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New Calibration Estimators of Population Variance in the Presence of Random Non-Response under Successive Sampling Scheme

Nuevos estimadores de calibración de la varianza poblacional en presencia de falta de respuesta aleatoria bajo un esquema de muestreo sucesivo

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Abstract

This research addresses the challenge of estimating population variance in surveys conducted over two-occasion (successive) sampling, particularly when dealing with non-response. The study introduces a traditional estimator and two new calibration-based estimators to mitigate the impact of non-response. These calibration estimators are designed to improve the accuracy and reliability of estimates derived from successive sampling surveys, where non-sampling errors can significantly distort the data and the resulting population parameters. The study provides expressions for the proposed estimators and analyzes their statistical properties. Simulation studies reveal that the calibration estimators outperform the traditional estimator in terms of bias, mean squared error, and relative absolute bias, especially when non-response rates are high.

***Keywords*:** Calibration estimator; Mean square error; Random non-response; Two-occasion sampling.

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Resumen

Esta investigación aborda el reto de estimar la varianza poblacional en encuestas realizadas con muestreos sucesivos, en particular en el caso de la falta de respuesta. El estudio introduce un estimador tradicional y dos nuevos estimadores basados en calibración para mitigar el impacto de la falta de respuesta. Estos estimadores de calibración están diseñados para mejorar la precisión y la fiabilidad de las estimaciones derivadas de encuestas con muestreos sucesivos, donde los errores no muestrales pueden distorsionar significativamente los datos y los parámetros poblacionales resultantes. El estudio proporciona expresiones para los estimadores propuestos y analiza sus propiedades estadísticas. Estudios de simulación revelan que los estimadores de calibración superan al estimador tradicional en términos de sesgo, error cuadrático medio y sesgo absoluto relativo, especialmente cuando las tasas de falta de respuesta son altas.

***Palabras clave*:** Error cuadrático medio; Estimador de calibración; Muestreo en dos ocasiones; Respuesta no aleatoria.

1. Introduction

Estimating the variance of a population is of paramount significance in various domains, from business to manufacturing, services, pharmaceuticals, medical sciences, biology, and agriculture. In these real-world scenarios and applications, an accurate determination of the dispersion (variance) of a population is vital for decision making and efficient resource allocation (Ahmed et al., 2016; Naz et al., 2020; Muhammad et al., 2022). This highlights the importance and practical implications of accurate estimation of population variance across various industries.

Variance estimators have wide-ranging practical applications in diverse fields. In clinical trials, they help assess treatment variability and determine adequate sample sizes, while in economics, they are essential for measuring market volatility and other key indicators to support informed decision-making. In agriculture, variance estimation guides choices about crops and resource inputs by analyzing yield variability. Security professionals use variance estimators to assess risks such as cyber-attacks and natural disasters, and in forensic science, they support evidence analysis by evaluating the variability in DNA or fingerprint data. These applications highlight the critical importance of developing efficient and reliable population variance estimators. This has inspired continued research efforts and interest among scholars and statisticians, who strive to develop more efficient estimators for population variance and coefficient of variation. Several authors have work extensively in this direction.

Singh & Homa (2013) explored effective rotation patterns in successive sampling over two occasions, providing improved variance estimation methods that accounted for repeated measures and correlation structures, enhancing efficiency in survey data collected over time. Gupta & Shabbir (2016) developed variance estimators within two-phase sampling frameworks, incorporating auxiliary information to reduce sampling errors and improve precision, particularly when second-phase samples are more informative. Shabbir & Gupta (2017) extended these ideas to two-stage cluster sampling, proposing improved estimators that effectively used information from primary and secondary sampling units to better estimate population variance. Khan & Ahmad (2017) presented efficient variance estimators for two-phase sampling schemes, showing how to leverage additional auxiliary variables to reduce estimator variance and bias under practical survey conditions. Özgür & Ş. Çavuş (2018) studied dual-frame estimators for population variance, allowing for more robust estimation when data come from overlapping survey frames, addressing under-coverage or duplication in sample surveys. Singh & Jafa (2019) proposed improved variance estimators for two-phase sampling by combining multiple sources of auxiliary data, achieving better efficiency and reducing mean squared error compared to conventional estimators. Shakeel & Shabbir (2019) investigated ratio-type variance estimators in two-phase sampling, further refining the use of auxiliary information to improve the estimation of population variance. Kadilar & Cingi (2020) examined variance estimation in two-phase sampling with partially defective or missing items, developing estimators robust to non-response and item non-availability. Bhushan & Singh (2021) introduced improved variance estimators in two-stage cluster sampling, offering enhancements in survey efficiency by optimizing the use of auxiliary variables across both cluster and element levels. Singh, Jafa & Goyal (2021) and Singh, Khalid & Kim (2021) advanced improved variance estimation methods for two-phase sampling, and also investigated imputation strategies to handle missing data, helping maintain estimator efficiency under non-response and measurement errors. Muhammad et al. (2022) developed generalized estimators for population variance using measurable and cost-effective auxiliary characteristics, aiming for practical applications in resource-limited surveys. Zaman & Bulut (2023) explored robust calibration techniques for estimating the population mean in stratified random sampling, focusing on strategies to reduce the influence of outliers and improve estimator stability in practical survey applications. Shahzad et al. (2023) developed calibrated estimators for estimating the coefficient of variation within a double stratified random sampling setting, demonstrating through theoretical and empirical results that their methods offer substantial gains in efficiency and precision compared to conventional estimators. Audu et al. (2025) and Audu, Lekganyane, Ishaq & Aremu (2024) proposed calibration-based estimators under successive sampling and mail surveys with non-response, applying new weighting schemes to improve the estimation of population variance in the presence of measurement errors and missing data. Ali et al. (2024) proposed improved variance estimators by applying transformations to auxiliary variables under simple random sampling, demonstrating through simulation studies that the transformed approach significantly reduces the estimator's bias and mean squared error. Pandey, Singh, Zaman, Al Mutairi & Mustafa (2024) developed improved variance estimators in stratified successive sampling frameworks by employing calibrated weights under non-response, showing how such calibration can address both sampling and non-sampling errors while increasing estimator efficiency. Pandey, Singh, Zaman, Mutairi & Mustafa (2024) extended this line of work to define a general class of variance estimators under stratified sampling, specifically targeting non-sampling errors through calibrated weights, providing flexible estimators that adapt to different survey error structures. Although many variance estimators have been proposed in the literature as discussed above, they generally rely on the assumption of full response from all sampled units, often within simple random sampling frameworks. However, in practical survey applications, especially in successive sampling with repeated measurements in which cost of taking sample is minimized, non-response is common and introduces substantial non-sampling errors. As a result, estimators designed under the assumption of complete response may perform poorly or become inefficient when faced with missing data.

The unbiased traditional estimator of population variance in the absence of non-response under successive sampling can be defined as in Equation (1).

$$s_y^{*2} = \frac{(u-1)\, s_{yu}^2 + (m-1)\, s_{ym}^2}{u+m-1}, \tag{1}$$

where $s_{yu}^2 = (u-1)^{-1} \sum\limits_{i=1}^{u} \left(y_i - \overline{Y}_u\right)^2,\ s_{ym}^2 = (m-1)^{-1} \sum\limits_{i=1}^{m} \left(y_i - \overline{Y}_m\right)^2.$

Despite the development of numerous estimators and methods in the previous literature, many of these techniques assume a complete response from all sampling units and have been studied in the context of simple random sampling. However, this assumption may not hold true in real-world survey situations, where non-response is a significant source of non-sampling error, particularly in successive sampling designs due to the repetitive nature of the surveys over time. This raises a crucial issue, as variance estimation techniques developed for complete response data may no longer be valid or efficient when confronted with missing data due to non-response. According to Little & Rubin (2019). Missing data can be classified into three main categories: Missing Completely at Random (MCAR), Missing at Random (MAR), and Missing Not at Random (MNAR). The study is based specifically on MCAR.

This paper explores the development of new variance calibration estimators for use in successive sampling surveys to address the challenges posed by random non-response. These proposed estimators leverage information from both the matched and unmatched sample of auxiliary variable to provide more reliable and efficient variance estimates, even in the presence of non-response.

The paper is organized as follows: Section 1 discusses the background and significance of the study, as well as the importance of the proposed methods It introduces basic notations, sample structures, and random non-response models/probability functions. Section 2 presents a new conventional estimator of population variance in the presence of random non-response along with its variance. Section 3 presents the procedure and results for calibration estimators. Section 4 compares the performance of the proposed calibration estimators with the proposed conventional estimator of population variance numerically through simulation studies. Finally, Section 5 provides the conclusion and offers some recommendations based on the findings.

1.1. Basic Notations and Definitions

Let $U = \{U_1, U_2, \ldots, U_N\}$ be finite population of size N under study to be sampled over two occasions. Let the study variable be denoted by Y and auxiliary variable be denoted by X. Assume that random non-response occurs on Y and X for both occasions.

On the first occasion, a preliminary simple random sampling without replacement (SRSWOR) sample of size n is drawn from the population, where r_3 units do not respond. From the responding part of this sample, a subsample sample of size $m = n\lambda$ is drawn from n using SRSWOR, where r_1 units do not respond. This sample is matched or retained for the first occasion, and information on Y and X is collected. Additionally, a fresh sample of size is drawn from the population using SRSWOR at the second stage, and information on Y and X is collected again, where r_2 units do not respond.

Then, the following notations were used.

$\overline{Y}, \overline{X}$: are the means of the population under study for Y and X, respectively.

S_x^2, S_y^2: are the population variances of X and Y, respectively.

$\overline{Y}_{m-r_2} = (m - r_2)^{-1} \sum_{i=1}^{m-r_2} y_i$, $\overline{Y}_{u-r_3} = (u - r_3)^{-1} \sum_{i=1}^{u-r_3} y_i$: are the sample means of Y based on the $m - r_1$ and $u - r_2$, respectively.

$s^2_{y_{m-r_1}} = (m - r_1 - 1)^{-1} \sum_{i=1}^{m-r_1} \left(y_i - \overline{Y}_{m-r_1}\right)^2$,

$s^2_{y_{u-r_2}} = (u - r_2 - 1)^{-1} \sum_{i=1}^{u-r_2} \left(y_i - \overline{Y}_{u-r_2}\right)^2$: are the sample variances of Y based on the $m - r_1$ and $u - r_2$, respectively.

$\overline{X}_{m-r_1} = (m - r_1)^{-1} \sum_{i=1}^{m-r_1} x_i$,

$\overline{X}_{u-r_2} = (u - r_2)^{-1} \sum_{i=1}^{u-r_2} x_i$: are the sample means of X based on the $m - r_1$ and $u - r_2$, respectively.

$s^2_{x_{m-r_1}} = (m - r_1 - 1)^{-1} \sum_{i=1}^{m-r_1} \left(x_i - \overline{X}_{m-r_1}\right)^2$,

$s^2_{x_{u-r_2}} = (u - r_2 - 1)^{-1} \sum_{i=1}^{u-r_2} \left(x_i - \overline{X}_{u-r_2}\right)^2$: are the sample variances of X based on the $m - r_1$ and $u - r_2$, respectively.

1.2. Probability Models for r_1, r_2 and r_3

Let take $r_3 \{r_3 = 0, 1, 2, \ldots, n - 2\}$ as the number of units in the sample S_n of size n on which information on X and Y could not be collected due to random non-response. Take $r_1 \{r_1 = 0, 1, 2, \ldots, m - 2\}$ as the number of units in the matched sample S_m of size m on which information on X and Y could not be collected due to random non-response. Finally, consider $r_2 \{r_2 = 0, 1, 2, \ldots, u - 2\}$ as the number of units in the unmatched sample S_u of size u on which information on X and Y could not be collected due to random non-response. We assume that $0 \leq r_1 \leq (m - 2)$, $0 \leq r_2 \leq (u - 2)$ and $r_3 = r_1 + r_2$. If p_1, p_2, p_3 are the probabilities of non-response among the $(m - 2)$, $(u - 2)$ and $(n - 2)$ are possible values of non-responses respectively, then r_1, r_2 and r_3 have the following discrete probability distributions in Equations (2), (3) and (4), respectively.

$$p\left(r_1\right)=\frac{m-r_1}{nq_1+2p_1}\times\frac{(m-2)!}{r_1!\left(m-r_1-2\right)!}p_1^{r_1}q_1^{m-r_1-2},\ \ r_1=0,1,2,\ldots,m-2 \quad (2)$$

$$p\left(r_2\right)=\frac{u-r_2}{nq_2+2p_2}\times\frac{(u-2)!}{r_2!\left(u-r_2-2\right)!}p_2^{r_2}q_2^{u-r_2-2},\ \ r_2=0,1,2,\ldots,u-2 \quad (3)$$

$$p\left(r_3\right)=\frac{n-r_3}{nq_3+2p_3}\times\frac{(n-2)!}{r_3!\left(n-r_3-2\right)!}p_3^{r_3}q_3^{n-r_3-2},\ \ r_3=0,1,2,\ldots,n-2 \quad (4)$$

2. Materials and Methods

2.1. New proposed Conventional Estimator of Population Variance in the Presence of Non-Response

Motivated by the work of Singh et al. (2019) and Audu, Singh, Ishaq, Khare, Singh & Adewara (2024), we proposed conventional estimator of population variance in the presence of random non-response under successive sampling as in Equation (5).

$$s_{y(NR)}^{*2}=\frac{\left(u-r_2-1\right)s_{y\left(u-r_2\right)}^2+\left(m-r_1\right)s_{y\left(m-r_1\right)}^2}{n-r_3-1}, \quad (5)$$

where $s_{y(u-r_2)}^2=\frac{1}{u-r_2-1}\sum_{i=1}^{u-r_2}\left(y_i-\overline{Y}_u\right)^2,\ s_{y(m-r_1)}^2=\frac{1}{m-r_1-1}\sum_{i=1}^{m-r_1}\left(y_i-\overline{Y}_m\right)^2.$

2.1.1. Properties of the Proposed Estimators $s_{y(NR)}^{*2}$

Theorem 1. *The proposed alternative estimator $s_{y(NR)}^{*2}$ is unbiased, that is* $E\left(s_{y(NR)}^{*2}\right)=S_Y^2$.

Proof.

$$E\left(s_{y(NR)}^{*2}\right)=\frac{\left(u-r_2-1\right)E_u\left(s_{y\left(u-r_2\right)}^2\right)+\left(m-r_1\right)E_m\left(s_{y\left(m-r_1\right)}^2\right)}{n-r_3-1}, \quad (6)$$

where E_u is the expectation based on an unmatched sample and E_m is the expectation based on the matched sample.

Consider $E_u\left(s^{*2}_{y(u-r_2)}\right)$ in Equation (6), then Equation (7) is obtained:

$$\begin{aligned} E_u\left(s^{*2}_{y(u-r_2)}\right) &= \frac{E_u\left(\sum_{i=1}^{u-r} y_i^2 - \frac{1}{u-r_2}\left(\sum_{i=1}^{u-r_2} y_i\right)^2\right)}{u-r_2-1} \\ &= \frac{E_u\left(\left(1-\frac{1}{u-r_2}\right)\sum_{i=1}^{u-r} y_i^2 - \frac{1}{u-r_2}\sum_{i\neq j=1}^{u-r} y_i y_j\right)}{u-r_2-1}. \end{aligned} \tag{7}$$

Equation (7) can be expressed in terms of population values as in Equation (8):

$$E_u\left(s^{*2}_{y(u-r_2)}\right) = \frac{E_u\left(\left(1-\frac{1}{u-r_2}\right)\sum_{i=1}^{N} a_{(u)i}Y_i^2 - \frac{1}{u-r_2}\sum_{i\neq j=1}^{N} a_{(u)i}a_{(u)j}Y_iY_j\right)}{u-r_2-1}, \tag{8}$$

where $a_{(u)i} = \begin{cases} 1, & if \quad Y_i \in S_{(u-r_2)} \\ 0, & otherwise \end{cases}$, $a_{(u)i} \sim binomial\left(u-r_2,\, N^{-1}\right)$, $E_u\left(a_{(u)i}\right) = \frac{u-r_2}{N}$, $E_u\left(a_{(u)i}a_{(u)j}\right) = \frac{u-r_2}{N}\frac{u-r_2-1}{N-1}$.

Simplify the right-hand side of Equation (8), Equation (9) is obtained:

$$\begin{aligned} E_u\left(s^{*2}_{y(u-r_2)}\right) &= \frac{1}{N}\left(\sum_{i=1}^{N} Y_i^2 - \frac{1}{N-1}\sum_{i\neq j=1}^{N} Y_iY_j\right) \\ &= \frac{1}{N}\left(\left(1+\frac{1}{N-1}\right)\sum_{i=1}^{N} Y_i^2 - \frac{1}{N-1}\left(\sum_{i=1}^{N} Y_i\right)^2\right) = S_Y^2. \end{aligned} \tag{9}$$

Similarly, considering $E_m\left(s^{*2}_{y(m-r_1)}\right)$ in Equation (6), then Equation (10) is obtained:

$$\begin{aligned} E_m\left(s^{*2}_{y(m-r_1)}\right) &= \frac{E_m\left(\sum_{i=1}^{m-r_1} y_i^2 - \frac{1}{m-r_1}\left(\sum_{i=1}^{m-r_1} y_i\right)^2\right)}{m-r_1-1} \\ &= \frac{E_m\left(\left(1-\frac{1}{m-r_1}\right)\sum_{i=1}^{m-r_1} y_i^2 - \frac{1}{m-r_3}\sum_{i\neq j=1}^{m-r_1} y_i y_j\right)}{m-r_1-1} \end{aligned} \tag{10}$$

$$E_m\left(s^{*2}_{y(m-r_1)}\right) = \frac{E_m\left(\left(1-\frac{1}{m-r_1}\right)\sum_{i=1}^{N} a_{(m)i}Y_i^2 - \frac{1}{m-r_1}\sum_{i\neq j=1}^{N} a_{(m)i}a_{(m)j}Y_iY_j\right)}{m-r_1-1}, \tag{11}$$

where $a_{(m)i} = \begin{cases} 1, & if \quad Y_i \in S_{(m-r_1)} \\ 0, & otherwise \end{cases}$, $a_{(m)i} \sim binomial\left(m-r_1,\, N^{-1}\right)$, $E_m\left(a_{(m)i}\right) = \frac{m-r_1}{N}$, $E_m\left(a_{(m)i}a_{(m)j}\right) = \frac{m-r_1}{N} \times \frac{m-r_1-1}{N-1}$.

Simplify the right-hand side of Equation (11), Equation (12) is obtained:

$$\begin{aligned} E_m\left(s^{*2}_{y(m-r_1)}\right) &= \frac{1}{N}\left(\sum_{i=1}^{N} Y_i^2 - \frac{1}{N-1}\sum_{i\neq j=1}^{N} Y_iY_j\right) \\ &= \frac{1}{N}\left(\left(1+\frac{1}{N-1}\right)\sum_{i=1}^{N} Y_i^2 - \frac{1}{N-1}\left(\sum_{i=1}^{N} Y_i\right)^2\right) = S_Y^2. \end{aligned} \tag{12}$$

Substituting the results of $E_u\left(s^{*2}_{y(u-r_2)}\right)$ and $E_m\left(s^{2}_{y(m-r_1)}\right)$ obtained in Equation (9) and Equation (12) respectively into Equation (6), Theorem 1 is proved. □

Theorem 2. *The variance of $s^{*2}_{y(NR)}$ is given in Equation (13):*

$$\operatorname{var}\left(s^{*2}_{y(NR)}\right) = \begin{pmatrix} \left(\frac{u-r_2-1}{n-r_3-1}\right)^2\left(\frac{1}{uq_2+2p_2}-\frac{1}{N}\right) \\ +\left(\frac{m-r_1-1}{n-r_3-1}\right)^2\left(\frac{1}{mq_1+2p_1}-\frac{1}{N}\right) \end{pmatrix} S_y^4\left(\beta_{40}-1\right), \tag{13}$$

where $\beta_{rs} = \frac{\mu_{rs}}{\mu_{20}^{r/2}\mu_{02}^{s/2}},$, $\mu_{rs} = \frac{1}{N-1}\sum_{i=1}^{N}\left(Y_i-\overline{Y}\right)^r\left(X_i-\overline{X}\right)^s$.

***Proof*.**

$$\operatorname{var}\left(s^{*2}_{y(NR)}\right) = \frac{\left(u-r_2-1\right)^2 \operatorname{var}_u\left(s^2_{y(u-r_2)}\right) + \left(m-r_1\right)^2 \operatorname{var}_m\left(s^2_{y(m-r_1)}\right)}{\left(n-r_3-1\right)^2}. \tag{14}$$

From the results of Singh et al. (2019), $\operatorname{var}_u\left(s^2_{y(u-r_2)}\right)$ and $\operatorname{var}_m\left(s^2_{y(m-r_1)}\right)$ are given in Equation (15) and Equation (16), respectively:

$$\operatorname{var}\left(s^2_{y(u-r_2)}\right) = \left(\frac{1}{uq_2+2p_2}-\frac{1}{N}\right) S_y^4\left(\beta_{40}-1\right) \tag{15}$$

$$\operatorname{var}\left(s^2_{y(m-r_1)}\right) = \left(\frac{1}{mq_1+2p_1}-\frac{1}{N}\right) S_y^4\left(\beta_{40}-1\right). \tag{16}$$

Substitute Equation (15) and Equation (16) in Equation (14), Equation (13) is obtained, hence, the proof. □

2.2. New Proposed Calibration Estimators

The conventional estimator proposed in Equation (8) can be formulated as in Equation (17).

$$s^{*2}_{y(NR)} = \sum_{i=1}^{u-r_2} w_1^*\left(y_i-\overline{Y}_{(u-r_2)}\right)^2 + \sum_{i=1}^{m-r_1} w_2^*\left(y_i-\overline{Y}_{(m-r_1)}\right)^2, \tag{17}$$

where $w_1^* = \frac{1}{n-r_3-1}$, $w_2^* = \frac{m-r_1}{(n-r_3-1)(m-r_1-1)}$.

2.2.1. Proposed Calibration Estimator Based on Non-Linear Constraint of Auxiliary Variable

The proposed calibrated estimator of population variance under random non-response is given in Equation (18):

$$T^*_{1(NR)} = \sum_{i=1}^{u-r_2} \omega^*_{1i} \left(y_i - \overline{Y}_{(u-r_2)}\right)^2 + \sum_{i=1}^{m-r_1} \omega^*_{2i} \left(y_i - \overline{Y}_{(m-r_1)}\right)^2, \tag{18}$$

where ω^*_{1i}, ω^*_{2i} are the new calibration weights to be obtained by minimizing the chi-square distance defined in Equation (19) subject to given constraint.

$$\left.\begin{array}{ll} \min & Z^*_1 = 2^{-1} \sum\limits_{i=1}^{u-r_2} \left(\omega^*_{1i} - w^*_1\right)^2 / w^*_1 \varphi_{1i} + 2^{-1} \sum\limits_{i=1}^{m-r_1} \left(\omega^*_{2i} - w^*_2\right)^2 / w^*_2 \varphi_{2i} \\ s.t. & \sum\limits_{i=1}^{u-r_2} \omega^*_{1i} \left(x_i - \overline{X}_{(u-r_2)}\right)^2 + \sum\limits_{i=1}^{m-r_1} \omega^*_{2i} \left(x_i - \overline{X}_{(m-r_1)}\right)^2 = S^2_X \end{array}\right\} \tag{19}$$

To compute new calibrated weights ω^*_{1i}, $i = 1, 2, \ldots, u-r_2$ and ω^*_{2i}, $i = 1, 2, \ldots, m-r_1$, the Lagrange function L_1 of the formed in Equation (20) is defined:

$$\begin{aligned} L_1 = & \sum_{i=1}^{u-r_2} \frac{\left(\omega^*_{1i} - w^*_1\right)^2}{2 w^*_1 \varphi_{1i}} \\ & + \sum_{i=1}^{m-r_1} \frac{\left(\omega^*_{2i} - w^*_2\right)^2}{2 w^*_2 \varphi_{2i}} - \lambda_1 \left(\begin{array}{l} \sum\limits_{i=1}^{u-r_2} \omega^*_{1i} \left(x_i - \overline{X}_{(u-r_2)}\right)^2 \\ + \sum\limits_{i=1}^{m-r_1} \omega^*_{2i} \left(x_i - \overline{X}_{(m-r_1)}\right)^2 - S^2_X \end{array} \right). \end{aligned} \tag{20}$$

Differentiate Equation (20) partially with respect to ω^*_{1i}, ω^*_{2i}, λ_1 and equates the results to zeros, Equation (21), Equation (22) and Equation (23) are obtained, respectively:

$$\omega^*_{1i} = w^*_1 + \lambda_1 w^*_1 \varphi_{1i} \left(x_i - \overline{X}_{(u-r_2)}\right)^2 \tag{21}$$

$$\omega^*_{2i} = w^*_2 + \lambda_1 w^*_2 \varphi_{2i} \left(x_i - \overline{X}_{(m-r_1)}\right)^2 \tag{22}$$

$$\sum_{i=1}^{u-r_2} \omega^*_{1i} \left(x_i - \overline{X}_{(u-r_2)}\right)^2 + \sum_{i=1}^{m-r_1} \omega^*_{2i} \left(x_i - \overline{X}_{(m-r_1)}\right)^2 = S^2_X. \tag{23}$$

Substitute both Equation (21) and Equation (22) in Equation (23) and solve for λ_1, Equation (24) is obtained:

$$\lambda_1 = \frac{S^2_X - s^{*2}_{x(NR)}}{\sum\limits_{i=1}^{u-r_2} w^*_1 \varphi_{1i} \left(x_i - \overline{X}_{(u-r_2)}\right)^4 + \sum\limits_{i=1}^{m-r_1} w^*_2 \varphi_{2i} \left(x_i - \overline{X}_{(m-r_1)}\right)^4} \tag{24}$$

where $s^{*2}_{x(NR)} = \sum\limits_{i=1}^{u-r_2} w^*_1 \left(x_i - \overline{X}_{(u-r_2)}\right)^2 + \sum\limits_{i=1}^{m-r_1} w^*_2 \left(x_i - \overline{X}_{(m-r_1)}\right)^2$.

Substitute the expression for λ_1 in Equation (21) and Equation (22), the expressions for $\omega_{1i}^*,\ \omega_{2i}^*$ are obtained as in Equation (25) and Equation (26):

$$\omega_{1i}^* = w_1^*\left(1+\frac{\varphi_{1i}\left(x_i-\overline{X}_{(u-r_2)}\right)^2\left(S_X^2-s_{x(NR)}^{*2}\right)}{\sum_{i=1}^{u-r_2} w_1^*\varphi_{1i}\left(x_i-\overline{X}_{(u-r_2)}\right)^4+\sum_{i=1}^{m-r_1} w_2^*\varphi_{2i}\left(x_i-\overline{X}_{(m-r_1)}\right)^4}\right) \tag{25}$$

$$\omega_{2i}^* = w_2^*\left(1+\frac{\varphi_{2i}\left(x_i-\overline{X}_{(m-r_1)}\right)^2\left(S_X^2-s_{x(NR)}^{*2}\right)}{\sum_{i=1}^{u-r_2} w_1^*\varphi_{1i}\left(x_i-\overline{X}_{(u-r_2)}\right)^4+\sum_{i=1}^{m-r_1} w_2^*\varphi_{2i}\left(x_i-\overline{X}_{(m-r_1)}\right)^4}\right). \tag{26}$$

Substitute the results of $\omega_{1i}^*,\ \omega_{2i}^*$ in Equation (18), the proposed calibration estimator becomes Equation (27):

$$T_1^* = s_{y(NR)}^{*2} + \hat{b}_1\left(S_X^2 - s_{x(NR)}^{*2}\right) \tag{27}$$

where

$$\hat{b}_1 = \frac{\sum_{i=1}^{u-r_2} w_1^*\varphi_{1i}\left(x_i-\overline{X}_{(u-r_2)}\right)^2\left(y_i-\overline{Y}_{(u-r_2)}\right)^2+\sum_{i=1}^{m-r_1} w_2^*\varphi_{2i}\left(x_i-\overline{X}_{(m-r_1)}\right)^2\left(y_i-\overline{Y}_{(m-r_1)}\right)^2}{\sum_{i=1}^{u-r_2} w_1^*\varphi_{1i}\left(x_i-\overline{X}_{(u-r_2)}\right)^4+\sum_{i=1}^{m-r_1} w_2^*\varphi_{2i}\left(x_i-\overline{X}_{(m-r_1)}\right)^4}.$$

The members of the proposed calibrated estimator T_1^* are obtained as below.

Case A: Setting $\varphi_{1i}=1$ and $\varphi_{2i}=1$ in $\hat{b}_1$, the first member of T_1^* denoted by T_{11}^* is obtained as in Equation (28):

$$T_{11}^* = s_{y(NR)}^{*2} + \hat{b}_{11}\left(S_X^2 - s_{x(NR)}^{*2}\right) \tag{28}$$

where $\hat{b}_{11} = \frac{\sum_{i=1}^{u-r_2} w_1^*\left(x_i-\overline{X}_{(u-r_2)}\right)^2\left(y_i-\overline{Y}_{(u-r_2)}\right)^2+\sum_{i=1}^{m-r_1} w_2^*\left(x_i-\overline{X}_{(m-r_1)}\right)^2\left(y_i-\overline{Y}_{(m-r_1)}\right)^2}{\sum_{i=1}^{u-r_2} w_1^*\left(x_i-\overline{X}_{(u-r_2)}\right)^4+\sum_{i=1}^{m-r_1} w_2^*\left(x_i-\overline{X}_{(m-r_1)}\right)^4}$.

Case B: Setting $\varphi_{1i}=\left(x_i-\overline{X}_{(u-r_2)}\right)^{-1}$ and $\varphi_{2i}=\left(x_i-\overline{X}_{(m-r_1)}\right)^{-1}$ in $\hat{b}_1$, the second member of T_1^* denoted by T_{12}^* is obtained as in Equation (29):

$$T_{12}^* = s_{y(NR)}^{*2} + \hat{b}_{12}\left(S_X^2 - s_{x(NR)}^{*2}\right), \tag{29}$$

where $\hat{b}_{12} = \frac{\sum_{i=1}^{u-r_2} w_1^*\left(y_i-\overline{Y}_{(u-r_2)}\right)^2+\sum_{i=1}^{m-r_1} w_2^*\left(y_i-\overline{Y}_{(m-r_1)}\right)^2}{\sum_{i=1}^{u-r_2} w_1^*\left(x_i-\overline{X}_{(u-r_2)}\right)^2+\sum_{i=1}^{m-r_1} w_2^*\left(x_i-\overline{X}_{(m-r_1)}\right)^2}$.

The MSE of T_1^* denoted by $MSE\left(T_1^*\right)$ can be obtained using the function defined in Equation (30):

$$MSE\left(T_1^*\right) = \Theta_1\Sigma_1\Theta_1^T, \tag{30}$$

where $\Theta_1$is $1x2$ matrix given as

$$\Theta_1 = \left(\left. \frac{\partial T_1^*}{\partial s_{y(NR)}^{*2}} \right|_{s_{y(NR)}^{*2}=S_Y^2,\, s_{x(NR)}^{*2}=S_X^2,\, \hat{b}_1=\beta_1} \quad \left. \frac{\partial T_1^*}{\partial s_{x(NR)}^{*2}} \right|_{s_{y(NR)}^{*2}=S_Y^2,\, s_{x(NR)}^{*2}=S_X^2,\, \hat{b}_1=\beta_1} \right),$$

Σ_1 is the variance-covariance matrix of order 2×2 defined as

$$\Sigma_1 = \begin{pmatrix} \operatorname{var}\left(s_{y(NR)}^{*2}\right) & \operatorname{cov}\left(s_{y(NR)}^{*2} s_{x(NR)}^{*2}\right) \\ \operatorname{cov}\left(s_{x(NR)}^{*2} s_{y(NR)}^{*2}\right) & \operatorname{var}\left(s_{x(NR)}^{*2}\right) \end{pmatrix},$$ Θ_1^T is the transpose of Θ_1 and $\beta_1 = \sum_{i=1}^{N} \left(X_i - \overline{X}\right)^2 \left(Y_i - \overline{Y}\right)^2 / \sum_{i=1}^{N} \left(X_i - \overline{X}\right)^4$.

Simplifying Equation (30), $MSE\left(T_1^*\right)$ is obtained as in Equation (31):

$$MSE\left(T_1^*\right) = \Phi\left(S_y^4\left(\beta_{40}-1\right) + \beta_1^2 S_x^4\left(\beta_{04}-1\right) - 2\beta_1 S_y^2 S_x^2\left(\beta_{22}-1\right)\right), \tag{31}$$

where $\Phi = \left(\frac{u-r_2-1}{n-r_3-1}\right)^2\left(\frac{1}{uq_2+2p_2} - \frac{1}{N}\right) + \left(\frac{m-r_1-1}{n-r_3-1}\right)^2\left(\frac{1}{mq_1+2p_1} - \frac{1}{N}\right)$.

2.2.2. Proposed Calibration Estimator Based on Linear Constraint of Auxiliary Variable

The second proposed calibrated estimator of population variance under random non-response is given in Equation (32):

$$T_{2(NR)}^* = \sum_{i=1}^{u-r_2} \varpi_{1i}^*\left(y_i - \overline{Y}_{(u-r_2)}\right)^2 + \sum_{i=1}^{m-r_1} \varpi_{2i}^*\left(y_i - \overline{Y}_{(m-r_1)}\right)^2, \tag{32}$$

where ϖ_{1i}^*, ϖ_{2i}^* are the new calibration weights to be obtained by minimizing the chi-square distance Z_2^* defined in Equation (33).

$$\left.\begin{array}{ll} \min & Z_2^* = 2^{-1}\sum_{i=1}^{u-r_2}\left(\varpi_{1i}^* - w_1^*\right)^2 / w_1^*\varphi_{1i} + 2^{-1}\sum_{i=1}^{m-r_1}\left(\varpi_{2i}^* - w_2^*\right)^2 / w_2^*\varphi_{2i} \\ s.t. & \frac{n-r_3-1}{n-r_3}\left(\sum_{i=1}^{u-r_2}\varpi_{1i}^* x_i + \frac{m-r_1-1}{m-r_1}\sum_{i=1}^{m-r_1}\varpi_{2i}^* x_i\right) = \overline{X} \end{array}\right\} \tag{33}$$

To compute new calibrated weights ϖ_{1i}^*, $i = 1, 2, \ldots, u - r_2$ and ϖ_{2i}^*, $i = 1, 2, \ldots,$ $m - r_1$, the defined Lagrange function L_2 of the formed in Equation (34) is defined:

$$\begin{aligned} L_2 = & \sum_{i=1}^{u-r_2}\frac{\left(\varpi_{1i}^* - w_1^*\right)^2}{2w_1^*\varphi_{1i}} + \sum_{i=1}^{m-r_1}\frac{\left(\varpi_{2i}^* - w_2^*\right)^2}{2w_2^*\varphi_{2i}} \\ & -\lambda_2\left(\frac{n-r_3-1}{n-r_3}\left(\sum_{i=1}^{u-r_2}\varpi_{1i}^* x_i + \frac{m-r_1-1}{m-r_1}\sum_{i=1}^{m-r_1}\varpi_{2i}^* x_i\right) - \overline{X}\right). \end{aligned} \tag{34}$$

Differentiate Equation (34) partially with respect to ϖ_{1i}^*, ϖ_{2i}^*, λ_2 and equates the results to zeros, Equation (35), Equation (36) and Equation (37) are obtained, respectively:

$$\varpi_{1i}^* = w_1^* + \frac{n-r_3-1}{n-r_3}\lambda_2 w_1^*\varphi_{1i}x_1 \tag{35}$$

$$\varpi_{2i}^{*} = w_2^{*} + \frac{(n-r_3-1)(m-r_1-1)}{(n-r_3)(m-r_1)}\lambda_2 w_2^{*}\varphi_{2i}x_i \tag{36}$$

$$\frac{n-r_3-1}{n-r_3}\left(\sum_{i=1}^{u-r_2}\varpi_{1i}^{*}x_i + \frac{m-r_1-1}{m-r_1}\sum_{i=1}^{m-r_1}\varpi_{2i}^{*}x_i\right) = \overline{X}. \tag{37}$$

Substitute both Equation (35) and Equation (36) in Equation (37) and solve for λ_2, Equation (38) is obtained:

$$\lambda_2 = \frac{\overline{X} - \overline{X}^{*}_{(NS)}}{\left(\frac{n-r_3-1}{n-r_3}\right)^2\left(\sum\limits_{i=1}^{u-r_2} w_1^{*}\varphi_{1i}x_i^2 + \left(\frac{m-r_1-1}{m-r_1}\right)^2\sum\limits_{i=1}^{m-r_1} w_2^{*}\varphi_{2i}x_i^2\right)}, \tag{38}$$

where $\overline{X}^{*}_{(NR)} = \frac{n-r_3-1}{n-r_3}\left(\sum\limits_{i=1}^{u-r_2} w_1^{*}x_i + \frac{m-r_1-1}{m-r_1}\sum\limits_{i=1}^{m-r_1} w_2^{*}x_i\right)$.

Substitute the expression for λ_2 in Equation (35) and Equation (36), the expressions for ϖ_{1i}^{*}, $\varpi_{2i}^{*}$are obtained as in Equation (39) and Equation (40), respectively.

$$\varpi_{1i}^{*} = w_1^{*}\left(1 + \frac{\varphi_{1i}x_i\left(\overline{X} - \overline{X}^{*}_{(NS)}\right)}{\left(\frac{n-r_3-1}{n-r_3}\right)\left(\sum\limits_{i=1}^{u-r_2} w_1^{*}\varphi_{1i}x_i^2 + \left(\frac{m-r_1-1}{m-r_1}\right)^2\sum\limits_{i=1}^{m-r_1} w_2^{*}\varphi_{2i}x_i^2\right)}\right) \tag{39}$$

$$\varpi_{2i}^{*} = w_2^{*}\left(1 + \frac{(m-r_1-1)\left(\overline{X} - \overline{X}^{*}_{(NR)}\right)\varphi_{2i}x_i}{(m-r_1)\left(\frac{n-r_3-1}{n-r_3}\right)\left(\sum\limits_{i=1}^{u-r_2} w_1^{*}\varphi_{1i}x_i^2 + \left(\frac{m-r_1-1}{m-r_1}\right)^2\sum\limits_{i=1}^{m-r_1} w_2^{*}\varphi_{2i}x_i^2\right)}\right). \tag{40}$$

Substitute the results of ϖ_{1i}^{*}, ϖ_{2i}^{*} in Equation (32), the second proposed calibration estimator becomes Equation (41):

$$T_2^{*} = s^{*2}_{y(NR)} + \hat{b}_2\left(\overline{X} - \overline{X}^{*}_{(NS)}\right), \tag{41}$$

where $\hat{b}_2 = \frac{\left(\sum\limits_{i=1}^{u-r_2} w_1^{*}\varphi_{1i}x_i\left(y_i - \overline{Y}_{(u-r_2)}\right)^2 + \frac{m-r_1-1}{m-r_1}\sum\limits_{i=1}^{m-r_1} w_2^{*}\varphi_{2i}x_i\left(y_i - \overline{Y}_{(m-r_1)}\right)^2\right)}{\left(\frac{n-r_3-1}{n-r_3}\right)\left(\sum\limits_{i=1}^{u-r_2} w_1^{*}\varphi_{1i}x_i^2 + \left(\frac{m-r_1-1}{m-r_1}\right)^2\sum\limits_{i=1}^{m-r_1} w_2^{*}\varphi_{2i}x_i^2\right)}$.

The members of the proposed calibrated estimator T_2^{*} are obtained as below.

Case A: Setting $\varphi_{1i} = 1$ and $\varphi_{2i} = 1$ in $\hat{b}_2$, the first member of T_2^{*} denoted by T_{21}^{*} is obtained as in Equation (42):

$$T_{21}^{*} = s^{*2}_{y(NR)} + \hat{b}_{21}\left(\overline{X} - \overline{X}^{*}_{(NS)}\right), \tag{42}$$

where $\hat{b}_{21} = \frac{\left(\sum\limits_{i=1}^{u-r_2} w_1^{*}x_i\left(y_i - \overline{Y}_{(u-r_2)}\right)^2 + \frac{m-r_1-1}{m-r_1}\sum\limits_{i=1}^{m-r_1} w_2^{*}x_i\left(y_i - \overline{Y}_{(m-r_1)}\right)^2\right)}{\left(\frac{n-r_3-1}{n-r_3}\right)\left(\sum\limits_{i=1}^{u-r_2} w_1^{*}x_i^2 + \left(\frac{m-r_1-1}{m-r_1}\right)^2\sum\limits_{i=1}^{m-r_1} w_2^{*}x_i^2\right)}$.

Case B: Setting $\varphi_{1i} = x_i^{-1}$ and $\varphi_{2i} = x_i^{-1}$ in $\hat{b}_2$, the second member of T_2^* denoted by T_{22}^* is obtained as in Equation (43):

$$T_{22}^* = s_{y(NR)}^{*2} + \hat{b}_{22}\left(\overline{X} - \overline{X}_{(NS)}^*\right), \tag{43}$$

where $\hat{b}_{22} = \dfrac{\left(\sum_{i=1}^{u-r_2} w_1^*\left(y_i - \overline{Y}_{(u-r_2)}\right)^2 + \frac{m-r_1-1}{m-r_1}\sum_{i=1}^{m-r_1} w_2^*\left(y_i - \overline{Y}_{(m-r_1)}\right)^2\right)}{\left(\frac{n-r_3-1}{n-r_3}\right)\left(\sum_{i=1}^{u-r_2} w_1^* x_i + \left(\frac{m-r_1-1}{m-r_1}\right)^2 \sum_{i=1}^{m-r_1} w_2^* x_i\right)}$.

The MSE of T_2^* denoted by $MSE\left(T_2^*\right)$, can be obtained using the function defined in Equation (44):

$$MSE\left(T_2^*\right) = \Theta_2 \Sigma_2 \Theta_2^T, \tag{44}$$

where Θ_2 is 1×2 matrix given as

$\Theta_2 = \left(\left.\frac{\partial T_2^*}{\partial s_{y(NR)}^{*2}}\right|_{s_{y(NR)}^{*2}=S_Y^2,\, \overline{X}_{(NR)}^*=\overline{X},\, \hat{b}_2=\beta_2} \quad \left.\frac{\partial T_2^*}{\partial \overline{X}_{(NR)}^*}\right|_{s_{y(NR)}^{*2}=S_Y^2,\, \overline{X}_{(NR)}^*=\overline{X},\, \hat{b}_2=\beta_2} \right)$,

Σ_2 is the variance-covariance matrix of order 2×2 defined as

$\Sigma_2 = \begin{pmatrix} \operatorname{var}\left(s_{y(NR)}^{*2}\right) & \operatorname{cov}\left(s_{y(NR)}^{*2}\overline{X}_{(NR)}^*\right) \\ \operatorname{cov}\left(\overline{X}_{(NR)}^* s_{y(NR)}^{*2}\right) & \operatorname{var}\left(\overline{X}_{(NR)}^*\right) \end{pmatrix}$, Θ_2^T is the transpose of Θ_2

and $\beta_2 = \sum_{i=1}^{N} X_i \left(Y_i - \overline{Y}\right)^2 / \sum_{i=1}^{N} X_i^2$.

Simplifying Equation (44), $MSE\left(T_2^*\right)$ is obtained as in Equation (45):

$$MSE\left(T_2^*\right) = \Phi\left(S_y^4\left(\beta_{40} - 1\right) + \beta_2^2 S_x^4\left(\beta_{04} - 1\right) - 2\beta_2 S_y^2 S_x^2\left(\beta_{22} - 1\right)\right). \tag{45}$$

Proposition 1. *For practical purposes, the estimate of* $MSE\left(T_{ij}^*\right),\ i,j = 1,2$ *denoted by* $\widehat{MSE}\left(T_{ij}^*\right),\ i,j = 1,2$ *is as given in Equation (46).*

$$\widehat{MSE}\left(T_{ij}^*\right) = \Phi\left(\left(\tau_{40} - 1\right)s_{y(NR)}^{*4} + \hat{b}_{ij}^2\left(\tau_{04} - 1\right)s_{x(NR)}^{*4} - 2\hat{b}_{ij}s_{y(NR)}^{*2}s_{x(NR)}^{*2}\left(\tau_{22} - 1\right)\right), \tag{46}$$

where

$$\tau_{rs} = \frac{\hat{\mu}_{rs}}{\hat{\mu}_{20}^{r/2}\hat{\mu}_{02}^{s/2}},$$

$$\hat{\mu}_{rs} = \frac{\sum_{i=1}^{u-r_2}\left(y_i - \overline{Y}_{(u-r_2)}\right)^r\left(x_i - \overline{X}_{(u-r_2)}\right)^s}{u - r_2} + \frac{\sum_{i=1}^{m-r_1}\left(y_i - \overline{Y}_{(m-r_1)}\right)^r\left(x_i - \overline{X}_{(m-r_1)}\right)^s}{m - r_1}.$$

3. Results and Discussion

3.1. Efficiency Comparisons

To assess the performance of the proposed estimators $T_{ij}^*,\ i,j = 1,2$, with respect to s_y^{*2}, absolute relative bias (ARB), mean square error (MSE) and percentage relative efficiency (PRE) of the estimators were computed using Equation (47), Equation (48) and Equation (49), respectively:

$$ARB\left(\Phi\right)=\frac{E\left|\Phi-S_y^2\right|}{S_y^2} \tag{47}$$

$$MSE\left(\Phi\right)=E\left(\Phi-S_y^2\right)^2 \tag{48}$$

$$PRE\left(\Phi\right)=E\left(s_y^{*2}-S_y^2\right)^2\left(E\left(\Phi-S_y^2\right)^2\right)^{-1}\times\ 100 \tag{49}$$

where Φ is any estimator considered in the study, $E\left(\Phi-S_y^2\right)=M^{-1}\sum\limits_{j=1}^{M}\left(\Phi_j-S_y^2\right)$ and $E\left(\Phi-S_y^2\right)^2=M^{-1}\sum\limits_{j=1}^{M}\left(\Phi_j-S_y^2\right)^2$.

Simulation studies were conducted to evaluate the superiority of the proposed estimators compared to other estimators considered in the present study. For this purpose, 3 different populations of size $N=1000$ were generated. The data for study and auxiliary variables were generated using the distributions defined in Table 1. The nature of the data generated was presented in the 1. A sample of sizes 50, 100 and 200 were selected 1000 times using successful sampling without replacement method from each population generated for computations of ARBs, MSEs and PREs of the estimators under consideration. The probabilities of non-respondents for matched and unmatched samples units considered in the study are $p_1=\{0.1,0.2,0.3\}$ and $p_2=\{0.1,0.2,0.3\}$, respectively.

Table 1: Distributions of Populations used for Empirical Study

Population	Auxiliary Variable	Study variable
I	$f\left(X\right)=\frac{1}{b-a},\quad a=1,\ b=10$	$Y_i=X_i+\varepsilon_i,$
II	$f\left(X\right)=\lambda e^{-\lambda x},\quad \lambda=1$	$\varepsilon\sim N\left(0,1\right)$
III	$f\left(X\right)=\frac{1}{2^{\alpha}\Gamma\left(\alpha/2\right)}x^{\frac{\alpha}{2}-1}e^{-\frac{x}{2}},\ \ \alpha=1$	

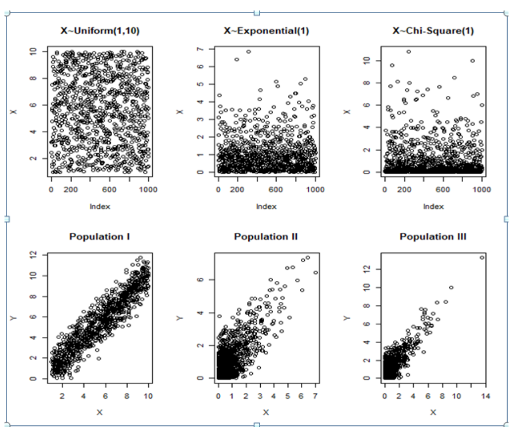

Figure 1: Please write your figure caption here.

TABLE 2: ARB, MSE and PRE of $s^{*2}_{y(NR)}$ and $T^{*}_{ij},\ i,j=1,2$, using Population I when $n=50$.

	$p_1=0.1, p_2=0.1$			$p_1=0.1, p_2=0.2$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.0802	0.6398	100	0.0973	0.6090	100
$T^*_{11(NR)}$	0.0530	0.2814	227.3952	0.0891	0.5103	119.3349
$T^*_{12(NR)}$	0.0535	0.2860	223.7363	0.0867	0.4836	125.9230
$T^*_{21(NR)}$	0.0628	0.4018	159.2360	0.0901	0.5227	116.5126
$T^*_{22(NR)}$	0.0601	0.3737	171.2178	0.0882	0.5011	121.5402
	$p_1=0.1, p_2=0.3$			$p_1=0.2, p_2=0.1$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.1280	0.9042	100	0.1264	1.3311	100
$T^*_{11(NR)}$	0.0523	0.1507	599.8423	0.0785	0.5841	227.8836
$T^*_{12(NR)}$	0.0492	0.1337	676.0332	0.0816	0.6413	207.5565
$T^*_{21(NR)}$	0.0610	0.2050	441.1529	0.1056	1.0116	131.5779
$T^*_{22(NR)}$	0.0475	0.1247	725.3194	0.1028	0.9699	137.2397
	$p_1=0.2, p_2=0.2$			$p_1=0.2, p_2=0.3$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.1449	2.1230	100	0.2273	4.9408	100
$T^*_{11(NR)}$	0.0943	0.8311	255.4364	0.1305	1.6625	297.1881
$T^*_{12(NR)}$	0.0961	0.8576	247.5566	0.1290	1.6501	299.4162
$T^*_{21(NR)}$	0.0935	0.8353	254.1708	0.1352	1.7485	282.5665
$T^*_{22(NR)}$	0.0978	0.8710	243.7396	0.1319	1.6907	292.2344
	$p_1=0.3, p_2=0.1$			$p_1=0.3, p_2=0.2$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.1213	1.4660	100	0.1560	2.3646	100
$T^*_{11(NR)}$	0.0682	0.4614	317.7615	0.0968	0.8733	270.7665
$T^*_{12(NR)}$	0.0691	0.4772	307.1784	0.0982	0.8965	263.7591
$T^*_{21(NR)}$	0.0848	0.7423	197.4976	0.0984	0.9026	261.9897
$T^*_{22(NR)}$	0.0804	0.6804	215.4533	0.0986	0.8792	268.9348

Table 3: ARB, MSE and PRE of $s^{*2}_{y(NR)}$ and $T^*_{ij},\ i,j=1,2$, using Population I when $n=100$.

	$p_1=0.1, p_2=0.1$			$p_1=0.1, p_2=0.2$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.0918	0.7131	100	0.0969	0.8616	100
$T^*_{11(NR)}$	0.0585	0.2839	251.1535	0.0625	0.3642	236.5754
$T^*_{12(NR)}$	0.0586	0.2878	247.7660	0.0632	0.3701	232.7938
$T^*_{21(NR)}$	0.0712	0.4306	165.6015	0.0691	0.4429	194.5246
$T^*_{22(NR)}$	0.0679	0.3915	182.1384	0.0647	0.3938	218.7850
	$p_1=0.1, p_2=0.3$			$p_1=0.2, p_2=0.1$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.1115	1.2000	100	0.0856	0.6442	100
$T^*_{11(NR)}$	0.0774	0.6000	200.0002	0.0563	0.2854	225.7144
$T^*_{12(NR)}$	0.0790	0.6253	191.9167	0.0571	0.2922	220.4387
$T^*_{21(NR)}$	0.0764	0.5920	202.6987	0.0650	0.3746	171.9638
$T^*_{22(NR)}$	0.0761	0.5779	207.6284	0.0623	0.3445	186.9936
	$p_1=0.2, p_2=0.2$			$p_1=0.2, p_2=0.3$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.0991	0.8716	100	0.1355	1.5911	100
$T^*_{11(NR)}$	0.0669	0.4058	214.7917	0.0885	0.6940	229.2513
$T^*_{12(NR)}$	0.0677	0.4219	206.6075	0.0906	0.7257	219.2455
$T^*_{21(NR)}$	0.0696	0.4263	204.4619	0.0911	0.7423	214.3567
$T^*_{22(NR)}$	0.0668	0.3963	219.9274	0.0879	0.6907	230.3607
	$p_1=0.3, p_2=0.1$			$p_1=0.3, p_2=0.2$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.0880	0.6667	100	0.1067	1.0501	100
$T^*_{11(NR)}$	0.0570	0.2859	233.2094	0.0672	0.4141	253.5671
$T^*_{12(NR)}$	0.0580	0.2928	227.6905	0.0682	0.4280	245.3531
$T^*_{21(NR)}$	0.0650	0.3697	180.3550	0.0715	0.4744	221.3569
$T^*_{22(NR)}$	0.0621	0.3418	195.0366	0.0682	0.4284	245.1342

TABLE 4: ARB, MSE and PRE of $s^{*2}_{y(NR)}$ and T^{*}_{ij}, $i,j=1,2$, using Population I when $n=200$.

	$p_1=0.1, p_2=0.1$			$p_1=0.1, p_2=0.2$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.0550	0.2567	100	0.0620	0.3180	100
$T^*_{11(NR)}$	0.0385	0.1278	200.7808	0.0400	0.1322	240.6176
$T^*_{12(NR)}$	0.0390	0.1314	195.2919	0.0402	0.1337	237.8629
$T^*_{21(NR)}$	0.0387	0.1267	202.5197	0.0407	0.1385	229.6608
$T^*_{22(NR)}$	0.0384	0.1260	203.7248	0.0406	0.1379	230.5964
	$p_1=0.1, p_2=0.3$			$p_1=0.2, p_2=0.1$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.0650	0.3748	100	0.0566	0.2949	100
$T^*_{11(NR)}$	0.0411	0.1501	249.6727	0.0356	0.1179	250.0917
$T^*_{12(NR)}$	0.0418	0.1552	241.5234	0.0359	0.1201	245.5379
$T^*_{21(NR)}$	0.0415	0.1565	239.5143	0.0363	0.1213	243.0318
$T^*_{22(NR)}$	0.0412	0.1517	247.1184	0.0359	0.1185	248.8769
	$p_1=0.2, p_2=0.2$			$p_1=0.2, p_2=0.3$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.0582	0.3185	100	0.0641	0.3693	100
$T^*_{11(NR)}$	0.0386	0.1360	234.1134	0.0399	0.1413	261.3181
$T^*_{12(NR)}$	0.0390	0.1397	227.9544	0.0401	0.1437	257.0726
$T^*_{21(NR)}$	0.0391	0.1418	224.6057	0.0414	0.1526	242.0196
$T^*_{22(NR)}$	0.0395	0.1437	221.5554	0.0407	0.1467	251.8384
	$p_1=0.3, p_2=0.1$			$p_1=0.3, p_2=0.2$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.0559	0.3214	100	0.0614	0.3046	100
$T^*_{11(NR)}$	0.0372	0.1438	223.5358	0.0375	0.1178	258.5044
$T^*_{12(NR)}$	0.0375	0.1458	220.4009	0.0377	0.1197	254.3790
$T^*_{21(NR)}$	0.0371	0.1430	224.7829	0.0397	0.1300	234.3559
$T^*_{22(NR)}$	0.0372	0.1435	224.0330	0.0385	0.1225	248.5923

TABLE 5: ARB, MSE and PRE of $s^{*2}_{y(NR)}$ and $T^*_{ij},\ i,j=1,2$, using Population II when $n=50$.

	$p_1=0.1, p_2=0.1$			$p_1=0.1, p_2=0.2$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.2432	0.0976	100	0.2292	0.0648	100
$T^*_{11(NR)}$	0.2156	0.0767	127.2418	0.1610	0.0320	202.7316
$T^*_{12(NR)}$	0.2073	0.0709	137.6908	0.1114	0.0153	423.6120
$T^*_{21(NR)}$	0.1079	0.0192	507.8410	0.1651	0.0336	192.8433
$T^*_{22(NR)}$	1.4220	3.3349	2.9257	4.9257	29.9208	0.2165
	$p_1=0.1, p_2=0.3$			$p_1=0.2, p_2=0.1$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	100.0000	2.3009	100	0.2576	0.1441	100
$T^*_{11(NR)}$	2981.408	0.0772	2981.408	0.1574	0.0547	263.6959
$T^*_{12(NR)}$	2806.148	0.0820	2806.148	0.1620	0.0641	224.7197
$T^*_{21(NR)}$	995.2102	0.2312	995.2102	0.3253	0.2686	53.6636
$T^*_{22(NR)}$	0.0530	218.5032	1.0530	2.8354	36.7168	0.3926
	$p_1=0.2, p_2=0.2$			$p_1=0.2, p_2=0.3$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.2625	0.1629	100	0.4291	0.3396	100
$T^*_{11(NR)}$	0.2098	0.1217	133.8399	0.2771	0.1389	244.5466
$T^*_{12(NR)}$	0.2259	0.1250	130.2719	0.2955	0.1593	213.1932
$T^*_{21(NR)}$	0.4058	0.3684	44.2137	0.6597	0.8136	41.7449
$T^*_{22(NR)}$	5.9883	74.8749	0.2175	9.3561	159.8159	0.2125
	$p_1=0.3, p_2=0.1$			$p_1=0.3, p_2=0.2$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.2669	0.2875	100	0.2709	0.1679	100
$T^*_{11(NR)}$	0.1623	0.0876	328.3237	0.2058	0.1020	164.5798
$T^*_{12(NR)}$	0.1891	0.1249	230.2593	0.2403	0.1410	119.0900
$T^*_{21(NR)}$	0.3466	1.1174	25.7344	0.5070	0.6291	26.6889
$T^*_{22(NR)}$	4.1803	93.5518	0.3074	7.5446	135.9564	0.1235

TABLE 6: ARB, MSE and PRE of $s^{*2}_{y(NR)}$ and T^*_{ij}, $i, j = 1, 2$, using Population II when $n = 100$.

	$p_1 = 0.1, p_2 = 0.1$			$p_1 = 0.1, p_2 = 0.2$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.1500	0.0437	100	0.1971	0.0826	100
$T^*_{11(NR)}$	0.1083	0.0213	205.1354	0.1223	0.0324	255.3149
$T^*_{12(NR)}$	0.1245	0.0298	146.6409	0.1543	0.0513	160.9716
$T^*_{21(NR)}$	0.1391	0.0422	103.4282	0.1621	0.0619	133.4819
$T^*_{22(NR)}$	0.1742	0.0636	68.6216	0.2057	0.0950	86.9949
	$p_1 = 0.1, p_2 = 0.3$			$p_1 = 0.2, p_2 = 0.1$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.2171	0.0795	100	0.2261	0.0999	100
$T^*_{11(NR)}$	0.1332	0.0312	254.7296	0.1452	0.0406	246.2596
$T^*_{12(NR)}$	0.1681	0.0498	159.8051	0.1932	0.0723	138.1073
$T^*_{21(NR)}$	0.2177	0.0826	96.3240	0.2224	0.0981	101.8897
$T^*_{22(NR)}$	0.2861	0.1343	59.1660	0.2999	0.1797	55.5879
	$p_1 = 0.2, p_2 = 0.2$			$p_1 = 0.2, p_2 = 0.3$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.1931	0.0590	100	0.1994	0.0802	100
$T^*_{11(NR)}$	0.1268	0.0307	192.1558	0.1383	0.0416	192.6723
$T^*_{12(NR)}$	0.1449	0.0386	152.8956	0.1723	0.0591	135.8495
$T^*_{21(NR)}$	0.1650	0.0502	117.4567	0.1801	0.0656	122.2772
$T^*_{22(NR)}$	0.2168	0.0839	70.3826	0.2371	0.1057	75.8969
	$p_1 = 0.3, p_2 = 0.1$			$p_1 = 0.3, p_2 = 0.2$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.2238	0.0932	100	0.2113	0.0933	100
$T^*_{11(NR)}$	0.1359	0.0393	237.3015	0.1536	0.0470	198.2505
$T^*_{12(NR)}$	0.1572	0.0527	176.9367	0.1718	0.0592	157.6750
$T^*_{21(NR)}$	0.1865	0.0730	127.6986	0.1862	0.0713	130.8113
$T^*_{22(NR)}$	0.2344	0.1061	87.8530	0.2367	0.1076	86.7026

Table 7: ARB, MSE and PRE of $s^{*2}_{y(NR)}$ and $T^*_{ij},\ i,j=1,2$, using Population II when $n=200$.

	$p_1=0.1, p_2=0.1$			$p_1=0.1, p_2=0.2$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.1278	0.0431	100	0.1329	0.0500	100
$T^*_{11(NR)}$	0.0717	0.0139	309.5852	0.0870	0.0228	218.7593
$T^*_{12(NR)}$	0.0779	0.0168	256.0251	0.0918	0.0256	195.3060
$T^*_{21(NR)}$	0.1057	0.0331	130.1570	0.1401	0.0604	82.6849
$T^*_{22(NR)}$	0.0814	0.0179	240.3696	0.1120	0.0372	134.4824
	$p_1=0.1, p_2=0.3$			$p_1=0.2, p_2=0.1$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.1634	0.0736	100	0.1487	0.0651	100
$T^*_{11(NR)}$	0.0876	0.0254	289.9308	0.0916	0.0274	237.1616
$T^*_{12(NR)}$	0.1006	0.0354	207.7972	0.1132	0.0425	153.2436
$T^*_{21(NR)}$	0.1142	0.0496	148.4646	0.1247	0.0519	125.5623
$T^*_{22(NR)}$	0.1205	0.0583	126.2820	0.1339	0.0612	106.2925
	$p_1=0.2, p_2=0.2$			$p_1=0.2, p_2=0.3$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.1508	0.0635	100	0.1606	0.0761	100
$T^*_{11(NR)}$	0.0810	0.0201	316.1914	0.0890	0.0254	299.4094
$T^*_{12(NR)}$	0.0965	0.0314	202.2671	0.1105	0.0410	185.6101
$T^*_{21(NR)}$	0.1106	0.0453	140.2840	0.1188	0.0481	158.1644
$T^*_{22(NR)}$	0.1153	0.0508	124.9049	0.1296	0.0580	131.1980
	$p_1=0.3, p_2=0.1$			$p_1=0.3, p_2=0.2$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.1521	0.0659	100	0.1407	0.0545	100
$T^*_{11(NR)}$	0.0819	0.0206	320.6563	0.0865	0.0247	220.6680
$T^*_{12(NR)}$	0.0931	0.0286	230.5833	0.1102	0.0404	134.9909
$T^*_{21(NR)}$	0.1075	0.0436	151.0586	0.1223	0.0514	106.0156
$T^*_{22(NR)}$	0.1127	0.0491	134.2650	0.1315	0.0616	88.4742

Table 8: ARB, MSE and PRE of $s^{*2}_{y(NR)}$ and $T^{*}_{ij},\ i,j=1,2$, using Population III when $n=50$.

	$p_1=0.1, p_2=0.1$			$p_1=0.1, p_2=0.2$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.2439	0.1085	100	0.2292	0.0740	100
$T^*_{11(NR)}$	0.1916	0.0672	161.5436	0.1768	0.0492	150.4066
$T^*_{12(NR)}$	0.2110	0.0839	129.2394	0.1830	0.0570	129.8940
$T^*_{21(NR)}$	0.2308	0.1048	103.5182	0.1952	0.0718	102.8036
$T^*_{22(NR)}$	0.2725	0.1553	69.8746	0.2395	0.1127	65.6300
	$p_1=0.1, p_2=0.3$			$p_1=0.2, p_2=0.1$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.2559	0.1284	100	0.2695	0.1604	100
$T^*_{11(NR)}$	0.2011	0.0820	156.5610	0.2152	0.0946	169.5440
$T^*_{12(NR)}$	0.2177	0.0964	133.3257	0.2389	0.1181	135.8137
$T^*_{21(NR)}$	0.2406	0.1199	107.0633	0.2601	0.1456	110.2101
$T^*_{22(NR)}$	0.2782	0.1677	76.5753	0.2960	0.1971	81.4192
	$p_1=0.2, p_2=0.2$			$p_1=0.2, p_2=0.3$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.2261	0.0946	100	0.2305	0.0751	100
$T^*_{11(NR)}$	0.1728	0.0563	167.8582	0.1780	0.0487	154.3194
$T^*_{12(NR)}$	0.1917	0.0705	134.2646	0.1883	0.0565	133.0603
$T^*_{21(NR)}$	0.2076	0.0849	111.3676	0.2051	0.0720	104.3385
$T^*_{22(NR)}$	0.2411	0.1183	79.9777	0.2428	0.1090	68.8530
	$p_1=0.3, p_2=0.1$			$p_1=0.3, p_2=0.2$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.2447	0.1121	100	0.2553	0.1407	100
$T^*_{11(NR)}$	0.1903	0.0738	151.8855	0.2020	0.0828	169.9543
$T^*_{12(NR)}$	0.2082	0.0872	128.6194	0.2180	0.1006	139.8041
$T^*_{21(NR)}$	0.2300	0.1110	100.9811	0.2396	0.1267	111.0382
$T^*_{22(NR)}$	0.2691	0.1604	69.9307	0.2805	0.1761	79.9029

Table 9: ARB, MSE and PRE of $s^{*2}_{y(NR)}$ and $T^*_{ij},\ i,j=1,2$, using Population III when $n=100$.

	$p_1=0.1, p_2=0.1$			$p_1=0.1, p_2=0.2$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.1322	0.0479	100	0.1408	0.0589	100
$T^*_{11(NR)}$	0.0887	0.0264	181.6291	0.0911	0.0277	212.9202
$T^*_{12(NR)}$	0.0942	0.0308	155.5489	0.1086	0.0420	140.2293
$T^*_{21(NR)}$	0.1182	0.0448	107.0453	0.1249	0.0550	107.0917
$T^*_{22(NR)}$	0.1298	0.0521	91.9876	0.1330	0.0607	97.0463
	$p_1=0.1, p_2=0.3$			$p_1=0.2, p_2=0.$‘		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.1547	0.0718	100	0.1604	0.0795	100
$T^*_{11(NR)}$	0.0960	0.0318	225.7351	0.0954	0.0334	237.9292
$T^*_{12(NR)}$	0.1097	0.0422	170.1261	0.1187	0.0504	157.7815
$T^*_{21(NR)}$	0.1254	0.0556	129.2922	0.1325	0.0627	126.8241
$T^*_{22(NR)}$	0.1382	0.0668	107.4516	0.1410	0.0720	110.4175
	$p_1=0.2, p_2=0.2$			$p_1=0.2, p_2=0.3$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.1480	0.0649	100	0.1496	0.0715	100
$T^*_{11(NR)}$	0.0936	0.0301	215.3302	0.0924	0.0320	223.7469
$T^*_{12(NR)}$	0.1080	0.0410	158.3456	0.1156	0.0486	147.1165
$T^*_{21(NR)}$	0.1210	0.0520	124.6625	0.1273	0.0584	122.3421
$T^*_{22(NR)}$	0.1349	0.0635	102.1417	0.1367	0.0690	103.6230
	$p_1=0.3, p_2=0.1$			$p_1=0.3, p_2=0.2$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.1426	0.0583	100	0.1452	0.0672	100
$T^*_{11(NR)}$	0.0898	0.0270	215.7295	0.0911	0.0284	236.5713
$T^*_{12(NR)}$	0.1061	0.0397	146.8732	0.1108	0.0459	146.4742
$T^*_{21(NR)}$	0.1196	0.0504	115.6705	0.1245	0.0559	120.2409
$T^*_{22(NR)}$	0.1315	0.0602	96.8707	0.1334	0.0647	103.8840

TABLE 10: ARB, MSE and PRE of $s^{*2}_{y(NR)}$ and T^*_{ij}, $i, j = 1, 2$, using Population III when $n = 200$.

	$p_1 = 0.1, p_2 = 0.1$			$p_1 = 0.1, p_2 = 0.2$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.2027	0.0684	100	0.2173	0.0906	100
$T^*_{11(NR)}$	0.1433	0.0342	199.9614	0.1510	0.0436	207.5062
$T^*_{12(NR)}$	0.1625	0.0455	150.4815	0.1739	0.0606	149.3966
$T^*_{21(NR)}$	0.1806	0.0586	116.6720	0.1912	0.0747	121.1966
$T^*_{22(NR)}$	0.2131	0.0839	81.4849	0.2285	0.1075	84.2310
	$p_1 = 0.1, p_2 = 0.3$			$p_1 = 0.2, p_2 = 0.1$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.2485	0.1206	100	0.2548	0.1523	100
$T^*_{11(NR)}$	0.1730	0.0559	215.6917	0.1859	0.0703	216.8571
$T^*_{12(NR)}$	0.1915	0.0699	172.6250	0.2014	0.0876	173.8465
$T^*_{21(NR)}$	0.2120	0.0892	135.1360	0.2212	0.1085	140.3732
$T^*_{22(NR)}$	0.2425	0.1162	103.8149	0.2516	0.1442	105.6212
	$p_1 = 0.2, p_2 = 0.2$			$p_1 = 0.2, p_2 = 0.3$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.2287	0.0948	100	0.2425	0.1286	100
$T^*_{11(NR)}$	0.1579	0.0474	200.1013	0.1692	0.0609	211.1896
$T^*_{12(NR)}$	0.1768	0.0607	156.1066	0.1908	0.0782	164.5179
$T^*_{21(NR)}$	0.1964	0.0772	122.7543	0.2062	0.0948	135.5730
$T^*_{22(NR)}$	0.2294	0.1024	92.5996	0.2377	0.1215	105.8429
	$p_1 = 0.3, p_2 = 0.1$			$p_1 = 0.3, p_2 = 0.2$		
Estimators	MAPE	MSEs	PREs	MAPE	MSEs	PREs
$s^2_{y(NR)}$	0.2371	0.1042	100	0.2486	0.1391	100
$T^*_{11(NR)}$	0.1649	0.0519	200.9405	0.1766	0.0657	211.4741
$T^*_{12(NR)}$	0.1830	0.0655	159.1759	0.1970	0.0836	166.2947
$T^*_{21(NR)}$	0.2037	0.0846	123.1905	0.2156	0.1029	135.1205
$T^*_{22(NR)}$	0.2345	0.1100	94.7302	0.2437	0.1301	106.9428

From the empirical results, the following were observed.

1. Tables 1 - 10 showed the results of the ARB, MSE and PRE of the proposed conventional estimator $s_{y(NR)}^{*2}$ and that of proposed calibration estimators $T_{ij(NR)}^{*},\ i,j=1,2$ of population variance under successive sampling in the situation when both the study and auxiliary variables are characterized with random non-response. The performances evaluation of the estimators using ARB, MSE and PRE under the equal and unequal probabilities of non-responses on the matched and unmatched samples are presented in Tables 1 to 10. The probabilities of non-responses on the matched samples are taken as $p_1 = 0.1, 0.2, 0.3$ while the probabilities of non-responses on the matched samples are taken as $p_2 = 0.1, 0.2, 0.3$.

2. Tables 2, 3 and 4 showed that the results of ARB, MSE and PRE of the population I for the proposed conventional estimator $s_{y(NR)}^{*2}$ and that of proposed calibration estimators $T_{ij(NR)}^{*},\ i,j=1,2$ for different probabilities of non-responses on matched (first occasion) and unmatched sample (second occasion). ARB, MSE and PRE of the estimators were computed for the values of $p_1 = 0.1, 0.2, 0.3$ and $p_2 = 0.1, 0.2, 0.3$. The results revealed that the proposed calibration estimators $T_{ij(NR)}^{*},\ i,j=1,2$ have minimum ARB, minimum MSE and higher PRE compared to the proposed conventional estimator $s_{y(NR)}^{*2}$ in all cases.

3. Tables 5, 6 and 7 showed that the results of ARB, MSE and PRE of the population II for the proposed conventional estimator $s_{y(NR)}^{*2}$ and that of proposed calibration estimators $T_{ij(NR)}^{*},\ i,j=1,2$ for different probabilities of non-responses on matched (first occasion) and unmatched sample (second occasion). ARB, MSE and PRE of the estimators were computed for the values of $p_1 = 0.1, 0.2, 0.3$ and $p_2 = 0.1, 0.2, 0.3$. The results revealed that the proposed calibration estimators $T_{ij(NR)}^{*},\ i,j=1,2$ have minimum ARB, minimum MSE and higher PRE compared to the proposed conventional estimator $s_{y(NR)}^{*2}$. However, proposed calibration estimators $T_{21(NR)}^{*}, T_{22(NR)}^{*}$ compared to $s_{y(NR)}^{*2}$ performed below the standard in some cases when $n = 50$, $T_{21(NR)}^{*}$ performed below the standard for $p_1 = 0.1, p_2 = 0.1$ and $p_1 = 0.1, p_2 = 0.2$ when $n = 100$, $T_{22(NR)}^{*}$ performed below the standard for $p_1 = 0.1, p_2 = 0.2$ when $n = 100$ while $T_{21(NR)}^{*}$ performed below the standard for $p_1 = 0.1, p_2 = 0.2$ when $n = 200$.

4. Tables 8, 9 and 10 showed that the results of ARB, MSE and PRE of the population III for the proposed conventional estimator $s_{y(NR)}^{*2}$ and that of proposed calibration estimators $T_{ij(NR)}^{*},\ i,j=1,2$ for different probabilities of non-responses on matched (first occasion) and unmatched sample (second occasion). ARB, MSE and PRE of the estimators were computed for the values of $p_1 = 0.1, 0.2, 0.3$ and $p_2 = 0.1, 0.2, 0.3$. The results revealed that the proposed calibration estimators $T_{11(NR)}^{*}, T_{12(NR)}^{*}$ have minimum ARB, minimum MSE and higher PRE compared to the proposed conventional estimator $s_{y(NR)}^{*2}$ in all cases while $T_{21(NR)}^{*}, T_{22(NR)}^{*}$ performed below the standard.

4. Conclusions

The methodological advancements proposed in this research seek to improve the reliability and validity of population parameter estimates in successive sampling surveys, where non-sampling errors can significantly impact the quality of data and inference. Analytical expressions for the proposed estimators are derived, allowing for the quantification of their statistical properties and performance. Simulation studies demonstrate the superior performance of the proposed conventional and calibration estimators, especially in situations where non-response is substantial. The proposed methods have the potential to enhance the quality and efficiency of estimation in successive sampling surveys, thus improving the accuracy of inferences drawn from such surveys.

From the discussion above, it can be concluded that the proposed calibration estimators $T^*_{11(NR)}$, $T^*_{12(NR)}$ demonstrated high level of efficiency in estimating the population variance when the variable of interest and auxiliary variable are characterized by non-response under successive sampling scheme. We may therefore recommend the estimators to statisticians and survey users for use in practical situations described in the introduction section. The proposed traditional estimator $s^{*2}_{y(NR)}$ is also recommended for use in practical situations when only information on the study variable is available.

The present study is limited to the development of conventional and calibration estimators of population variance in the presence of random non-response. The study can be extended and improved using ratio, exponential, logarithmic and two-step calibration transformations/techniques.

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