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Inverse Exponential Logistical Distribution Lehmann Type II: Application to Right Censored Data

Distribución logística exponencial inversa Lehmann tipo II: aplicación a datos censurados por la derecha

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Abstract

In this article, we introduce a new continuous probability distribution called the Inverse Exponential Logistic Lehmann Type II distribution, derived from the Lehmann Type II alternative. The main objective is to apply this new distribution to survival analysis, specifically with right-censored data. We discuss various properties of the proposed distribution, including quantiles, skewness, kurtosis, moments, order statistics, and Rényi entropy. The distribution exhibits a hazard rate function with different shapes depending on the parameter values. Simulation studies were conducted to evaluate the performance of maximum likelihood estimates under a right-censoring scheme. Finally, we illustrate the usefulness and flexibility of the proposed distribution by applying it to two real datasets and comparing its performance with that of other distributions.

***Keywords*:** Inverse Exponential Logistics; Lehamann's alternative; Right-censored data; Survival analysis.

Resumen

En este artículo, presentamos una nueva distribución de probabilidad continua denominada distribución Exponencial Inversa Logística Lehmann Tipo II, derivada de la alternativa Lehmann Tipo II. El objetivo principal es

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aplicar esta nueva distribución al análisis de supervivencia, específicamente con datos censurados por la derecha. Discutimos diversas propiedades de la distribución propuesta, incluyendo cuantiles, asimetría, curtosis, momentos, estadísticos de orden y entropía de Rényi. La distribución exhibe una función de riesgo con diferentes formas dependiendo de los valores de los parámetros. Se realizaron estudios de simulación para evaluar el desempeño de las estimaciones obtenidas por el método de máxima verosimilitud bajo un esquema de censura por la derecha. Finalmente, ilustramos la utilidad y flexibilidad de la distribución propuesta mediante aplicaciones a dos conjuntos de datos reales, comparando su desempeño con otras distribuciones existentes.

***Palabras clave*:** Alternativa de Lehmann; Análisis de supervivencia; Datos censurados por la derecha; Logística exponencial inversa.

1. Introduction

Statistical modeling is a fundamental pillar for describing and understanding empirical phenomena observed in the real world. In particular, the identification and selection of an appropriate probability distribution is indispensable for the correct analysis, interpretation, and inference of data. An adequate selection of the underlying distributions is crucial to ensure the validity of statistical inferences and the robustness of the conclusions obtained. However, the inherently complex nature of many datasets imposes significant limitations on the descriptive capacity of traditional models.

Such limitations become particularly evident in Survival Analysis. In clinical studies, for example, survival time in individuals with colon cancer or the lifespan of patients with breast cancer frequently exhibit complex risk patterns that classical distributions, such as the exponential, cannot adequately capture. This challenge inherent to data complexity has motivated a robust and growing line of research focused on developing statistical distributions with greater flexibility and precision in their fitting. Recent academic research has focused on generalizing and extending classical probabilistic models, culminating in the creation of new families of distributions.

It is possible to mention some recent articles that work in this line, such as Mansoor et al. (2019), who proposed the Marshall-Olkin Exponential Logistics distribution; Okorie et al. (2017), who proposed the exponential Log-Logistica Adjusted distribution; George & Thobias (2019), who carried out a study of the Marshall-Olkin Exponential distribution Kumaraswamy; Nassar et al. (2019), who introduced the Marshall-Olkin Exponential alpha power distribution family; and Basheer (2019), who applied this development to the Inverse Exponential distribution. Fallah & Kazemi (2020) performed an inferential study of the generalized weighted exponential distribution; Chaudhary & Kumar (2020) worked with the Inverse Exponential Logistics distribution; Sobhi & Mashail (2020) studied the Power Inverse Exponential Logistic distribution in the context of insurance data; Eghwerido et al. (2022), extended the generalized exponential alpha power distribution; and Ikechukwu & Eghwerido (2022) propose the transmuted shifted exponential distribution.

In this work, the Lehmann Type II alternative will be proposed for the Inverse Exponential Logistic (IEL) distribution (Chaudhary & Kumar, 2020), to obtain a more flexible distribution for analyzing survival data with right-censoring. Therefore, consider X be the Inverse Exponential Logistic random variable, $X \sim \text{IEL}(\gamma, \lambda)$, whose cumulative distribution function, probability density function, and hazard rate are given, respectively, by

$$F_X(x;\gamma,\lambda) = \frac{1}{1+(\exp(\lambda/x)-1)^{\gamma}},\ x>0 \text{ and } \gamma,\lambda>0,$$

$$f_X(x;\gamma,\lambda) = \frac{\gamma\lambda}{x^2}\frac{\exp(\lambda/x)(\exp(\lambda/x)-1)^{\gamma-1}}{[1+(\exp(\lambda/x)-1)^{\gamma}]^2},\ x>0 \text{ and } \gamma,\lambda>0,$$

and

$$h_X(x;\gamma,\lambda) = \frac{\gamma\lambda}{x^2}\frac{\exp(\lambda/x)}{[1+(\exp\ (\lambda/x)-1)^{\gamma}][\exp(\lambda/x)-1]},\ x>0 \text{ and } \gamma,\lambda>0.$$

To obtain the Type II Lehmann alternative, the transformation is based on the cumulative distribution function of an absolutely continuous random variable $F_T(t) = 1-[1-F(t)]^{\alpha}$, $\alpha>0$, where $F(\cdot)$ is the distribution function of an absolutely continuous random variable. The density and hazard functions associated with $F_T(t)$ are given, respectively, by

$$f_T(y;\alpha) = \alpha f(t)[1-F(t)]^{\alpha-1} \quad \text{and} \quad h_T(t;\alpha) = \frac{\alpha f(t)}{1-F(t)}.$$

For other developments that apply Lehmann alternatives, readers can consult works such as Chaubey & Zhang (2015), who proposed a Lehmann alternative for the Chen distribution family; Tomazella et al. (2020), who applied a Lehmann alternative for the inverse Weibull distribution; Awodutire et al. (2020), who developed a study for the half-Logistic distribution; Ogunde et al. (2020), who proposed a Lehmann alternative for the extended Type-II Gumbel distribution, while Ogunde et al. (2021), the authors developed a Lehmann alternative for the Frèchet-Poisson distribution.

The article is organized as follows. In Section 2, we present the main characteristics of the Inverse Exponential Logistic Lehmann Type II (IELL2) distribution, including its quantiles, skewness, kurtosis, and moments. In addition, we derive the expressions for the score function and the Fisher information matrix. Section 3 reports the results of the Monte Carlo simulation study conducted to assess the performance of the maximum likelihood estimators for small and large sample sizes. In Section 4, we illustrate the practical applicability of the IELL2 model by fitting it to two real datasets. Finally, Section 5 concludes the paper with some remarks and possible directions for future research.

2. Inverse Exponential Logistic Distribution Lehmann Type II

2.1. Definition

Let T be a recurring variable with Lehmann Type II Inverse Logistic distribution, denoted by $T \sim \text{IELL2}(\alpha, \gamma, \lambda)$, its cumulative distribution function is given by

$$F_T(t; \alpha, \gamma, \lambda) = 1 - \left[\frac{(\exp(\lambda/t) - 1)^\gamma}{1 + (\exp(\lambda/t) - 1)^\gamma} \right]^\alpha, \ t > 0 \text{ and } \alpha, \gamma, \lambda > 0, \qquad (1)$$

and its density function is given by

$$f_T(t; \alpha, \gamma, \lambda) = \frac{\alpha\gamma\lambda}{t^2} \frac{\exp(\lambda/t)(\exp(\lambda/t) - 1)^{\alpha\gamma - 1}}{[1 + (\exp(\lambda/t) - 1)^\gamma]^{\alpha + 1}}, \ t > 0 \text{ and } \alpha, \gamma, \lambda > 0. \qquad (2)$$

For $\alpha = 1$, we have the Inverse Exponential distribution as a particular case (Chaudhary & Kumar, 2020), and when $\alpha = \gamma = 1$ we obtain the Inverse Exponential distribution as a particular case (Keller et al., 1982).

The hazard function is represented by

$$h_T(t; \alpha, \gamma, \lambda) = \frac{\alpha\gamma\lambda}{t^2} \frac{\exp(\lambda/t)}{[1 + (\exp(\lambda/t) - 1)^\gamma][\exp(\lambda/t) - 1]}, \ t > 0 \text{ and } \alpha, \gamma, \lambda > 0.$$

In Figure 1, we present the probability density and hazard functions for the IELL2 distribution. Note that density may assume different forms (asymmetry and kurtosis), which is useful in practical applications.

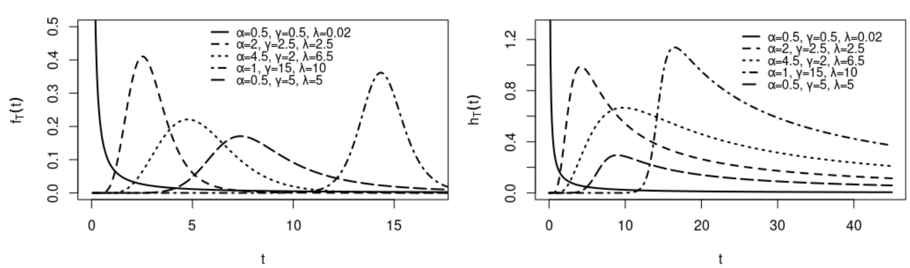

FIGURE 1: Probability density function (left) and hazard function (right), for the Inverse Exponential Logistic Lehmann Type II distribution, considering some values for the parameters.

2.2. Quantiles, Asymmetry and Kurtosis

The quantile function of the IELL2 distribution, denoted by $Q(u)$ for $u \in (0, 1)$, can be obtained by inverting the cumulative distribution function presented in Equation (1). Thus, the quantile function is given by

$$Q(u) = \frac{\lambda}{\log\left[1 + \left(\frac{1}{1-(1-u)^{1/\alpha}} - 1\right)^{1/\gamma}\right]}, \ \alpha, \gamma, \lambda > 0, \tag{3}$$

If there is particular interest in the median, it can be found by evaluating $Q(u)$ at $u = 0.5$. Hence, the median (Med) is given by

$$\text{Med} = \frac{\lambda}{\log\left[1 + \left(\frac{1}{1-2^{-1/\alpha}} - 1\right)^{1/\gamma}\right]}, \ \alpha, \gamma, \lambda > 0.$$

Asymmetry is calculated using Bowley's quantile asymmetry coefficient (Kenney & Keeping, 1962):

$$S = \frac{Q(0.25) - 2Q(0.5) + Q(0.75)}{Q(0.75) - Q(0.25)},$$

while kurtosis is quantified using the quantile kurtosis coefficient proposed by Moors (1988):

$$K = \frac{Q(0.875) - Q(0.625) + Q(0.375) - Q(0.125)}{Q(0.75) - Q(0.25)}.$$

In Figure 2, the behavior of asymmetry and kurtosis is observed considering some parameter values. Note that for smaller values of the parameter α, both skewness and kurtosis tend to increase in their respective amounts.

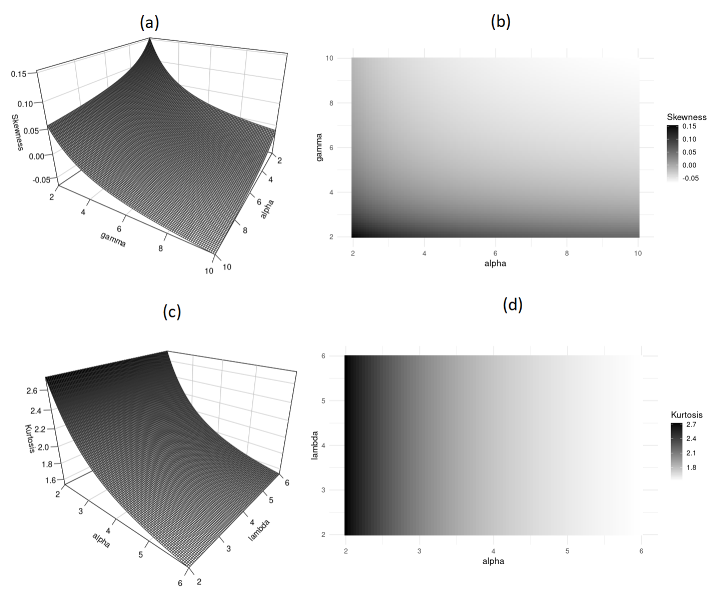

FIGURE 2: Behavior of asymmetry (a-b) and kurtosis (c-d) for certain values of the parameters of the Inverse Logistic Exponential Lehmann Type II distribution.

2.3. Moments

Let T be a random variable following a distribution IELL2$(\alpha, \gamma, \lambda)$. The r-th moment is given by

$$\begin{aligned}\mathbb{E}(T^r) &= \int_0^\infty t^r \frac{\alpha\gamma\lambda}{t^2} \frac{\exp(\lambda/t)(\exp(\lambda/t)-1)^{\alpha\gamma-1}}{[1+(\exp(\lambda/x)-1)^\gamma]^{\alpha+1}} dt \\ &= \alpha\gamma\lambda \int_0^\infty t^{r-2} \frac{\exp(\lambda/t)(\exp(\lambda/t)-1)^{\alpha\gamma-1}}{[1+(\exp(\lambda/t)-1)^\gamma]^{\alpha+1}} dt.\end{aligned}$$

For non-negative ramdom variables, the r-th moment admits the following alternative representation:

$$\mathbb{E}[T^r] = r \int_0^\infty t^{r-1} S_T(t) dt,$$

and therefore,

$$\mathbb{E}[T^r] = r \int_0^\infty t^{r-1} \left[\frac{(\exp(\lambda/t)-1)^\gamma}{1+(\exp(\lambda/t)-1)^\gamma} \right]^\alpha dt. \tag{4}$$

Observe that

$$\frac{(\exp(\lambda/t)-1)^\gamma}{1+(\exp(\lambda/t)-1)^\gamma} = 1 - \frac{1}{1+(\exp(\lambda/t)-1)^\gamma},$$

and, by defining

$$w = \frac{1}{1+(\exp(\lambda/t)-1)^\gamma},$$

and applying the Generalized Binomial Series, which is valid for $|w| < 1$, we obtain:

$$\begin{aligned}(1-w)^\alpha &= \sum_{k=0}^\infty (-1)^k \binom{\alpha}{k} w^k \\ &= \sum_{k=0}^\infty (-1)^k \binom{\alpha}{k} \left(1+(\exp(\lambda/t)-1)^\gamma\right)^{-k}.\end{aligned}$$

Finally, performing the substituition $z = \lambda/t$, we arrive at:

$$\mathbf{E}[T^r] = r\lambda^r \sum_{k=0}^\infty (-1)^k \binom{\alpha}{k} \int_0^\infty z^{-(r+1)} \left[1+(\exp(z)-1)^\gamma\right]^{-k} dz. \tag{5}$$

It is important to emphasize that the convergence of the integral defining the moment $\mathbb{E}[T^r]$ requires the shape parameter γ to satisfy the condition $\gamma > r$. For the distribution proposed in this work, obtaining closed-form analytical expressions for the moments is not feasible; thus, numerical and computational approaches are required. In what follows, we describe three computational schemes for approximating the ordinary r-th moment and illustrate their application by numerically evaluating the first and second moments.

- **Method 1. Monte Carlo Simulation:** We employed the inverse transform method, whose applicability is ensured by the closed-form expression of the quantile function of the proposed distribution (Equation (3)). We generated $n = 10^7$ samples from a uniform distribution $U(0, 1)$, allowing the estimation of empirical moments through

$$\widehat{\mathbb{E}}[T^r] = \frac{1}{n}\sum_{i=1}^{n} T_i^r,$$

where $T_i = Q(U_i)$ denotes the realizations of the random variable.

- **Method 2. Direct Numerical Integration:** Numerical integration was performed using the `integrate()` function in R, applied to the ordinary r-th moment given in Equation (4). This approach yields highly accurate approximations when the integration is carried out with appropriate error tolerances. Specifically, we adopted relative and absolute tolerances of 10^{-10} to ensure robust convergence.

- **Method 3. Truncated Series Expansion:** We applied the approximation given in Equation (5) truncating the upper limit of the series at $K = 500$, which ensures adequate numerical convergence.

Several combinations of the parameters α, γ e λ were examined. The results show that the Monte Carlo and the numerical integration methods converge to similar values. The truncated series method, in contrast, presents convergence issues for small α, which limits its applicability. Table 1 reports the estimates of the first two moments for $\alpha = 3$, $\gamma = 4$ e $\lambda = 5$.

TABLE 1: Approximations of the first two moments for $\alpha = 3$, $\gamma = 4$ e $\lambda = 5$, obtained using three computational methods: Monte Carlo, Numerical integration, and Truncated Series.

	Monte Carlo	**Numerical Integration**	**Truncated Series**
$\mathbb{E}[T]$	5.7702	5.7421	5.7703
$\mathbb{E}[T^2]$	34.9430	34.9435	34.7320

2.4. Order Statistics

Let $T_1, T_2, \ldots, T_n$ be a random sample of $T \sim \text{IELL2}(\alpha, \gamma, \lambda)$, $\alpha,\ \gamma,\ \lambda > 0$. Consider their respective order statistics $T_{(1)}, T_{(2)}, \ldots, T_{(n)}$. The density function of the l-th ($1 \leq l \leq n$) order statistic is given by

$$\begin{aligned} f_{T_{(l)}}(t) &= \frac{n!}{(l-1)!(n-l)!} f_T(t)[F_T(t)]^{l-1}[1 - F_T(t)]^{n-l} \\ &= \frac{n!}{(l-1)!(n-l)!}\frac{\alpha\gamma\lambda}{t^2}\frac{\exp(\lambda/t)(\exp(\lambda/t)-1)^{\alpha\gamma[1+n-l]-1}}{[1+(\exp(\lambda/t)-1)^{\gamma}]^{\alpha[n-l+1]+1}} \\ &\times \left[1 - \frac{(\exp(\lambda/t)-1)^{\alpha\gamma}}{[1+(\exp(\lambda/t)-1)^{\gamma}]^{\alpha}}\right]^{l-1}. \end{aligned}$$

Using binomial expansion, it is possible to rewrite the density function of the l-th order statistic as

$$f_{T_{(l)}}(t) = \frac{n!}{(l-1)!(n-l)!} f_T(t)[1 - F_T(t)]^{n-l} \sum_{k=0}^{l-1} (-1)^k \binom{l-1}{k} [1 - F_T(t)]^k.$$

Through an algebraic organization, the density function of the l-th statistic becomes

$$f_{T_{(l)}}(t) = \frac{n!}{(l-1)!(n-l)!} \sum_{k=0}^{l-1} (-1)^k \binom{l-1}{k} f_T(t; \alpha^*, \gamma, \lambda),$$

where $f_T(t; \alpha^*, \gamma, \lambda)$ is the density function of $T \sim \text{IELL2}(\alpha^*, \gamma, \lambda)$, being $\alpha^* = n + k - l + 1$.

2.5. Rényi's Entropy

Let T be a random variable with density function $f_T(t)$, define the Rényi entropy (Rényi, 1961) of order ρ as

$$H_\rho = \frac{1}{1-\rho} \log \left[\int_{-\infty}^{\infty} [f_T(t)]^\rho dt \right], \ \rho > 0, \ \rho \neq 1.$$

In case of $\rho \uparrow 1$, we have the Shannon entropy (Shannon, 1948) as a particular case. Considering the distribution $T \sim \text{IELL2}(\alpha, \gamma, \lambda)$ the Rényi entropy is given by

$$\begin{aligned} H_\rho &= \frac{1}{1-\rho} \log \left[\int_0^\infty \frac{(\alpha\gamma\lambda)^\rho}{t^{2\rho}} \frac{\exp(\rho\lambda/t)(\exp(\lambda/t) - 1)^{\alpha\gamma\rho - \rho}}{[1 + (\exp(\lambda/t) - 1)^\gamma]^{\rho(\alpha+1)}} dt \right] \\ &= \frac{1}{1-\rho} \log \left[(\alpha\gamma\lambda)^\rho \int_0^\infty (1/t)^{2\rho} \frac{\exp(\rho\lambda/t)(\exp(\lambda/t) - 1)^{\alpha\gamma\rho - \rho}}{[1 + (\exp(\lambda/t) - 1)^\gamma]^{\rho(\alpha+1)}} dt \right]. \end{aligned}$$

Considering the substituition $u = \lambda/t$, we obtain $t = \lambda/u$, $dt = -\lambda/u^2\, du$. Applying this change of variables, we can rewrite the integral as

$$\int_0^\infty [f_T(t)]^\rho dt = (\alpha\gamma)^\rho \lambda \int_0^\infty u^{2\rho-2} \frac{\exp(\rho u)(\exp(u) - 1)^{\rho(\alpha\gamma-1)}}{[1 + (\exp(u) - 1)^\gamma]^{\rho(\alpha+1)}} du.$$

Therefore, the Rényi entropy can be written as

$$H_\rho = \frac{\rho \log(\alpha\gamma) + \log(\lambda) + \log(I_\rho(\alpha, \gamma))}{1-\rho},$$

where

$$I_\rho(\alpha, \gamma) = \int_0^\infty u^{2\rho-2} \frac{\exp(\rho u)(\exp(u) - 1)^{\rho(\alpha\gamma-1)}}{[1 + (\exp(u) - 1)^\gamma]^{\rho(\alpha+1)}} du.$$

Let us analyze the analysis of the general case. Consider

$$\begin{aligned} H_\rho &= \frac{1}{1-\rho} \log \int_0^\infty \alpha^\rho \, [f(t)]^\rho \, [1-F(t)]^{\rho(\alpha-1)} \, dt \\ &= \frac{\rho \log \alpha}{1-\rho} + \frac{1}{1-\rho} \log \int_0^\infty [f(t)]^\rho \, [1-F(t)]^{\rho(\alpha-1)} \, dt, \end{aligned}$$

where $f(\cdot)$ and $F(\cdot)$ denote, respectively, the probability density function and the cumulative distribution of the IEL (the baseline distribution). Applying a series expansion, we obtain

$$[1-F(t)]^{\rho(\alpha-1)} = \sum_{k=0}^{\infty} \binom{\rho(\alpha-1)}{k} (-1)^k [F(t)]^k.$$

After some algebric manipulations, the Rényi entropy can be rewritten as

$$H_\rho = \frac{\rho \log \alpha}{1-\rho} + \frac{1}{1-\rho} \log \left[\sum_{k=0}^{\infty} \binom{\rho(\alpha-1)}{k} (-1)^k J_k^{(\rho)} \right],$$

where

$$J_k^{(\rho)} = \int_0^\infty [f(t)]^\rho [F(t)]^k \, dt.$$

The term $J_k^{(\rho)}$ qauntifies the interaction between the power ρ of the local density $[f(t)]^\rho$, and the power k of the accumulated information $[F(t)]^k$, capturing how the system that has already occurred up to time t (through $F(t)$) relates to the intensity of the distribution at each point (through $[f(t)]^\rho$). In other words, $J_k^{(\rho)}$ measures the coupling between two dimensions of the distributions: (i) the density concentration represented by $[f(t)]^\rho$, and (ii) the cumulative progression represented by $[F(t)]^k$. The greater its value, the greater the "synchronization" between the present state and the set of states previously observed in the system.

2.6. Estimation via Maximum Likelihood in the Presence of Right-Censored Data

Let Y_i be a random variable that represents the failure time of the $i-$th unit, and C_i be the random variable that represents the censoring time of the respective unit. The variables Y_i and C_i are assumed independent, Y_i has a IELL2 distribution parameterized by $\boldsymbol{\theta} = (\alpha, \gamma, \lambda)^\top$. Observing the data pairs (T_i, δ_i), where $T_i = \min(Y_i, C_i)$, with $\delta_i = 1$, where $Y_i \leq C_i$ and $\delta_i = 0$ otherwise.

Given a sample of size n of pairs (T_i, δ_i), the logarithm of the likelihood function is given by

$$\begin{aligned}\ell(\boldsymbol{\theta}) &= \sum_{i=1}^{n}\delta_i\log[f_T(t_i)] + \sum_{i=1}^{n}(1-\delta_i)\log[S_T(t_i)]\\ &= \log(\alpha\gamma\lambda)\sum_{i=1}^{n}\delta_i - 2\sum_{i=1}^{n}\delta_i\log(t_i) + \lambda\sum_{i=1}^{n}\frac{\delta_i}{t_i}\\ &\quad + \sum_{i=1}^{n}(\alpha\gamma-\delta_i)\log(\exp(\lambda/t_i)-1)\\ &\quad - \sum_{i=1}^{n}(\alpha+\delta_i)\log(1+(\exp(\lambda/t_i)-1)^{\gamma}),\end{aligned}$$

where $f_T(\cdot)$ is given by (2) and $S_T(\cdot) = 1 - F_T(\cdot)$ with $F_T(\cdot)$ given by (1).

The score function for the parameters is the form

$$\begin{aligned}U_\alpha(\boldsymbol{\theta}) &= \frac{1}{\alpha}\sum_{i=1}^{n}\delta_i + \gamma\sum_{i=1}^{n}\log(\eta_i) - \sum_{i=1}^{n}\log(1+\eta_i^{\gamma}),\\ U_\gamma(\boldsymbol{\theta}) &= \frac{1}{\gamma}\sum_{i=1}^{n}\delta_i + \alpha\sum_{i=1}^{n}\log(\eta_i) - \sum_{i=1}^{n}(\alpha+\delta_i)\frac{\log(\eta_i)\eta_i^{\gamma}}{1+\eta_i^{\gamma}},\\ U_\lambda(\boldsymbol{\theta}) &= \frac{1}{\lambda}\sum_{i=1}^{n}\delta_i + \sum_{i=1}^{n}\frac{\delta_i}{t_i} + \sum_{i=1}^{n}\frac{(\alpha\gamma-\delta_i)(1+\eta_i)}{t_i\eta_i} - \gamma\sum_{i=1}^{n}\frac{(\alpha+\delta_i)\eta_i^{\gamma-1}(1+\eta_i)}{t_i(1+\eta_i^{\gamma})},\end{aligned}$$

where $\eta_i = \exp(\lambda/t_i) - 1$. The maximum likelihood estimators are obtained by solving the system of equations $U_\alpha(\boldsymbol{\theta}) = 0$, $U_\gamma(\boldsymbol{\theta}) = 0$ and $U_\lambda(\boldsymbol{\theta}) = 0$. As this system does not have a closed analytical solution, computational routines were used in R Core Team (2025) using the *optim* function, which in turn uses the Nelder & Mead (1965) optimization method to find a numerical solution. Asymptotically, and under the condition that the parameters are interior points of the parameter space, $\widehat{\boldsymbol{\theta}}$ has a trivariate Normal distribution given by $\sqrt{n}(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}) \sim \mathcal{N}_3(\mathbf{0}, \boldsymbol{I}^{-1}(\boldsymbol{\theta}))$ when $n \to \infty$, where $\mathbf{I}(\boldsymbol{\theta})$ is expected Fisher information matrix.

Such behavior can still be expected if the information matrix $\mathbf{I}(\boldsymbol{\theta})$ is replaced by the observed information matrix evaluated at $\widehat{\boldsymbol{\theta}}$, $\mathbf{J}(\widehat{\boldsymbol{\theta}})$, where $\mathbf{J}(\widehat{\boldsymbol{\theta}}) = -\frac{\partial^2\ell(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^\top}\Big|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}}$ is represented by

$$\mathbf{J}(\widehat{\boldsymbol{\theta}}) = \begin{pmatrix} \frac{d^2\ell(\boldsymbol{\theta})}{d\alpha^2} & \frac{d^2\ell(\boldsymbol{\theta})}{d\alpha d\gamma} & \frac{d^2\ell(\boldsymbol{\theta})}{d\alpha d\lambda}\\ . & \frac{d^2\ell(\boldsymbol{\theta})}{d\gamma^2} & \frac{d^2\ell(\boldsymbol{\theta})}{d\gamma d\lambda}\\ . & . & \frac{d^2\ell(\boldsymbol{\theta})}{d\lambda^2}\end{pmatrix},$$

where

$$
\begin{aligned}
\frac{d^2\ell(\boldsymbol{\theta})}{d\alpha^2} &= -\frac{1}{\alpha^2}\sum_{i=1}^{n}\delta_i,\\
\frac{d^2\ell(\boldsymbol{\theta})}{d\alpha d\gamma} &= \sum_{i=1}^{n}\log(\eta_i) - \sum_{i=1}^{n}\frac{\log(\eta_i)\eta_i^{\gamma}}{1+\eta_i^{\gamma}},\\
\frac{d^2\ell(\boldsymbol{\theta})}{d\alpha d\lambda} &= \sum_{i=1}^{n}\frac{\gamma(1+\eta_i)}{t_i\eta_i} - \sum_{i=1}^{n}\frac{\gamma\eta_i^{\gamma-1}(1+\eta_i)}{t_i(1+\eta_i^{\gamma})},\\
\frac{d^2\ell(\boldsymbol{\theta})}{d\gamma^2} &= -\frac{1}{\gamma^2}\sum_{i=1}^{n}\delta_i - \sum_{i=1}^{n}\frac{(\alpha+\delta_i)\log^2(\eta_i)\eta_i^{\gamma}}{(1+\eta_i^{\gamma})^2},\\
\frac{d^2\ell(\boldsymbol{\theta})}{d\gamma d\lambda} &= \sum_{i=1}^{n}\frac{\alpha(1+\eta_i)}{t_i\eta_i} - \sum_{i=1}^{n}\frac{(\alpha+\delta_i)\eta_i^{\gamma-1}(1+\eta_i)(\gamma\log(\eta_i)+\eta_i^{\gamma}+1)}{t_i(1+\eta_i^{\gamma})^2},\\
\frac{d^2\ell(\boldsymbol{\theta})}{d\lambda^2} &= -\frac{1}{\lambda^2}\sum_{i=1}^{n}\delta_i - \sum_{i=1}^{n}\frac{(\alpha\gamma-\delta_i)(1+\eta_i)}{t_i^2\eta_i^2}\\
&\quad -\sum_{i=1}^{n}\frac{(\alpha+\delta_i)\gamma\eta_i^{\gamma-2}(1+\eta_i)(\gamma(1+\eta_i)-\eta_i^{\gamma}-1)}{t_i^2(1+\eta_i^{\gamma})^2},
\end{aligned}
$$

with $\eta_i = \exp(\lambda/t_i) - 1$.

3. Simulation Study

To evaluate the maximum likelihood estimates under both large and small sample sizes, a Monte Carlo simulation study was conducted. To obtain samples from the IELL2 distribution, the inverse transformation method was used from

$$
t = \lambda\left\{\log\left[\left(\frac{1}{1-(1-\mathcal{U})^{1/\alpha}}-1\right)^{1/\gamma}+1\right]\right\}^{-1},
$$

where $\mathcal{U}$ represents a standard uniform random variable. Thus, $N = 5000$ Monte Carlo replicas of the distribution IELL2$(\alpha, \gamma, \lambda)$ were taken for the sample sizes $n = 50, 100, 200, 500, 1000, 2000$, in addition to evaluating for the following censorship rates $5\%, 10\%$ and 20%. The censoring variables, δ_i, were obtained from the Bernoulli random variable(p), where p are the censoring rates. For each sample size and censoring rate, bias and mean squared error were observed. Furthermore, the proportion of times that the Akaike information criterion given by AIC $= -2\ell(\widehat{\theta}) + 2p$ was lower when evaluated in the simulated case than in the case of previously fixed parameter values.

Parameters were fixed at $\alpha = 2$, $\gamma = 3$, and $\lambda = 5$, with the random seed set to 513776 for reproducibility. Maximum likelihood estimates were computed using the R statistical software (R Core Team, 2025) through the `maxLik` function, which provides efficient optimization algorithms for likelihood-based inference. Results are displayed in Table 2.

Table 2: Results of the simulation study for different sample sizes and censoring rates considering the IELL2 distribution with parameters $\alpha = 2, \gamma = 3, \lambda = 5$.

Censure	n	Bias			Standard Error		
		$\widehat{\alpha}$	$\widehat{\gamma}$	$\widehat{\lambda}$	$\widehat{\alpha}$	$\widehat{\gamma}$	$\widehat{\lambda}$
5%	50	4.834	−0.056	2.558	10.937	1.289	5.522
	100	2.694	−0.086	1.758	6.705	0.966	4.355
	200	1.561	−0.082	1.050	4.868	0.711	3.284
	500	0.496	−0.044	0.426	2.572	0.453	1.830
	1000	0.108	−0.025	0.134	1.183	0.258	0.796
	2000	−0.064	−0.003	0.029	0.306	0.164	0.274
10%	50	4.495	−0.039	2.766	9.792	1.507	5.641
	100	2.522	−0.121	1.778	6.515	0.937	4.447
	200	1.473	−0.115	1.107	4.699	0.702	3.248
	500	0.370	−0.047	0.410	2.420	0.432	1.829
	1000	−0.070	−0.008	0.091	0.680	0.251	0.562
	2000	−0.129	−0.011	0.047	0.329	0.167	0.288
20%	50	3.767	−0.004	2.664	9.415	1.484	5.661
	100	2.534	−0.130	1.962	6.437	0.986	4.642
	200	1.203	−0.114	1.262	4.246	0.762	3.602
	500	0.170	−0.049	0.467	2.495	0.465	2.109
	1000	−0.287	−0.009	0.150	1.222	0.297	1.059
	2000	−0.337	−0.001	0.046	0.590	0.192	0.451

Table 3: Results of the simulation study for different sample sizes and censoring rates considering the IELL2 distribution with parameters $\alpha = 0.4, \gamma = 7.3, \lambda = 4.6$.

Censure	n	Bias			Standard Error		
		$\widehat{\alpha}$	$\widehat{\gamma}$	$\widehat{\lambda}$	$\widehat{\alpha}$	$\widehat{\gamma}$	$\widehat{\lambda}$
45%	50	5.527	4.023	0.868	17.154	14.775	3.270
	100	4.102	0.819	0.567	15.034	5.049	1.941
	200	1.533	0.446	0.234	8.121	1.948	1.271
	500	0.426	0.294	0.023	0.144	1.224	0.222
	1000	0.425	0.046	0.028	0.098	0.810	0.159
	2000	0.408	0.029	0.007	0.059	0.551	0.102
50%	50	7.831	5.409	1.063	21.164	17.768	3.140
	100	3.175	1.153	0.472	12.968	6.027	1.673
	200	1.848	0.386	0.319	9.514	2.182	1.334
	500	0.232	0.222	0.057	3.236	1.177	0.481
	1000	0.015	0.131	0.012	0.117	0.849	0.182
	2000	0.004	0.076	−0.001	0.068	0.602	0.117
55%	50	12.750	4.532	1.464	25.865	15.584	3.079
	100	6.782	1.127	0.875	19.126	5.805	2.367
	200	2.509	0.471	0.367	11.689	2.065	1.553
	500	0.500	0.107	0.107	5.130	1.179	0.687
	1000	0.089	0.078	0.047	1.127	0.783	0.320
	2000	0.019	0.015	0.022	0.102	0.560	0.153

To assess the consistency of the application results with the simulation study, Table 3 presents the estimation results using the parameter values obtained from the real application in Section 4.1. As shown in Table 3, the proposed distribution exhibits the same favorable characteristics observed in the simulation study: bias and standard error decrease as the sample size increases. This consistency validates that the estimator's performance under realistic conditions (application) aligns with the theoretical behavior observed in controlled simulation scenarios.

4. Aplication

4.1. Time to Death in Individuals With Colon Cancer

In this section, we present the results of fitting the IELL2 model to the colon cancer survival dataset, which comprises 929 individuals from the *colon* dataset available in the `survival` library (Therneau, 2023). To obtain the data, the 'etype=2' column, referring to patients who experienced the event of interest during follow-up, was filtered.

Based on the results of adjusting the IELL2 model, it was compared with other developments found in the literature, such as Inverse Exponential Logistics (IEL) (Chaudhary & Kumar, 2020) and Marshall-Olkin Exponential Logistics (MOLE) (Mansoor et al., 2019) and their particular cases, Marshall-Olkin Exponential (MOE) and Logistics Exponential (LE).

To compare the different models, some information criterion measures were used, such as the Akaike information criterion given by $\text{AIC} = -2\ell(\widehat{\theta}) + 2p$ and the information criterion Bayesian given by $\text{BIC} = -2\ell(\widehat{\theta}) + p \times \log(n)$ where p and n are, respectively, the number of parameters and sample size.

Figure 3 presents the adjustment for the data set considering the IELL2 distribution and other distributions, in addition to the Kaplan-Meier versus IELL2 quantile plot. Based on this, it can be observed that the IELL2 distribution presents a more adequate fit compared to the other candidate distributions. This indicator is also observed in Table 4, it is noted that the proposed distribution presents the lowest values for the information criteria adopted. However, observing the results of the BIC information criterion, the IELL2 and IEL distributions present very close values. Carrying out a comparative analysis of these distributions through the likelihood ratio test for the hypotheses $\mathcal{H}_0 : \alpha = 1$ versus $\mathcal{H}_1 : \alpha \neq 1$, we have that model IELL2 is significant (p-value: 0.003).

Table 4: AIC and BIC criteria for the different distributions, considering the death time dataset in 929 individuals with colon cancer.

Distribution	AIC	BIC
IELL2	**2253.598**	**2268.100**
MOLE	2267.523	2282.025
LE	2268.521	2278.190
MOE	2289.171	2298.839
IEL	2260.016	2269.684

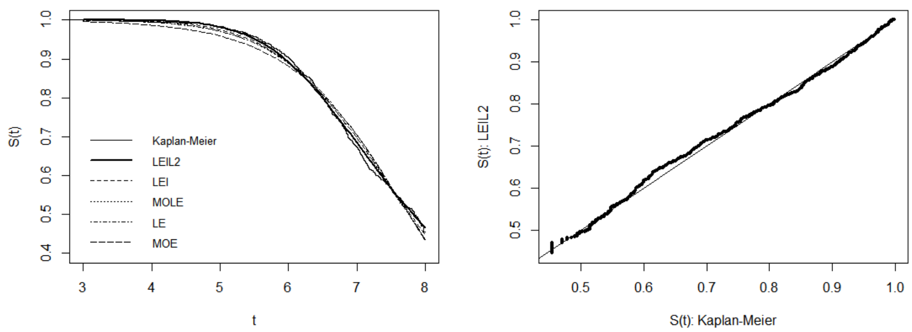

Figure 3: Left: Distribution adjustment and Kaplan-Meier adjustment, considering death time data in individuals with colon cancer. Right: Kaplan-Meier quantile plot versus Lehamann Inverse Exponential Logistics Type 2.

Table 5 presents the estimates and standard error for the parameters of the IELL2 distribution.

Table 5: Estimates and standard error of the maximum likelihood estimators of the IELL2 distribution parameters, considering the death time data set.

Parameters	Estimation	Standard Error
α	0.388	0.007
γ	7.319	0.466
λ	4.614	0.024

4.2. Lifetime of Breast Cancer Patients

In this second application, the IELL2 distribution was fitted to the dataset of lifetimes for 272 individuals with breast cancer. The description and full dataset can be found at `https://data.mendeley.com/datasets/2y5nr64gc2/2`.

The IELL2 distribution was compared with the same distributions considered previously, Inverse Exponential Logistics (IEL) (Chaudhary & Kumar, 2020) and Marshall-Olkin Exponential Logistics (MOLE) (Mansoor et al., 2019) and their particular cases, Marshall-Olkin Exponential (MOE) and Exponential Logistics (LE). The Akaike information criterion and the Bayesian information criterion were used. As can be seen in Table 6, there is evidence in favor of the IELL2 distribution, as it presents the lowest values for the adopted criteria adopted. Since the BIC information criterion again presents close values for the IELL2 and IEL distributions, a likelihood ratio test was carried out for the hypotheses $\mathcal{H}_0 : \alpha = 1$ versus $\mathcal{H}_1 : \alpha \neq 1$, we have that the IELL2 model is significant (p-value: 0.005).

Table 6: AIC and BIC criteria for different distributions, considering the lifetime dataset of breast cancer patients.

Distribution	AIC	BIC
IELL2	**655.98**	**666.79**
MOLE	671.04	681.86
LE	672.79	680.01
MOE	673.90	681.11
LAW	661.54	668.75

Figure 4 shows the adjustments for the distributions considered for the analysis, notice how well the IELL2 distribution fits the data. Table 7 presents the parameter estimates and the standard error of the IELL2 distribution parameters.

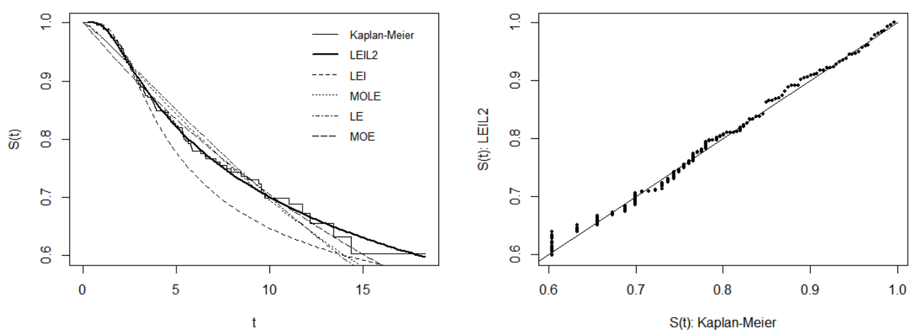

Figure 4: Left: Adjustment of distributions and Kaplan-Meier adjustment, considering lifetime data from breast cancer patients. Right: Kaplan-Meier quantile plot versus Lehamann Inverse Exponential Logistics Type 2.

Table 7: Estimates and standard error of the maximum likelihood estimators of the IELL2 distribution parameters, considering the lifetime dataset of breast cancer patients.

Parameters	Estimation	Standard Error
α	0.452	0.085
γ	0.764	0.196
λ	5.821	22.923

5. Conclusions

In this paper, we propose a new continuous probability distribution supported on the positive real line based on a Lehmann Type II alternative, which we call the Lehmann Type II Inverse Exponential Logistic distribution. The new distribution encompasses the Inverse Exponential and Inverse Exponential Logistic distributions as particular cases. We derive the main statistical properties of the new distribution.

Through simulation studies under right-censoring schemes, we verified that the maximum likelihood estimates performed satisfactorily, with bias and standard error decreasing with increasing sample size across different censoring rates. In applications to real data, we compared the IELL2 distribution and its IEL sub-model with other competing distributions from the literature. The proposed distribution proved to be a competitive alternative, as evidenced by the information criteria and hypothesis testing results.

The flexibility of the Lehmann Type II Inverse Exponential Logistic distribution in accommodating different hazard rate shapes makes it a valuable tool for practitioners in survival analysis. The proposed distribution extends the existing literature on lifetime models and provides an alternative framework for modeling complex survival phenomena in various application areas. Future research could explore Bayesian inference approaches, other censoring mechanisms, and potential extensions to multivariate settings, further broadening the applicability of this new distribution family. For future work, we plan to implement the results obtained in this article in R software, making them available to practitioners.

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