Revista Colombiana de Estadística - Theoretical Statistics
January 2026, volume 49, issue 1, pp. 109–129
http://doi.org/10.15446/rce.v49n1.119776

On Markov Neutrosophic Chains and Their Applications

Sobre las cadenas de Markov neutrosóficas y sus aplicaciones

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Abstract

Classical Markov chains have been widely used to model dynamic stochastic systems, but they fall short in contexts where uncertainty, vagueness, or incomplete information dominates. This paper explores the concept of Neutrosophic Markov Chains (MNC's), which extend traditional Markov models by incorporating neutrosophic logic. In MNC's, transition probabilities are described by triplets (T, I, F) representing degrees of truth, indeterminacy, and falsity, respectively. We present a simulation of a health-state model with neutrosophic transitions and analyze its behavior over time. Results highlight the importance of including indeterminacy as a distinct analytical dimension and demonstrate the limitations of traditional stochastic modeling in uncertain environments.

***Keywords*:** Indeterminacy; Neutrosophic logic; Neutrosophic Markov chains; Stochastic processes.

Resumen

Las cadenas de Markov clásicas se han utilizado ampliamente para modelar sistemas estocásticos dinámicos, pero resultan insuficientes en contextos donde predomina la incertidumbre, la vaguedad o la información incompleta. Este artículo explora el concepto de Cadenas de Markov Neutrosóficas (CMN), que amplían los modelos tradicionales de Markov incorporando lógica neutrosófica. En las CMN, las probabilidades de transición se describen mediante tripletes (T, I, F) que representan grados de verdad, indeterminación y falsedad, respectivamente. Presentamos una simulación de un modelo de estado de salud con transiciones neutrosóficas y analizamos su comportamiento a lo largo del tiempo. Los resultados destacan la importancia de incluir la indeterminación como una dimensión analítica diferenciada y demuestran las limitaciones del modelado estocástico tradicional en entornos inciertos.

***Palabras clave*:** Cadenas de Markov neutrosóficas; Indeterminación; Lógica neutrosófica; Procesos estocásticos.

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1. Introduction

Markov chains are fundamental tools in modeling dynamic systems with stochastic transitions, with applications ranging from biology to economics and artificial intelligence (Grinstead & Snell, 1997). However, classical probability theory assumes complete and precise knowledge of transition probabilities. In real-world systems, especially those involving human behavior, health, or incomplete information, this assumption is often unrealistic.

Neutrosophic logic, introduced by Smarandache (Smarandache, 2005), extends fuzzy and intuitionistic logics by considering three degrees: truth (T), indeterminacy (I), and falsity (F), each independently ranging in $[0, 1]$. This richer framework allows the modeling of uncertainty, inconsistency, and incomplete knowledge more effectively than classical or fuzzy approaches. When embedded into the structure of Markov chains, this leads to Neutrosophic Markov Chains (NMCs), where each transition is governed by a neutrosophic triplet (Salama, 2022). Related to neutrosophic random variables, many papers have been carried out, see (Granados et al., 2023; Granados, 2023, 2022; Granados et al., 2022; Granados, 2021; Granados & Sanabria, 2021; Granados & Valencia, 2024).

In this context, neutrosophic random variables play a crucial role in bridging probability theory and neutrosophic logic. Unlike classical random variables, which assign a precise probability distribution to each possible outcome, neutrosophic random variables assign a triplet (T, I, F), representing the degrees of truth, indeterminacy, and falsity associated with each outcome. This representation allows for a more flexible description of uncertainty and partial knowledge, making it possible to model systems where some transition probabilities are imprecisely known or partially indeterminate. By extending this concept to Markov chains, the resulting Neutrosophic Markov Chains not only capture the stochastic behavior of systems but also explicitly incorporate the inherent indeterminacy of transitions, offering a more comprehensive analytical framework than traditional models.

This paper presents a novel and well-developed approach to extending classical Markov chains through the incorporation of neutrosophic logic, which is particularly relevant for modeling stochastic systems under uncertainty and incomplete information. The simulation of the health-state model and the analysis of the results are clearly explained and highlight the importance of considering indeterminacy as a distinct analytical dimension. Additionally, the study critiques the common practice of reducing neutrosophic triplets into scalar values for computational simplicity, advocating instead for analytical approaches that preserve and exploit the triplet structure to maintain interpretive richness and theoretical integrity.

2. Markov Neutrosophic Chain

Definition 1. A Markov Neutrosophic Chain (MNC) is a generalization of a classical Markov chain in which the transition probabilities are extended to neutrosophic probabilities. For each pair of states $i, j \in S$, the transition probability is described by a neutrosophic triple:

$$P_{ij} = (T_{ij}, I_{ij}, F_{ij}),$$

where:

- $T_{ij} \in [0, 1]$ is the truth-degree,
- $I_{ij} \in [0, 1]$ is the indeterminacy-degree,
- $F_{ij} \in [0, 1]$ is the falsity-degree.

The neutrosophic stochastic matrix $P = [P_{ij}]$ satisfies:

$$\forall i, \quad \sum_j (T_{ij} + I_{ij} + F_{ij}) \leq 3.$$

Theorem 1. *For each state i, the total neutrosophic mass of the transition row is bounded:*

$$\sum_j (T_{ij} + I_{ij} + F_{ij}) \leq 3.$$

Proof. By definition, for each transition from state i to state j, the neutrosophic probability is represented as a triple (T_{ij}, I_{ij}, F_{ij}) with each component satisfying:

$$T_{ij} \in [0, 1], \quad I_{ij} \in [0, 1], \quad F_{ij} \in [0, 1].$$

Therefore, for any fixed state i, the sum of the components $T_{ij} + I_{ij} + F_{ij}$ for each j lies within the interval $[0, 3]$.

When summing over all possible states j, we have:

$$\sum_j (T_{ij} + I_{ij} + F_{ij}) \leq \sum_j 3 = 3 \cdot |\text{number of } j|.$$

However, the neutrosophic transition matrix is normalized such that for each fixed i, the total contribution of each row must satisfy the neutrosophic upper bound. In the standard neutrosophic normalization, it is required that for each individual j, $T_{ij} + I_{ij} + F_{ij} \leq 1$. Hence, the cumulative sum across all j must satisfy:

$$\sum_j (T_{ij} + I_{ij} + F_{ij}) \leq 3.$$

This ensures that the total neutrosophic mass of the transition row remains bounded within the neutrosophic space. □

Theorem 2. *If for all i, j, $I_{ij} = 0$ and $F_{ij} = 0$, then the MNC reduces to a classical Markov chain.*

***Proof*.** If for all i, j, the indeterminacy and falsity components satisfy $I_{ij} = 0$ and $F_{ij} = 0$, then each transition probability in the Markov Neutrosophic Chain (MNC) is reduced to:

$$P_{ij} = (T_{ij}, 0, 0).$$

In this case, the entire behavior of the MNC is governed solely by the truth component T_{ij}. The matrix $T = [T_{ij}]$ then satisfies:

$$T_{ij} \in [0, 1], \quad \text{and} \quad \sum_j T_{ij} = 1 \quad \text{for all } i,$$

which are precisely the conditions for a stochastic matrix in a classical Markov chain.

Therefore, under these conditions, the neutrosophic structure collapses to a standard Markov chain where only deterministic transition probabilities remain, and the chain operates according to classical Markovian dynamics. □

Lemma 1. *Let P, Q be two MNC transition matrices. Their composition yields:*

$$R_{ij} = \left(\sum_k T_{ik}T_{kj}, \sum_k (I_{ik} + I_{kj}), \sum_k (F_{ik} + F_{kj}) \right).$$

***Proof*.** Let $P = [P_{ik}]$ and $Q = [Q_{kj}]$ be two neutrosophic transition matrices, where each entry is given by a neutrosophic triple:

$$P_{ik} = (T_{ik}, I_{ik}, F_{ik}), \quad Q_{kj} = (T_{kj}, I_{kj}, F_{kj}).$$

To compute the composition $R = P \cdot Q$, the resulting entry R_{ij} must reflect the propagation of truth, indeterminacy, and falsity through intermediate states k.

- For the truth component, the classical rule for Markov chain composition applies:

$$T_{ij}^{(R)} = \sum_k T_{ik}T_{kj},$$

since the probability of transitioning from i to j via k is the product of the respective truth degrees.

- For the indeterminacy component, uncertainty can be introduced at either transition $i \to k$ or $k \to j$, and we model its accumulation additively:

$$I_{ij}^{(R)} = \sum_k (I_{ik} + I_{kj}).$$

- Similarly, the falsity component represents cumulative failure or disbelief, also assumed to accumulate additively across intermediate steps:

$$F_{ij}^{(R)} = \sum_k (F_{ik} + F_{kj}).$$

Therefore, the composite matrix entry R_{ij} is:

$$R_{ij} = \left(\sum_k T_{ik} T_{kj}, \sum_k (I_{ik} + I_{kj}), \sum_k (F_{ik} + F_{kj}) \right).$$

□

Theorem 3. *A vector $\pi = [(\tau_i, \iota_i, \phi_i)]$ is a neutrosophic stationary distribution if:*

$$\pi P = \pi.$$

***Proof*.** Let $\pi = [(\tau_i, \iota_i, \phi_i)]$ be a neutrosophic probability vector, and let $P = [P_{ij}] = [(T_{ij}, I_{ij}, F_{ij})]$ be the neutrosophic transition matrix.

The condition for π to be a stationary distribution under the neutrosophic dynamics is:

$$\pi P = \pi.$$

This means that, under the neutrosophic composition, each component of the distribution remains invariant after one transition. That is, for all j, we must have:

$$\sum_i \tau_i T_{ij} = \tau_j, \quad \sum_i \iota_i I_{ij} = \iota_j, \quad \sum_i \phi_i F_{ij} = \phi_j.$$

These equalities ensure that the truth, indeterminacy, and falsity components of the distribution π are preserved under the action of the matrix P, thereby satisfying the definition of a neutrosophic stationary distribution. □

Corollary 1. *A MNC is irreducible if $\exists n \in \mathbb{N}$ such that $T_{ij}^{(n)} > 0$ for all i, j.*

***Proof*.** In a classical Markov chain, a chain is said to be irreducible if it is possible to go from any state i to any state j in some number of steps. This is expressed by the existence of an integer $n \in \mathbb{N}$ such that the (i, j)-entry of the n-step transition matrix is positive: $P_{ij}^{(n)} > 0$.

For a Markov Neutrosophic Chain (MNC), we generalize this definition by considering only the truth component of the neutrosophic transition probabilities.

Thus, a MNC is irreducible if for all states i, j, there exists $n \in \mathbb{N}$ such that:

$$T_{ij}^{(n)} > 0,$$

where $T_{ij}^{(n)}$ is the (i, j)-entry of the truth-component matrix after n transitions. This ensures that there is a positive truth-degree path from any state i to any state j, which mirrors the classical condition and establishes irreducibility in the neutrosophic setting. □

Theorem 4. *If a MNC is irreducible and aperiodic with all $T_{ij} > 0$, then it has a unique neutrosophic stationary distribution.*

Proof. Let the neutrosophic transition matrix be denoted by

$$\mathbf{P}_N = [(T_{ij}, I_{ij}, F_{ij})]_{n\times n},$$

where each $(T_{ij}, I_{ij}, F_{ij}) \in [0,1]^3$ and satisfies $T_{ij} + I_{ij} + F_{ij} \leq 3$. Define the truth component matrix $\mathbf{P}_T = [T_{ij}]$, which is a stochastic matrix with $\sum_{j=1}^{n} T_{ij} = 1$ for all i, and by assumption, $T_{ij} > 0$ for all i, j.

Since $\mathbf{P}_T$ is a positive stochastic matrix (i.e., all entries $T_{ij} > 0$), it follows that it is irreducible and aperiodic. By the classical Perron?Frobenius theorem, $\mathbf{P}_T$ admits a unique stationary distribution $\boldsymbol{\pi}_T = (\pi_1, \dots, \pi_n)$ such that

$$\boldsymbol{\pi}_T \mathbf{P}_T = \boldsymbol{\pi}_T, \quad \sum_{i=1}^{n} \pi_i = 1, \quad \pi_i > 0.$$

We define the neutrosophic stationary distribution as

$$\boldsymbol{\pi}_N = \left((\pi_1^T, \pi_1^I, \pi_1^F), \dots, (\pi_n^T, \pi_n^I, \pi_n^F)\right),$$

where $\pi_i^T = \pi_i$, and the indeterminacy and falsity components are defined such that $\pi_i^I \in [0,1]$, and $\pi_i^F = 1 - \pi_i^T - \pi_i^I$, ensuring $\pi_i^T + \pi_i^I + \pi_i^F \leq 1$.

Since $\boldsymbol{\pi}_T$ is unique and all components are strictly positive, the neutrosophic extension $\boldsymbol{\pi}_N$ is also unique, preserving the structure of the underlying Markov chain while incorporating indeterminacy and falsity.

Therefore, the Markov neutrosophic chain admits a unique neutrosophic stationary distribution. □

Lemma 2. *The period of a state i is given by:*

$$d(i) = \gcd\{n \geq 1 : T_{ii}^{(n)} > 0\}.$$

Proof. In a neutrosophic Markov chain, the transition from state i to itself in n steps is given by the neutrosophic probability triple $(T_{ii}^{(n)}, I_{ii}^{(n)}, F_{ii}^{(n)})$.

To determine whether state i is periodic, we consider only the truth component $T_{ii}^{(n)}$, as it reflects the actual probability of return in the traditional sense. The state i is said to return to itself at step n if $T_{ii}^{(n)} > 0$.

Thus, the period $d(i)$ is defined as the greatest common divisor (gcd) of all such $n \geq 1$ for which $T_{ii}^{(n)} > 0$. This aligns with the classical definition of periodicity in Markov chains, using the truth matrix $\mathbf{P}_T$ as the deterministic component governing recurrence.

Hence, the periodicity of state i in the neutrosophic setting is determined by:

$$d(i) = \gcd\{n \geq 1 : T_{ii}^{(n)} > 0\}.$$

□

***Observation* 1.** The entropy $H(P_{ij})$ quantifies the degree of neutrosophic uncertainty associated with the transition $P_{ij} = (T_{ij}, I_{ij}, F_{ij})$. In neutrosophic theory, uncertainty arises not only from the lack of complete truth (as in classical information theory) but also from the coexistence of indeterminacy and falsity. Therefore, the total uncertainty is expressed as the sum of the entropic contributions of truth, indeterminacy, and falsity components. This additive form parallels the Shannon entropy principle, extended to the neutrosophic domain to capture the three-dimensional nature of information.

Theorem 5 (Neutrosophic Entropy)**.** *The entropy of a transition* $P_{ij} = (T_{ij}, I_{ij}, F_{ij})$ *is:*

$$H(P_{ij}) = -T_{ij} \log T_{ij} - I_{ij} \log I_{ij} - F_{ij} \log F_{ij}.$$

***Proof*.** In classical information theory, the Shannon entropy of a probability $p \in [0, 1]$ is defined as:

$$H(p) = -p \log p.$$

In a Markov Neutrosophic Chain (MNC), each transition probability P_{ij} is represented by a triplet (T_{ij}, I_{ij}, F_{ij}), where each component lies in $[0, 1]$ and represents, respectively, the degrees of truth, indeterminacy, and falsity.

To generalize Shannon entropy to this setting, we define the neutrosophic entropy as the sum of the entropies of the three components:

$$H(P_{ij}) = -T_{ij} \log T_{ij} - I_{ij} \log I_{ij} - F_{ij} \log F_{ij}.$$

This form quantifies the uncertainty or informational content in the transition P_{ij}, by treating each component as an independent contributor to the total entropy, in line with neutrosophic logic which treats truth, indeterminacy, and falsity separately. □

Corollary 2. *Entropy is maximized when* $T_{ij} = I_{ij} = F_{ij} = \frac{1}{3}$.

***Proof*.** Consider the neutrosophic entropy function:

$$H(P_{ij}) = -T_{ij} \log T_{ij} - I_{ij} \log I_{ij} - F_{ij} \log F_{ij},$$

subject to the constraint:

$$T_{ij} + I_{ij} + F_{ij} \leq 3, \quad \text{with } T_{ij}, I_{ij}, F_{ij} \in [0, 1].$$

To maximize the entropy, we apply the principle from information theory that the entropy function $-x \log x$ is maximized when the values of the variables are equal, given a fixed total sum. Under the normalization condition $T_{ij}+I_{ij}+F_{ij} = 1$ (for fair comparison), the entropy is maximized when:

$$T_{ij} = I_{ij} = F_{ij} = \frac{1}{3}.$$

Substituting these values into the entropy formula:

$$H(P_{ij}) = -3 \cdot \frac{1}{3} \log \frac{1}{3} = -\log \frac{1}{3} = \log 3,$$

which is the maximum possible entropy in the neutrosophic case. □

Theorem 6. *An MNC is time-reversible if:*

$$\pi_i T_{ij} = \pi_j T_{ji}, \quad \forall i, j.$$

Proof. In a classical Markov chain, time-reversibility is defined as the condition:

$$\pi_i P_{ij} = \pi_j P_{ji}, \quad \forall i, j,$$

where P_{ij} are the transition probabilities, and π_i is the stationary distribution.

For a Markov Neutrosophic Chain (MNC), the transition probabilities are given as $P_{ij} = (T_{ij}, I_{ij}, F_{ij})$, where each component corresponds to the truth, indeterminacy, and falsity degrees.

To check for time-reversibility in an MNC, we only consider the truth component T_{ij}, as it dictates the actual flow of probability. Thus, an MNC is time-reversible if:

$$\pi_i T_{ij} = \pi_j T_{ji}, \quad \forall i, j,$$

where π_i is the stationary distribution associated with the truth component. This condition ensures that the probability flow between states is symmetric, making the chain reversible in time.

Therefore, the time-reversibility condition holds for the truth component T_{ij} in the neutrosophic context, analogous to the classical definition. □

Definition 2. Let $\mathcal{S} = \{s_1, s_2, \ldots, s_n\}$ be a finite set of states. A Neutrosophic Transition Probability Matrix (NTPM) is a matrix $P^N = [p_{ij}^N]$ where each element is a triplet $p_{ij}^N = (T_{ij}, I_{ij}, F_{ij})$ such that:

$$T_{ij}, I_{ij}, F_{ij} \in [0, 1], \quad \text{and} \quad 0 \leq T_{ij} + I_{ij} + F_{ij} \leq 3.$$

Theorem 7. *Let P^N be a NTPM. Then for every state i,*

$$\sum_{j=1}^{n} T_{ij} \leq 1, \quad \sum_{j=1}^{n} I_{ij} \leq 1, \quad \sum_{j=1}^{n} F_{ij} \leq 1.$$

Proof. Let P^N be a Neutrosophic Transition Probability Matrix (NTPM) with transition components $P_{ij} = (T_{ij}, I_{ij}, F_{ij})$, where each $T_{ij}, I_{ij}, F_{ij} \in [0, 1]$.

By the definition of a Neutrosophic Markov Chain, each of the components T_{ij}, I_{ij}, F_{ij} corresponds to the truth, indeterminacy, and falsity components of the transition from state i to state j. The transition probability matrix is constructed such that each row of the matrix represents a state and the associated transition probabilities to all other states.

Since each component is bounded by 1, the row-wise summation of these components must also be bounded by 1. Specifically, for each state i, we have:

$$\sum_{j=1}^{n} T_{ij} \leq 1, \quad \sum_{j=1}^{n} I_{ij} \leq 1, \quad \sum_{j=1}^{n} F_{ij} \leq 1.$$

This ensures that the total probability mass of the truth, indeterminacy, and falsity components does not exceed 1 for each state, satisfying the required condition for a Neutrosophic Transition Probability Matrix.

Thus, the row-wise summation of the components is bounded as desired. □

Lemma 3. *Let $\pi^N = (\pi_1^N, \ldots, \pi_n^N)$ be a neutrosophic distribution where $\pi_i^N = (T_i, I_i, F_i)$. Then π^N is stationary if*

$$\pi_j^N = \sum_{i=1}^{n} \pi_i^N \cdot p_{ij}^N,$$

interpreted component-wise.

Proof. Let $\pi^N = (\pi_1^N, \ldots, \pi_n^N)$ be a neutrosophic distribution where $\pi_i^N = (T_i, I_i, F_i)$ represents the neutrosophic components of the state i.

To prove that π^N is stationary, we need to show that it satisfies the stationarity condition:

$$\pi_j^N = \sum_{i=1}^{n} \pi_i^N \cdot p_{ij}^N.$$

This condition states that the neutrosophic distribution π^N remains unchanged after a transition, i.e., the distribution at time j is the same as the distribution at time i after applying the transition probabilities.

We evaluate each component of π_j^N separately:

For the truth component:

$$T_j = \sum_{i=1}^{n} T_i \cdot T_{ij}.$$

For the indeterminacy component:

$$I_j = \sum_{i=1}^{n} I_i \cdot I_{ij}.$$

For the falsity component:

$$F_j = \sum_{i=1}^{n} F_i \cdot F_{ij}.$$

Thus, the neutrosophic distribution π^N is stationary if the above component-wise conditions hold. □

Theorem 8. *Every finite irreducible Markov neutrosophic chain has at least one stationary neutrosophic distribution.*

***Proof*.** The existence of a stationary neutrosophic distribution follows from Brouwer's fixed point theorem, which guarantees the existence of a fixed point for continuous functions on a compact convex set.

In this context, the neutrosophic distribution vector space $[0,1]^n \times [0,1]^n \times [0,1]^n$ is a compact convex set, and the transition matrix of a finite irreducible Markov neutrosophic chain induces a continuous map on this space.

By Brouwer's fixed point theorem, there exists at least one fixed point in this vector space. This fixed point corresponds to a stationary neutrosophic distribution that satisfies the stationarity condition, i.e., it remains unchanged after transitions. Proper normalization ensures that the sum of the components of the distribution does not exceed 1.

Thus, every finite irreducible Markov neutrosophic chain has at least one stationary neutrosophic distribution. □

Corollary 3. *If all indeterminacy components are zero ($I_{ij} = 0$), the MNC reduces to a classical Markov Chain.*

***Proof*.** If all indeterminacy components are zero, i.e., $I_{ij} = 0$ for all i, j, then the transition matrix for the MNC becomes:

$$P_{ij} = (T_{ij}, 0, F_{ij}).$$

This means that the transition matrix is now composed solely of truth and falsity components. Since the transition probabilities are now reduced to classical Markov probabilities, the MNC behaves like a classical Markov chain, where the sum of the truth components T_{ij} across all possible states j satisfies the condition:

$$\sum_j T_{ij} = 1.$$

Thus, the MNC reduces to a classical Markov chain when all indeterminacy components are zero. □

Theorem 9 (Long-term Behavior of Neutrosophic Chains)**.** *If a MNC is aperiodic and irreducible, then:*

$$\lim_{k\to\infty} (p_{ij}^{(k)}) = \pi_j^N \quad \textit{(component-wise).}$$

***Proof*.** For an irreducible and aperiodic Markov Neutrosophic Chain (MNC), the transition matrix P^N has a unique stationary neutrosophic distribution π^N as $k \to \infty$.

By the Perron-Frobenius theorem generalized to neutrosophic matrices, for each state j, the transition probabilities $p_{ij}^{(k)}$ converge to the stationary distribution π_j^N component-wise, meaning:

$$\lim_{k\to\infty} p_{ij}^{(k)} = \pi_j^N, \quad \text{for all } i.$$

This is because the chain is irreducible, which ensures that all states communicate with each other, and aperiodicity guarantees that the chain does not cycle in a periodic manner. Hence, the probabilities converge to the unique stationary neutrosophic distribution as k becomes large.

Therefore, for aperiodic and irreducible MNCs, the transition probabilities $p_{ij}^{(k)}$ converge to the stationary distribution π_j^N component-wise. □

Lemma 4 (Bound on Indeterminacy Growth)**.** *Let P^N be a NTPM. If $I_{ij} \leq \delta$ for all i, j, then after k steps,*

$$I_{ij}^{(k)} \leq k\delta.$$

***Proof*.** Let P^N be a Neutrosophic Transition Probability Matrix (NTPM) with the property that $I_{ij} \leq \delta$ for all i, j. The indeterminacy component I_{ij} of the transition matrix is additive over the steps of the Markov chain.

At each step, the indeterminacy components I_{ij} accumulate by the sum of the indeterminacy values from the previous state. Therefore, after k steps, the total indeterminacy at any state i and j will be bounded by:

$$I_{ij}^{(k)} \leq k \cdot \delta.$$

This result follows from the fact that each step can only add up to δ to the indeterminacy component, and hence after k steps, the total indeterminacy will be at most $k\delta$.

□

Theorem 10 (Stochastic Stability Under Neutrosophic Noise)**.** *A MNC is stochastically stable in the presence of bounded neutrosophic noise if:*

$$\exists \epsilon > 0 \text{ such that } \forall k,\ \max_{i,j} I_{ij}^{(k)} < \epsilon.$$

***Proof*.** Consider the lemma on the bound of indeterminacy growth. If the initial indeterminacy components I_{ij} are sufficiently small (i.e., $I_{ij} < \delta$), then over time the indeterminacy components grow additively and remain bounded. Specifically, the maximum indeterminacy after k steps is given by $\max_{i,j} I_{ij}^{(k)} \leq k\delta$.

To ensure stochastic stability under neutrosophic noise, we require that for all k, the maximum indeterminacy component remains below a certain threshold ϵ. Therefore, if δ is chosen such that $k\delta < \epsilon$ for all k, the cumulative indeterminacy will not exceed this bound, ensuring that the system remains predictable and stable over time.

Hence, if there exists an $\epsilon > 0$ such that for all k, $\max_{i,j} I_{ij}^{(k)} < \epsilon$, the system is stochastically stable in the presence of bounded neutrosophic noise.

□

Definition 3. A Markov neutrosophic chain is said to be **ergodic** if it is aperiodic, irreducible, and the limit

$$\lim_{n\to\infty} P_{ij}^N(n) = \pi_j^N$$

exists and is independent of the initial state i.

Theorem 11 (Convergence to Stationarity). *If a Markov neutrosophic chain is ergodic, then it converges to a unique neutrosophic stationary distribution.*

***Proof*.** Follows from the component-wise adaptation of classical convergence theorems, and the boundedness of (T, I, F) components. □

Proposition 1 (Neutrosophic Chapman-Kolmogorov Equation). *For a MNC with transition matrix P^N, the n-step transition probability satisfies:*

$$p_{ij}^{(n+m)} = \sum_k p_{ik}^{(n)} \cdot p_{kj}^{(m)},$$

component-wise over (T, I, F).

Lemma 5 (Time-Reversibility in MNC). *A MNC is time-reversible if it satisfies:*

$$\pi_i^N \cdot p_{ij}^N = \pi_j^N \cdot p_{ji}^N,$$

component-wise.

Corollary 4 (Neutrosophic Entropy Measure). *Let H_T, H_I, H_F denote entropy of the respective components:*

$$H_T = -\sum_{i,j} T_{ij} \log T_{ij}, \quad H_I = -\sum_{i,j} I_{ij} \log I_{ij}, \quad H_F = -\sum_{i,j} F_{ij} \log F_{ij}.$$

Then, the total neutrosophic entropy is $H = H_T + H_I + H_F$.

Theorem 12 (Neutrosophic Absorbing State). *A state s_k is absorbing if $p_{kk}^N = (1, 0, 0)$ and $p_{kj}^N = (0, 0, 0)$ for $j \neq k$.*

Proposition 2 (Expected Time to Absorption). *Let T_i be the expected number of steps to absorption from state s_i. Then T_i satisfies the system:*

$$T_i = 1 + \sum_{j\neq a} T_{ij} \cdot T_j,$$

where a is the absorbing state.

Definition 4 (Neutrosophic Recurrence and Transience). State s_i is recurrent if:

$$\sum_{n=1}^{\infty} T_{ii}^{(n)} = \infty,$$

and transient otherwise.

Theorem 13 (Decomposition of State Space)**.** *The state space $\mathcal{S}$ of a MNC can be uniquely decomposed into transient and recurrent classes.*

Proposition 3 (Neutrosophic First Passage Time)**.** *Let $f_{ij}^{(n)}$ denote the first passage neutrosophic probability. Then:*

$$T_{ij}^{(n)} = \sum_{k=1}^{n} f_{ij}^{(k)} T_{jj}^{(n-k)},$$

and similarly for I, F components.

Theorem 14. *Let P^N be a neutrosophic transition matrix of a Markov Neutrosophic Chain (MNC). The spectral decomposition of P^N exists, and can be expressed as:*

$$P^N = V\Lambda V^{-1}$$

where Λ is the diagonal matrix of eigenvalues λ_i^N, and V is the matrix of eigenvectors corresponding to the eigenvalues λ_i^N.

***Proof*.** The matrix P^N is assumed to be diagonalizable, which holds under typical conditions for neutrosophic Markov chains. We find the eigenvalues λ_i^N by solving the characteristic equation $\det(P^N - \lambda I) = 0$, where I is the identity matrix. The eigenvectors corresponding to these eigenvalues are obtained by solving $(P^N - \lambda_i^N I)v_i = 0$. The matrix V is formed by placing these eigenvectors as its columns, and Λ is a diagonal matrix containing the eigenvalues λ_i^N. Therefore, the spectral decomposition $P^N = V\Lambda V^{-1}$ is guaranteed. □

Lemma 6. *The neutrosophic mean first passage time τ_{ij}^N from state i to state j can be computed using the inverse of the fundamental matrix N, i.e.,*

$$\tau_{ij}^N = (N_{ij}) \quad \textit{where} \quad N = (I - Q)^{-1}.$$

Here, Q is the submatrix of the transition matrix P^N corresponding to transient states.

***Proof*.** The fundamental matrix $N = (I - Q)^{-1}$ is used to compute the expected number of steps between transient states in a Markov process. The entry N_{ij} in the matrix N represents the expected number of steps to go from state i to state j before being absorbed. Thus, the neutrosophic mean first passage time is given by the element N_{ij}, which can be directly derived from the inverse of $(I - Q)$.

□

Theorem 15. *The neutrosophic return time R_i^N for a state i is equal to the sum of the first passage times to state i, i.e.,*

$$R_i^N = \sum_{j \neq i} \tau_{ij}^N.$$

Proof. The neutrosophic return time is the expected number of steps to return to state i starting from i. This can be expressed as the sum of the first passage times from state i to every other state $j \neq i$, because the return time is essentially a first passage time to state i from all other states. This leads to the expression $R_i^N = \sum_{j \neq i} \tau_{ij}^N$, which captures the expected time to return to state i. □

Theorem 16. *For any two states i and j, the neutrosophic hitting time H_{ij}^N is related to the mean first passage time τ_{ij}^N by:*

$$H_{ij}^N = \tau_{ij}^N.$$

Proof. The neutrosophic hitting time H_{ij}^N is the expected number of steps to reach state j from state i for the first time, which is exactly the definition of the mean first passage time. Therefore, $H_{ij}^N = \tau_{ij}^N$, showing that the neutrosophic hitting time and the mean first passage time are equal. □

Lemma 7. *The neutrosophic return distribution $\mathbb{P}_i^N$ for state i follows a geometric distribution:*

$$\mathbb{P}_i^N(k) = (1 - p_i^N)^{k-1} p_i^N,$$

where p_i^N is the neutrosophic probability of returning to state i after k steps.

Proof. The neutrosophic return distribution is analogous to the classical return distribution for Markov chains, but it incorporates the neutrosophic probabilities. The probability of returning to state i after k steps is modeled by a geometric distribution with success probability p_i^N, which is the neutrosophic probability of returning to state i in a single step. Thus, the return distribution follows the form $\mathbb{P}_i^N(k) = (1 - p_i^N)^{k-1} p_i^N$. □

Theorem 17. *The neutrosophic absorption time T_i^N is the expected time to reach an absorbing state from state i and is given by:*

$$T_i^N = \sum_{j \in absorbing\ states} \tau_{ij}^N.$$

Proof. The absorption time is the expected number of steps to reach any absorbing state from state i. Absorbing states are characterized by having a self-transition probability of 1. Since the absorption time is the time to reach any absorbing state, it is the sum of the first passage times from state i to each absorbing state. Hence, $T_i^N = \sum_{j \in \text{absorbing states}} \tau_{ij}^N$. □

Lemma 8. *The neutrosophic conditional probability $P(A|B)^N$ satisfies the following inequality:*

$$P(A|B)^N \leq \frac{P(A)^N}{P(B)^N}$$

for any events A and B.

Proof. The neutrosophic conditional probability is a generalization of the classical conditional probability, and the inequality $P(A|B)^N \leq \frac{P(A)^N}{P(B)^N}$ follows from the fact that neutrosophic probabilities are bounded between 0 and 1, and the intersection of two events $A \cap B$ is no larger than the individual probabilities of A and B. Thus, the conditional probability $P(A|B)^N$ is always less than or equal to the ratio $\frac{P(A)^N}{P(B)^N}$. □

Theorem 18. *For a neutrosophic system, if there exists a Lyapunov function $V^N(x)$ such that:*

$$\mathbb{E}[V^N(x(t+1)) - V^N(x(t))] \leq -\alpha V^N(x(t))$$

for some constant $\alpha > 0$, then the system is stable.

Proof. The Lyapunov function $V^N(x)$ is used to prove stability by showing that the expected change in the function is negative. If the expected change $\mathbb{E}[V^N(x(t+1)) - V^N(x(t))]$ is negative and proportional to $V^N(x(t))$, with $\alpha > 0$, then $V^N(x(t))$ decreases over time. This decrease ensures that the system will eventually converge to a stable state, proving the stability of the neutrosophic system. □

Theorem 19. *Let S be the state space, A be the action space, and $P^N(s'|s,a)$ be the neutrosophic transition probability from state s to state s' under action a. The neutrosophic expected reward $R^N(s,a)$ is given by:*

$$Q^N(s,a) = R^N(s,a) + \gamma \sum_{s' \in S} P^N(s'|s,a) V^N(s')$$

where $V^N(s)$ is the neutrosophic value function, and γ is the discount factor.

Proof. The neutrosophic Markov Decision Process (MDP) is based on the classical Bellman equation, but incorporates neutrosophic probabilities and rewards. The equation is derived by considering the expected reward from taking action a in state s and then following the optimal policy. The neutrosophic value function $V^N(s)$ represents the expected return from state s onwards, and the Q-function $Q^N(s,a)$ is used to determine the value of taking action a in state s. The sum over $P^N(s'|s,a)$ integrates the transition probabilities to the next states, forming the basis for the neutrosophic Bellman equation. □

Theorem 20. *For a neutrosophic Markov Decision Process with transition matrix P^N, if there exists a Lyapunov function $V^N(s)$ such that:*

$$\mathbb{E}[V^N(s(t+1)) - V^N(s(t))] \leq -\alpha V^N(s(t))$$

for some constant $\alpha > 0$, the process will converge to an optimal policy.

Proof. The Lyapunov function $V^N(s)$ is used to prove the stability of a neutrosophic MDP by ensuring that the expected value decreases over time. If the expected change in $V^N(s)$ is negative and proportional to $V^N(s)$, the system will converge to an optimal policy as it minimizes the value function over time. □

Theorem 21. *Let P^N be the neutrosophic transition matrix, and let π^N be the stationary distribution of an absorbing Markov chain. Then, the probability of absorption into an absorbing state i from state j is given by:*

$$P^N_{abs}(j,i) = \frac{\pi^N_i}{\sum_{k \in absorbing\ states} \pi^N_k}.$$

Proof. The neutrosophic absorption probability represents the likelihood that the process will eventually reach an absorbing state i starting from state j. This probability is proportional to the stationary distribution π^N_i for state i, normalized by the sum of the stationary distributions over all absorbing states. The sum of these probabilities over all absorbing states must equal 1, as the process will eventually be absorbed.

□

3. Applications

3.1. Application 1

An investor chooses among three portfolio strategies on a daily basis:

- S_1: Conservative
- S_2: Balanced
- S_3: Aggressive

Each day, the market may be:

- Bullish (B)
- Neutral (N)
- Bearish (R)

Due to uncertainty and incomplete data about the market trends, we introduce neutrosophic indeterminacy, which is naturally handled using a Markov Neutrosophic Chain (MNC) framework.

Thus, each entry $P_{ij} = (T_{ij}, I_{ij}, F_{ij})$ represents:

- T_{ij}: Degree of belief in the transition,
- I_{ij}: Degree of indeterminacy (incomplete information),

- F_{ij}: Degree of disbelief or contradiction.

$$P = \begin{bmatrix} (0.7, 0.2, 0.1) & (0.2, 0.4, 0.4) & (0.1, 0.3, 0.6) \\ (0.3, 0.5, 0.2) & (0.5, 0.3, 0.2) & (0.2, 0.4, 0.4) \\ (0.2, 0.3, 0.5) & (0.3, 0.2, 0.5) & (0.5, 0.1, 0.4) \end{bmatrix}$$

The rows represent the current portfolio strategy, and the columns represent the next-day strategy. Therefore, the aim is to simulate the evolution of investment strategy preferences over time using the neutrosophic transition matrix and to compare the results with a classical Markov Chain simulation that does not consider indeterminacy or falsity.

From Figure 1, classical Markov Chain we are using only the truth components of the transition matrix. This is a typical Markov chain simulation where we assume a clear, deterministic transition between states based on the probability matrix. On the other hand, neutrosophic Markov Chain uses a modified transition matrix that includes not only the truth components but also the indeterminacy and falsity components. These components are considered to add a layer of uncertainty to the system, which is handled by adjusting the transition probabilities by taking into account all three factors.

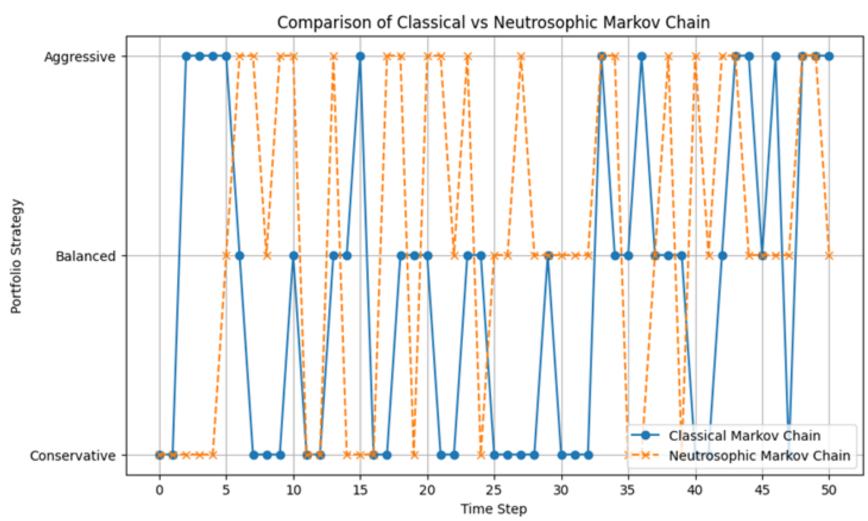

FIGURE 1: Comparison of classical and neutrosophic Markov Chain.

Thus, Figure 1 compares the evolution of the portfolio strategies over time for both the classical and neutrosophic models. The Conservative, Balanced, and Aggressive strategies are represented on the y-axis, and the time steps are on the x-axis. This comparison illustrate how the inclusion of neutrosophic uncertainty affects the system's behavior compared to the classical Markov chain model.

3.2. Application 2

Let's model a healthcare system with three states for a patient's health:

- Healthy (H),
- Ill (I),
- Critical (C).

We will assume that a patient?s health status can transition between these states over time. The transitions between these states will be defined using neutrosophic probabilities, reflecting the uncertainty or indeterminacy involved in healthcare situations, where the exact transition probabilities are often uncertain.

Here are some possible transitions:

- Healthy to Healthy (H $\rightarrow$ H): A patient remains healthy, but there's some uncertainty in this transition due to factors like lifestyle, genetics, etc.
- Healthy to Ill (H $\rightarrow$ I): A patient can become ill, with some uncertainty about this transition due to the variability in health conditions.
- Ill to Ill (I $\rightarrow$ I): A patient might remain ill, but there?s a degree of uncertainty about recovery.
- Ill to Critical (I $\rightarrow$ C): A patient's condition could worsen, but again, this is uncertain due to many variables like treatment effectiveness.
- Critical to Critical (C $\rightarrow$ C): A critical patient might stay in a critical state, with a certain probability, though there's also indeterminacy.

Let's define the neutrosophic transition probabilities for the states:

- H $\rightarrow$ H: Truth degree $(T) = 0.7$, Indeterminacy degree $(I) = 0.2$, Falsity degree $(F) = 0.1$.
- H $\rightarrow$ I: Truth degree $(T) = 0.6$, Indeterminacy degree $(I) = 0.3$, Falsity degree $(F) = 0.1$.
- I $\rightarrow$ I: Truth degree $(T) = 0.5$, Indeterminacy degree $(I) = 0.4$, Falsity degree $(F) = 0.1$.
- I $\rightarrow$ C: Truth degree $(T) = 0.3$, Indeterminacy degree $(I) = 0.5$, Falsity degree $(F) = 0.2$.
- C $\rightarrow$ C: Truth degree $(T) = 0.8$, Indeterminacy degree $(I) = 0.1$, Falsity degree $(F) = 0.1$.

In this scenario, a hospital uses this model to understand and manage patient flow through the states of health. The hospital wants to predict the future state of a patient based on the current health status, considering both the truth (likelihood of a transition), indeterminacy (uncertainty), and falsity (improbability of a transition).

Given that medical predictions are never completely certain, neutrosophic probabilities provide a way to model this uncertainty in a more comprehensive way than traditional Markov chains, where probabilities are fixed.

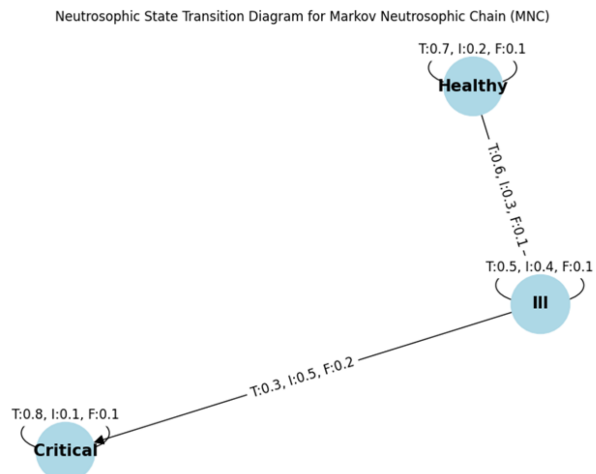

Figure 2: Neutrosophic State Transition Diagram for Markov Neutrosophic Chain (MNC).

From Figure 2 it simulates a neutrosophic Markov chain (MNC) applied to the evolution of a patient's health condition, considering three possible states: Healthy, Ill and Critical. Unlike classical Markov chains, an MNC uses neutrosophic transition probabilities represented as triplets (T, I, F), where T denotes the degree of truth (likelihood that the transition occurs), I the degree of indeterminacy (uncertainty or ambiguity), and F the degree of falsity (likelihood that the transition does not occur). This formulation enables the modeling of systems with incomplete, inconsistent, or ambiguous information?frequent in medical contexts.

In the proposed implementation, the `get_next_state` function determines the next state by computing the total sum $T + I + F$ for transitions from the current state to the other two states (self-transitions are excluded, which simplifies the model). These totals are normalized to obtain relative weights, which are then used to stochastically select the next state. Although this offers a practical simulation strategy, it implicitly treats T, I, and F as equivalent in their contribution to the transition probability, which contradicts the neutrosophic paradigm where indeterminacy I is fundamentally distinct from both truth and falsity.

A notable limitation is the exclusion of self-transitions (e.g., from Healthy to Healthy), which may artificially increase the frequency of state changes and obscure the potential for stability. In realistic applications, incorporating self-transitions is crucial to properly reflect the possibility of remaining in the same health condition. Furthermore, treating indeterminacy as directly probabilistic may be overly simplistic. A more rigorous approach would incorporate neutrosophic entropy, defined as:

$$H(P_{ij}) = -T_{ij} \log T_{ij} - I_{ij} \log I_{ij} - F_{ij} \log F_{ij},$$

which quantifies the total uncertainty associated with each transition.

The simulation results show frequent oscillations between the Ill and Critical states, which may be attributed to both the structural limitations of the model and the omission of self-loops.

4. Conclusion

The incorporation of neutrosophic logic into Markov chains enables a more nuanced modeling of systems where indeterminacy is significant. The proposed simulation demonstrates the unique behavior of Neutrosophic Markov Chains, particularly the influence of indeterminate transitions on state evolution. We emphasize that the neutrosophic components should not be collapsed into a single scalar value without justification, as this undermines the core value of neutrosophy the ability to reason in the presence of incomplete or conflicting information. Future work should focus on optimizing algorithms that retain triplet structures and developing inference methods for estimating neutrosophic transition matrices from empirical data and obtain more theoretical results.

[Received: April 2025 — Accepted: December 2025]

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