artificial Intelligence (2579-0021)* **6**.

Arik, A., Cairns, A. J., Dodd, E., Macdonald, A. S., Shao, A. & Streftaris, G. (2023), 'Insurance pricing for breast cancer under different multiple state models', *arXiv preprint arXiv:2311.15975* .

Bahnemann, D. (2015), 'Distributions for actuaries', *CAS monograph series* **2**, 1–200.

Bolstad, W. (2007), 'Introduction to bayesian statistics. america: A john wiley & sons'.

Clemente, C., Guerreiro, G. R. & Bravo, J. M. (2023), 'Modelling motor insurance claim frequency and severity using gradient boosting', *Risks* **11**(9), 163.

Cousineau, D. & Helie, S. (2013), 'Improving maximum likelihood estimation using prior probabilities: A tutorial on maximum a posteriori estimation and an examination of the weibull distribution', *Tutorials in Quantitative Methods for Psychology* **9**(2), 61–71.

Cox, D. (1972), 'Regression models and life tables. jr statist. soc', *B34* **187**.

Cronk, L. & Aktipis, A. (2021), 'Design principles for risk-pooling systems', *Nature Human Behaviour* **5**(7), 825–833.

Dempster, A. P., Laird, N. M. & Rubin, D. B. (1977), 'Maximum likelihood from incomplete data via the em algorithm', *Journal of the royal statistical society: series B (methodological)* **39**(1), 1–22.

Edwards, W., Lindman, H. & Savage, L. J. (1963), 'Bayesian statistical inference for psychological research.', *Psychological review* **70**(3), 193.

Ekberg, S. (2015), 'Claim-level loss reserving for workers compensation insurance'.

Eric, U., Olusola, O. M. O. & Eze, F. C. (2021), 'A study of properties and applications of gamma distribution', *African Journal of Mathematics and Statistics Studies* **4**(2), 52–65.

Feng, R. (2023), Economics of risk and insurance, *in* 'Decentralized insurance: Technical foundation of business models', Springer, pp. 55–84.

Gilenko, E. V. & Mironova, E. A. (2017), 'Modern claim frequency and claim severity models: An application to the russian motor own damage insurance market', *Cogent Economics & Finance* **5**(1), 1311097.

Gilks, W. R., Richardson, S. & Spiegelhalter, D. (1995), *Markov chain Monte Carlo in practice*, CRC press.

Hesse, C. A., Oduro, F. T., Ofosu, J. B. & Kpeglo, E. D. (2016), 'Assessing the risk of road traffic fatalities across sub-populations of a given geographical zone, using a modified smeed's model', *International Journal of Statistics and Probability* **5**(6), 121–132.

Jaroengeratikun, U., Bodhisuwan, W. & Thongteeraparp, A. (2012), 'A bayesian inference of non-life insurance based on claim counting process with periodic claim intensity', *Open Journal of Statistics* **2**(2), 177–183.

Kaplan, E. L. & Meier, P. (1958), 'Nonparametric estimation from incomplete observations', *Journal of the American statistical association* **53**(282), 457–481.

Kleinbaum, D. G. & Klein, M. (1996), *Survival analysis a self-learning text*, Springer.

Kochenburger, P. & Salve, P. (2023), An introduction to insurance regulation, *in* 'Research handbook on international insurance law and regulation', Edward Elgar Publishing, pp. 247–280.

Kundu, D., Gupta, R. D. & Manglick, A. (2005), 'Discriminating between the lognormal and generalized exponential distributions', *Journal of Statistical Planning and Inference* **127**(1-2), 213–227.

Landriault, D., Willmot, G. E. & Xu, D. (2014), 'On the analysis of time dependent claims in a class of birth process claim count models', *Insurance: Mathematics and Economics* **58**, 168–173.

Liedtke, P. M. (2007), 'What's insurance to a modern economy?', *The Geneva Papers on Risk and Insurance-Issues and Practice* **32**, 211–221.

Louzada, F. & Ramos, P. L. (2018), 'Efficient closed-form maximum a posteriori estimators for the gamma distribution', *Journal of Statistical Computation and Simulation* **88**(6), 1134–1146.

Mann, N. R., Schafer, R. E. & Singpurwalla, N. D. (1974), 'Methods for statistical analysis of reliability and life data(book)', *Research supported by the U. S. Air Force and Rockwell International Corp. New York, John Wiley and Sons, Inc., 1974. 573 p* .

Moumeesri, A., Klongdee, W. & Pongsart, T. (2020), 'Bayesian bonus-malus premium with poisson-lindley distributed claim frequency and lognormal-gamma distributed claim severity in automobile insurance', *WSEAS Transactions on Mathematics* **19**(46), 443–451.

Ndwandwe, L., Allison, J., Santana, L. & Visagie, J. (2024), 'Revisiting the memoryless property–testing for the pareto type i distribution', *arXiv preprint arXiv:2401.13777* .

Nocedal, J. & Wright, S. J. (1999), *Numerical optimization*, Springer.

Ofosu, J. & Hesse, C. (2011), *Introduction to probability and probability distributions*, EPP Books Services.

Omari, C. O., Nyambura, S. G. & Mwangi, J. M. W. (2018), 'Modeling the frequency and severity of auto insurance claims using statistical distributions'.

Press, W. H. (1992), *The art of scientific computing*, Cambridge university press.

Ramani, S., Jafari, H. & Kia, G. S. (2023), 'Ordering results of aggregate claim amounts from two heterogeneous portfolios', *Contemporary Mathematics* pp. 1279–1290.

Riaman, R., Octavian, A., Supian, S., Firman, S. & Saputra, J. (2023), 'Estimating the value-at-risk (var) in stock investment of insurance companies: An application of the extreme value theory', *Decision Science Letters* **12**, 749–758.

van der Heide, A. (2023), Life insurance in the age of finance, *in* 'Dealing in Uncertainty', Bristol University Press, pp. 1–18.

Yohandoko, S., Prabowo, A., Yakubu, U. A. & Wang, C. (2023), 'Life insurance aggregate claims distribution model estimation', *Life* **4**(4), 117–125.

Zaçaj, O., Raço, E., Haxhi, K., Llagami, E. & Hila, K. (2022), 'Bootstrap methods for claims reserving: R language approach', *WSEAS Transactions on Mathematics* **21**, 252–259.

Zakaria, Z., Azmi, N. M., Hassan, N. F. H. N., Salleh, W. A., Tajuddin, M. T. H. M., Sallem, N. R. M. & Noor, J. M. M. (2016), 'The intention to purchase life insurance: A case study of staff in public universities', *Procedia Economics and Finance* **37**, 358–365.

Zhou, L. (2024), Intelligent insurance actuarial model under machine learning and data mining, *in* '2024 IEEE 2nd International Conference on Image Processing and Computer Applications (ICIPCA)', IEEE, pp. 1726–1731.

Zuanetti, D. A., Diniz, C. A. & Leite, J. G. (2006), 'A lognormal model for insurance claims data', *REVSTAT-Statistical Journal* **4**(2), 131–142.

Appendix

Listings A1. Likelihood Function of μ and σ for the Lognormal Distribution

```
function y = MLE_lognormal(mu,sigma,x,n,xr,r)
y=(1-normcdf((log(xr))))^(n-r)*(1/sigma)^r*exp(-0.5*sum((log(x)-mu)/sigma)^2);
```

Listings A2. Metropolis Hastings in MATLAB using the Likelihood Function for Lognormal Distribution

```
% % Metropolis procedure to sample from the posterior distribution
% Component-wise updating. Use a normal proposal distribution
opts = spreadsheetImportOptions("NumVariables", 1);
% Specify sheet and range
opts.Sheet = "Sheet1";
opts.DataRange = "A1:A80";
% Specify column names and types
opts.VariableNames = "x";
opts.VariableTypes = "double";
% Import the data
x = readtable("C:\Users\USER\Lognormal.xlsx", opts, "UseExcel", false);
x=table2array(x);
r=length(x);
xr=x(80);
n=120;
```

```
% % Initialize the Metropolis sampler
T=5000; % Set the maximum number of iteration
propsigma=[0.014,0.006]; % standard deviation of proposal distribution
parametermin=[2,1]; % define minimum for alpha and beta
parametermax=[4,3]; % define maximum for alpha and beta
seed=1; rand( 'state' , seed ); randn('state',seed ); %#ok<RAND> % set the random
seed
state=zeros(2,T); % storage space for the state of the sampler
mu=unifrnd(parametermin(1),parametermax(1)); % Start value for mu
sigma=unifrnd(parametermin(2),parametermax(2)); % Start value for sigma
t=1; % initialize iteration at 1
state(1,t)=mu; % save the current state
state(2,t)=sigma;
% % Start sampling
while t<T % Iterate until we have T samples
t=t+1;
% % Propose a new value for mu
new_mu=normrnd(mu,propsigma(1));
pratio=MLE_lognormal(new_mu,sigma,x,n,xr,r)/MLE_lognormal(mu,sigma,x,n,xr,r);
a=min([1 pratio]); % Calculate the acceptance ratio
u=rand; % Draw a uniform deviate from [0 1]
if u<a % Do we accept this proposal?
mu=new_mu; % proposal becomes new value for mu
end
% % Propose a new value for sigma
new_sigma=normrnd(sigma,propsigma(2));
pratio=MLE_lognormal(mu,new_sigma,x,n,xr,r)/MLE_lognormal(mu,sigma,x,n,xr,r);
a=min([1 pratio]); % Calculate the acceptance ratio
u=rand; % Draw a uniform deviate from [0 1]
if u<a % Do we accept this proposal?
sigma=new_sigma; % proposal becomes new value for beta
end
% % Save state
state(1,t) = mu;
state(2,t) = sigma;
end
Mean=mean(state,2)
Mode=mode(state,2)
```

Listings A3. Likelihood function for λ

```
function y = MLE_Exponential(lambda,x,n,xr,r)
y=1/(lambda^r)*exp(-(sum(x)+(n-r)*xr)/lambda);
```

Listings A4. Metropolis Hastings in MATLAB using the Likelihood Function for exponential distribution

```
% % Metropolis procedure to sample from the posterior distribution
% Component-wise updating. Use a normal proposal distribution
opts = spreadsheetImportOptions("NumVariables", 1);
% Specify sheet and range
opts.Sheet = "Sheet1";
opts.DataRange = "A1:A80";
% Specify column names and types
opts.VariableNames = "x";
opts.VariableTypes = "double";
% Import the data
x = readtable("C:\Users\USER\Exp.xlsx", opts, "UseExcel", false);
x=table2array(x);
r=length(x);
xr=x(80);
n=120;
% % Initialize the Metropolis sampler
T=5000; % Set the maximum number of iteration
sigma = 0.5; % Set standard deviation of normal proposal density
lambdamin = 140; lambdamax = 160; % define a range for starting values
lambda = zeros( 1 , T ); % Init storage space for our samples
lambda(1) = unifrnd( lambdamin , lambdamax ); % Generate start value
%% Start sampling
t = 1;
while t < T % Iterate until we have T samples
t = t + 1;
% Propose a new value for theta using a normal proposal density
lambda_star = normrnd( lambda(t-1) , sigma );
% Calculate the acceptance ratio
alpha = min([1 MLE_Exponential(lambda_star,x,n,xr,r) / MLE_Exponential(lambda(t-
1),x,n,xr,r)]);
% Draw a uniform deviate from [ 0 1 ]
u = rand;
% Do we accept this proposal?
if u < alpha
lambda(t) = lambda_star; % If so, proposal becomes new state
else
lambda(t) = lambda(t-1); % If not, copy old state
end
end
Mean=mean(lambda)
Mode=mode(lambda)
```