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Dual to Ratio Type Class of Estimators for the Estimation of Population Mean Using Auxiliary Variable in Simple Random Sampling

Clase de estimadores duales al tipo razón para la estimación de la media poblacional utilizando una variable auxiliar en muestreo aleatorio simple

Housila P. Singh[1,a], Rajesh Tailor[1,b], Akanksha Agrawal[1,c]

[1]School of Studies in Statistics, Vikram University, Ujjain, India

Abstract

In this article, we introduce a dual to ratio-type class of estimators for population mean under simple random sampling using information on auxiliary variable, inspired by the estimator proposed by Srivenkataramana (1980) and Kumar & Siddiqui (2024). Our goal is to develop an efficient class of estimators for population mean. We have derived the bias and mean squared error of the proposed estimator up to the first order of approximation. We have derived the optimum condition at which the suggested class of estimators attained the minimum mean squared error. To assess the performance of the suggested class of estimators,we compare the MSE of our proposed estimator with that of several existing estimators. From the theoretical analysis, we demonstrate that the proposed dual to ratio-type class of estimators exhibits a lower mean squared error compared to some existing alternatives, indicating that it is more efficient. This suggests that the proposed class of estimators provides more accurate estimates for a given sample size. To validate our theoretical results,we have performed a simulation study, which confirms that the proposed estimator consistently outperforms with some existing estimators in terms of efficiency, regardless of whether the study is empirical or simulation based. These findings provide compelling evidence for the effectiveness of the proposed class of estimators in practical applications.

***Keywords*:** Dual to ratio type class of estimators; Empirical study; Mean squared error; Simple random sampling; Simulation study.

[a]Ph.D. E-mail: hpsujn@gmail.com
[b]Ph.D. E-mail: tailorraj@gmail.com
[c]Ph.D. E-mail: agrawal.akanksha8@gmail.com

Resumen

En este artículo, presentamos una clase de estimadores duales al tipo razón para la estimación de la media poblacional bajo muestreo aleatorio simple, utilizando información de una variable auxiliar, inspirados en el estimador propuesto por Srivenkataramana (1980) y Kumar & Siddiqui (2024). Nuestro objetivo es desarrollar una clase eficiente de estimadores para la media poblacional. Hemos derivado el sesgo y el error cuadrático medio del estimador propuesto hasta el primer orden de aproximación. También hemos determinado la condición óptima bajo la cual la clase sugerida de estimadores alcanza el error cuadrático medio mínimo. Para evaluar el rendimiento de la clase propuesta de estimadores, comparamos el ECM (error cuadrático medio) de nuestro estimador con el de varios estimadores existentes. A partir del análisis teórico, demostramos que la clase de estimadores dual al tipo razón propuesta presenta un error cuadrático medio menor en comparación con algunas alternativas existentes, lo que indica que es más eficiente. Esto sugiere que la clase de estimadores propuesta proporciona estimaciones más precisas para un tamaño de muestra dado. Para validar nuestros resultados teóricos, realizamos un estudio de simulación, el cual confirma que el estimador propuesto supera de manera constante a algunos estimadores existentes en términos de eficiencia, independientemente de si el estudio es empírico o basado en simulación. Estos hallazgos proporcionan evidencia contundente sobre la efectividad de la clase propuesta de estimadores en aplicaciones prácticas.

***Palabras clave*:** Clase de estimadores dual al tipo razón; Error cuadrático medio; Estudio de simulación; Estudio empírico; Muestreo aleatorio simple.

1. Introduction

In practice, the statistician's task is to get estimates of the population parameters, such as population mean or total, variance, coefficient of variation, correlation coefficient, etc., based on the sample data at hand. So, one of the primary goals of survey statisticians is to get improved estimates of the population parameter under study. For about the last nine decades, statisticians have been practicing the use of auxiliary information in the estimation procedures to develop precise estimates of the parameter under investigation. It is observed that the ratio method of estimation due to Cochran (1940) and the product method of estimation due to Robson (1957) and revisited by Murthy (1964) take advantage of the sign of the correlation coefficient between the study variable and the auxiliary variable. When there is a positive (high) correlation between the study and auxiliary variables, the ratio method of estimation is employed for better results. On the other hand, if this correlation is negative (high), the product method estimation can be employed quite effectively. However, the ratio and product estimators of population mean have the limitation of having at most the same efficiency as that of the linear regression estimator (Watson, 1937). In some very specific cases, the difference estimator is an alternative to ratio and product estimators.

In spite of its less practicability, regression estimate appears to occupy a unique position due to its sound theoretical basis. There have been various attempts to develop modified estimators of population mean that are improvements over usual mean per unit (sample mean), ratio, and product estimators using auxiliary information in order to provide better alternatives to regression estimators. When the population mean of the auxiliary variable x is known in advance, various authors have suggested modified versions of ratio and product estimators; for instance, see Srivastava (1967), Walsh (1970), Vos (1980), Adhvaryu & Gupta (1983), Bandyopadhyay (1980), Naik & Gupta (1991), Srivenkataramana (1980), Sisodia & Dwivedi (1981), Bahl & Tuteja (1991), Upadhyaya & Singh (1999), Kadilar (2016), Singh & Tailor (2003), Singh et al. (2009), etc. Using Searls (1964) procedure, Yadav & Kadilar (2013) improved Singh et al. (2009) estimator and studied the properties. Ijaz & Ali (2018) envisaged some improved ratio estimator for estimating the population mean under simple random sampling (SRS).

Yadav et al. (2019) utilized various auxiliary information and introduced a class of estimators of population mean along with properties under SRS. Kumar et al. (2024) suggested an enhanced class of estimators for estimating population mean under SRS and studied its properties up to the first order of approximation. Moreover, some more researchers such as Bhushan et al. (2021), Yadav et al. (2023), Kumar et al. (2024), Sher et al. (2025) and Javed et al. (2025), also suggested an estimator for estimating population mean under SRSWOR scheme. Using the same transformation as adopted by Bandyopadhyay (1980) and Srivenkataramana (1980), we have developed a novel dual-to-ratio-type class of estimators for estimating the finite population mean of the study variable y under the SRSWOR scheme, drawing inspiration from the class of estimators due to Kumar et al. (2024). This is how the rest of the article is structured. The methodology and notations are shown in Section 2 using the SRSWOR scheme. Reviews of some existing estimators are provided in Section 3. We have suggested a novel dual-to-ratio-type class of exponential estimators for population mean along with its proper under SRSWOR scheme in Section 4. The efficiency conditions are derived in Section 5. An empirical study based on three real population datasets has been conducted, and the results are provided in Section 6. In Section 7, a simulation study has been carried out based on four artificially generated populations. In Section 8, the conclusion of the work contained in the paper is provided.

2. Methodology and Notations

Let a population $U = \{U_1, U_2, \ldots, U_N\}$ be the configuration of the identifiable units with a finite size N. Let a sample of size n $(< N)$ be drawn from the population using a simple random sampling without replacement (SRSWOR) scheme. Let (y, x) denote the (study, auxiliary) variables, respectively. Let (y_i, x_i) the values of (study, auxiliary) be respectively for the i^{th} $(i = 1, 2, \ldots, N)$ unit from population U. Let us denote:

$\overline{Y} = \frac{1}{N}\sum_{i=1}^{N} y_i$: Population mean of the study variable y,

$\overline{X} = \frac{1}{N}\sum_{i=1}^{N} x_i$: Population mean of the auxiliary variable x,

$S_y^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(y_i - \overline{Y}\right)^2$: Population mean square of the study variable y,

$S_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(x_i - \overline{X}\right)^2$: Population mean square of the auxiliary variable x,
$S_{yx} = \frac{1}{N-1}\sum_{i=1}^{N}\left(y_i - \overline{Y}\right)\left(x_i - \overline{X}\right)$: Population covariance between the study variable y and the auxiliary variable x,
$\rho_{yx} = \frac{S_{yx}}{(S_y S_x)}$: Population correlation coefficient between the study variable y and the auxiliary variable x,
$C_y = \frac{S_y}{\overline{Y}}$: Population coefficient of variation of the study variable y,
$C_x = \frac{S_x}{\overline{X}}$: Population coefficient of variation of the auxiliary variable x,
M_x and M_y be the population medians of the auxiliary variable x and the study variable y,
S_x and S_y be the population standard deviations of the auxiliary variable x and the study variable y respectively,
$\beta_1(x) = \frac{\mu_3{}^2(x)}{\mu_2{}^3(x)}$: Population coefficient of skewness of the auxiliary variable x,
$\beta_2(x) = \frac{\mu_4(x)}{\mu_2{}^2(x)}$: Population coefficient of kurtosis of the auxiliary variable x,

$$\lambda_1 = \left(\frac{\overline{X}}{\overline{X}+C_x}\right), \lambda_2 = \left(\frac{\overline{X}}{\overline{X}+\beta_2(x)}\right), \lambda_3 = \left(\frac{C_x\overline{X}}{C_x\overline{X}+\beta_2(x)}\right), \lambda_4 = \left(\frac{\overline{X}}{\overline{X}+\rho_{yx}}\right),$$

$$\phi_1 = \left(\frac{\beta_2(x) M_x\overline{X}}{\beta_2(x) M_x\overline{X}+\rho_{yx}}\right),\ \phi_2 = \left(\frac{\beta_2(x) M_x\overline{X}}{\beta_2(x) M_x\overline{X}+\rho_{yx}C_x}\right),\ \phi_3 = \left(\frac{\beta_1(x) M_x\overline{X}}{\beta_1(x) M_x\overline{X}+\rho_{yx}}\right),$$

$$\phi_4 = \left(\frac{\beta_2(x) M_x\overline{X}}{\beta_2(x) M_x\overline{X}+\rho_{yx}C_x}\right),\ \phi_5 = \left(\frac{n\overline{X}}{n\overline{X}+\rho_{yx}}\right), \phi_6 = \left(\frac{n\overline{X}}{n\overline{X}+C_x}\right),$$

$$\phi_7 = \left(\frac{n\overline{X}}{n\overline{X}+\rho_{yx}C_x}\right),\quad \phi_8 = \left(\frac{n\rho_{yx}\overline{X}}{n\rho_{yx}\overline{X}+C_x}\right),\quad \phi_9 = \left(\frac{nC_x\overline{X}}{nC_x\overline{X}+\rho_{yx}}\right),$$

$$\mu_2(x) = \frac{1}{N}\sum_{i=1}^{N}\left(x_i - \overline{X}\right)^2,\ \mu_3(x) = \frac{1}{N}\sum_{i=1}^{N}\left(x_i - \overline{X}\right)^3,\ \mu_4(x) = \frac{1}{N}\sum_{i=1}^{N}\left(x_i - \overline{X}\right)^4,$$

$$R = \frac{\overline{Y}}{\overline{X}}, k_{yx} = \rho_{yx}\frac{C_y}{C_x}, \tau = \frac{u\overline{X}}{2\left(u\overline{X}+v\right)},$$

$\overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$: Sample mean of the study variable y,
$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$: Sample mean of the auxiliary variable x,

$$\overline{x}^* = \frac{\left(N\overline{X} - n\overline{x}\right)}{(N-n)},\ \theta = \frac{n}{(N-n)},$$

$s_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \overline{x}\right)^2$: Sample mean square of the auxiliary variable x,
$s_y^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(y_i - \overline{y}\right)^2$: Sample mean square of the study variable y,
$s_{yx} = \frac{1}{n-1}\sum_{i=1}^{n}\left(y_i - \overline{y}\right)\left(x_i - \overline{x}\right)$: Sample covariance between the study variable y and the auxiliary variable x.

To derive the bias and mean squared error (MSE) of the proposed class of estimators of the population mean $\overline{Y}$ of the study variable y, we write:

$$\overline{y} = \overline{Y}\left(1 + e_0\right), \quad \overline{x} = \overline{X}\left(1 + e_1\right)$$

such that

$$E\left(e_0\right) = E\left(e_1\right) = 0$$

and $E\left(e_0^2\right) = \psi C_y^2$, $E\left(e_1^2\right) = \psi C_x^2$, $E\left(e_0 e_1\right) = \psi \rho_{yx} C_y C_x$, where $\psi = \left(\frac{1}{n} - \frac{1}{N}\right)$.

3. Review of Some Existing Estimators

This section presents the mathematical description of some existing estimators along with their MSE/minimum MSE expressions under the SRSWOR scheme.

The conventional unbiased estimator of population mean $\overline{Y}$ of y is defined by

$$t_m = \overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i,$$

whose variance/MSE under SRSWOR is given by

$$MSE(t_m) = var\left(t_m\right) = \psi \overline{Y}^2 C_y^2. \tag{1}$$

The conventional ratio estimator, introduced by Cochran (1940), is expressed as

$$t_r = \overline{y}\left(\frac{\overline{X}}{\overline{x}}\right).$$

To the first order of approximation, the MSE of t_r is given by

$$MSE(t_r) = \psi \overline{Y}^2\left(C_x^2 + C_y^2 - 2\rho_{yx} C_y C_x\right). \tag{2}$$

The standard regression estimator proposed by Watson (1937), is given by

$$t_{reg} = \overline{y} + \beta\left(\overline{X} - \overline{x}\right),$$

where β is the preassigned constant.

The minimum MSE of the regression estimator t_{reg} at the optimum value $\beta_{opt} = R\rho_{yx}\left(\frac{C_y}{C_x}\right) = \frac{S_{yx}}{S_x^2}$ of β is given by

$$\min MSE(t_{reg}) = \psi S_y^2\left(1 - \rho_{yx}^2\right). \tag{3}$$

Srivastava (1967) proposed the power ratio estimator for $\overline{Y}$ of y as

$$t_s = \overline{y}\left(\frac{\overline{X}}{\overline{x}}\right)^{\delta},$$

where δ is a suitably chosen constant.

For the optimum value $\delta_{opt} = \rho_{yx}\left(\frac{C_y}{C_x}\right)$ of δ, the minimum MSE of t_s is given by

$$\min MSE(t_s) = \psi S_y^2\left(1-\rho_{yx}^2\right). \tag{4}$$

Walsh (1970) envisaged the generalized ratio-type estimator for $\overline{Y}$ of y as

$$t_w = \overline{y}\left(\frac{\overline{X}}{\overline{X}+\alpha\left(\overline{x}-\overline{X}\right)}\right),$$

where α is a suitably chosen scalar.

For the optimum value $\alpha_{opt} = \rho_{yx}\left(\frac{C_y}{C_x}\right)$, the minimum MSE of t_w is given by

$$\min MSE(t_w) = \psi S_y^2\left(1-\rho_{yx}^2\right). \tag{5}$$

Bandyopadhyay (1980) and Srivenkataramana (1980) suggested a dual to ratio estimator for the population mean $\overline{Y}$ of the study variable y as

$$\widehat{\overline{Y}}_a = \overline{y}\left(\frac{\overline{x}^*}{\overline{X}}\right),$$

where $\overline{x}^* = \frac{\left(N\overline{X}-n\overline{x}\right)}{(N-n)}$.

To the first order of approximation, the MSE of $\widehat{\overline{Y}}_a$ is given by

$$MSE(\widehat{\overline{Y}}_a) = \psi\overline{Y}^2\left(C_y^2+\theta^2C_x^2-2\theta\rho_{yx}C_yC_x\right), \tag{6}$$

where $\theta = \frac{n}{(N-n)}$.

Sisodia & Dwivedi (1981) leveraged auxiliary information and proposed a ratio estimator for $\overline{Y}$ of y as

$$t_{sd} = \overline{y}\left(\frac{\overline{X}+C_x}{\overline{x}+C_x}\right),$$

The MSE of t_{sd} to the first order of approximation is given by

$$MSE(t_{sd}) = \psi\overline{Y}^2\left(C_y^2+\lambda_1^2C_x^2-2\lambda_1\rho_{yx}C_yC_x\right), \tag{7}$$

where $\lambda_1 = \left(\frac{\overline{X}}{\overline{X}+C_x}\right)$.

Bahl & Tuteja (1991) introduced the ratio-type exponential ratio estimator for $\overline{Y}$ of y as

$$t_{re} = \overline{y}\exp\left(\frac{\overline{X}-\overline{x}}{\overline{X}+\overline{x}}\right).$$

The MSE of t_{re} to the first order of approximation is given by

$$MSE(t_{re}) = \psi\overline{Y}^2\left(C_y^2+\frac{1}{4}C_x^2-\rho_{yx}C_yC_x\right). \tag{8}$$

Singh & Kakran (1993) employed the coefficient of kurtosis to construct the estimator for $\overline{Y}$ of y as

$$t_{sk} = \overline{y}\left(\frac{\overline{X} + \beta_2(x)}{\overline{x} + \beta_2(x)}\right).$$

The MSE of t_{sk} to the first order of approximation is given by

$$MSE(t_{sk}) = \psi\overline{Y}^2\left(C_y^2 + \lambda_2^2 C_x^2 - 2\lambda_2\rho_{yx}C_yC_x\right), \tag{9}$$

where $\lambda_2 = \left(\frac{\overline{X}}{\overline{X}+\beta_2(x)}\right)$.

Upadhyaya & Singh (1999) used the transformed auxiliary variable and proposed the following estimators for $\overline{Y}$ of y as

$$t_{us1} = \overline{y}\left(\frac{C_x\overline{X} + \beta_2(x)}{C_x\overline{x} + \beta_2(x)}\right),$$

$$t_{us2} = \overline{y}\left(\frac{\beta_2(x)\overline{X} + C_x}{\beta_2(x)\overline{x} + C_x}\right).$$

The MSE of t_{us1} and t_{us2} to the first order of approximation, are respectively given by

$$MSE(t_{us1}) = \psi\overline{Y}^2\left(C_y^2 + \lambda_3^2 C_x^2 - 2\lambda_3\rho_{yx}C_yC_x\right) \tag{10}$$

$$MSE(t_{us2}) = \psi\overline{Y}^2\left(C_y^2 + \lambda_4^2 C_x^2 - 2\lambda_4\rho_{yx}C_yC_x\right), \tag{11}$$

where $\lambda_3 = \left(\frac{C_x\overline{X}}{C_x\overline{X}+\beta_2(x)}\right)$ and $\lambda_4 = \left(\frac{\beta_2(x)\overline{X}}{C_x+\overline{X}\beta_2(x)}\right)$.

Singh & Tailor (2003) presented the ratio estimator for $\overline{Y}$ of y as

$$t_{ST} = \left(\frac{\overline{X} + \rho_{yx}}{\overline{x} + \rho_{yx}}\right).$$

The MSE of t_{ST} to the first order of approximation is given by

$$MSE(t_{ST}) = \psi\overline{Y}^2\left(C_y^2 + \lambda_5^2 C_x^2 - 2\lambda_2\rho_{yx}C_yC_x\right), \tag{12}$$

where $\lambda_5 = \left(\frac{\overline{X}}{\overline{X}+\rho_{yx}}\right)$.

Vos (1980) and Ijaz & Ali (2018) proposed a class of estimators for $\overline{Y}$ of y as

$$t_v = w\overline{y} + (1-w)\,\overline{y}\frac{\overline{X}}{\overline{x}},$$

where 'w' is suitable chosen scalar.

For the optimum value $w_{opt} = \left(1 - \rho_{yx}\frac{C_y}{C_x}\right)$ of w, the minimum MSE of t_v is given by

$$\min MSE(t_v) = \psi\overline{Y}^2 C_y^2\left(1 - \rho_{yx}^2\right). \tag{13}$$

Singh et al. (2009) considered the class of exponential estimators for $\overline{Y}$ of y as

$$t_{sn} = \overline{y} exp\left[\frac{\left(u\overline{X}+v\right)-\left(u\overline{x}+v\right)}{\left(u\overline{X}+v\right)+\left(u\overline{x}+v\right)}\right],$$

where u and v are either real constants or known parametric values of the auxiliary variable x.

For different values of (u, v) we have the following members of the class of estimators t_{sn}:

$t_{sn1} = \overline{y}\exp\left(\frac{\overline{X}-\overline{x}}{\overline{X}+\overline{x}}\right)$, for $(u, v) = (1, 0)$

$t_{sn2} = \overline{y}\exp\left\{\frac{\left(\overline{X}-\overline{x}\right)}{\overline{x}+\overline{X}+2C_x}\right\}$, for $(u, v) = (1, C_x)$

$t_{sn3} = \overline{y}\exp\left\{\frac{\beta_2(x)\left(\overline{X}-\overline{x}\right)}{\beta_2(x)\left(\overline{X}+\overline{x}\right)+2C_x}\right\}$, for $(u, v) = (\beta_2(x),\, C_x)$

$t_{sn4} = \overline{y}\exp\left\{\frac{C_x\left(\overline{X}-\overline{x}\right)}{C_x\left(\overline{X}+\overline{x}\right)+2\beta_2(x)}\right\}$, for $(u, v) = (C_x, \beta_2(x))$

$t_{sn5} = \overline{y}\ exp\left\{\frac{\left(\overline{X}-\overline{x}\right)}{\overline{X}+\overline{x}+2\rho_{yx}}\right\}$, for $(u, v) = (1, \rho_{yx})$

$t_{sn6} = \overline{y}\exp\left\{\frac{\left(\overline{X}-\overline{x}\right)}{\overline{X}+\overline{x}+2\beta_2(x)}\right\}$, for $(u, v) = (1, \beta_2(x))$

$t_{sn7} = \overline{y}\exp\left\{\frac{C_x\left(\overline{X}-\overline{x}\right)}{C_x\left(\overline{X}+\overline{x}\right)+2\rho_{yx}}\right\}$, for $(u, v) = (C_x, \rho_{yx})$

$t_{sn8} = \overline{y}\exp\left\{\frac{\beta_2(x)\left(\overline{X}-\overline{x}\right)}{\beta_2(x)\left(\overline{X}+\overline{x}\right)+2\rho_{yx}}\right\}$, for $(u, v) = (\beta_2(x), \rho_{yx})$

The MSE of the class of estimators t_{sni} $(i = 1$ to $8)$ to the first order of approximation is given by

$$MSE(t_{sni}) = \psi\overline{Y}^2\left(C_y^2 + \tau^2 C_x^2 - 2\tau\rho_{yx}C_yC_x\right), \tag{14}$$

where $\tau = \frac{u\overline{X}}{2\left(u\overline{X}+v\right)}$.

Yadav & Kadilar (2013) proposed an enhanced class of ratio-type exponential estimators for $\overline{Y}$ of y as

$$t_{yk} = k\overline{y}\exp\left[\frac{\left(u\overline{X}+v\right)-\left(u\overline{x}+v\right)}{\left(u\overline{X}+v\right)+\left(u\overline{x}+v\right)}\right],$$

Here (u, v) are same as defined earlier.

For different values of (u, v) we have the following members of the class of estimators t_{yk}:

$t_{yk1} = k\overline{y}\exp\left(\frac{\overline{X}-\overline{x}}{\overline{X}+\overline{x}}\right)$, for $(u, v) = (1, 0)$

$t_{yk2} = k\overline{y}\exp\left\{\frac{\left(\overline{X}-\overline{x}\right)}{\overline{x}+\overline{X}+2C_x}\right\}$, for $(u, v) = (1, C_x)$

$$t_{yk3} = k\overline{y}\exp\left\{\frac{\beta_2(x)\left(\overline{X}-\overline{x}\right)}{\beta_2(x)\left(\overline{X}+\overline{x}\right)+2C_x}\right\}, \text{ for } (u,v) = (\beta_2(x),\, C_x)$$

$$t_{yk4} = k\overline{y}\exp\left\{\frac{C_x\left(\overline{X}-\overline{x}\right)}{C_x\left(\overline{X}+\overline{x}\right)+2\beta_2(x)}\right\}, \text{ for } (u,v) = (C_x, \beta_2(x))$$

$$t_{yk5} = k\overline{y}\exp\left\{\frac{\left(\overline{X}-\overline{x}\right)}{\overline{X}+\overline{x}+2\rho_{yx}}\right\}, \text{ for } (u,v) = (1, \rho_{yx})$$

$$t_{yk6} = k\overline{y}\exp\left\{\frac{\left(\overline{X}-\overline{x}\right)}{\overline{X}+\overline{x}+2\beta_2(x)}\right\}, \text{ for } (u,v) = (1, \beta_2(x))$$

$$t_{yk7} = k\overline{y}\exp\left\{\frac{C_x\left(\overline{X}-\overline{x}\right)}{C_x\left(\overline{X}+\overline{x}\right)+2\rho_{yx}}\right\}, \text{ for } (u,v) = (C_x,\ \rho_{yx})$$

$$t_{yk8} = k\overline{y}\exp\left\{\frac{\beta_2(x)\left(\overline{X}-\overline{x}\right)}{\beta_2(x)\left(\overline{X}+\overline{x}\right)+2\rho_{yx}}\right\}, \text{ for } (u,v) = (\beta_2(x),\ \rho_{yx})$$

The minimum MSE of the class of estimators $t_{yki}(i = 1 \text{ to } 8)$ at optimum value $k_{opt} = \frac{A_1}{B_1}$ of k is given by

$$\min MSE(t_{yki}) = \overline{Y}^2\left(1 - \frac{A_1^2}{B_1}\right), \tag{15}$$

where $A_1 = \left[1 + \psi\left\{\frac{3}{2}\tau^2 C_x^2 - \tau\rho_{yx}C_yC_x\right\}\right]$ and $B_1 = \left[1 + \psi\left\{C_x^2 + 4\tau^2 {C_x}^2 - 4\tau\rho_{yx}C_yC_x\right\}\right]$.

Kadilar (2016) proposed a class of estimators for $\overline{Y}$ of y as

$$t_{gk} = \overline{y}\left(\frac{\overline{X}}{\overline{x}}\right)^{\eta}\exp\left[\frac{\overline{X}-\overline{x}}{\overline{X}+\overline{x}}\right],$$

where 'η' being a suitable chosen constant.

For the optimum value $\eta_{opt} = \left(\rho_{yx}\frac{C_y}{C_x} - \frac{1}{2}\right)$ of η, the minimum MSE of t_{gk} is given by

$$\min MSE(t_{gk}) = \psi\overline{Y}^2 C_y^2\left(1 - \rho_{yx}^2\right). \tag{16}$$

Yadav et al. (2019) proposed the following estimators for $\overline{Y}$ of y as

$$t_{y_1} = \overline{y}\left[\frac{\beta_2(x)\,M_x\overline{X} + \rho_{yx}}{\beta_2(x)\,M_x\overline{x} + \rho_{yx}}\right],$$

$$t_{y_2} = \overline{y}\left[\frac{\beta_2(x)\,M_x\overline{X} + C_x\rho_{yx}}{\beta_2(x)\,M_x\overline{x} + C_x\rho_{yx}}\right],$$

$$t_{y_3} = \overline{y}\left[\frac{\beta_1(x)\,M_x\overline{X} + \rho_{yx}}{\beta_1(x)\,M_x\overline{x} + \rho_{yx}}\right],$$

$$t_{y_4} = \overline{y}\left[\frac{\beta_1(x)\,M_x\overline{X} + C_x\rho_{yx}}{\beta_1(x)\,M_x\overline{x} + C_x\rho_{yx}}\right],$$

$$t_{y_5} = \overline{y}\left[\frac{n\overline{X} + \rho_{yx}}{n\overline{x} + \rho_{yx}}\right],$$

$$t_{y_6} = \overline{y}\left[\frac{n\overline{X}+C_x}{n\overline{x}+C_x}\right],$$

$$t_{y_7} = \overline{y}\left[\frac{n\overline{X}+C_x\rho_{yx}}{n\overline{x}+C_x\rho_{yx}}\right],$$

$$t_{y_8} = \overline{y}\left[\frac{n\rho_{yx}\overline{X}+C_x}{n\rho_{yx}\overline{x}+C_x}\right],$$

$$t_{y_9} = \overline{y}\left[\frac{nC_x\overline{X}+\rho_{yx}}{nC_x\overline{x}+\rho_{yx}}\right].$$

To the first order of approximation, the MSEs of the estimators t_{y_j} ($j = 1$ to 9) are respectively given by:

$$MSE(t_{y_1}) = \psi\overline{Y}^2\left(C_y^2+\varphi_1^2C_x^2-2\varphi_1\rho_{yx}C_yC_x\right), \tag{17}$$

$$MSE(t_{y_2}) = \psi\overline{Y}^2\left(C_y^2+\varphi_2^2C_x^2-2\varphi_2\rho_{yx}C_yC_x\right), \tag{18}$$

$$MSE(t_{y_3}) = \psi\overline{Y}^2\left(C_y^2+\varphi_3^2C_x^2-2\varphi_3\rho_{yx}C_yC_x\right), \tag{19}$$

$$MSE(t_{y_4}) = \psi\overline{Y}^2\left(C_y^2+\varphi_4^2C_x^2-2\varphi_4\rho_{yx}C_yC_x\right), \tag{20}$$

$$MSE(t_{y_5}) = \psi\overline{Y}^2\left(C_y^2+\varphi_5^2C_x^2-2\varphi_5\rho_{yx}C_yC_x\right), \tag{21}$$

$$MSE(t_{y_6}) = \psi\overline{Y}^2\left(C_y^2+\varphi_6^2C_x^2-2\varphi_6\rho_{yx}C_yC_x\right). \tag{22}$$

$$MSE(t_{y_7}) = \psi\overline{Y}^2\left(C_y^2+\varphi_7^2C_x^2-2\varphi_7\rho_{yx}C_yC_x\right), \tag{23}$$

$$MSE(t_{y_8}) = \psi\overline{Y}^2\left(C_y^2+\varphi_8^2C_x^2-2\varphi_8\rho_{yx}C_yC_x\right), \tag{24}$$

$$MSE(t_{y_9}) = \psi\overline{Y}^2\left(C_y^2+\varphi_9^2C_x^2-2\varphi_9\rho_{yx}C_yC_x\right). \tag{25}$$

Recently, Kumar & Siddiqui (2024) proposed a class of estimators for $\overline{Y}$ of y as

$$t_{ks} = k_1\overline{y}\left(\frac{\overline{X}}{\overline{x}}\right)^{k_2}\exp\left\{\frac{\left(u\overline{X}+v\right)-\left(u\overline{x}+v\right)}{\left(u\overline{X}+v\right)+\left(u\overline{x}+v\right)}\right\},$$

where (u, v) are same as defined earlier and (k_1, k_2) are constants to determine the minimum MSE of t_{ks}. Some members of the class of estimators t_{ks} are given in Table 1.

For the optimum value $k_{1(opt)} = \frac{A_2}{B_2}$ of k_1, the minimum MSE of t_{ks} is given by

$$\min MSE(t_{ks}) = \overline{Y}^2\left(1-\frac{A_2^2}{B_2}\right), \tag{26}$$

where $A_2 = \left[1+\psi\left\{\left(\frac{k_2(k_2+1)}{2}+k_2\tau+\frac{3}{2}\tau^2\right)C_x^2-\left(k_2+\tau\right)\rho_{yx}C_yC_x\right\}\right]$ and

$$B_2 = \left[1+\psi\left\{C_y^2+\left(k_2\left(2k_2+1\right)+4\tau^2+4k_2\tau\right)C_x^2-4\left(k_2+\tau\right)\rho_{yx}C_yC_x\right\}\right].$$

TABLE 1: Some members of the class of estimators $\boldsymbol{t_{ks}}$.

S. No.	**Estimator**	$\boldsymbol{u}$	$\boldsymbol{v}$
1.	$t_{ks_1} = k_1\overline{y}\left(\frac{\overline{X}}{\overline{x}}\right)^{k_2} \exp\left(\frac{\overline{X}-\overline{x}}{\overline{X}+\overline{x}}\right)$	1	0
2.	$t_{ks_2} = k_1\overline{y}\left(\frac{\overline{X}}{\overline{x}}\right)^{k_2} \exp\left\{\frac{\overline{X}-\overline{x}}{\overline{x}+\overline{X}+2C_x}\right\}$	1	C_x
3.	$t_{ks_3} = k_1\overline{y}\left(\frac{\overline{X}}{\overline{x}}\right)^{k_2} \exp\left\{\frac{\beta_2(x)(\overline{X}-\overline{x})}{\beta_2(x)(\overline{X}+\overline{x})+2C_x}\right\}$	$\beta_2(x)$	C_x
4.	$t_{ks_4} = k_1\overline{y}\left(\frac{\overline{X}}{\overline{x}}\right)^{k_2} \exp\left\{\frac{C_x(\overline{X}-\overline{x})}{C_x(\overline{X}+\overline{x})+2\beta_2(x)}\right\}$	C_x	$\beta_2(x)$
5.	$t_{ks_5} = k_1\overline{y}\left(\frac{\overline{X}}{\overline{x}}\right)^{k_2} \exp\left\{\frac{\overline{X}-\overline{x}}{\overline{X}+\overline{x}+2\rho_{yx}}\right\}$	1	ρ_{yx}
6.	$t_{ks_6} = k_1\overline{y}\left(\frac{\overline{X}}{\overline{x}}\right)^{k_2} \exp\left\{\frac{\overline{X}-\overline{x}}{\overline{X}+\overline{x}+2\beta_2(x)}\right\}$	1	$\beta_2(x)$
7.	$t_{ks_7} = k_1\overline{y}\left(\frac{\overline{X}}{\overline{x}}\right)^{k_2} \exp\left\{\frac{C_x(\overline{X}-\overline{x})}{C_x(\overline{X}+\overline{x})+2\rho_{yx}}\right\}$	C_x	ρ_{yx}
8.	$t_{ks_8} = k_1\overline{y}\left(\frac{\overline{X}}{\overline{x}}\right)^{k_2} \exp\left\{\frac{\beta_2(x)(\overline{X}-\overline{x})}{\beta_2(x)(\overline{X}+\overline{x})+2\rho_{yx}}\right\}$	$\beta_2(x)$	ρ_{yx}

4. Proposed Class of Dual to Ratio-Type Exponential Estimators

Using the transformation $x_i^* = (1+\theta)\,\overline{X} - \theta x_i$; $i = 1, 2, 3 \ldots N$; with $\theta = \frac{N}{(N-n)}$ as adapted by Srivenkataramana (1980) we propose the class of dual to ratio-type estimators for $\overline{Y}$ of y as

$$T_P = k_1\overline{y}\left(\frac{\overline{x}^*}{\overline{X}}\right)^{k_2} \exp\left\{\frac{(u\overline{x}^* + v) - (u\overline{X} + v)}{(u\overline{x}^* + v) + (u\overline{X} + v)}\right\}, \tag{27}$$

where (k_1, k_2) and (u, v) are same as defined earlier. Some members of the proposed class of estimators T_P are shown in Table 2.

TABLE 2: Some members of the class of estimators $\boldsymbol{T_P}$.

S. No.	**Estimator**	$\boldsymbol{u}$	$\boldsymbol{v}$
1.	$T_{P_1} = k_1\overline{y}\left(\frac{\overline{x}^*}{\overline{X}}\right)^{k_2} \exp\left(\frac{\overline{x}^*-\overline{X}}{\overline{x}^*+\overline{X}}\right)$	1	0
2.	$T_{P_2} = k_1\overline{y}\left(\frac{\overline{x}^*}{\overline{X}}\right)^{k_2} \exp\left\{\frac{\overline{x}^*-\overline{X}}{\overline{x}^*+\overline{X}+2C_x}\right\}$	1	C_x
3.	$T_{P_3} = k_1\overline{y}\left(\frac{\overline{x}^*}{\overline{X}}\right)^{k_2} \exp\left\{\frac{\beta_2(x)(\overline{x}^*-\overline{X})}{\beta_2(x)(\overline{X}+\overline{x}^*)+2C_x}\right\}$	$\beta_2(x)$	C_x
4.	$T_{P_4} = k_1\overline{y}\left(\frac{\overline{x}^*}{\overline{X}}\right)^{k_2} \exp\left\{\frac{C_x(\overline{x}^*-\overline{X})}{C_x(\overline{X}+\overline{x}^*)+2\beta_2(x)}\right\}$	C_x	$\beta_2(x)$
5.	$T_{P_5} = k_1\overline{y}\left(\frac{\overline{x}^*}{\overline{X}}\right)^{k_2} \exp\left\{\frac{\overline{x}^*-\overline{X}}{\overline{X}+\overline{x}^*+2\rho_{yx}}\right\}$	1	ρ_{yx}
6.	$T_{P_6} = k_1\overline{y}\left(\frac{\overline{x}^*}{\overline{X}}\right)^{k_2} \exp\left\{\frac{\overline{x}^*-\overline{X}}{\overline{X}+\overline{x}^*+2\beta_2(x)}\right\}$	1	$\beta_2(x)$
7.	$T_{P_7} = k_1\overline{y}\left(\frac{\overline{x}^*}{\overline{X}}\right)^{k_2} \exp\left\{\frac{C_x(\overline{x}^*-\overline{X})}{C_x(\overline{X}+\overline{x}^*)+2\rho_{yx}}\right\}$	C_x	ρ_{yx}
8.	$T_{P_8} = k_1\overline{y}\left(\frac{\overline{x}^*}{\overline{X}}\right)^{k_2} \exp\left\{\frac{\beta_2(x)(\overline{x}^*-\overline{X})}{\beta_2(x)(\overline{X}+\overline{x}^*)+2\rho_{yx}}\right\}$	$\beta_2(x)$	ρ_{yx}

Expressing (27) in terms of e's we have

$$\begin{aligned} T_p &= k_1\overline{Y}\left(1+e_0\right)\left(1-\theta e_1\right)^{k_2}\exp\left\{-\frac{u\theta\overline{X}e_1}{2\left(u\overline{X}+v\right)-u\theta\overline{X}e_1}\right\}, \\ &= k_1\overline{Y}\left(1+e_0\right)\left(1-\theta e_1\right)^{k_2}\exp\left\{-\tau\theta e_1(1-\tau\theta e_1)^{-1}\right\}. \end{aligned} \tag{28}$$

Expanding the right-hand side of (28), multiplying out and neglecting terms of e's having power greater than two we have

$$T_P = \overline{Y}k_1\Bigg[1+e_0-\left(k_2+\tau\right)\theta e_1-\left(k_2+\tau\right)\theta e_0 e_1 + \left\{\frac{k_2\left(k_2-1\right)}{2}+k_2\tau-\frac{\tau^2}{2}\right\}\theta^2 e_1^2\Bigg]$$

or

$$\left(T_P-\overline{Y}\right) = \overline{Y}\Bigg[k_1\Big\{1+e_0-\left(k_2+\tau\right)\theta\left(e_1+e_0e_1\right) + \left(\frac{k_2(k_2-1)}{2}+k_2\tau-\frac{\tau^2}{2}\right)\theta^2 e_1^2\Big\}-1\Bigg]. \tag{29}$$

Taking expectation of both sides of (29), we get the bias of t_p to the first order of approximation as

$$\mathrm{Bias}(T_P) = \overline{Y}\Bigg[k_1\Big\{1+\psi\bigg(\left(\frac{k_2(k_2-1)}{2}+k_2\tau-\frac{\tau^2}{2}\right)\theta^2C_x^2 - (k_2+\tau)\theta\rho_{yx}C_yC_x\bigg)\Big\}-1\Bigg]. \tag{30}$$

Squaring both sides of (29) and neglecting terms of e's having power greater than two we have

$$\begin{aligned} \left(T_P-\overline{Y}\right)^2 = \overline{Y}^2\Bigg[&1+k_1^2\left\{1+2e_0-2(k_2+\tau)\theta e_1+e_0^2-4(k_2+\tau)\theta e_0e_1\right\} \\ &+k_2(2k_2+4\tau-1)\theta^2e_1^2-2k_1\Big\{1+e_0-(k_2+\tau)\theta(e_1+e_0e_1) \\ &+\left(\frac{k_2(k_2-1)}{2}+k_2\tau-\frac{\tau^2}{2}\right)\theta^2e_1^2\Big\}\Bigg]. \end{aligned} \tag{31}$$

Taking expectation of both sides of (31) we get the MSE of T_p to the first order of approximation as

$$\mathrm{MSE}(T_P) = \overline{Y}^2\left(1+k_1^2B_3-2k_1A_3\right) \tag{32}$$

where

$$\begin{aligned} A_3 &= \left[1+\psi\left\{C_x^2\theta^2\left(k_2\tau+\frac{k_2(k_2-1)}{2}-\frac{\tau^2}{2}\right)-\theta\rho_{yx}C_yC_x(k_2+\tau)\right\}\right], \\ B_3 &= \left[1+\psi\left\{C_y^2-4\rho_{yx}C_yC_x\theta\left(k_2+\tau\right)+C_x^2\theta^2k_2\left(2k_2+4\tau-1\right)\right\}\right] \end{aligned}$$

The MSE T_p at (32) is minimized for $k_1 = \frac{A_3}{B_3} = k_{1(\text{opt})}$, (say).

Substitution of (33) in (32) yields the minimum MSE of (T_P) as

$$\min MSE(T_P) = \overline{Y}^2 \left(1 - \frac{A_3^2}{B_3}\right). \tag{33}$$

which is non-negative if

$$0 < \frac{A_3^2}{B_3} < 1 \quad \text{and} \quad B_3 > 0 \tag{34}$$

5. Efficiency Comparison

In this section, we compare the minimum MSE of the proposed class of estimators T_P with that of existing estimators to identify the conditions under which the proposed estimator T_P exhibits a lower MSE compared to other existing estimators.

From (1) and (33) we note that $MSE(t_m) > \min MSE(T_P)$ if

$$\left(\frac{A_3^2}{B_3} + \psi C_y^2\right) > 1. \tag{35}$$

From (2) and (33) we have that MSE $(t_r) > \min MSE(T_P)$ if

$$\left[\frac{A_3^2}{B_3} + \psi(C_y^2 + C_x^2 - 2\rho_{yx} C_y C_x)\right] > 1. \tag{36}$$

It is observed from (3), (4), (5), (13), (16) and (33) that $\min MSE(t_{reg}$ or t_s or t_w or t_v or $t_{gk}) > \min MSE(T_P)$ if

$$\left[\frac{A_3^2}{B_3} + \psi C_y^2(1 - \rho_{yx}^2)\right] > 1. \tag{37}$$

It is observed from (6) and (8) that the proposed estimator T_P will dominate over the estimator $\widehat{\overline{Y}}_a$ i.e.if $MSE(\widehat{\overline{Y}}_a) > \min MSE(T_P)$ if

$$\left[\frac{A_3^2}{B_3} + \psi(C_y^2 + \theta^2 C_x^2 - 2\theta\rho_{yx} C_y C_x)\right] > 1. \tag{38}$$

From (7) and (8) we have that $MSE(t_{sd}) > \min MSE(T_P)$ if

$$\left[\frac{A_3^2}{B_3} + \psi(C_y^2 + \lambda_1^2 C_x^2 - 2\lambda_1\rho_{yx} C_y C_x)\right] > 1. \tag{39}$$

We note from (8) and (33) that MSE $(t_{re}) > \min MSE(T_P)$ if

$$\left[\frac{A_3^2}{B_3} + \psi(C_y^2 + \frac{1}{4} C_x^2 - \rho_{yx} C_y C_x)\right] > 1. \tag{40}$$

From (9) and (33) it is looked upon that the proposed class of estimators T_P is more precise than the estimator $t_{sk} MSE(t_{sk}) > \min MS(T_P)$ if

$$\left[\frac{A_3^2}{B_3} + \psi(C_y^2 + \lambda_2^2 C_x^2 - 2\lambda_2 \rho_{yx} C_y C_x)\right] > 1. \tag{41}$$

We have from (10) and (33) that $MSE(t_{us1}) > \min MSE(T_P)$ if

$$\left[\frac{A_3^2}{B_3} + \psi(C_y^2 + \lambda_3^2 C_x^2 - 2\lambda_3 \rho_{yx} C_y C_x)\right] > 1. \tag{42}$$

From (11) that, (33) we have that $MSE(t_{us2}) > \min MSE(T_P)$ if

$$\left[\frac{A_3^2}{B_3} + \psi(C_y^2 + \lambda_4^2 C_x^2 - 2\lambda_4 \rho_{yx} C_y C_x)\right] > 1. \tag{43}$$

It is observed from (12) and (33) that the proposed class of estimators T_p is better than the estimator t_{ST} if $MSE(t_{ST}) > \min MSE(T_P)$ if

$$\left[\frac{A_3^2}{B_3} + \psi(C_y^2 + \lambda_5^2 C_x^2 - 2\lambda_5 \rho_{yx} C_y C_x)\right] > 1. \tag{44}$$

From (14) and (33) we have that $MSE(t_{sn}) > \min MSE(T_P)$ if

$$\left[\frac{A_3^2}{B_3} + \psi(C_y^2 + \tau^2 C_x^2 - 2\tau \rho_{yx} C_y C_x)\right] > 1. \tag{45}$$

We note from (15) and (33) that $\min MSE(t_{yk}) > \min MSE(T_P)$ if

$$\frac{A_3^2}{B_3} > \frac{A_1^2}{B_1}. \tag{46}$$

From (17), (18), (19), (20), (21), (22), (23), (24), (25), and (33) we have that $\min MSE(t_{y_i}) > \min MSE(T_P)$, ($i = 1$ to 9) if

$$\left[\frac{A_3^2}{B_3} + \psi\left(C_y^2 + \varphi_i^2 C_x^2 - 2\varphi_i \rho_{yx} C_y C_x\right)\right] > 1 \tag{47}$$

Further, from (26) and (33), it can be seen that the proposed class of estimators T_P will dominate the Kumar & Siddiqui (2024) estimator t_{ks}; $\min MSE(t_{ks}) > \min MSE(T_p)$

$$\frac{A_3^2}{B_3} > \frac{A_2^2}{B_2}. \tag{48}$$

6. Empirical Study

In this section, we have conducted an empirical study to demonstrate that the proposed estimator T_P outperforms existing estimators discussed here. We have utilized three real population datasets and calculated the MSE and PRE for each estimator. The results clearly show that the proposed estimator is more efficient, as it exhibits a lower MSE in comparison to all other estimators and achieves higher PRE relative to the conventional sample mean estimator. The descriptive information of the populations (I–III) is given below.

Population-I Source: (Singh & Chaudhary, 1986)
y = Total number of seats (CS82) in municipal council in 1982.
x = Number of conservative seats (CS82) in municipal council in 1982.

$N = 22$	$n = 5$	$\overline{Y} = 22.6201$	$\overline{X} = 1467.5455$	$M_x = 534.5000$
$C_y = 1.4601$	C_x =1.7459	$\rho_{yx} = 0.9021$	$\beta_1(x) = 3.3914$	$\beta_2(x) = 13.3693$

Population-II Source: (Cochran, 1977)
y = Seasonal average price per pound during 1967.
x = Seasonal average price per pound during 1966.

$N = 49$	$n = 20$	$\overline{Y} = 127.7959$	$\overline{X} = 103.1429$	$M_x = 64$
$C_y = 0.9634$	$C_x = 1.0122$	$\rho_{yx} = 0.9817$	$\beta_1(x) = 2.2553$	$\beta_2(x) = 7.5114$

Population-III Source: (Kadilar & Cingi, 2006)
y = Apple production amount in 1999.
x = Number of apple trees in 1999.

$N = 106$	$n = 40$	$\overline{Y} = 2212.5943$	$\overline{X} = 27421.6981$	$M_x = 14543.8$
$C_y = 5.1961$	$C_x = 1.7459$	$\rho_{yx} = 0.8560$	$\beta_1(x) = 5.1238$	$\beta_2(x) = 34.5723$

Table 3: MSE and PRE of Estimators of $\overline{\boldsymbol{Y}}$ with respect to $\boldsymbol{t_m}$.

S. No.	Estimators	MSE	PRE $(.,t_m)$
		Population-I	
1.	$\boldsymbol{T}_{P_1}$	**0.038676068**	**851.881364**
2.	$\boldsymbol{T}_{P_8}$	**0.038676269**	**851.876941**
3.	$\boldsymbol{T}_{P_3}$	**0.038676457**	**851.872804**
4.	$\boldsymbol{T}_{P_7}$	**0.038677605**	**851.847515**
5.	$\boldsymbol{T}_{P_5}$	**0.03867875**	**851.8223**
6.	$\boldsymbol{T}_{P_2}$	**0.038681252**	**851.767196**
7.	$\boldsymbol{T}_{P_4}$	**0.038698618**	**851.384977**
8.	$\boldsymbol{T}_{P_6}$	**0.038715125**	**851.021968**
9.	$\boldsymbol{t}_{ks_1}$	0.051756499	636.585216
10.	$\boldsymbol{t}_{ks_3}$	0.051758239	636.563811
11.	$\boldsymbol{t}_{ks_7}$	0.05176338	636.500587
12.	$\boldsymbol{t}_{ks_8}$	0.05176338	636.500587
13.	$\boldsymbol{t}_{ks_5}$	0.051768506	636.437571
14.	$\boldsymbol{t}_{ks_4}$	0.051857298	635.347831
15.	$\boldsymbol{t}_{ks_6}$	0.051930856	634.44789

Continued

Table 3. *Continuation*

S. No.	Estimators	MSE	PRE $(.,t_m)$
16.	t_{reg}	0.061353236	537.011966
17.	t_s	0.061353236	537.011966
18.	t_w	0.061353236	537.011966
19.	t_{gk}	0.061353236	537.011966
20.	t_v	0.061353236	537.011966
21.	t_{yk1}	0.084281641	390.92051
22.	t_{yk8}	0.084287505	390.893313
23.	t_{yk3}	0.08429299	390.867878
24.	t_{yk7}	0.084326539	390.712372
25.	t_{yk5}	0.084360019	390.557308
26.	t_{yk2}	0.084433293	390.218369
27.	t_{ks2}	0.051779704	390.218369
28.	t_{yk4}	0.084945597	387.864977
29.	t_{yk6}	0.085438829	385.625859
30.	t_{sk}	0.087711637	375.633417
31.	t_{us1}	0.088573672	371.977601
32.	t_{sd}	0.089487713	368.178167
33.	t_{ST}	0.089620016	367.634636
34.	t_{y_8}	0.089700997	367.302740
35.	t_{y_6}	0.089706962	367.278316
36.	t_{y_7}	0.089712344	367.256282
37.	t_{y_5}	0.089733539	367.169536
38.	t_{us2}	0.089741390	367.137415
39.	t_{y_9}	0.089745684	367.119848
40.	t_{y_4}	0.089761836	367.053789
41.	t_{y_3}	0.089761894	367.053550
42.	t_{y_2}	0.089761938	367.053371
43.	t_{y_1}	0.089761953	367.053310
44.	$\widehat{\overline{Y}}_a$	0.161168474	367.053229
45.	t_{re}	0.091848015	358.716756
46.	t_{sn1}	0.091848015	358.716756
47.	t_{sn8}	0.091853526	358.695234
48.	t_{sn3}	0.091858680	358.675106
49.	t_{sn7}	0.091890214	358.552022
50.	t_{sn5}	0.091921690	358.429246
51.	t_{sn2}	0.091990602	358.160741
52.	t_{sn4}	0.092473342	356.291026
53.	t_{sn6}	0.092939642	354.503431
54.	t_r	0.089761973	204.428454
55.	t_m	0.329474220	100
	Population-II		
1.	$\boldsymbol{T}_{P_1}$	**0.000488896**	**5617.828205**
2.	$\boldsymbol{T}_{P_8}$	**0.000489111**	**5615.360870**
3.	$\boldsymbol{T}_{P_3}$	**0.000489118**	**5615.284457**
4.	$\boldsymbol{T}_{P_7}$	**0.000490464**	**5599.874638**
5.	$\boldsymbol{T}_{P_5}$	**0.000490482**	**5599.660605**
6.	$\boldsymbol{T}_{P_2}$	**0.000490531**	**5599.109442**
7.	$\boldsymbol{T}_{P_4}$	**0.000499481**	**5498.778971**
8.	$\boldsymbol{T}_{\boldsymbol{P_6}}$	**0.000499592**	**5497.557388**
9.	$\boldsymbol{T}_{\boldsymbol{ks_1}}$	0.000992337	2767.745743

Continued

Table 3. *Continuation*

S. No.	Estimators	MSE	PRE (.,t_m)
10.	$\boldsymbol{T_{ks_8}}$	0.000992373	2767.643586
11.	$\boldsymbol{t_{ks_3}}$	0.000992374	2767.640429
12.	$\boldsymbol{t_{ks_7}}$	0.0009926	2767.011645
13.	$\boldsymbol{t_{ks_5}}$	0.000992603	2767.003024
14.	$\boldsymbol{t_{ks_4}}$	0.000993933	2763.301562
15.	$\boldsymbol{t_{ks_6}}$	0.000993947	2763.261392
16.	t_{reg}	0.000996034	2757.4713
17.	t_s	0.000996034	2757.4713
18.	t_w	0.000996034	2757.4713
19.	t_{gk}	0.000996034	2757.4713
20.	t_v	0.000996034	2757.4713
21.	t_{us1}	0.000996101	2757.285248
22.	t_{sk}	0.000996188	2757.045636
23.	t_{sd}	0.001090811	2517.882933
24.	t_{ST}	0.001091797	2515.609101
25.	t_{us2}	0.001121481	2449.024805
26.	t_{y_8}	0.001124642	2442.14287
27.	t_{y_6}	0.001124678	2442.064504
28.	t_{y_7}	0.001124713	2441.987565
29.	t_{y_5}	0.001124736	2441.937814
30.	t_{y_9}	0.001124759	2441.88866
31.	t_{y_4}	0.001126357	2438.424324
32.	t_{y_3}	0.00112636	2438.417401
33.	t_{y_2}	0.001126542	2438.022315
34.	t_{y_1}	0.001126543	2438.020236
35.	$\widehat{\overline{Y}}_a$	0.002811662	2437.849762
36.	t_r	0.001126622	976.8370688
37.	t_{yk1}	0.006701057	409.8660188
38.	t_{re}	0.006716418	408.9286207
39.	t_{sn1}	0.006716418	408.9286207
40.	t_{yk8}	0.006717689	408.8512321
41.	t_{yk3}	0.006718205	408.8198026
42.	t_{sn8}	0.006733096	407.9156838
43.	t_{sn3}	0.006733614	407.8843111
44.	t_{yk7}	0.006824044	402.4791741
45.	t_{yk5}	0.006825538	402.3910688
46.	t_{yk2}	0.006829388	402.1641811
47.	t_{ks2}	0.000992611	402.1641811
48.	t_{sn7}	0.006839755	401.5546308
49.	t_{sn5}	0.006841254	401.4666702
50.	t_{sn2}	0.006845116	401.2401545
51.	t_{yk4}	0.007615867	360.6333383
52.	t_{yk6}	0.007626652	360.1233335
53.	t_{sn4}	0.00763447	359.7545714
54.	t_{sn6}	0.007645303	359.244834
55.	t_m	0.027465354	100
		Population-III	
1.	$\boldsymbol{T_{P_1}}$	**0.102149206**	**411.4320166**
2.	$\boldsymbol{T_{P_8}}$	**0.102149208**	**411.4320106**
3.	$\boldsymbol{T_{P_3}}$	**0.10214921**	**411.4320019**

Continued

Table 3. *Continuation*

S. No.	Estimators	MSE	PRE $(.,t_m)$
4.	$\boldsymbol{T_{P_7}}$	**0.102149231**	**411.4319167**
5.	$\boldsymbol{T_{P_5}}$	**0.102149258**	**411.4318082**
6.	$\boldsymbol{T_{P_2}}$	**0.102149332**	**411.4315089**
7.	$\boldsymbol{T_{P_4}}$	**0.102150207**	**411.427985**
8.	$\boldsymbol{T_{P_6}}$	**0.102151291**	**411.4236201**
9.	$\boldsymbol{t_{ks_1}}$	0.111340905	377.4664302
10.	$\boldsymbol{t_{ks_8}}$	0.111340906	377.466427
11.	$\boldsymbol{t_{ks_3}}$	0.111340907	377.4664223
12.	$\boldsymbol{t_{ks_7}}$	0.111340921	377.4663767
13.	$\boldsymbol{t_{ks_5}}$	0.111340938	377.4663186
14.	$\boldsymbol{t_{ks_4}}$	0.111341542	377.4642715
15.	$\boldsymbol{t_{ks_6}}$	0.111342231	377.461935
16.	t_{reg}	0.112324254	374.1618774
17.	t_s	0.112324254	374.1618774
18.	t_w	0.112324254	374.1618774
19.	t_{gk}	0.112324254	374.1618774
20.	t_v	0.112324254	374.1618774
21.	$\widehat{\overline{Y}}_a$	0.270122792	210.9870665
22.	t_{y_1}	0.199194456	210.9870665
23.	t_{y_2}	0.199194456	210.9870665
24.	t_{y_3}	0.199194456	210.9870665
25.	t_{y_4}	0.199194456	210.9870664
26.	t_{y_9}	0.199194513	210.9870057
27.	t_{y_5}	0.199194575	210.9869397
28.	t_{y_7}	0.199194705	210.9868021
29.	t_{y_6}	0.199194747	210.9867576
30.	t_{us2}	0.199194793	210.9867091
31.	t_{y_8}	0.199194796	210.9867057
32.	t_{ST}	0.199199243	210.9819954
33.	t_{sd}	0.19920612	210.9747123
34.	t_{us1}	0.199287148	210.8889328
35.	t_{sk}	0.199387694	210.7825868
36.	t_{yk1}	0.242344507	173.4202866
37.	t_{yk8}	0.242344564	173.420246
38.	t_{yk3}	0.242344645	173.4201877
39.	t_{yk7}	0.242345448	173.4196134
40.	t_{yk5}	0.242346469	173.4188826
41.	t_{yk2}	0.242349287	173.4168662
42.	t_{ks2}	0.111340985	173.4168662
43.	t_{yk4}	0.24238248	173.3931178
44.	t_{yk6}	0.242423642	173.3636763
45.	t_r	0.199194456	155.586478
46.	t_{re}	0.292809115	143.5319178
47.	t_{sn1}	0.292809115	143.5319178
48.	t_{sn8}	0.292809215	143.5318689
49.	t_{sn3}	0.292809358	143.5317986
50.	t_{sn7}	0.29281077	143.5311068
51.	t_{sn5}	0.292812566	143.5302264
52.	t_{sn2}	0.292817522	143.5277972
53.	t_{sn4}	0.292875907	143.4991848

Continued

Table 3. *Continuation*

S. No.	Estimators	MSE	PRE (.,t_m)
54.	t_{sn6}	0.292948332	143.4637079
55.	t_m	0.420274539	100

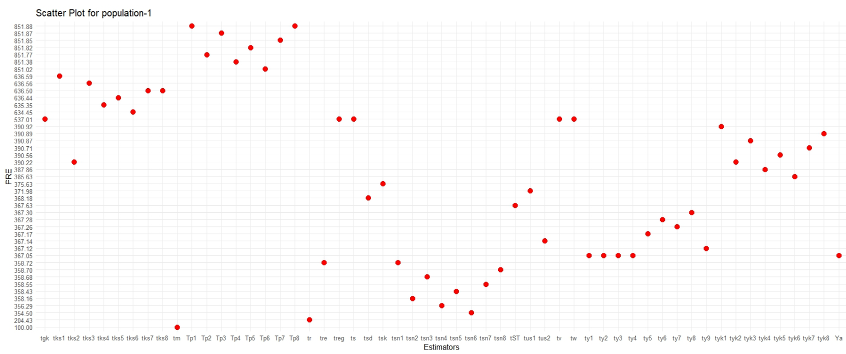

FIGURE 1: PREs of the estimators for population-I.

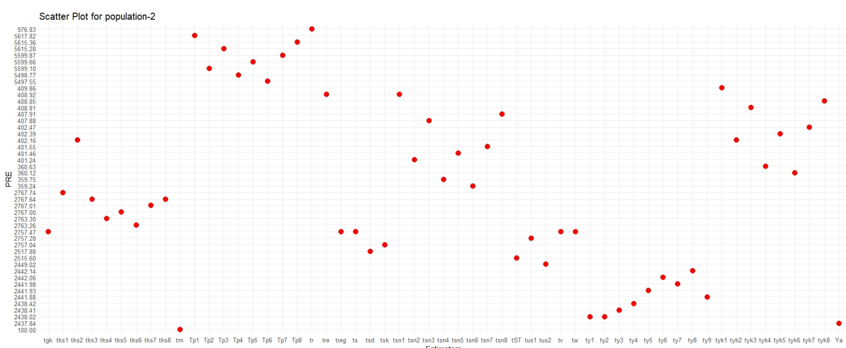

FIGURE 2: PREs of the estimators for population-II.

TABLE 4: Optimum value of k_1 and k_2 for different Population for proposed estimator.

Estimators	Population-I	Population-II	Population-III
$\boldsymbol{T_{P_1}}$	(1.1814, 2.065)	(0.8548, 1.0230)	(1.1191, 3.0190)
$\boldsymbol{T_{P_2}}$	(1.1814, 2.0656)	(0.8596, 1.0230)	(1.1191, 3.0190)
$\boldsymbol{T_{P_3}}$	(1.18145, 2.0651)	(0.8554, 1.0230)	(1.1191, 3.0190)
$\boldsymbol{T_{P_4}}$	(1.18135, 2.0676)	(0.8883, 1.0227)	(1.1191, 3.0193)
$\boldsymbol{T_{P_5}}$	(1.181446, 2.0653)	(0.8595, 1.0230)	(1.1191, 3.0190)
$\boldsymbol{T_{P_6}}$	(1.1812, 2.06956)	(0.8887, 1.022)	(1.1191, 3.0196)
$\boldsymbol{T_{P_7}}$	(1.18145, 2.06523)	(0.8594, 1.0230)	(1.1191, 3.0190)
$\boldsymbol{T_{P_8}}$	(1.18145, 2.0650)	(0.8554, 1.0230)	(1.1191, 3.0190)

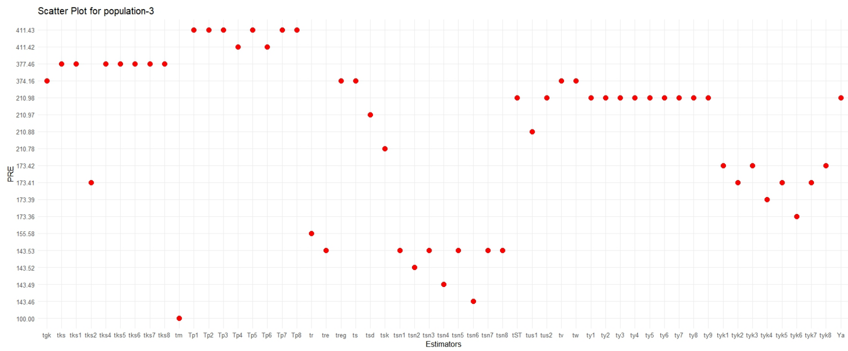

Figure 3: PREs of the estimators for population-III.

Table 5: Range of k_1 t the fixed value of k_2 for different populations for existing estimators.

Estimators	Population-I ($k_2 = 2.065$)	Population-II ($k_2 = 1.0230$)	Population-III ($k_2 = 3.01905$)
t_m	(0.53, 1.83)	(0.85, 1.19)	(0.45, 1.78)
$\widehat{\bar{Y}}_a$	(0.75, 1.60)	(0.97, 1.07)	(0.63, 1.60)
t_r	(0.90, 1.45)	(0.99, 1.04)	(0.75, 1.48)
t_{reg}	(1, 1.36)	(0.99, 1.04)	(1, 1.23)
t_s	(1, 1.36)	(0.99, 1.04)	(1, 1.23)
t_w	(1, 1.36)	(0.99, 1.04)	(1, 1.23)
t_{sd}	(0.90, 1.45)	(0.99, 1.04)	(0.75, 1.48)
t_{re}	(0.90, 1.45)	(0.94, 1.10)	(0.60, 1.63)
t_{sk}	(0.91, 1.44)	(0.99, 1.04)	(0.75, 1.48)
t_{us1}	(0.90, 1.45)	(0.99, 1.04)	(0.75, 1.48)
t_{us2}	(0.90, 1.45)	(0.99, 1.04)	(0.75, 1.48)
t_{ST}	(0.90, 1.45)	(0.99, 1.04)	(0.75, 1.48)
t_{sn1}	(0.90, 1.45)	(0.94, 1.10)	(0.60, 1.63)
t_{sn2}	(0.90, 1.45)	(0.94, 1.10)	(0.60, 1.63)
t_{sn3}	(0.90, 1.45)	(0.94, 1.10)	(0.60, 1.63)
t_{sn4}	(0.89, 1.46)	(0.93, 1.10)	(0.60, 1.63)
t_{sn5}	(0.90, 1.45)	(0.94, 1.10)	(0.60, 1.63)
t_{sn6}	(0.89, 1.46)	(0.93, 1.10)	(0.60, 1.63)
t_{sn7}	(0.90, 1.45)	(0.94, 1.10)	(0.60, 1.63)
t_{sn8}	(0.90, 1.45)	(0.94, 1.10)	(0.60, 1.63)
t_{yk1}	(0.92, 1.43)	(0.94, 1.10)	(0.67, 1.56)
t_{yk2}	(0.92, 1.43)	(0.94, 1.10)	(0.67, 1.56)
t_{yk3}	(0.92, 1.43)	(0.94, 1.10)	(0.67, 1.56)
t_{yk4}	(0.92, 1.43)	(0.93, 1.10)	(0.67, 1.56)
t_{yk5}	(0.92, 1.43)	(0.94, 1.10)	(0.67, 1.56)
t_{yk6}	(0.92, 1.44)	(0.93, 1.10)	(0.67, 1.56)
t_{yk7}	(0.92, 1.43)	(0.94, 1.10)	(0.67, 1.56)
t_{yk8}	(0.92, 1.43)	(0.94, 1.10)	(0.67, 1.56)
t_{gk}	(1.11, 1.24)	(0.99, 1.04)	(1, 1.23)
t_v	(1.11, 1.24)	(0.99, 1.04)	(1, 1.23)
t_{y_1}	(0.90, 1.45)	(0.99, 1.04)	(0.75, 1.48)
t_{y_2}	(0.90, 1.45)	(0.99, 1.04)	(0.75, 1.48)

Continued

Table 5. *Continuation*

Estimators	Population-I ($k_2 = 2.065$)	Population-II ($k_2 = 1.0230$)	Population-III ($k_2 = 3.01905$)
$\boldsymbol{t_{y_3}}$	(0.90, 1.45)	(0.99, 1.04)	(0.75, 1.48)
$\boldsymbol{t_{y_4}}$	(0.90, 1.45)	(0.99, 1.04)	(0.75, 1.48)
$\boldsymbol{t_{y_5}}$	(0.90, 1.45)	(0.99, 1.04)	(0.75, 1.48)
$\boldsymbol{t_{y_6}}$	(0.90, 1.45)	(0.99, 1.04)	(0.75, 1.48)
$\boldsymbol{t_{y_7}}$	(0.90, 1.45)	(0.99, 1.04)	(0.75, 1.48)
$\boldsymbol{t_{y_8}}$	(0.90, 1.45)	(0.99, 1.04)	(0.75, 1.48)
$\boldsymbol{t_{y_9}}$	(0.90, 1.45)	(0.99, 1.04)	(0.75, 1.48)
$\boldsymbol{t_{ks_1}}$	(1.04, 1.31)	(0.99, 1.04)	(1, 1.23)
$\boldsymbol{t_{ks_2}}$	(1.04, 1.31)	(0.99, 1.04)	(1, 1.23)
$\boldsymbol{t_{ks_3}}$	(1.04, 1.31)	(0.99, 1.04)	(1, 1.23)
$\boldsymbol{t_{ks_4}}$	(1.04, 1.31)	(0.99, 1.04)	(1, 1.23)
$\boldsymbol{t_{ks_5}}$	(1.04, 1.31)	(0.99, 1.04)	(1, 1.23)
$\boldsymbol{t_{ks_6}}$	(1.04, 1.31)	(0.99, 1.04)	(1, 1.23)
$\boldsymbol{t_{ks_7}}$	(1.04, 1.31)	(0.99, 1.04)	(1, 1.23)
$\boldsymbol{t_{ks_8}}$	(1.04, 1.31)	(0.99, 1.04)	(1, 1.23)

Table 6: Range of k_2 at the fixed value of k_1 for different populations for existing estimators.

Estimators	Population-I ($k_1 = 1.18146$)	Population-II ($k_1 = 0.8548$)	Population-III ($k_1 = 1.11914$)
$\boldsymbol{t_m}$	(0.025, 4.23)	(-0.45, 2.18)	(0.028, 6.10)
$\boldsymbol{t_a}$	(0.76, 3.49)	(0.47, 1.25)	(0.85, 5.27)
$\boldsymbol{t_r}$	(1.24, 3.01)	(0.66, 1.06)	(1.38, 4.74)
$\boldsymbol{t_{reg}}$	(1.54, 2.72)	(0.68, 1.04)	(2.52, 3.61)
$\boldsymbol{t_s}$	(1.54, 2.72)	(0.68, 1.04)	(2.52, 3.61)
$\boldsymbol{t_w}$	(1.54, 2.72)	(0.68, 1.04)	(2.52, 3.61)
$\boldsymbol{t_{sd}}$	(1.24, 3.01)	(0.66, 1.06)	(1.38, 4.74)
$\boldsymbol{t_{re}}$	(1.22, 3.03)	(0.23, 1.50)	(0.71, 5.41)
$\boldsymbol{t_{sk}}$	(1.26, 2.99)	(0.68, 1.04)	(1.38, 4.74)
$\boldsymbol{t_{us1}}$	(1.25, 3.00)	(0.68, 1.04)	(1.38, 4.74)
$\boldsymbol{t_{us2}}$	(1.24, 3.01)	(0.66, 1.06)	(1.38, 4.74)
$\boldsymbol{t_{ST}}$	(1.24, 3.01)	(0.66, 1.06)	(1.38, 4.74)
$\boldsymbol{t_{sn1}}$	(1.22, 3.03)	(0.23, 1.50)	(0.71, 5.41)
$\boldsymbol{t_{sn2}}$	(1.22, 3.03)	(0.23, 1.50)	(0.71, 5.41)
$\boldsymbol{t_{sn3}}$	(1.22, 3.03)	(0.23, 1.50)	(0.71, 5.41)
$\boldsymbol{t_{sn4}}$	(1.22, 3.03)	(0.18, 1.54)	(0.71, 5.41)
$\boldsymbol{t_{sn5}}$	(1.22, 3.03)	(0.23, 1.50)	(0.71, 5.41)
$\boldsymbol{t_{sn6}}$	(1.22, 3.03)	(0.18, 1.54)	(0.71, 5.41)
$\boldsymbol{t_{sn7}}$	(1.22, 3.03)	(0.23, 1.50)	(0.71, 5.41)
$\boldsymbol{t_{sn8}}$	(1.22, 3.03)	(0.23, 1.50)	(0.71, 5.41)
$\boldsymbol{t_{yk1}}$	(1.29, 2.96)	(0.23, 1.50)	(1.04, 5.08)
$\boldsymbol{t_{yk2}}$	(1.29, 2.96)	(0.23, 1.50)	(1.04, 5.08)
$\boldsymbol{t_{yk3}}$	(1.29, 2.96)	(0.23, 1.50)	(1.04, 5.08)
$\boldsymbol{t_{yk4}}$	(1.29, 2.96)	(0.18, 1.54)	(1.04, 5.08)
$\boldsymbol{t_{yk5}}$	(1.29, 2.96)	(0.23, 1.50)	(1.04, 5.08)
$\boldsymbol{t_{yk6}}$	(1.29, 2.96)	(0.18, 1.54)	(1.04, 5.08)

Continued

Table 6. *Continuation*

Estimators	Population-I ($k_1 = 1.18146$)	Population-II ($k_1 = 0.8548$)	Population-III ($k_1 = 1.11914$)
t_{yk7}	(1.29, 2.96)	(0.23, 1.50)	(1.04, 5.08)
t_{yk8}	(1.29, 2.96)	(0.23, 1.50)	(1.04, 5.08)
t_{gk}	(1.54, 2.72)	(0.23, 1.50)	(2.52, 3.61)
t_v	(1.54, 2.72)	(0.23, 1.50)	(2.52, 3.61)
t_{y_1}	(1.24, 3.01)	(0.66, 1.06)	(1.38, 4.74)
t_{y_2}	(1.24, 3.01)	(0.66, 1.06)	(1.38, 4.74)
t_{y_3}	(1.24, 3.01)	(0.66, 1.06)	(1.38, 4.74)
t_{y_4}	(1.24, 3.01)	(0.66, 1.06)	(1.38, 4.74)
t_{y_5}	(1.24, 3.01)	(0.66, 1.06)	(1.38, 4.74)
t_{y_6}	(1.24, 3.01)	(0.66, 1.06)	(1.38, 4.74)
t_{y_7}	(1.24, 3.01)	(0.66, 1.06)	(1.38, 4.74)
t_{y_8}	(1.24, 3.01)	(0.66, 1.06)	(1.38, 4.74)
t_{y_9}	(1.24, 3.01)	(0.66, 1.06)	(1.38, 4.74)
t_{ks_1}	(1.68, 2.58)	(0.68, 1.04)	(2.52, 3.61)
t_{ks_2}	(1.68, 2.58)	(0.68, 1.04)	(2.52, 3.61)
t_{ks_3}	(1.68, 2.58)	(0.68, 1.04)	(2.52, 3.61)
t_{ks_4}	(1.68, 2.58)	(0.68, 1.04)	(2.52, 3.61)
t_{ks_5}	(1.68, 2.58)	(0.68, 1.04)	(2.52, 3.61)
t_{ks_6}	(1.68, 2.58)	(0.68, 1.04)	(2.52, 3.61)
t_{ks_7}	(1.68, 2.58)	(0.68, 1.04)	(2.52, 3.61)
t_{ks_8}	(1.68, 2.58)	(0.68, 1.04)	(2.52, 3.61)

The empirical analysis, conducted on three real population datasets, demonstrates that the proposed class of estimators significantly outperforms all other existing estimators. There is considerable gain in efficiency by using proposed estimators over all the existing estimators discussed here for all population datasets (I-III). The findings reveal that the developed class of estimators not only exhibits lower MSE but also achieves the higher PRE rather than all other estimators. Empirical study clearly underscores the superior efficiency and effectiveness of the proposed estimators, highlighting their advantage over the existing available alternatives. It is noticed from Tables 5 and 6 that there is enough scope for selecting the scalers k_1 and k_2 to obtain better estimators than the existing estimators. It is further seen from Tables 4, 5, and 6 that from the ranges of $(k_1,\ k_2)$, for all the populations (I-III), it is looked upon that there is a wide range of $(k_1,\ k_2)$, for obtaining better estimators from the suggested class of estimators even if the suggested values of $(k_1,\ k_2)$, depart from their exact optimum values. We have further generated scatter plots in R Studio for the aforementioned real population datasets. Below, we present three scatter plots corresponding to each dataset. Upon analysing these plots, we observed that the proposed class of estimators outperforms all existing estimators in terms of PRE.

7. Simulation Study

In this section, we have evaluated the MSE and PRE of the proposed estimator, alongside other existing estimators, with respect to the sample mean estimator. We have already implemented the analysis in R Studio, where we have generated an artificial population dataset and have conducted a simulation study. The results show that our proposed estimator exhibits the lowest MSE compared to the other existing estimators. Additionally, when calculating the PRE, we found that the proposed estimator outperforms all others, achieving the highest PRE when compared to the usual sample mean estimator. The simulation study clearly demonstrates that the proposed estimator performs exceptionally well in comparison to the existing alternatives. For this study, we generated four distinct normal artificial population sets, as outlined below.

Population-1

$N = 1000$	$n = 60$
$X = rnorm(N, 9, 5)$	$Y = X + rnorm(N, 5, 1)$

Population-II

$N = 2000$	$n = 200$
$X = rnorm(N, 35, 10)$	$Y = X + rnorm(N, 25, 3)$

Population-III

$N = 4000$	$n = 370$
$X = rnorm(N, 25, 18)$	$Y = X + rnorm(N, 10, 5)$

Population-IV

$N = 1500$	$n = 160$
$X = rnorm(N, 20, 8)$	$Y = X + rnorm(N, 15, 3)$

Table 7: MSE and PRE of Estimators for Simulated Population.

S. No.	Estimators	MSE	PRE $(., t_m)$
	Population-I		
1.	$\boldsymbol{T_{P_1}}$	**0.008995871**	**3817.526**
2.	$\boldsymbol{T_{P_8}}$	**0.008995874**	**3817.524**
3.	$\boldsymbol{T_{P_2}}$	**0.008995877**	**3817.523**
4.	$\boldsymbol{T_{P_5}}$	**0.00899588**	**3817.522**
5.	$\boldsymbol{T_{P_7}}$	**0.008995884**	**3817.52**
6.	$\boldsymbol{T_{P_6}}$	**0.00899589**	**3817.518**
7.	$\boldsymbol{T_{P_4}}$	**0.008995894**	**3817.516**
8.	$\boldsymbol{T_{P_3}}$	**0.008996759**	**3817.149**
9.	t_w	0.009333495	3679.433
10.	t_{ks4}	0.009650311	3558.638
11.	t_{ks6}	0.009796858	3505.407
12.	t_s	0.009899586	3469.031

Continued

Table 7. *Continuation*

S. No.	Estimators	MSE	PRE $(.,t_m)$
13.	t_{gk}	0.009928625	3458.885
14.	t_{ks7}	0.00995834	3448.564
15.	t_{ks5}	0.01008337	3405.802
16.	t_{ks8}	0.01015216	3382.726
17.	t_{ks2}	0.01017092	3376.487
18.	t_{sk}	0.01019803	3367.51
19.	t_{reg}	0.01025971	3347.264
20.	t_{ks3}	0.01026976	3343.988
21.	t_{ks1}	0.01032299	3326.748
22.	t_v	0.0106634	3220.546
23.	t_{us1}	0.01104571	3109.078
24.	t_{yk1}	0.03307938	1038.169
25.	t_{re}	0.03388585	1013.46
26.	t_{sn1}	0.03388585	1013.46
27.	t_{yk3}	0.03569017	962.2248
28.	t_{ST}	0.03606542	952.2131
29.	t_{sn3}	0.0364882	941.1801
30.	t_{yk8}	0.03756263	914.2589
31.	t_{sn8}	0.03835483	895.3752
32.	t_{yk2}	0.04125276	832.4769
33.	t_{sn2}	0.04203488	816.9874
34.	t_{sd}	0.04676346	734.3761
35.	t_{yk5}	0.047121	728.8039
36.	t_{sn5}	0.04789318	717.0535
37.	t_{yk7}	0.05754673	596.7667
38.	t_{sn7}	0.0583268	588.7854
39.	t_{us2}	0.05959652	576.2412
40.	t_{y_1}	0.06530907	525.8377
41.	t_{Y_9}	0.06559054	523.5811
42.	t_{Y_2}	0.06591943	520.9688
43.	t_{Y_5}	0.06608187	519.6882
44.	t_{Y_8}	0.06634945	517.5923
45.	t_{Y_6}	0.06635719	517.532
46.	t_{Y_7}	0.06636478	517.4728
47.	t_r	0.06674571	514.5195
48.	t_{yk6}	0.07648264	449.0165
49.	t_{sn6}	0.07737936	443.813
50.	t_{Y_4}	0.1000611	343.21
51.	t_{yk4}	0.1041683	329.6778
52.	t_{sn4}	0.1054916	325.5424
53.	t_{Y_3}	0.1342466	255.8127
54.	$\widehat{\overline{Y}}_a$	0.2803107	122.5139
55.	t_m	0.3434197	100
	Population-II		
1.	$\boldsymbol{T_{p3}}$	0.03806378	786.1318
2.	$\boldsymbol{T_{p1}}$	0.03806387	786.13
3.	$\boldsymbol{T_{p2}}$	**0.03806391**	**786.1292**
4.	$\boldsymbol{T_{p8}}$	**0.03806392**	**786.1291**
5.	$\boldsymbol{T_{p5}}$	**0.038064**	**786.1273**
6.	$\boldsymbol{T_{p6}}$	**0.03806424**	**786.1223**

Continued

Table 7. *Continuation*

S. No.	Estimators	MSE	PRE $(.,t_m)$
7.	$\boldsymbol{T_{p7}}$	**0.03806426**	**786.122**
8.	$\boldsymbol{T_{p4}}$	**0.03806476**	**786.1117**
9.	t_{reg}	0.03818973	783.5393
10.	t_{ks1}	0.03821337	783.0546
11.	t_{ks3}	0.03821424	783.0367
12.	t_{ks2}	0.03821607	782.9991
13.	t_{ks5}	0.03822212	782.8753
14.	t_{ks8}	0.03822393	782.8383
15.	t_{ks6}	0.03824065	782.4959
16.	t_{ks7}	0.03824203	782.4677
17.	t_{ks4}	0.03829521	781.381
18.	t_w	0.03835841	780.0937
19.	t_{gk}	0.03836788	779.901
20.	t_s	0.03836869	779.8846
21	t_v	0.03838893	779.4734
22	t_{re}	0.04050315	738.7859
23	t_{sn1}	0.04050315	738.7859
24	t_{sn3}	0.04063854	736.3246
25	t_{yk1}	0.04085042	732.5054
26	t_{sn2}	0.04093091	731.065
27	t_{sn8}	0.04095443	730.645
28	t_{yk3}	0.04098518	730.0969
29	t_{yk2}	0.04127621	724.9491
30	t_{yk8}	0.04129963	724.5381
31	t_{sn5}	0.04198016	712.7927
32	t_{yk5}	0.04232077	707.0559
33	t_{sn6}	0.04594792	651.2407
34	t_{yk6}	0.04627329	646.6614
35	t_{sn7}	0.04628668	646.4744
36	t_{yk7}	0.04661095	641.9769
37	t_{sn4}	0.06334823	472.3597
38	t_{yk4}	0.06365345	470.0948
39	t_{us1}	0.07464869	400.853
40	t_{sk}	0.1473708	203.0467
41	t_{ST}	0.1835231	163.0484
42	t_{sd}	0.1968464	152.0127
43	t_{us2}	0.2010407	148.8412
44	t_{y_9}	0.202719	147.609
45	t_{y_1}	0.2028818	147.4906
46	t_{y_5}	0.2029674	147.4283
47	t_{y_2}	0.2030154	147.3935
48	t_{y_8}	0.2030381	147.3771
49	t_{y_6}	0.2030393	147.3761
50	t_{y_7}	0.2030406	147.3752
51	t_r	0.203071	147.3531
52	$\widehat{\overline{Y}}_a$	0.2062015	145.1161
53	t_{y_4}	0.2112733	141.6324
54	t_{y_3}	0.2325807	128.6571
55	t_m	0.2992315	100
	Population-III		

Continued

Table 7. *Continuation*

S. No.	Estimators	MSE	PRE $(.,t_m)$
1.	$\boldsymbol{T_{p1}}$	**0.05151544**	**2051.674**
2.	$\boldsymbol{T_{p8}}$	**0.05151561**	**2051.667**
3.	$\boldsymbol{T_{p2}}$	**0.05151582**	**2051.659**
4.	$\boldsymbol{T_{p5}}$	**0.05151593**	**2051.655**
5.	$\boldsymbol{T_{p7}}$	**0.0515161**	**2051.648**
6.	$\boldsymbol{T_{p6}}$	**0.05151677**	**2051.621**
7.	$\boldsymbol{T_{p3}}$	**0.051517**	**2051.61**
8.	$\boldsymbol{T_{p4}}$	**0.05151712**	**2051.607**
9.	t_{reg}	0.05313504	1989.138
10.	t_{y_3}	0.05377525	1965.456
11.	t_w	0.05449519	1939.491
12.	t_{y_4}	0.05575415	1895.696
13.	t_{ks4}	0.05640496	1873.823
14.	t_{gk}	0.05658514	1867.856
15.	t_{ks6}	0.05659883	1867.404
16.	t_s	0.05661105	1867.001
17.	t_{ks7}	0.05695803	1855.628
18.	t_{ks5}	0.05704651	1852.75
19.	t_{ks2}	0.05710737	1850.775
20.	t_{ks8}	0.05715663	1849.18
21.	t_{ks3}	0.05724126	1846.446
22.	t_{ks1}	0.05731234	1844.156
23.	t_v	0.0590103	1791.093
24.	t_{us1}	0.1145731	922.4932
25.	t_{yk1}	0.1280028	825.7077
26.	t_{re}	0.128889	820.0305
27.	t_{sn1}	0.128889	820.0305
28.	t_{yk3}	0.1317587	802.1705
29.	t_{sn3}	0.1326361	796.8637
30.	t_{yk8}	0.1329659	794.8875
31.	t_{sn8}	0.1338407	789.6915
32.	t_{yk2}	0.1393125	758.6749
33.	t_{sn2}	0.1401756	754.0035
34.	t_{sk}	0.1410223	749.4768
35.	t_{yk5}	0.1429677	739.2782
36.	t_{sn5}	0.1438253	734.8701
37.	t_{yk7}	0.1485484	711.5048
38.	t_{sn7}	0.1493992	707.4529
39	t_{yk6}	0.1749864	604.0064
40	t_{sn6}	0.1758288	601.1127
41	t_{yk4}	0.1923653	549.4386
42	t_{sn4}	0.1932201	547.0077
43	t_{ST}	0.2101384	502.968
44	t_{sd}	0.2201884	480.0112
45	t_{us2}	0.2427297	435.4346
46	t_{y_1}	0.2542563	415.6944
47	t_{y_2}	0.2544335	415.4048
48	t_{y_9}	0.2547324	414.9174
49	t_{y_5}	0.2547816	414.8373
50	t_{y_8}	0.2548102	414.7907
51	t_{y_6}	0.2548139	414.7847

Continued

Table 7. *Continuation*

S. No.	Estimators	MSE	PRE $(.,t_m)$
52	t_{y_7}	0.2548175	414.7788
53	t_r	0.254915	414.6202
54	$\widehat{\overline{Y}}_a$	0.7731309	136.7076
55	t_m	1.056929	100
	Population-IV		
1	$\boldsymbol{T}_{p1}$	**0.03722767**	**1211.874**
2	$\boldsymbol{T}_{p8}$	**0.03722781**	**1211.870**
3	$\boldsymbol{T}_{p2}$	**0.03722785**	**1211.868**
4	$\boldsymbol{T}_{p5}$	**0.03722807**	**1211.861**
5	$\boldsymbol{T}_{p7}$	**0.03722852**	**1211.846**
6	$\boldsymbol{T}_{p6}$	**0.03722873**	**1211.839**
7	$\boldsymbol{T}_{p3}$	**0.03722878**	**1211.838**
8	$\boldsymbol{T}_{p4}$	**0.0372295**	**1211.815**
9	t_w	0.03806715	1185.149
10	t_{reg}	0.03812789	1183.261
11	t_{ks4}	0.03888643	1160.180
12	t_s	0.03926109	1149.108
13	t_{gk}	0.03926356	1149.036
14	t_{ks6}	0.03932843	1147.141
15	t_{ks7}	0.03943863	1143.936
16	t_{ks5}	0.03966292	1137.467
17	t_{ks8}	0.03972397	1135.719
18	t_{ks2}	0.03976511	1134.544
19	t_{ks3}	0.03982258	1132.906
20	t_{ks1}	0.03985167	1132.079
21	t_v	0.04051303	1113.598
22.	t_{yk1}	0.0501139	900.2543
23.	t_{re}	0.05093796	885.6901
24.	t_{sn1}	0.05093796	885.6901
25.	t_{yk3}	0.05094107	885.636
26.	t_{sn3}	0.05174491	871.878
27.	t_{yk8}	0.05201465	867.3565
28.	t_{yk2}	0.05267302	856.5154
29.	t_{sn8}	0.05279386	854.5549
30.	t_{sn2}	0.05343793	844.2551
31.	t_{us1}	0.05416187	832.9707
32.	t_{yk5}	0.05609443	804.2732
33.	t_{sn5}	0.0567936	794.3721
34.	t_{yk7}	0.06535085	690.3545
35.	t_{sn7}	0.06592195	684.3737
36.	t_{yk6}	0.07092687	636.0812
37.	t_{sn6}	0.07144372	631.4796
38.	t_{yk4}	0.1028256	438.7549
39.	t_{sn4}	0.1031899	437.2059
40.	t_{sk}	0.1137486	396.6226
41.	t_{ST}	0.1738662	259.4826
42.	t_{sd}	0.1939792	232.5777
43.	t_{us2}	0.2055893	219.4436
44.	t_{y_1}	0.2108651	213.9531
45.	t_{y_9}	0.2109122	213.9054

Continued

Table 7. *Continuation*

S. No.	Estimators	MSE	PRE $(.,t_m)$
46.	t_{y_2}	0.2112635	213.5496
47.	t_{y_5}	0.211283	213.5299
48.	t_{y_8}	0.211422	213.3896
49.	t_{y_6}	0.2114294	213.3821
50.	t_{y_7}	0.2114364	213.375
51.	t_r	0.2115444	213.2661
52.	t_{y_4}	0.2221578	203.0775
53.	t_{y_3}	0.2382411	189.368
54.	$\widehat{\overline{Y}}_a$	0.3039298	148.4397
55.	t_m	0.4511525	100

8. Conclusion

In this section, we conclude the result of present paper by summarizing the key findings. The proposed estimator is a class of dual-to-ratio-type estimators that utilizes information on auxiliary variables in simple random sampling. We derived the bias term, MSE term, and the minimum MSE of the suggested class of estimators. We have conducted an efficiency comparison and identified the conditions under which the proposed estimator outperforms the existing estimators. We also performed an empirical study using three real population datasets, and the results are presented in Table 3. The findings closed in Table 3 demonstrate that the proposed estimator is more efficient than all other existing estimators, as it achieves lower MSE and higher PRE. Additionally, Table 6 shows the optimal values of k_1 and k_2. Tables 5 and 6 display the ranges of k_1 when k_2 is fixed for all populations and the ranges of k_2 when k_1 is fixed for all population datasets, respectively. We have further conducted a simulation study, also showing that the proposed estimator performs significantly better than existing estimators. For the simulation study, we used R Studio for coding, and the results are presented in Table 7. The findings of Table 7 confirm that the proposed estimator has lower MSE and higher PRE compared to all other existing estimators. Based on both the empirical and simulation studies, we conclude that our proposed estimator is more efficient than the existing alternatives. This fact can also be seen in graphical representation.

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References

Adhvaryu, D. & Gupta, P. C. (1983), 'On some alternative strategies using auxiliary information', *Metrika* **30**, 217–226.

Bahl, S. & Tuteja, R. K. (1991), 'Ratio and product type exponential estimators', *Journal of Information and Optimization Sciences* **12**(1), 159–164.

Bandyopadhyay, S. (1980), 'Improved ratio and product estimator', *Sankhya: Series C* **42**, 45–49.

Bhushan, S., Kumar, A., Singh, S. & Kumar, S. (2021), 'An improved class of estimators of population mean under simple random sampling', *Philippine Statistician* **70**(1), 33–47.

Cochran, W. G. (1940), 'The estimation of yields of cereal experiments by sampling for ratio of grain to total produce', *Journal of Agricultural Science* **30**, 262–275.

Cochran, W. G. (1977), *Sampling techniques*, 3 edn, John Wiley & Sons, New York.

Ijaz, M. & Ali, H. (2018), 'Some improved ratio estimators for estimating mean of finite population', *Research and Reviews: Journal of Statistics and Mathematical Sciences* **4**, 18–23.

Javed, M., Irfan, M., Shongwe, S. C., Hussain, M. A. & Meetei, Z. (2025), 'Difference-cum-exponential-type estimators for estimation of finite population mean in survey sampling', *PLoS ONE* **20**(1), e0313712.

Kadilar, C. & Cingi, H. (2006), 'Improvement in estimating the population mean in simple random sampling', *Applied Mathematics Letters* **19**, 75–79.

Kadilar, G. O. (2016), 'A new exponential type estimator for the population mean in simple random sampling', *Journal of Modern Applied Statistical Methods* **15**, 207–214.

Kumar, A. & Siddiqui, A. S. (2024), 'Enhanced estimation of population mean using simple random sampling', *Research in Statistics* **2**(1).

Kumar, A., Siddiqui, A. S., Mustafa, M. S., Hossam, E., Aljohani, H. M. & Almulhim, F. (2024), 'Mean estimation using an efficient class of estimators based on simple random sampling: Simulation and applications', *Alexandria Engineering Journal* pp. 197–203.

Murthy, M. N. (1964), 'Product method of estimator', *The Indian Journal of Statistics* **26**, 69–74.

Naik, V. D. & Gupta, P. C. (1991), 'A general class of estimators for estimating population mean using auxiliary information', *Metrika* **38**, 11–17.

Robson, D. S. (1957), 'Applications of multivariate polykays to the theory of unbiased ratio-type estimation', *Journal of the American Statistical Association* **52**, 511–522.

Searls, D. T. (1964), 'Utilization of a known coefficient of variation in the estimation procedure', *Journal of the American Statistical Association* **59**, 1225–1226.

Sher, K., Iqbal, M., Ali, H., Iftikhar, S., Aphane, M. & Audu, A. (2025), 'Novel efficient estimators of finite population mean in simple random sampling', *Scientific African* **27**, e02598.

Singh, D. & Chaudhary, F. (1986), *Theory and Analysis of Sample Survey Design*, New Age Publications, New Delhi.

Singh, H. P. & Kakran, M. S. (1993), A modified ratio estimator using known coefficient of kurtosis of an auxiliary character. Unpublished manuscript.

Singh, H. P., Solanki, R. K. S. & Singh, A. (2009), 'Improvement in estimating the population mean using exponential estimator in simple random sampling', *Bulletin of Economics and Statistics* **3**, 13–18.

Singh, H. P. & Tailor, R. (2003), 'Use of known correlation coefficient in estimating the finite population mean', *Statistics in Transition* **6**(4), 555–560.

Sisodia, B. V. S. & Dwivedi, V. K. (1981), 'A modified ratio estimator using coefficient of variation of auxiliary variable', *Journal of the Indian Society of Agricultural Statistics* **33**, 13–18.

Srivastava, S. K. (1967), 'An estimator using auxiliary information', *Calcutta Statistical Association Bulletin* **16**, 121–132.

Srivenkataramana, T. (1980), 'A dual to ratio estimator in sample surveys', *Biometrika Journal* **67**(1), 199–204.

Upadhyaya, L. N. & Singh, H. P. (1999), 'An estimator of population variance that utilizes the kurtosis of an auxiliary variable in sample surveys', *Vikram Mathematical Journal* **19**, 14–17.

Vos, J. W. E. (1980), 'Mixing of direct, ratio and product method estimators', *Statistica Neerlandica* **34**, 209–218.

Walsh, J. E. (1970), 'Generalization of ratio estimator for population total', *Sankhya: Series A* **32**, 99–106.

Watson, D. J. (1937), 'The estimation of leaf area in field crops', *Journal of Agricultural Science* **27**, 474–483.

Yadav, S., Arya, D., Vishwakarma, G. & Verma, M. (2023), 'Generalized ratio-cum-exponential-log ratio type estimators of population mean under simple random sampling scheme', *Lobachevskii Journal of Mathematics* **44**, 3889–3901.

Yadav, S. K., Dixit, M. K., Dungana, H. N. & Mishra, S. S. (2019), 'Improved estimators for estimating average yield using auxiliary variable', *International Journal of Mathematical, Engineering and Management Sciences* **4**, 1228–1238.

Yadav, S. K. & Kadilar, C. (2013), 'Efficient family of exponential estimators for the population mean', *Hacettepe Journal of Mathematics and Statistics* **42**, 671–677.