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E-bayesian and H-bayesian Estimation of Inverse Power Lomax Distribution under Different Loss Functions with an Application of Clinical Data

Estimación e-bayesiana y h-bayesiana de la distribución de potencia inversa de Lomax bajo diferentes funciones de pérdida con una aplicación de datos clínicos

Hemani Sharma[1,a], Subhankar Dutta[2,b]

[1]Department of Statistics, University of Jammu, Jammu and Kashmir, India

[2]Department of mathematics, Bioinformatics and Computer Applications, Maulana Azad National Institute of Technology, Bhopal, Madhya Pradesh, India

Abstract

In this paper, Expected Bayesian and Hierarchical Bayesian techniques have been discussed to estimate the shape parameter of the inverse power Lomax distribution. The proposed estimates for the shape parameter are obtained by using an informative gamma prior based on squared error, entropy and weighted balance loss functions. The definitions of the proposed estimators as well as their characteristics are provided. A Monte Carlo simulation is executed to compare the performance of the proposed estimators in terms of mean squared error. Finally a real life data set has been analyzed for further illustrations.

***Key words*:** Bayesian estimation; e-bayesian estimation; h-bayesian estimation; inverse power lomax distribution; mean squared error.

Resumen

En este artículo, se analizan las técnicas bayesiana esperada y bayesiana jerárquica para estimar el parámetro de forma de la distribución Lomax de potencia inversa. Las estimaciones propuestas para el parámetro de forma se obtienen utilizando una distribución gamma previa informativa basada en el error cuadrático, la entropía y las funciones de pérdida de balance ponderadas. Se proporcionan las definiciones de los estimadores propuestos, así como sus características. Se ejecuta una simulación de Monte Carlo para comparar el rendimiento de los estimadores propuestos en términos de error cuadrático medio. Finalmente, se analiza un conjunto de datos reales para obtener más ejemplos.

***Palabras clave*:** Distribución Lomax de potencia inversa; error cuadrático medio; estimación bayesiana; estimación e-bayesiana; estimación h-bayesiana.

[a]Ph.D. E-mail: hemanisharma124@gmail.com
[b]Ph.D. E-mail: subhankar.dta@gmail.com

1. Introduction

In life testing, the Lomax distribution is one of the most often used probability distributions. In the fields of economics and actuarial science, this distribution has been used. Harris (1968) models company failure data using the Lomax distribution for income and wealth. Because of its broad applicability, the Lomax distribution is widely employed in biological investigations and the distribution of computer file sizes. According to Rady et al. (2016), the Power Lomax (PL) distribution can be derived from the Lomax distribution by using the power transformation $Z = Y^{\frac{1}{\beta}}$, where Y follows the Lomax distribution. The probability density function (pdf) of PL distribution is defined as

$$f(z;\alpha,\lambda,\beta) = \alpha\beta\lambda^{\alpha}z^{\beta-1}(\lambda + z^{\beta})^{-\alpha-1}, z,\alpha,\lambda,\beta > 0 \tag{1}$$

Then using the transformation $X = \frac{1}{Z}$, the pdf of inverse power Lomax (IPL) distribution can be obtained as

$$f(x;\alpha,\lambda,\beta) = \frac{\alpha\beta}{\lambda}x^{-(\beta+1)}\left(1 + \frac{x^{-\beta}}{\lambda}\right)^{-\alpha-1},\ x,\alpha,\lambda,\beta > 0, \tag{2}$$

where α, λ, β are shape, location and scale parameters, respectively. The cumulative distribution function (cdf) of the IPL distribution is given by

$$F(x;\alpha,\lambda,\beta) = \left(1 + \frac{x^{-\beta}}{\lambda}\right)^{-\alpha},\ x,\alpha,\lambda,\beta > 0. \tag{3}$$

The reliability function of the IPL distribution is given as follows

$$R(t) = 1 - \left(1 + \frac{t^{-\beta}}{\lambda}\right)^{-\alpha},\ t > 0,\alpha,\lambda,\beta > 0. \tag{4}$$

The IPL is very adaptable when it comes to analysing scenarios that have a failure rate that is not monotone. As a result, there is a broad variety of contexts in which the IPL distribution might be useful, including econometrics, biology, survey sampling, engineering sciences, medicine, and even life testing.

In addition to the Bayesian methodology, we have also studied Hierarchical Bayesian (H-Bayesian) estimation technique, which was introduced by Lindley & Smith (1972). The H-Bayesian approach is more resilient than Bayesian methods since it comprises two stages for constructing the prior distribution. The complicated integrals contained in the estimator formulation are one of the technique's drawbacks. It must be solved using a combination of numerical approximation methods. As an immediate outcome of this, doing the computation takes up a greater amount of time and is more stressful. In order to make up for the shortcomings of the H-Bayesian approach, the expected Bayesian (E-Bayesian) estimation method was developed. When contrasted to the H-Bayesian approach, E-Bayesian seems very simple. In recent years E- and H- Bayesian estimation have gained a lot of attention by several researchers. Han & Ding (2004) introduced E-Bayesian estimation technique. Since then, there has been a lot of work in the literature on inference using E-Bayesian estimation. For instance, Han (2009) developed E- and H- Bayesian estimates for the failure rate of exponential distributions and investigated their features. Jaheen & Okasha (2011) discussed the E-Bayesian estimation for the Burr type XII distribution based on Type II censored data. Han (2017) also discussed the E- and H-Bayesian estimators for Pareto distribution under various loss functions. E-Bayesian and Hierarchical Bayesian estimation of the parameter and reliability function of the Inverse Weibull distribution were discussed by Basheer et al. (2021).The E- and H-Bayesian estimates of the scale parameter and inverted hazard rate of the inverse Rayleigh distribution were studied by Abdul-Sathar & Athirakrishnan (2021). Due to the importance of IPL distribution in lifetime analysis and the usefulness of E- and H-Bayesian estimation over Bayesian estimation, we have considered the E- and H- Bayesian estimation for IPL distribution under some different loss functions. These methods offer advantages in handling different types of data and loss functions, leading to more robust and efficient inference. A Monte Carlo simulation study has been carried out to compare the performance of the proposed estimates. A real life data set has been analyzed. The work seeks to establish a guideline for selecting the best method of estimation if the item/subgroup quality characteristic follows IPL distribution, which we believe would be of profound interest to applied statisticians and quality control engineers.

The following section structure is presented in this article. In Section 2, we derive Bayesian estimators of the shape parameter using various loss functions. Section 3 focuses on the derivation of E-Bayesian estimators of the shape parameter under different loss functions. In Section 4, we obtain H-Bayesian estimators for various loss functions. The properties of the E- and H-Bayesian estimators are discussed in Section 5. A simulation study is conducted using R to assess the performance of the estimators under different loss functions in Section 6. Additionally, in Section 7, we analyze a real-life dataset for illustrative purposes. Finally, in Section 8, we provide conclusions based on the findings of this study.

2. Bayesian Estimation

In this section, we derived the Bayesian estimators of shape parameter α of Inverse Power Lomax (IPL) distribution using the Squared Error loss function (SELF), Entropy loss function (ELF) and Weighted balance loss function (WBLF).

Let $\mathbf{x} = \{x_1, \ldots, x_n\}$ be a random sample of size n taken from the IPL distribution. Then the likelihood function for the given sample observations is given by

$$L(\alpha|\mathbf{x}) = \frac{\alpha^n \beta^n}{\lambda^n} \prod_{i=1}^{n} \frac{x_i^{(\beta+1)}}{\left(1 + \frac{x_i^{-\beta}}{\lambda}\right)} \exp\left(-\alpha \sum_{i=1}^{n} \log\left(1 + \frac{x_i^{-\beta}}{\lambda}\right)\right). \tag{5}$$

2.1. Prior and Posterior Distribution

Consider the prior distribution of α follows gamma distribution with hyperparameters a and b, respectively. Then prior distribution function is

$$g_1(\alpha|a,b) = \frac{b^a\alpha^{a-1}}{\Gamma(a)}e^{-b\alpha}; \alpha, a, b > 0 \tag{6}$$

Using the likelihood function (5) and the prior (6), the posterior distribution for the parameter α becomes

$$\begin{aligned}
P_1(\alpha|\mathbf{x}) &= \frac{L(\alpha|\mathbf{x}) * g_1(\alpha|a,b)}{\int_0^\infty L(\alpha|\mathbf{x}) * g_1(\alpha|a,b)d\alpha} \\
&= \frac{\frac{\alpha^n\beta^n}{\lambda^n}\prod_{i=1}^n \frac{x_i^{(\beta+1)}}{\left(1+\frac{x_i^{-\beta}}{\lambda}\right)} \exp\left[-\alpha\sum_{i=1}^n log\left(1+\frac{x_i^{-\beta}}{\lambda}\right)\right] * \frac{b^a\alpha^{a-1}}{\Gamma(a)}\exp\left(-b\alpha\right)}{\int_0^\infty \frac{\alpha^n\beta^n}{\lambda^n}\prod_{i=1}^n \frac{x_i^{(\beta+1)}}{\left(1+\frac{x_i^{-\beta}}{\lambda}\right)} \exp\left[-\alpha\sum_{i=1}^n log\left(1+\frac{x_i^{-\beta}}{\lambda}\right)\right] * \frac{b^a\alpha^{a-1}}{\Gamma(a)}\exp\left(-b\alpha\right)d\alpha} \\
P_1(\alpha|\mathbf{x}) &= \frac{S^{n+a}\alpha^{(n+a)-1}\exp\left(-\alpha S\right)}{\Gamma(n+a)},
\end{aligned}$$

where $S = b + \sum_{i=1}^n log\left(1+\frac{x_i^{-\beta}}{\lambda}\right)$.

2.2. Bayesian Estimate of α under SELF

The SELF is defined as

$$L(\alpha) = c(\alpha - \hat{\alpha})^2 \tag{7}$$

where $\hat{\alpha}$ is an estimator of α. Then the associated Bayes estimate of α is written as

$$\hat{\alpha}_{BS} = E[\alpha|\mathbf{x}], \tag{8}$$

provided that $E[\alpha|\mathbf{x}]$ exists and finite.

Theorem 1. *For a sample $\mathbf{x} = \{x_1, \ldots, x_n\}$, the Bayes estimator of α under SELF is given by*

$$\hat{\alpha_{BS}} = \frac{n+a}{S}.$$

***Proof*.** By using SELF, the Bayes estimator is given by

$$\begin{aligned}
E[\alpha|\mathbf{x}] &= \int_0^\infty \alpha.P_1(\alpha|\mathbf{x}) \\
&= \int_0^\infty \frac{1}{\alpha}\frac{S^{n+a}\alpha^{(n+a)-1}\exp\left(-\alpha S\right)}{\Gamma(n+a)}d\alpha \\
&= \frac{S^{n+a}}{\Gamma(n+a)}.\frac{\Gamma(n+a+1)}{S^{n+a+1}} = \frac{n+a}{S}
\end{aligned}$$

Therefore,

$$\alpha_{\hat{BS}} = E[\alpha|\mathbf{x}] = \frac{n+a}{S}. \tag{9}$$

□

2.3. Bayesian Estimate of α under ELF

Dey et al. (1987) discussed the ELF of the form:

$$L(\alpha) = \left[\left(\frac{\hat{\alpha}}{\alpha}\right) - \log\left(\frac{\hat{\alpha}}{\alpha}\right) - 1\right], \tag{10}$$

where $\hat{\alpha}$ is an estimator of α. Then the associated Bayes estimate of α is written as

$$\hat{\alpha}_{BE} = E[\alpha^{-1}|\mathbf{x}]^{-1}, \tag{11}$$

provided that $E[\alpha^{-1}|\mathbf{x}]^{-1}$ exists and finite.

Theorem 2. *For a sample* $\mathbf{x} = \{x_1, \ldots, x_n\}$*, the Bayes estimator of* α *under ELF is given by*

$$\alpha_{\hat{BE}} = \frac{n+a-1}{S}.$$

Proof. By using ELF, the Bayes estimator is given by

$$\begin{aligned} E[\alpha^{-1}|\mathbf{x}] &= \int_0^\infty \frac{1}{\alpha}.P_1(\alpha|x) \\ &= \int_0^\infty \frac{1}{\alpha}\frac{S^{n+a}\alpha^{(n+a)-1}\exp\left(-\alpha S\right)}{\Gamma(n+a)}d\alpha \\ &= \frac{S^{n+a}}{\Gamma(n+a)}.\frac{\Gamma(n+a-1)}{S^{n+a-1}} = \frac{S}{n+a-1} \end{aligned}$$

Therefore,

$$\alpha_{\hat{BE}} = E[\alpha^{-1}|\mathbf{x}]^{-1} = \frac{n+a-1}{S}. \tag{12}$$

□

2.4. Bayesian Estimate of α under WBLF

The WBLF can be expressed as (see Nasir & Aslam, 2015)

$$L(\alpha) = \left(\frac{\alpha - \hat{\alpha}}{\hat{\alpha}}\right)^2, \tag{13}$$

and the associated Bayes estimator of α is given by

$$\hat{\alpha}_{BW} = \frac{E[\alpha^2|\mathbf{x}]}{E[\alpha|\mathbf{x}]}, \tag{14}$$

provided that the expectation $E[\alpha^2|\mathbf{x}]$, $E[\alpha|\mathbf{x}]$ exists and finite.

Theorem 3. *For a sample* $\mathbf{x} = \{x_1, \ldots, x_n\}$*, the Bayes estimator of* α *under WBLF is given by*

$$\alpha_{\hat{BW}} = \frac{n+a+1}{S}.$$

Proof. By using WBLF, the Bayes estimator is given by

$$\begin{aligned} E[\alpha^2|\mathbf{x}] &= \int_0^\infty \alpha^2.P_1(\alpha|x) \\ &= \int_0^\infty \alpha^2 \frac{S^{n+a}\alpha^{(n+a)-1}\exp{(-\alpha S)}}{\Gamma(n+a)} d\alpha \\ &= \frac{S^{n+a}}{\Gamma(n+a)}.\frac{\Gamma(n+a+2)}{S^{n+a+2}} = \frac{(n+a)(n+a+1)}{S^2}, \end{aligned}$$

and

$$E[\alpha|\mathbf{x}] = \frac{n+a}{S},$$

Then,

$$\hat{\alpha}_{BW} = \frac{E[\alpha^2|\mathbf{x}]}{E[\alpha|\mathbf{x}]} = \frac{(n+a+1)}{S}. \tag{15}$$

□

3. Expected-Bayesian Estimation

The prior parameters 'a' and 'b' should be chosen such that the prior provided in (6) is a decreasing function of α, as suggested by Han (1997).

$$\frac{d}{d\alpha} g_1(\alpha|a,b) = \frac{(a^b)}{\Gamma(a)} \alpha^{(b-2)} e^{-\alpha a}((b-1) - a\alpha). \tag{16}$$

Thus, the prior distribution given in (6) becomes a decreasing function of α, for $0 < a < 1$ and $b > 0$. The E-Bayesian estimate of α is given by

$$\hat{\alpha}_{EB} = \int_0^1 \int_0^p \hat{\alpha}_B * \pi(\alpha, a, b)\ dadb, \tag{17}$$

where the domain of the first and second integral is the domain of the hyper parameters 'a' and 'b' respectively for which our prior density function is the decreasing function of α and $\hat{\alpha}_B$ denotes the Bayesian estimate of α.

Now for the E-Bayesian estimates of α, we choose the prior distribution of the hyper parameters 'a' and 'b'. These distributions are primarily used to study the effect of various prior distributions on the E-Bayesian estimates of α. The distributions of hyper parameters 'a' and 'b' are as follows

$$\pi_1(a,b) = \frac{2(q-b)}{q^2}, \quad 0 < a < 1, 0 < b < q, \tag{18}$$

$$\pi_2(a,b) = \frac{1}{q}, \quad 0 < a < 1, 0 < b < q, \tag{19}$$

and

$$\pi_3(a,b) = \frac{2b}{q^2}, \quad 0 < a < 1, 0 < b < q. \tag{20}$$

3.1. E-Bayesian Estimation of α under SELF

Theorem 4. *For a sample $\underline{x} = \{x_1, \ldots, x_n\}$ of IPL distribution, using the priors given in (18), (19) and (20), the E-Bayesian estimators of α under SELF are given by*

$$\begin{aligned}
\hat{\alpha}_{EBS_1} &= \frac{(2n+1)}{q^2}\left[(q+F)\log\left(\frac{q+F}{F}\right) - q\right],\\
\hat{\alpha}_{EBS_2} &= \frac{(2n+1)}{2q}\left[\log\left(\frac{q+F}{F}\right)\right],\\
\hat{\alpha}_{EBS_3} &= \frac{(2n+1)}{q^2}.\left[F\log\left(\frac{F}{q+F}\right) + q\right],
\end{aligned}$$

where $F = S - b$.

***Proof*.** E-Bayes estimate of α under $\pi_1(a,b)$, is obtained as

$$\begin{aligned}
\hat{\alpha}_{EBS_1} &= \int_0^1 \int_0^q \hat{\alpha}_{BS} * \pi_1(a,b)\ dadb\\
&= \frac{2}{q^2}\int_0^1 \int_0^p \frac{n+a}{\left[b + \sum_{i=1}^n log\left(1 + \frac{x_i^{-\beta}}{\lambda}\right)\right]}.(q-b)\ dadb\\
&= \frac{2}{p^2}\int_0^1 (n+a)da * \int_0^q \frac{(q-b)}{b + \sum_{i=1}^n log\left(1 + \frac{x_i^{-\beta}}{\lambda}\right)}\ db\\
&= \frac{2}{q^2}\left[\frac{(2n+1)}{2} * (q+F)\log(q+F) - q\log F - q - F\log F\right]\\
&= \frac{(2n+1)}{q^2}\left[(q+F)\log\left(\frac{q+F}{F}\right) - q\right].
\end{aligned} \tag{21}$$

Now, the E-Bayes estimate of α under $\pi_2(a,b)$, is written as

$$\begin{aligned}
\hat{\alpha}_{EBS_2} &= \int_0^1 \int_0^q \hat{\alpha}_{BS} * \pi_2(a,b)\ dadb \\
&= \int_0^1 \int_0^q \frac{n+a}{\left[b + \sum_{i=1}^n log\left(1 + \frac{x_i^{-\beta}}{\lambda}\right)\right]} . \frac{1}{q}\ dadb \\
&= \frac{1}{q}\left[\int_0^1 (n+a)da \int_0^q \left(\frac{1}{b+F}\right)\ db\right] \\
&= \frac{1}{q}\left[\frac{2n+1}{2} * \log\left(\frac{q+F}{F}\right)\right] \\
&= \frac{(2n+1)}{2q}\left[\log\left(\frac{q+F}{F}\right)\right]. \qquad (22)
\end{aligned}$$

Further, the E-Bayes estimate of α under $\pi_3(a,b)$, is given as

$$\begin{aligned}
\hat{\alpha}_{EBS_3} &= \int_0^1 \int_0^q \hat{\alpha}_{BS} * \pi_3(a,b)\ dadb \\
&= \int_0^1 \int_0^q \frac{n+a}{\left[b + \sum_{i=1}^n log\left(1 + \frac{x_i^{-\beta}}{\lambda}\right)\right]} . \frac{2b}{q^2}\ dadb \\
&= \frac{2}{q^2}\left[\int_0^1 (n+a)da \int_0^q \left(\frac{b}{b+F}\right)\ db\right] \qquad (23) \\
&= \frac{2}{q^2}\left[\frac{2n+1}{2} * F[\log F - \log(q+F) + q\right] \\
&= \frac{(2n+1)}{q^2} . \left[F \log\left(\frac{F}{q+F}\right) + q\right].
\end{aligned}$$

□

3.2. E-Bayesian Estimation of α under ELF

Theorem 5. *For a sample $\mathbf{x} = \{x_1, \ldots, x_n\}$ of IPL distribution, using the priors given in (18), (19) and (20), the E-Bayesian estimators of α under ELF are given by*

$$\begin{aligned}
\hat{\alpha}_{EBE_1} &= \frac{(2n-1)}{q^2}\left[(q+F)\log\left(\frac{q+F}{F}\right) - q\right], \\
\hat{\alpha}_{EBE_2} &= \frac{(2n-1)}{2q}\left[\log\left(\frac{q+F}{F}\right)\right], \\
\hat{\alpha}_{EBE_3} &= \frac{(2n-1)}{q^2} . \left[F \log\left(\frac{F}{q+F}\right) + q\right],
\end{aligned}$$

where $F = S - b$.

Proof. E-Bayes estimate of α under $\pi_1(a,b)$, is written as

$$\begin{aligned}
\hat{\alpha}_{EBE_1} &= \int_0^1 \int_0^q \hat{\alpha}_{BE} * \pi_1(a,b)\ dadb \\
&= \frac{2}{q^2} \int_0^1 \int_0^q \frac{n+a-1}{\left[b + \sum_{i=1}^n log\left(1 + \frac{x_i^{-\beta}}{\lambda}\right)\right]} .(q-b)\ dadb \\
&= \frac{2}{q^2} \int_0^1 (n+a-1)da * \int_0^q \frac{(q-b)}{b + \sum_{i=1}^n log\left(1 + \frac{x_i^{-\beta}}{\lambda}\right)}\ db \\
&= \frac{2}{q^2} \left[\frac{(2n-1)}{2} * (q+F)\log(q+F) - q\log F - q - F\log F\right] \\
&= \frac{(2n-1)}{q^2} \left[(q+F)\log\left(\frac{q+F}{F}\right) - q\right].
\end{aligned} \tag{24}$$

Now, the E-Bayes estimate of α under $\pi_2(a,b)$, is expressed as

$$\begin{aligned}
\hat{\alpha}_{EBE_2} &= \int_0^1 \int_0^p \hat{\alpha}_{BE} * \pi_2(a,b)\ dadb \\
&= \int_0^1 \int_0^q \frac{n+a-1}{\left[b + \sum_{i=1}^n log\left(1 + \frac{x_i^{-\beta}}{\lambda}\right)\right]} . \frac{1}{q}\ dadb \\
&= \frac{1}{q} \left[\int_0^1 (n+a-1)da \int_0^q \left(\frac{1}{b+F}\right)\ db\right] \\
&= \frac{1}{q} \left[\int_0^1 (n+a-1)da \int_0^q \left(\frac{1}{b+F}\right)\ db\right] \\
&= \frac{(2n-1)}{2q} \left[\log\left(\frac{q+F}{F}\right)\right].
\end{aligned} \tag{25}$$

Further, the E-Bayes estimate of α under $\pi_3(a,b)$, is given by

$$\begin{aligned}
\hat{\alpha}_{EBE_3} &= \int_0^1 \int_0^q \hat{\alpha}_{BE} * \pi_3(a,b)\ dadb \\
&= \int_0^1 \int_0^q \frac{n+a-1}{\left[b + \sum_{i=1}^n log\left(1 + \frac{x_i^{-\beta}}{\lambda}\right)\right]} . \frac{2b}{q^2}\ dadb \\
&= \frac{2}{q^2} \left[\int_0^1 (n+a-1)da \int_0^q \left(\frac{b}{b+F}\right)\ db\right] \\
&= \frac{2}{q^2} \left[\frac{2n-1}{2} F\log F - F\log(q+F) + q\right] \\
&= \frac{(2n-1)}{q^2} . \left[F\log\left(\frac{F}{q+F}\right) + q\right].
\end{aligned} \tag{26}$$

□

3.3. E-Bayesian Estimation of α under WBLF

Theorem 6. *For a sample $\mathbf{x} = \{x_1, \ldots, x_n\}$ of IPL distribution, using the priors given in (18), (19) and (20), the E-Bayesian estimators of α under WBLF are given by*

$$\hat{\alpha}_{EBW_1} = \frac{(2n+3)}{q^2}\left[(q+F)\log\left(\frac{q+F}{F}\right) - q\right],$$
$$\hat{\alpha}_{EBW_2} = \frac{(2n+3)}{2q}\left[\log\left(\frac{q+F}{F}\right)\right],$$
$$\hat{\alpha}_{EBW_3} = \frac{(2n+3)}{q^2}.\left[F\log\left(\frac{F}{q+F}\right) + q\right],$$

where $F = S - b$.

Proof. E-Bayes estimate of α under $\pi_1(a,b)$, is written as

$$\begin{aligned}
\hat{\alpha}_{EBW_1} &= \int_0^1\int_0^q \hat{\alpha}_{BW} * \pi_1(a,b)\ dadb \\
&= \frac{2}{q^2}\int_0^1\int_0^q \frac{n+a+1}{\left[b+\sum_{i=1}^n log\left(1+\frac{x_i^{-\beta}}{\lambda}\right)\right]}.(q-b)\ dadb \\
&= \frac{2}{q^2}\int_0^1 (n+a+1)da * \int_0^q \frac{(q-b)}{b+\sum_{i=1}^n log\left(1+\frac{x_i^{-\beta}}{\lambda}\right)}\ db \qquad (27)\\
&= \frac{2}{q^2}\left[\frac{(2n+3)}{2} * (q+F)\log(q+F) - p\log F - q - F\log F\right] \\
&= \frac{(2n+3)}{q^2}\left[(q+F)\log\left(\frac{q+F}{F}\right) - q\right].
\end{aligned}$$

Now, E-Bayes estimate of α under $\pi_2(a,b)$, is written as

$$\begin{aligned}
\hat{\alpha}_{EBW_2} &= \int_0^1\int_0^q \hat{\alpha}_{BW} * \pi_2(a,b)\ dadb \\
&= \int_0^1\int_0^q \frac{n+a+1}{\left[b+\sum_{i=1}^n log\left(1+\frac{x_i^{-\beta}}{\lambda}\right)\right]}.\frac{1}{q}\ dadb \\
&= \frac{1}{q}\left[\int_0^1 (n+a+1)da \int_0^q \left(\frac{1}{b+F}\right)\ db\right] \qquad (28)\\
&= \frac{1}{q}\left[\frac{2n+3}{2}.\log\left(\frac{q+F}{F}\right)\right] \\
&= \frac{(2n+3)}{2q}\left[\log\left(\frac{q+F}{F}\right)\right].
\end{aligned}$$

And, E-Bayes estimate of α under $\pi_3(a,b)$, is obtained as

$$\begin{aligned}
\hat{\alpha}_{EBW_3} &= \int_0^1 \int_0^q \hat{\alpha}_{BW} * \pi_3(a,b)\ dadb \\
&= \int_0^1 \int_0^q \frac{n+a+1}{\left[b+\sum_{i=1}^n log\left(1+\frac{x_i^{-\beta}}{\lambda}\right)\right]} . \frac{2b}{q^2}\ dadb \\
&= \frac{2}{q^2}\left[\int_0^1 (n+a+1)da \int_0^q \left(\frac{b}{b+F}\right)\ db\right] \\
&= \frac{2}{q^2}\left[\frac{2n+3}{2} * F\log\left(\frac{F}{q+F}\right)+q\right] \\
&= \frac{(2n+3)}{q^2} . \left[F\log\left(\frac{F}{q+F}\right)+q\right].
\end{aligned} \tag{29}$$

□

4. Hierarchical Bayesian Estimation

This section addresses the H-Bayesian estimation for the unknown parameter α of IPL distribution are obtained based on SELF, ELF, and WBLF. According to Lindley & Smith (1972), if a and b denotes hyper parameters in $g_1(\alpha|a,b)$ which is given in (6) and the hyper prior distributions of a, b, which are given in (18), (19) and (20). The associated hierarchical prior distributions of α have the following forms:

$$\begin{aligned}
\pi_4(\alpha) &= \int_0^1 \int_0^q g_1(\alpha|a,b)\pi_1(a,b)dbda \\
&= \int_0^1 \int_0 \frac{b^a\alpha^{a-1}}{\Gamma(a)} \exp\left(-b\alpha\right)\frac{2(q-b)}{q^2}\ dbda,
\end{aligned} \tag{30}$$

$$\pi_5(\alpha) = \int_0^1 \int_0^q g_1(\alpha|a,b)\pi_1(a,b)dbda = \int_0^1 \int_0 \frac{b^a\alpha^{a-1}}{\Gamma(a)} \exp\left(-b\alpha\right)\frac{1}{q}\ dbda, \tag{31}$$

and

$$\pi_6(\alpha) = \int_0^1 \int_0^q g_1(\alpha|a,b)\pi_1(a,b)dbda = \int_0^1 \int_0 \frac{b^a\alpha^{a-1}}{\Gamma(a)} \exp\left(-b\alpha\right)\frac{2b}{q^2}\ dbda. \tag{32}$$

Using Bayes theorem and equations (5), (30), (31) and (32), the hierarchical posterior distributions of α is written as

$$
\begin{aligned}
\pi_1(\alpha|x) &= \frac{L(\alpha|x)\pi_4(\alpha)}{\int_0^\infty L(\alpha|x)\pi_4(\alpha)\ d\alpha} \\
&= \frac{\begin{array}{c}\int_0^1 \int_0^q \frac{\alpha^n \beta^n}{\lambda^n} \prod_{i=1}^n \frac{x_i^{(\beta+1)}}{\left(1+\frac{x_i^{-\beta}}{\lambda}\right)} \times \\ \exp\left(-\alpha \sum_{i=1}^n \log\left(1+\frac{x_i^{-\beta}}{\lambda}\right)\right) \frac{b^a \alpha^{a-1}}{\Gamma(a)} \exp\left(-b\alpha\right) \frac{2(q-b)}{q^2}\ dbda\end{array}}{\begin{array}{c}\int_0^1 \int_0^p \int_0^\infty \frac{\alpha^n \beta^n}{\lambda^n} \prod_{i=1}^n \frac{x_i^{(\beta+1)}}{\left(1+\frac{x_i^{-\beta}}{\lambda}\right)} \times \\ \exp\left(-\alpha \sum_{i=1}^n \log\left(1+\frac{x_i^{-\beta}}{\lambda}\right)\right) \frac{b^a \alpha^{a-1}}{\Gamma(a)} \exp\left(-b\alpha\right) \frac{2(q-b)}{q^2}\ d\alpha dbda\end{array}} \\
&= \frac{\int_0^1 \int_0^p \frac{2(p-b)}{p^2} \frac{b^a}{\Gamma(a)} \alpha^{n+a-1} \exp\left(-\alpha S\right)\ dbda}{\int_0^1 \int_0^p \int_0^\infty \frac{2(p-b)}{p^2} \frac{b^a}{\Gamma(a)} \alpha^{n+a-1} \exp\left(-\alpha S\right)\ d\alpha dbda}, \\
&= \frac{\int_0^1 \int_0^p (p-b) \frac{b^a}{\Gamma(a)} \alpha^{n+a-1} \exp\left(-\alpha S\right)\ dbda}{\int_0^1 \int_0^p (p-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a)}{S^{n+a}}\ dbda},
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
\pi_2(\alpha|x) &= \frac{L(\alpha|x)\pi_5(\alpha)}{\int_0^\infty L(\alpha|x)\pi_5(\alpha)\ d\alpha} \\
&= \frac{\begin{array}{c}\int_0^1 \int_0^q \frac{\alpha^n \beta^n}{\lambda^n} \prod_{i=1}^n \frac{x_i^{(\beta+1)}}{\left(1+\frac{x_i^{-\beta}}{\lambda}\right)} \times \\ \exp\left(-\alpha \sum_{i=1}^n \log\left(1+\frac{x_i^{-\beta}}{\lambda}\right)\right) \frac{b^a \alpha^{a-1}}{\Gamma(a)} \exp\left(-b\alpha\right) \frac{1}{q}\ dbda\end{array}}{\begin{array}{c}\int_0^1 \int_0^q \int_0^\infty \frac{\alpha^n \beta^n}{\lambda^n} \prod_{i=1}^n \frac{x_i^{(\beta+1)}}{\left(1+\frac{x_i^{-\beta}}{\lambda}\right)} \times \\ \exp\left(-\alpha \sum_{i=1}^n \log\left(1+\frac{x_i^{-\beta}}{\lambda}\right)\right) \frac{b^a \alpha^{a-1}}{\Gamma(a)} \exp\left(-b\alpha\right) \frac{1}{q}\ d\alpha dbda\end{array}} \\
&= \frac{\int_0^1 \int_0^p \frac{1}{q} \frac{b^a}{\Gamma(a)} \alpha^{n+a-1} \exp\left(-\alpha S\right)\ dbda}{\int_0^1 \int_0^q \int_0^\infty \frac{1}{p} \frac{b^a}{\Gamma(a)} \alpha^{n+a-1} \exp\left(-\alpha S\right)\ d\alpha dbda}, \\
&= \frac{\int_0^1 \int_0^q \frac{b^a}{\Gamma(a)} \alpha^{n+a-1} \exp\left(-\alpha S\right)\ dbda}{\int_0^1 \int_0^q \frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a)}{S^{n+a}}\ dbda}
\end{aligned}
\tag{34}
$$

and

$$
\begin{aligned}
\pi_3(\alpha|x) &= \frac{L(\alpha|x)\pi_6(\alpha)}{\int_0^\infty L(\alpha|x)\pi_6(\alpha)\ d\alpha} \\
&= \frac{\begin{array}{c}\int_0^1\int_0^q \frac{\alpha^n\beta^n}{\lambda^n}\prod_{i=1}^n \frac{x_i^{(\beta+1)}}{\left(1+\frac{x_i^{-\beta}}{\lambda}\right)} \exp\left(-\alpha\sum_{i=1}^n log\left(1+\frac{x_i^{-\beta}}{\lambda}\right)\right)\frac{b^a\alpha^{a-1}}{\Gamma(a)} \\ \exp\left(-b\alpha\right)\frac{2b}{q^2}\ dbda\end{array}}{\begin{array}{c}\int_0^1\int_0^q\int_0^\infty \frac{\alpha^n\beta^n}{\lambda^n}\prod_{i=1}^n \frac{x_i^{(\beta+1)}}{\left(1+\frac{x_i^{-\beta}}{\lambda}\right)}\times \\ \exp\left(-\alpha\sum_{i=1}^n log\left(1+\frac{x_i^{-\beta}}{\lambda}\right)\right)\frac{b^a\alpha^{a-1}}{\Gamma(a)}\exp\left(-b\alpha\right)\frac{2b}{q^2}\ d\alpha dbda\end{array}} \\
&= \frac{\int_0^1\int_0^q \frac{2b}{q^2}\frac{b^a}{\Gamma(a)}\alpha^{n+a-1}\exp\left(-\alpha S\right)\ dbda}{\int_0^1\int_0^q\int_0^\infty \frac{2b}{q^2}\frac{b^a}{\Gamma(a)}\alpha^{n+a-1}\exp\left(-\alpha S\right)\ d\alpha dbda} \\
&= \frac{\int_0^1\int_0^q \frac{b^{a+1}}{\Gamma(a)}\alpha^{n+a-1}\exp\left(-\alpha S\right)\ dbda}{\int_0^1\int_0^q \frac{b^{a+1}}{\Gamma(a)}\frac{\Gamma(n+a)}{S^{n+a}}\ dbda}.
\end{aligned} \tag{35}
$$

4.1. H-Bayesian Estimation under SELF

Using SELF and the H-posterior distributions which are defined respectively in (7), (33), (34) and (35), the H-Bayesian estimates $\hat{\alpha}_{HS_1}$, $\hat{\alpha}_{HS_2}$, $\hat{\alpha}_{HS_3}$ of α can be defined as

$$\hat{\alpha}_{HS_j} = E[(\alpha|x)]; j = 1, 2, 3.$$

Here,

$$
\begin{aligned}
E[(\alpha|x)] &= \int_0^\infty \alpha.\pi_1(\alpha|x)\ d\alpha \\
&= \frac{\int_0^1\int_0^q\int_0^\infty (q-b)\frac{b^a}{\Gamma(a)}\alpha^{(n+a+1)-1}\exp\left(-\alpha S\right)\ d\alpha dbda}{\int_0^1\int_0^q (q-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(n+a)}{S^{n+a}}\ dbda} \\
&= \frac{\int_0^1\int_0^q (q-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(n+a+1)}{S^{n+a+1}}\ dbda}{\int_0^1\int_0^q (q-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(n+a)}{S^{n+a}}\ dbda},
\end{aligned}
$$

$$
\begin{aligned}
E[(\alpha|x)] &= \int_0^\infty \alpha.\pi_2(\alpha|x)\ d\alpha \\
&= \frac{\int_0^1\int_0^q\int_0^\infty \frac{b^a}{\Gamma(a)}\alpha^{(n+a+1)-1}\exp\left(-\alpha S\right)\ d\alpha dbda}{\int_0^1\int_0^q \frac{b^a}{\Gamma(a)}\frac{\Gamma(n+a)}{S^{n+a}}\ dbda} \\
&= \frac{\int_0^1\int_0^q \frac{b^a}{\Gamma(a)}\frac{\Gamma(n+a+1)}{S^{n+a+1}}dbda}{\int_0^1\int_0^q \frac{b^a}{\Gamma(a)}\frac{\Gamma(n+a)}{S^{n+a}}dbda},
\end{aligned}
$$

$$
\begin{aligned}
E[(\alpha|x)] &= \int_0^\infty \alpha.\pi_3(\alpha|x)\ d\alpha \\
&= \frac{\int_0^1\int_0^q\int_0^\infty b\frac{b^a}{\Gamma(a)}\alpha^{(n+a+1)-1}\exp\left(-\alpha S\right)\ d\alpha dbda}{\int_0^1\int_0^q b\frac{b^a}{\Gamma(a)}\frac{\Gamma(n+a)}{S^{n+a}}\ dbda} \\
&= \frac{\int_0^1\int_0^q \frac{b^{a+1}}{\Gamma(a)}\frac{\Gamma(n+a+1)}{S^{n+a+1}}\ dbda}{\int_0^1\int_0^q \frac{b^{a+1}}{\Gamma(a)}\frac{\Gamma(n+a)}{S^{n+a}}\ dbda},
\end{aligned}
$$

then we get,

$$
\hat{\alpha}_{HS_1} = \frac{\int_0^1\int_0^q (q-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(n+a+1)}{S^{n+a+1}}\ dbda}{\int_0^1\int_0^q (q-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(n+a)}{S^{n+a}}\ dbda}, \tag{36}
$$

$$
\hat{\alpha}_{HS_2} = \frac{\int_0^1\int_0^q \frac{b^a}{\Gamma(a)}\frac{\Gamma(n+a+1)}{S^{n+a+1}}\ dbda}{\int_0^1\int_0^q \frac{b^a}{\Gamma(a)}\frac{\Gamma(n+a)}{S^{n+a}}\ dbda}, \tag{37}
$$

$$
\hat{\alpha}_{HS_3} = \frac{\int_0^1\int_0^q \frac{b^{a+1}}{\Gamma(a)}\frac{\Gamma(n+a+1)}{S^{n+a+1}}\ dbda}{\int_0^1\int_0^q \frac{b^{a+1}}{\Gamma(a)}\frac{\Gamma(n+a)}{S^{n+a}}\ dbda}. \tag{38}
$$

4.2. H-Bayesian Estimation under ELF

Using ELF and the H-posterior distributions which are defined respectively in (10), (33), (34) and (35), the H-Bayesian estimates $\hat{\alpha}_{HE_1}$, $\hat{\alpha}_{HE_2}$, $\hat{\alpha}_{HE_3}$ of α can be defined as

$$
\hat{\alpha}_{HE_j} = E[(\alpha^{-1}|x)]^{-1}; j = 1, 2, 3.
$$

Here,

$$
\begin{aligned}
E[(\alpha^{-1}|x)] &= \int_0^\infty \frac{1}{\alpha}.\pi_1(\alpha|x)\ d\alpha \\
&= \frac{\int_0^1\int_0^q\int_0^\infty (q-b)\frac{b^a}{\Gamma(a)}\alpha^{(n+a-1)-1}\exp\left(-\alpha S\right)\ d\alpha dbda}{\int_0^1\int_0^q (q-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(n+a)}{S^{n+a}}\ dbda} \\
&= \frac{\int_0^1\int_0^q (q-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(n+a-1)}{S^{n+a-1}}\ dbda}{\int_0^1\int_0^q (q-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(n+a)}{S^{n+a}}\ dbda},
\end{aligned}
$$

$$\begin{aligned}
E[(\alpha^{-1}|x)] &= \int_0^\infty \frac{1}{\alpha}.\pi_2(\alpha|x)\ d\alpha \\
&= \frac{\int_0^1 \int_0^q \int_0^\infty \frac{b^a}{\Gamma(a)} \alpha^{(n+a-1)-1} \exp\left(-\alpha S\right)\ d\alpha db da}{\int_0^1 \int_0^q \frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a)}{S^{n+a}}\ dbda} \\
&= \frac{\int_0^1 \int_0^q \frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a-1)}{S^{n+a-1}} dbda}{\int_0^1 \int_0^q \frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a)}{S^{n+a}} dbda}, \\
E[(\alpha^{-1}|x)] &= \int_0^\infty \frac{1}{\alpha}.\pi_3(\alpha|x)\ d\alpha \\
&= \frac{\int_0^1 \int_0^q \int_0^\infty b\frac{b^a}{\Gamma(a)} \alpha^{(n+a-1)-1} \exp\left(-\alpha S\right)\ d\alpha db da}{\int_0^1 \int_0^q b\frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a)}{S^{n+a}}\ dbda} \\
&= \frac{\int_0^1 \int_0^q \frac{b^{a+1}}{\Gamma(a)} \frac{\Gamma(n+a-1)}{S^{n+a-1}}\ dbda}{\int_0^1 \int_0^q \frac{b^{a+1}}{\Gamma(a)} \frac{\Gamma(n+a)}{S^{n+a}}\ dbda},
\end{aligned}$$

then we get,

$$\hat{\alpha}_{HE_1} = \frac{\int_0^1 \int_0^q (q-b)\frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a)}{S^{n+a}}\ dbda}{\int_0^1 \int_0^q (q-b)\frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a-1)}{S^{n+a-1}}\ dbda}, \tag{39}$$

$$\hat{\alpha}_{HE_2} = \frac{\int_0^1 \int_0^q \frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a)}{S^{n+a}}\ dbda}{\int_0^1 \int_0^q \frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a-1)}{S^{n+a-1}}\ dbda}, \tag{40}$$

$$\hat{\alpha}_{HE_3} = \frac{\int_0^1 \int_0^q \frac{b^{a+1}}{\Gamma(a)} \frac{\Gamma(n+a)}{S^{n+a}}\ dbda}{\int_0^1 \int_0^q \frac{b^{a+1}}{\Gamma(a)} \frac{\Gamma(n+a-1)}{S^{n+a-1}}\ dbda}. \tag{41}$$

4.3. H-Bayesian Estimation under WBLF

Assuming WBLF as defined in (13), and using the priors defined in (33), (34) and (35), the H-Bayes estimates $\hat{\alpha}_{HW_1}$, $\hat{\alpha}_{HW_2}$, $\hat{\alpha}_{HW_3}$ of α are

$$\hat{\alpha}_{HW_j} = \frac{E[(\alpha^2|x)]}{E[(\alpha|x)]}; j = 1, 2, 3.$$

Here,

$$\begin{aligned}
E[(\alpha^2|x)] &= \int_0^\infty \alpha^2 \pi_1(\alpha|x)\ d\alpha \\
&= \frac{\int_0^1 \int_0^q \int_0^\infty (q-b)\frac{b^a}{\Gamma(a)} \alpha^{(n+a+2)-1} \exp\left(-\alpha S\right)\ d\alpha db da}{\int_0^1 \int_0^q (q-b)\frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a)}{S^{n+a}}\ dbda} \\
&= \frac{\int_0^1 \int_0^q (q-b)\frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a+2)}{S^{n+a+2}}\ dbda}{\int_0^1 \int_0^q (q-b)\frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a)}{S^{n+a}}\ dbda},
\end{aligned}$$

$$E[(\alpha|x)] = \int_0^\infty \alpha\pi_1(\alpha|x)\ d\alpha$$
$$= \frac{\int_0^1\int_0^q (q-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(n+a+1)}{S^{n+a+1}}\ dbda}{\int_0^1\int_0^q (q-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(n+a)}{S^{n+a}}\ dbda},$$

$$E[(\alpha^2|x)] = \int_0^\infty \alpha^2\pi_2(\alpha|x)\ d\alpha$$
$$= \frac{\int_0^1\int_0^q \frac{b^a}{\Gamma(a)}\frac{\Gamma(n+a+2)}{S^{n+a+2}}\ dbda}{\int_0^1\int_0^q \frac{b^a}{\Gamma(a)}\frac{\Gamma(n+a)}{S^{n+a}}\ dbda},$$
$$E[(\alpha|x)] = \int_0^\infty \alpha\pi_2(\alpha|x)\ d\alpha$$
$$= \frac{\int_0^1\int_0^q \frac{b^a}{\Gamma(a)}\frac{\Gamma(n+a+1)}{S^{n+a+1}}\ dbda}{\int_0^1\int_0^q \frac{b^a}{\Gamma(a)}\frac{\Gamma(n+a)}{S^{n+a}}\ dbda},$$

then we get,

$$\hat{\alpha}_{HW_1} = \frac{\int_0^1\int_0^q (q-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(n+a+2)}{S^{n+a+2}}\ dbda}{\int_0^1\int_0^q (q-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(n+a+1)}{S^{n+a+1}}\ dbda}, \tag{42}$$
$$\hat{\alpha}_{HW_2} = \frac{\int_0^1\int_0^q \frac{b^a}{\Gamma(a)}\frac{\Gamma(n+a+2)}{S^{n+a+2}}\ dbda}{\int_0^1\int_0^q \frac{b^a}{\Gamma(a)}\frac{\Gamma(n+a+1)}{S^{n+a+1}}\ dbda}, \tag{43}$$
$$\hat{\alpha}_{HW_3} = \frac{\int_0^1\int_0^q \frac{b^{a+1}}{\Gamma(a)}\frac{\Gamma(n+a+2)}{S^{n+a+2}}\ dbda}{\int_0^1\int_0^q \frac{b^{a+1}}{\Gamma(a)}\frac{\Gamma(n+a+1)}{S^{n+a+1}}\ dbda}. \tag{44}$$

5. Properties of E-Bayesian and H-Bayesian Estimation of α

In this section, we have discussed the properties of E-Bayesian estimates and the relations among the E-Bayesian and H-Bayesian estimates.

5.1. The Relations Between the E-Bayesian Estimates Under Different Loss Functions

Theorem 7. *It follows from (21), (22) and (23) that*
(1) $\hat{\alpha}_{EBS_3} < \hat{\alpha}_{EBS_2} < \hat{\alpha}_{EBS_1}$
(2) $\lim_{F\to\infty}\hat{\alpha}_{EBS_j} = 0, j = 1, 2, 3.$

Proof. (1) Using (21), (22), and (23), we have

$$\begin{aligned}\hat{\alpha}_{EBS_1} - \hat{\alpha}_{EBS_2} &= \frac{(2n+1)}{q^2}\left[(q+F)log\left(1+\frac{q}{F}\right)-q\right] - \frac{(2n-1)}{2q}\left[\log\left(1+\frac{q}{F}\right)\right] \\ &= \frac{(2n+1)}{q}\left[\log\left(1+\frac{q}{F}\right)\left(\frac{F}{q}+\frac{1}{2}\right)-1\right]\end{aligned}$$

and

$$\begin{aligned}\hat{\alpha}_{EBS_2} - \hat{\alpha}_{EBS_3} &= \frac{(2n+1)}{2q}\left[\log\left(1+\frac{q}{F}\right)\right] - \frac{(2n+1)}{p^2}\left[F\log\left(\frac{F}{q+F}\right)+q\right] \\ &= \frac{(2n+1)}{q}\left[\log\left(1+\frac{q}{F}\right)\left(\frac{F}{q}+\frac{1}{2}\right)-1\right].\end{aligned}$$

Therefore,

$$\hat{\alpha}_{EBS_1} - \hat{\alpha}_{EBS_2} = \hat{\alpha}_{EBS_2} - \hat{\alpha}_{EBS_3} = \frac{(2n+1)}{q}\left[\log\left(1+\frac{q}{F}\right)\left(\frac{F}{q}+\frac{1}{2}\right)-1\right]$$

For $-1 < t \leq 1$, we have $\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{5} - \cdots = \sum_{i=1}^{\infty}(-1)^{i+1}\frac{t^i}{i}$.

Let $t = \frac{q}{F}$, when $0 < q < F$, $0 < \frac{q}{F} < 1$, we get

$$\begin{aligned}&\left[\log\left(1+\frac{q}{F}\right)\left(\frac{F}{q}+\frac{1}{2}\right)-1\right] = \\ &\quad \left(\frac{F}{q}+\frac{1}{2}\right)\left[\left(\frac{q}{F}\right)-\frac{1}{2}\left(\frac{q}{F}\right)^2+\frac{1}{3}\left(\frac{q}{F}\right)^3-\frac{1}{4}\left(\frac{q}{F}\right)^4+\frac{1}{5}\left(\frac{q}{F}\right)^5-\cdots\right]-1 \\ &\quad = \left[1-\frac{1}{2}\left(\frac{q}{F}\right)+\frac{1}{3}\left(\frac{q}{F}\right)^2-\frac{1}{4}\left(\frac{q}{F}\right)^3+\frac{1}{5}\left(\frac{q}{F}\right)^4-\frac{1}{6}\left(\frac{q}{F}\right)^5+\cdots\right] \\ &\quad\quad + \left[\frac{1}{2}\left(\frac{q}{F}\right)+\frac{1}{4}\left(\frac{q}{F}\right)^2+\frac{1}{6}\left(\frac{q}{F}\right)^3-\frac{1}{8}\left(\frac{q}{F}\right)^4+\cdots\right]-1 \\ &\quad = \frac{1}{12}\left(\frac{q}{F}\right)^2\left[1-\frac{q}{F}\right]+\frac{3}{40}\left(\frac{q}{F}\right)^4\left[1-\frac{8}{9}\frac{q}{F}\right]+\cdots,\end{aligned}$$

where $0 < \frac{q}{F} < 1$. Thus,

$$\hat{\alpha}_{EBS_1} - \hat{\alpha}_{EBE_2} = \hat{\alpha}_{EBS_2} - \hat{\alpha}_{EBS_3} = \frac{(2n+1)}{q}\left[\log\left(1+\frac{q}{F}\right)\left(\frac{F}{q}+\frac{1}{2}\right)-1\right] > 0,$$

which yields that, $\hat{\alpha}_{EBS_3} < \hat{\alpha}_{EBS_S} < \hat{\alpha}_{EBS_1}$.

(2) From (1), we get

$$\begin{aligned}\lim_{F\to\infty}(\hat{\alpha}_{EBS_1} - \hat{\alpha}_{EBS_2}) &= \lim_{F\to\infty}(\hat{\alpha}_{EBS_2} - \hat{\alpha}_{EBS_3}) \\ &= \frac{(2n+1)}{q} \\ &\quad \times \lim_{F\to\infty}\left[\frac{1}{12}\frac{q^2}{F^2}\left(1-\frac{q}{F}\right)+\frac{3}{40}\frac{q^4}{F^4}\left(1-\frac{8}{9}\frac{q}{F}\right)+\cdots\right].\end{aligned}$$

So it can be easily obtained that

$$\begin{aligned}\lim_{F\to\infty}\hat{\alpha}_{EBS_1} &= \lim_{F\to\infty}\hat{\alpha}_{EBS_2}\\ &= \lim_{F\to\infty}\hat{\alpha}_{EBS_3}.\end{aligned}$$

From (22) and from the proof of (1), we have

$$\begin{aligned}\lim_{F\to\infty}\hat{\alpha}_{EBS_1} &= \lim_{F\to\infty}\frac{2n+1}{q}\left[\frac{1}{2}\left(\frac{q}{F}\right)-\frac{1}{6}\left(\frac{q}{F}\right)^2+\frac{1}{12}\left(\frac{q}{F}\right)^3-\frac{1}{20}\left(\frac{q}{F}\right)^4+\cdots\right]\\ &= 0\end{aligned}$$

Using (2), we get

$$\lim_{F\to\infty}\hat{\alpha}_{EBS_j} = 0,\ j = 1,2,3.$$

$\square$

Theorem 8. *It follows from (24), (25) and (26) that*
(1) $\hat{\alpha}_{EBE_3} < \hat{\alpha}_{EBE_2} < \hat{\alpha}_{EBE_1}$
(2) $\lim_{F\to\infty}\hat{\alpha}_{EBE_j} = 0, j = 1,2,3.$

***Proof*.** (1) From Equations (22), (23) and (24), we have

$$\begin{aligned}\hat{\alpha}_{EBE_1}-\hat{\alpha}_{EBE_2} &= \frac{(2n-1)}{q^2}\left[(q+F)log\left(1+\frac{q}{F}\right)-q\right]-\frac{(2n-1)}{2q}\left[\log\left(1+\frac{q}{F}\right)\right]\\ &= \frac{(2n-1)}{q}\left[\log\left(1+\frac{q}{F}\right)\left(\frac{F}{q}+\frac{1}{2}\right)-1\right]\end{aligned}$$

and

$$\begin{aligned}\hat{\alpha}_{EBE_2}-\hat{\alpha}_{EBE_3} &= \frac{(2n-1)}{2q}\left[\log\left(1+\frac{q}{F}\right)\right]-\frac{(2n-1)}{q^2}\left[F\log\left(\frac{F}{q+F}\right)+q\right]\\ &= \frac{(2n-1)}{q}\left[\log\left(1+\frac{q}{F}\right)\left(\frac{F}{q}+\frac{1}{2}\right)-1\right].\end{aligned}$$

Therefore,

$$\hat{\alpha}_{EBE_1}-\hat{\alpha}_{EBE_2} = \hat{\alpha}_{EBE_2}-\hat{\alpha}_{EBE_3} = \frac{(2n-1)}{q}\left[\log\left(1+\frac{q}{F}\right)\left(\frac{F}{q}+\frac{1}{2}\right)-1\right]$$

For $-1 < t \leq 1$, we have $\ln(1+t) = t-\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}+\frac{t^5}{5}-... = \sum_{i=1}^{\infty}(-1)^{i+1}\frac{t^i}{i}$.

Let $t = \frac{q}{F}$, when $0 < q < F$, $0 < \frac{q}{F} < 1$, we get

$$\begin{aligned}
&\left[\log\left(1+\frac{q}{F}\right)\left(\frac{F}{q}+\frac{1}{2}\right)-1\right] = \\
&\quad \left(\frac{F}{q}+\frac{1}{2}\right)\left[\left(\frac{q}{F}\right)-\frac{1}{2}\left(\frac{q}{F}\right)^2+\frac{1}{3}\left(\frac{q}{F}\right)^3-\frac{1}{4}\left(\frac{q}{F}\right)^4+\frac{1}{5}\left(\frac{q}{F}\right)^5-\cdots\right]-1 \\
&\quad = \left[1-\frac{1}{2}\left(\frac{q}{F}\right)+\frac{1}{3}\left(\frac{q}{F}\right)^2-\frac{1}{4}\left(\frac{q}{F}\right)^3+\frac{1}{5}\left(\frac{q}{F}\right)^4-\frac{1}{6}\left(\frac{q}{F}\right)^5+\cdots\right] \\
&\qquad + \left[\frac{1}{2}\left(\frac{q}{F}\right)+\frac{1}{4}\left(\frac{q}{F}\right)^2+\frac{1}{6}\left(\frac{q}{F}\right)^3-\frac{1}{8}\left(\frac{q}{F}\right)^4+\cdots\right]-1 \\
&\qquad = \frac{1}{12}\left(\frac{q}{F}\right)^2\left[1-\frac{q}{F}\right]+\frac{3}{40}\left(\frac{q}{F}\right)^4\left[1-\frac{8}{9}\frac{q}{F}\right]+\cdots,
\end{aligned}$$

where $0 < \frac{p}{F} < 1$. Thus,

$$\hat{\alpha}_{EBE_1}-\hat{\alpha}_{EBE_2}=\hat{\alpha}_{EBE_2}-\hat{\alpha}_{EBE_3}=\frac{(2n-1)}{q}\left[\log\left(1+\frac{q}{F}\right)\left(\frac{F}{q}+\frac{1}{2}\right)-1\right]>0,$$

which yields that, $\hat{\alpha}_{EBE_3} < \hat{\alpha}_{EBE_2} < \hat{\alpha}_{EBE_1}$.
(2) From (1), we get

$$\begin{aligned}
\lim_{F\to\infty}(\hat{\alpha}_{EBE_1}-\hat{\alpha}_{EBE_2}) &= \lim_{F\to\infty}(\hat{\alpha}_{EBE_2}-\hat{\alpha}_{EBE_3}) \\
&= \frac{(2n-1)}{p} \\
&\quad \times \lim_{F\to\infty}\left[\frac{1}{12}\frac{p^2}{F^2}\left(1-\frac{p}{F}\right)+\frac{3}{40}\frac{p^4}{F^4}\left(1-\frac{8}{9}\frac{p}{F}\right)+\cdots\right].
\end{aligned}$$

So it can be easily obtained that

$$\lim_{F\to\infty}\hat{\alpha}_{EBE_1}=\lim_{F\to\infty}\hat{\alpha}_{EBE_2}=\lim_{F\to\infty}\hat{\alpha}_{EBE_3}.$$

From (22) and from the proof of (1), we have

$$\lim_{F\to\infty}\hat{\alpha}_{EBE_1}=\lim_{F\to\infty}\frac{2n-1}{q}\left[\frac{1}{2}\left(\frac{q}{F}\right)-\frac{1}{6}\left(\frac{q}{F}\right)^2+\frac{1}{12}\left(\frac{q}{F}\right)^3-\frac{1}{20}\left(\frac{q}{F}\right)^4+\cdots\right]=0$$

Using (2), we get

$$\lim_{F\to\infty}\hat{\alpha}_{EBE_j}=0,\ j=1,2,3.$$

$\square$

Theorem 9. *It follows from (27), (28) and (29) that*
(1) $\hat{\alpha}_{EBW_3} < \hat{\alpha}_{EBW_2} < \hat{\alpha}_{EBW_1}$,
(2) $\lim_{F\to\infty}\hat{\alpha}_{EBW_j}=0, j=1,2,3.$

Proof . (1) From (25), (26) and (27), we have

$$\begin{aligned}\hat{\alpha}_{EBW_1} - \hat{\alpha}_{EBW_2} &= \frac{(2n+3)}{q^2}\left[(q+F)log\left(1+\frac{q}{F}\right)-q\right] - \frac{(2n+3)}{2q}\left[\log\left(1+\frac{q}{F}\right)\right] \\ &= \frac{(2n+3)}{q}\left[\log\left(1+\frac{q}{F}\right)\left(\frac{F}{q}+\frac{1}{2}\right)-1\right],\end{aligned}$$

and

$$\begin{aligned}\hat{\alpha}_{EBW_2} - \hat{\alpha}_{EBW_3} &= \frac{(2n+3)}{2q}\left[\log\left(1+\frac{q}{F}\right)\right] - \frac{(2n+3)}{q^2}\left[F\log\left(\frac{F}{q+F}\right)+q\right] \\ &= \frac{(2n+3)}{q}\left[\log\left(1+\frac{q}{F}\right)\left(\frac{F}{q}+\frac{1}{2}\right)-1\right].\end{aligned}$$

Therefore,

$$\hat{\alpha}_{EBW_1} - \hat{\alpha}_{EBW_2} = \hat{\alpha}_{EBW_2} - \hat{\alpha}_{EBW_3} = \frac{(2n+3)}{q}\left[\log\left(1+\frac{q}{F}\right)\left(\frac{F}{q}+\frac{1}{2}\right)-1\right].$$

For $-1 < t \leq 1$, we have $\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{5} - ... = \sum_{i=1}^{\infty}(-1)^{i+1}\frac{t^i}{i}$. Let $t = \frac{q}{F}$, when $0 < q < F$, $0 < \frac{q}{F} < 1$, we get

$$\begin{aligned}&\left[\log\left(1+\frac{q}{F}\right)\left(\frac{F}{q}+\frac{1}{2}\right)-1\right] = \\ &\quad\left(\frac{F}{q}+\frac{1}{2}\right)\left[\left(\frac{q}{F}\right)-\frac{1}{2}\left(\frac{q}{F}\right)^2+\frac{1}{3}\left(\frac{q}{F}\right)^3-\frac{1}{4}\left(\frac{q}{F}\right)^4+\frac{1}{5}\left(\frac{q}{F}\right)^5-\cdots\right]-1 \\ &\quad= \left[1-\frac{1}{2}\left(\frac{q}{F}\right)+\frac{1}{3}\left(\frac{q}{F}\right)^2-\frac{1}{4}\left(\frac{q}{F}\right)^3+\frac{1}{5}\left(\frac{q}{F}\right)^4-\frac{1}{6}\left(\frac{q}{F}\right)^5+\cdots\right] \\ &\quad\quad+\left[\frac{1}{2}\left(\frac{q}{F}\right)+\frac{1}{4}\left(\frac{q}{F}\right)^2+\frac{1}{6}\left(\frac{q}{F}\right)^3-\frac{1}{8}\left(\frac{q}{F}\right)^4+\cdots\right]-1 \\ &\quad= \frac{1}{12}\left(\frac{q}{F}\right)^2\left[1-\frac{q}{F}\right]+\frac{3}{40}\left(\frac{q}{F}\right)^4\left[1-\frac{8}{9}\frac{q}{F}\right]+\cdots,\end{aligned}$$

Then,

$$\hat{\alpha}_{EBW_1} - \hat{\alpha}_{EBW_2} = \hat{\alpha}_{EBW_2} - \hat{\alpha}_{EBW_3} = \frac{(2n+3)}{q}\left[\log\left(1+\frac{q}{F}\right)\left(\frac{F}{q}+\frac{1}{2}\right)-1\right] > 0,$$

This shows that

$$\hat{\alpha}_{EBW_3} < \hat{\alpha}_{EBW_2} < \hat{\alpha}_{EBW_1}.$$

(2) From (1), we get

$$\begin{aligned}\lim_{F\to\infty}\left(\hat{\alpha}_{EBW_1} - \hat{\alpha}_{EBW_2}\right) &= \lim_{F\to\infty}\left(\hat{\alpha}_{EBW_2} - \hat{\alpha}_{EBW_3}\right) \\ &= \frac{(2n+3)}{q} \\ &\quad\times \lim_{F\to\infty}\left[\frac{1}{12}\frac{q^2}{F^2}\left(1-\frac{q}{F}\right)+\frac{3}{40}\frac{q^4}{F^4}\left(1-\frac{8}{9}\frac{q}{F}\right)+\cdots\right]\end{aligned}$$

This yields that,

$$\lim_{F\to\infty} \hat{\alpha}_{EBW_1} = \lim_{F\to\infty} \hat{\alpha}_{EBW_2} = \lim_{F\to\infty} \hat{\alpha}_{EBW_3}.$$

From (25) and from the proof of (1), we have $\lim_{F\to\infty} \hat{\alpha}_{EBW_1} = 0$.
Using (2), we get

$$\lim_{F\to\infty} \hat{\alpha}_{EBW_j} = 0, \quad \text{for} \quad j = 1, 2, 3.$$

□

5.2. The Relations Between the H-Bayesian Estimates under Different Loss Functions

Theorem 10. *It follows from (36), (37) and (38) that*

$$\lim_{F\to\infty} \hat{\alpha}_{HS_j} = 0, \text{ for } j = 1, 2, 3.$$

Proof. Based on SELF, the H-Bayesian estimate of α can be expressed as

$$\hat{\alpha}_{HS_1} = \frac{\int_0^1 \int_0^q (q-b)\frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a+1)}{S^{n+a+1}}\, dbda}{\int_0^1 \int_0^q (q-b)\frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a)}{S^{n+a}}\, dbda}.$$

Using the result $\Gamma(n+a+1) = (n+a)\Gamma(n+a)$, we have,

$$\int_0^1 \int_0^q (q-b)\frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a+1)}{S^{n+a+1}}\, dbda = \int_0^1 \int_0^q (q-b)\frac{b^a}{\Gamma(a)} \frac{(n+a)\Gamma(n+a)}{S^{n+a+1}}\, dbda.$$

As $S = b + F$, and For $a \in (0,1)$, $b \in (0,q)$, $(n+a)(b+F)^{-1}$ is continuous and $\frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a)}{S^{n+a}} > 0$ and, hence, using the generalized mean value theorem, we can find at least one number $a_2 \in (0,1)$ and $b_2 \in (0,q)$ such that,

$$\int_0^1 \int_0^q (q-b)\frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a+1)}{S^{n+a+1}}\, dbda = \frac{(n+a_2)}{(b_2+F)} \int_0^1 \int_0^q (q-b)\frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a)}{S^{n+a}}\, dbda.$$

Therefore,

$$\hat{\alpha}_{HS_1} = \frac{\frac{(n+a_2)}{(b_2+F)} \int_0^1 \int_0^q (q-b)\frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a)}{S^{n+a}}\, dbda}{\int_0^1 \int_0^q (q-b)\frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a)}{S^{n+a}}\, dbda} = \frac{(n+a_2)}{(b_2+F)}.$$

Taking limit as $F \to \infty$, it has been noticed that,

$$\lim_{F\to\infty} \hat{\alpha}_{HS_1} = 0.$$

Similarly, we have

$$\lim_{F\to\infty} \hat{\alpha}_{HS_2} = \lim_{F\to\infty} \hat{\alpha}_{HS_3} = 0.$$

Thus,

$$\lim_{F\to\infty} \hat{\alpha}_{HS_j} = 0, \text{ for } j = 1, 2, 3.$$

□

Theorem 11. *It follows from (39), (40) and (41) that*

$$\lim_{F\to\infty} \hat{\alpha}_{HE_j} = 0, \textit{ for } j = 1, 2, 3.$$

Proof . Based on ELF, the H-Bayesian estimate of α can be expressed as

$$\hat{\alpha}_{HE_1} = \frac{\int_0^1 \int_0^q (q-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(n+a)}{S^{n+a}}\ dbda}{\int_0^1 \int_0^q (q-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(n+a-1)}{S^{n+a-1}}\ dbda}.$$

Using the result $\Gamma(n+a) = (n+a-1)\Gamma(n+a-1)$, we have,

$$\int_0^1 \int_0^q (q-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(n+a)}{S^{n+a}}\ dbda = \int_0^1 \int_0^q (q-b)\frac{b^a}{\Gamma(a)}\frac{(n+a-1)\Gamma(n+a-1)}{S^{n+a}}\ dbda.$$

As $S = b + F$, and For $a \in (0,1)$, $b \in (0,q)$, $(n+a-1)(b+F)^{-1}$ is continuous and $\frac{b^a}{\Gamma(a)}\frac{\Gamma(n+a-1)}{S^{n+a-1}} > 0$ and, hence, using the generalized mean value theorem, we can find at least one number $a_5 \in (0,1)$ and $b_5 \in (0,q)$ such that,

$$\int_0^1 \int_0^q (q-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(n+a)}{S^{n+a}}\ dbda = \frac{(n+a_5-1)}{(b_5+F)}\int_0^1 \int_0^q (q-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(n+a-1)}{S^{n+a-1}}\ dbda.$$

Therefore,

$$\hat{\alpha}_{HE_1} = \frac{\frac{(n+a_5-1)}{(b_5+F)}\int_0^1 \int_0^q (q-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(n+a-1)}{S^{n+a-1}}\ dbda}{\int_0^1 \int_0^q (q-b)\frac{b^a}{\Gamma(a)}\frac{\Gamma(n+a-1)}{S^{n+a-1}}\ dbda} = \frac{(n+a_5-1)}{(b_5+F)}.$$

Taking limit as $F \to \infty$, it has been noticed that,

$$\lim_{F\to\infty} \hat{\alpha}_{HE_1} = 0.$$

Similarly, we have

$$\lim_{F\to\infty} \hat{\alpha}_{HE_2} = \lim_{F\to\infty} \hat{\alpha}_{HE_3} = 0.$$

Thus,

$$\lim_{F\to\infty} \hat{\alpha}_{HE_j} = 0, \text{ for } j = 1, 2, 3.$$

□

Theorem 12. *It follows from (42), (43) and (44) that*

$$\lim_{F\to\infty} \hat{\alpha}_{HW_j} = 0, \text{ for } j = 1, 2, 3.$$

Proof. Under WBLF, the H-Bayesian estimate of α can be expressed as

$$\hat{\alpha}_{HW_1} = \frac{\int_0^1 \int_0^q (q-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a+2)}{S^{n+a+2}}\, dbda}{\int_0^1 \int_0^q (q-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a+1)}{S^{n+a+1}}\, dbda}.$$

Using the result, $\Gamma(n+a+2) = (n+a+1)\Gamma(n+a+1)$, we have,

$$\int_0^1 \int_0^q (q-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a+2)}{S^{n+a+2}}\, dbda = \int_0^1 \int_0^q (q-b) \frac{b^a}{\Gamma(a)} \frac{(n+a+1)\Gamma(n+a+1)}{S^{n+a+2}}\, dbda.$$

As $S = b + F$, and for $a \in (0,1)$, $b \in (0,q)$, $(n+a+1)(b+F)^{-1}$ is continuous and $\frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a+1)}{S^{n+a+1}} > 0$ and, hence, using the generalized mean value theorem, we can find at least one number $a_8 \in (0,1)$ and $b_8 \in (0,q)$ such that,

$$\int_0^1 \int_0^q (q-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a+2)}{S^{n+a+2}}\, dbda = \frac{(n+a_8+1)}{(b_8+F)} \int_0^1 \int_0^q (q-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a+1)}{S^{n+a+1}}\, dbda.$$

Therefor,

$$\hat{\alpha}_{HW_1} = \frac{\frac{(n+a_8+1)}{(b_8+F)} \int_0^1 \int_0^q (q-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a+1)}{S^{n+a+1}}\, dbda}{\int_0^1 \int_0^q (q-b) \frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a+1)}{S^{n+a+1}}\, dbda} = \frac{(n+a_8+1)}{(b_8+F)}.$$

Taking limit as $F \to \infty$, it has been obtained that

$$\lim_{F\to\infty} \hat{\alpha}_{HW_1} = 0.$$

Similarly, we have

$$\lim_{F\to\infty} \hat{\alpha}_{HW_2} = \lim_{F\to\infty} \hat{\alpha}_{HW_3} = 0$$

Thus,

$$\lim_{F\to\infty} \hat{\alpha}_{HW_j} = 0, \text{ for } j = 1, 2, 3.$$

□

6. Simulation Study

In this section, the performance of the proposed estimates for the shape parameter of IPL distribution has been compared using a Monte Carlo simulation study on the basis of MSEs. Mean Squared Error (MSE) can be used within a Bayesian context because it provides a useful metric for evaluating and comparing the performance of models in a Bayesian framework. While Bayesian models often deal with probability distributions rather than point estimates, MSE can be used to quantify the expected squared difference between the model's predictions and the true values, serving as a valuable tool for model selection and assessment.The framework of the simulation has been provided in the following steps:

1. Set the true value of α, β and λ at $(\alpha = 1.5)$, $(\beta = 0.25)$ and $(\lambda = 1)$.
2. Generate random sample of sizes $n = 20, 40, 70, 100$ from inverse Power Lomax distribution using the quantile function i.e. $\lambda(u^{\frac{-1}{\alpha}} - 1)^{\frac{-1}{\beta}}$, where $u \sim U(0, 1)$.
3. Fix the value of $q = 0.4, 0.7, 1$, and $a = 0.3$, $b = 0.2$.
4. Repeat steps 10000 times and compute the value of estimates and the associated mean squared error (MSE).

All the simulated values of the average estimates and the associated MSEs are tabulated in Tables 1-3. The performance of the proposed estimates have been compared based on MSE. From Tables 1-3 the following points are observed:

- When the value of n increases, the value of MSEs decreases for a given value of q.
- The MSE value decreases for a given n as the value of q increases.
- In terms of MSE, Bayes estimates often do better under ELF. Similar performance has been noticed for E-Bayes and H-Bayes estimates.
- In terms of MSE, E-Bayes estimates often do better than the H-Bayes estimates under a fixed loss function.

From these above observations, one can make a conclusion that the performance of the proposed estimates gives quite a satisfaction. The E-Bayes estimates under ELF should be a better choice than other proposed estimates.

TABLE 1: Average values and MSEs (in parentheses) of the proposed estimates based on SELF.

n	Bayes	q	E-Bayesian			H-Bayesian		
	$\hat{\alpha}_{BS}$		$\hat{\alpha}_{EBS_1}$	$\hat{\alpha}_{EBS_2}$	$\hat{\alpha}_{EBS_3}$	$\hat{\alpha}_{HS_1}$	$\hat{\alpha}_{HS_2}$	$\hat{\alpha}_{HS_3}$
20	1.6003	0.4	1.5984	1.5775	1.5680	1.5992	1.5782	1.5791
	(0.1461)		(0.1352)	(0.1291)	(0.1339)	(0.1367)	(0.1301)	(0.1392)
		0.7	1.5892	1.5640	1.5550	1.5910	1.5672	1.5631
			(0.1345)	(0.1336)	(0.1213)	(0.1454)	(0.1341)	(0.1320)
		1	1.5781	1.5594	1.5355	1.5821	1.5623	1.5414
			(0.1314)	(0.1236)	(0.1149)	(0.1421)	(0.1319)	(0.1290)
40	1.5939	0.4	1.5494	1.5439	1.5406	1.5526	1.5492	1.5481
	(0.0626)		(0.0637)	(0.0619)	(0.0696)	(0.0649)	(0.0628)	(0.0700)
		0.7	1.5476	1.5316	1.5297	1.5491	1.5402	1.5332
			(0.0590)	(0.0599)	(0.0617)	(0.0601)	(0.0609)	(0.0621)
		1	1.5310	1.5216	1.5120	1.5420	1.5392	1.5249
			(0.0569)	(0.0548)	(0.0535)	(0.0574)	(0.0556)	(0.0541)
70	1.4929	0.4	1.5414	1.5363	1.5280	1.5431	1.5324	1.5303
	(0.0382)		(0.0353)	(0.0349)	(0.0345)	(0.0371)	(0.0357)	(0.0352)
		0.7	1.5282	1.5203	1.5189	1.5342	1.5314	1.5262
			(0.0341)	(0.0331)	(0.0324)	(0.0365)	(0.0350)	(0.0349)
		1	1.5225	1.5136	1.5114	1.5298	1.5241	1.5201
			(0.0333)	(0.0318)	(0.0317)	(0.0349)	(0.0332)	(0.0341)
100	1.5228	0.4	1.5167	1.5159	1.5151	1.5219	1.5172	1.5192
	(0.0222)		(0.0247)	(0.0215)	(0.0269)	(0.0292)	(0.0226)	(0.0283)
		0.7	1.5098	1.5085	1.5070	1.5147	1.5120	1.5086
			(0.0228)	(0.0232)	(0.0225)	(0.0232)	(0.0222)	(0.0231)
		1	1.5060	1.5058	1.5047	1.5094	1.5038	1.5082
			(0.0234)	(0.0230)	(0.0229)	(0.0242)	(0.0237)	(0.0230)

Table 2: Average values and MSEs (in parentheses) of the proposed estimates based on ELF.

n	Bayes	q	E-Bayesian			H-Bayesian		
	$\hat{\alpha}_{BE}$		$\hat{\alpha}_{EBE_1}$	$\hat{\alpha}_{EBE_2}$	$\hat{\alpha}_{EBE_3}$	$\hat{\alpha}_{HE_1}$	$\hat{\alpha}_{HE_2}$	$\hat{\alpha}_{HE_3}$
20	1.4768	0 .4	1.5328	1.5152	1.5071	1.5398	1.5168	1.5078
	(0.1420)		(0.1259)	(0.1226)	(0.1221)	(0.1327)	(0.1262)	(0.1235)
		0 .7	1.5295	1.5073	1.4937	1.5274	1.5202	1.5148
			(0.1224)	(0.1183)	(0.1179)	(0.1231)	(0.1202)	(0.1199)
		1	1.4990	1.4910	1.4881	1.4956	1.4992	1.4805
			(0.1061)	(0.1107)	(0.1123)	(0.1078)	(0.1057)	(0.1190)
40	1.4879	0 .4	1.5171	1.5165	1.4986	1.5187	1.5042	1.5032
	(0.0621)		(0.0602)	(0.0618)	(0.0535)	(0.0634)	(0.0627)	(0.0610)
		0 .7	1.5051	1.4962	1.4847	1.5063	1.5071	1.4809
			(0.0522)	(0.0531)	(0.0584)	(0.0564)	(0.0572)	(0.0613)
		1	1.4984	1.4916	1.4841	1.5061	1.5028	1.4801
			(0.0512)	(0.0509)	(0.0592)	(0.0519)	(0.0530)	(0.0640)
70	1.4929	0 .4	1.5083	1.5030	1.4925	1.5067	1.5055	1.4915
	(0.0051)		(0.0350)	(0.0326)	(0.0324)	(0.0370)	(0.0341)	(0.0339)
		0 .7	1.5019	1.4997	1.4910	1.4965	1.4930	1.4900
			(0.0317)	(0.0314)	(0.0332)	(0.0350)	(0.0342)	(0.0348)
		1	1.4950	1.4906	1.4942	1.4964	1.4936	1.5019
			(0.0304)	(0.0296)	(0.0292)	(0.0298)	(0.0310)	(0.0292)
100	1.5093	0 .4	1.5033	1.4986	1.4979	1.5060	1.5019	1.4971
	(0.0267)		(0.0222)	(0.0216)	(0.0229)	(0.0231)	(0.0227)	(0.0234)
		0 .7	1.4994	1.4974	1.4953	1.4958	1.4952	1.4917
			(0.0210)	(0.0213)	(0.0211)	(0.0219)	(0.0215)	(0.0213)
		1	1.4979	1.4953	1.4918	1.4970	1.4949	1.4905
			(0.0219)	(0.0214)	(0.0211)	(0.0202)	(0.0219)	(0.0209)

TABLE 3: Average values and MSEs (in parentheses) of the proposed estimates based on WBLF.

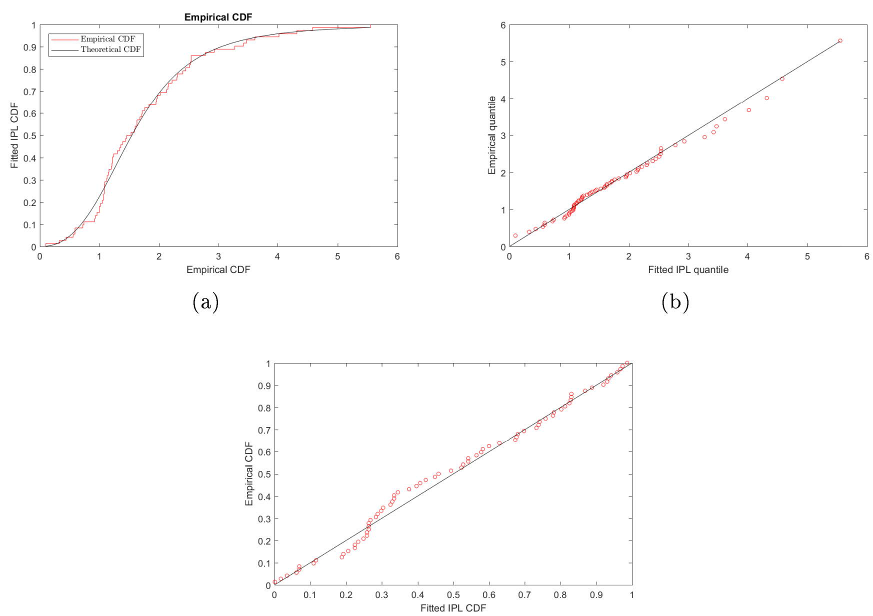
n	Bayes $\hat{\alpha}_{BE}$	q	E-Bayesian $\hat{\alpha}_{EBE_1}$	$\hat{\alpha}_{EBE_2}$	$\hat{\alpha}_{EBE_3}$	H-Bayesian $\hat{\alpha}_{HE_1}$	$\hat{\alpha}_{HE_2}$	$\hat{\alpha}_{HE_3}$
20	1.6883	0 .4	1.6792	1.6756	1.6611	1.6857	1.6748	1.6598
	(0.1883)		(0.2002)	(0.1854)	(0.1727)	(0.1852)	(0.1738)	(0.1528)
		0 .7	1.6667	1.6603	1.6403	1.6669	1.6621	1.6564
			(0.1782)	(0.1724)	(0.1412)	(0.1792)	(0.1735)	(0.1519)
		1	1.6531	1.6489	1.6306	1.6592	1.6571	1.6432
			(0.1632)	(0.1523)	(0.1401)	(0.1681)	(0.1620)	(0.1492)
40	1.5939	0 .4	1.5856	1.5764	1.5656	1.5906	1.5811	1.5763
	(0.0937)		(0.0856)	(0.0764)	(0.0696)	(0.0890)	(0.0862)	(0.0721)
		0 .7	1.5829	1.5748	1.5679	1.5891	1.5763	1.5709
			(0.0694)	(0.0691)	(0.0646)	(0.0710)	(0.0677)	(0.0659)
		1	1.5666	1.5615	1.5591	1.5779	1.5675	1.5620
			(0.0690)	(0.0652)	(0.0636)	(0.0702)	(0.0661)	(0.0629)
70	1.4929	0 .4	1.5540	1.5483	1.5440	1.5595	1.5521	1.5515
	(0.0398)		(0.0381)	(0.0364)	(0.0352)	(0.0392)	(0.0372)	(0.0361)
		0 .7	1.5523	1.5472	1.5423	1.5557	1.5499	1.5430
			(0.0377)	(0.0354)	(0.0346)	(0.0401)	(0.0362)	(0.0356)
		1	1.5414	1.5363	1.5280	1.5470	1.5398	1.5309
			(0.0371)	(0.0346)	(0.0324)	(0.0392)	(0.0351)	(0.0342)
100	1.5424	0 .4	1.5351	1.5313	1.5242	1.5401	1.5392	1.5304
	(0.0260)		(0.0254)	(0.0243)	(0.0269)	(0.0274)	(0.0269)	(0.0263)
		0 .7	1.5326	1.5288	1.5281	1.5364	1.5312	1.5251
			(0.0242)	(0.0231)	(0.0251)	(0.0257)	(0.0260)	(0.0259)
		1	1.5214	1.5207	1.5138	1.5309	1.5292	1.5234
			(0.0217)	(0.0210)	(0.0203)	(0.0227)	(0.0219)	(0.0210)

7. Real Data Analysis

A real life data set is used to illustrate the proposed method in this section. The survival periods (in days) of 72 guinea pigs infected with virulent tubercle bacilli are represented in this data set. Bjerkedal (1960) observed and reported this data collection as follows:
0.1, 0.33, 0.44, 0.56, 0.59, 0.59, 0.72, 0.74, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34,1.36,1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83,

1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78,2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

Hassan & Abd-Allah (2019) used the above data set to fit the inverse power Lomax distribution. The Kolmogorov-Smirnov distance and the corresponding p-value for the real data set are 0.0771 and 0.7569. These values yields that the given real data set can be fitted with IPL distribution. Figure 1 indicates the empirical CDF, quantile-quantile (Q-Q) and probability-probability (P-P) plots against the real data set. From Figure 1, it is observed that the given data set fits IPL distribution very well. Under similar set up as taken in simulation studies, the average values and the standard errors(SEs) are computed and listed in Table 4. From Table 4, it has been noticed that Bayes estimates under ELF perform better than other two Bayes estimates. Under any fixed loss function, E-Bayesian estimates perform better than Bayes and H-Bayesian estimates. For any fixed loss function, the values of SEs decrease with the increase of q values. The values of E-Bayesian estimates satisfy the results discussed in Theorem 5.1, 5.2, and 5.3.



(a) (b) (c)

FIGURE 1: (a) ECDF, (b) Q-Q and (c) P-P plots for the IPL distribution fitted to the given real data set.

TABLE 4: Average values and SEs (in parentheses) of the proposed estimates on different loss functions for real data.

Loss function	Bayes	q	E-Bayesian			H-Bayesian		
	$\hat{\alpha}_B$		$\hat{\alpha}_{EB_1}$	$\hat{\alpha}_{EB_2}$	$\hat{\alpha}_{EB_3}$	$\hat{\alpha}_{HB_1}$	$\hat{\alpha}_{HB_2}$	$\hat{\alpha}_{HB_3}$
SELF	1.5495 (0.00245)	0.4	1.5493 (0.00243)	1.5471 (0.00222)	1.5449 (0.00449)	1.5486 (0.00236)	1.5491 (0.00241)	1.5495 (0.00245)
		0.7	1.5460 (0.00212)	1.5422 (0.00178)	1.5384 (0.00147)	1.5493 (0.00243)	1.5498 (0.00248)	1.5502 (0.00252)
		1	1.5428 (0.00183)	1.5471 (0.00139)	1.5449 (0.00102)	1.5498 (0.00248)	1.5503 (0.00253)	1.5506 (0.00256)
ELF	1.5286 (0.00076)	0.4	1.5279 (0.00017)	1.5258 (0.00067)	1.5236 (0.00055)	1.5272 (0.00074)	1.5278 (0.00077)	1.5281 (0.00079)
		0.7	1.5247 (0.00061)	1.5209 (0.00044)	1.5172 (0.00029)	1.5280 (0.00078)	1.5285 (0.00081)	1.5288 (0.00083)
		1	1.5215 (0.00046)	1.5161 (0.00026)	1.5108 (0.00011)	1.5284 (0.00081)	1.5289 (0.000833)	1.5292 (0.00085)
WBLF	1.5712 (0.00512)	0.4	1.5707 (0.00500)	1.5685 (0.00469)	1.5662 (0.00439)	1.57000 (0.00490)	1.5705 (0.00497)	1.5708 (0.00502)
		0.7	1.5674 (0.00454)	1.5635 (0.00403)	1.5596 (0.00355)	1.5707 (0.00500)	1.5712 (0.00507)	1.5715 (0.00512)
		1	1.5640 (0.00410)	1.5585 (0.00343)	1.5530 (0.00281)	1.5712 (0.00507)	1.5716 (0.00514)	1.5719 (0.00518)

8. Conclusion

In this paper, the Bayesian, E-Bayesian and H-Bayesian are used to estimate the shape parameter of IPL distribution. An important finding of this article is that the suggested estimators are shown to be superior than the existing estimators as shown in Theorem 3.1-3.3 and Theorem 4.1-4.3. The proposed estimates and the associated MSEs are computed by a Monte Carlo simulation. In terms of MSE values, E-Bayesian estimates perform better than the Bayes and H-Bayesian estimates. Although the H-Bayesian estimates are very close to E-Bayesian estimates, the H-Bayesian estimates involve complicated integrals which cannot be solved explicitly. So E-Bayesian estimation technique is easy to use and preferable over H-Bayesian estimation. Furthermore, the estimates under ELF perform better than the estimates under SELF and WBLF. Finally, a real life data has been analyzed to illustrate the performance of proposed estimates.

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