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On the Moment Characteristics for the Univariate Compound Poisson and Bivariate Compound Poisson Processes with Applications

Sobre las características de los momentos de los procesos de Poisson compuestos univariados y bivariados con aplicaciones

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Abstract

The univariate and bivariate compound Poisson process (CPP and BCPP, respectively) ensure a better description than the homogeneous Poisson process for clustering of events. In this paper, new explicit representations of the moment characteristics (general, central, factorial, binomial and ordinary moments, factorial cumulants) and some covariance structures are derived for the CPP and BCPP. Then, the skewness and kurtosis of the univariate CPP are obtained for the first time and special cases of the CPP are studied in detail. Applications to two real data sets are given to illustrate the usage of these processes.

***Key words*:** Bivariate distribution, Compound Poisson process, Cumulant, Factorial moments, Moment.

Resumen

Los procesos univariados y bivariados compuestos de Poisson (CPP y BCPP, por sus siglas en inglés respectivamente) permiten una mejor descripción que los procesos homogéneos de Poisson para agrupamiento de eventos. En este artículo, se muestran específicamente las representaciones de las características de momentos (general, central, factorial, momentos binomiales y ordinarios, acumuladas factoriales) y algunas estructuras de covarianza para los CPP y BCPP. Adicionalmente, el sesgo y la curtosis de los procesos univariados CPP son presentados y casos especiales son estudiados en detalle. La aplicación a dos conjuntos de datos reales es usada con el fin de ilustrar el uso de estos procesos.

***Palabras clave*:** acumuladas factoriales, conjuntas, distribución bivariada, distribución compuesta de Poisson, momento.

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1. Introduction

Let $\{N_t, t \geq 0\}$ be a homogeneous Poisson process with parameter $\lambda > 0$ and let X_i, $i = 1, 2, \ldots$, be identically and independent distributed (i.i.d.) non-negative, integer-valued random variables, independent of $\{N_t, t \geq 0\}$. Then, $\{S_t, t \geq 0\}$ has a univariate CPP if it is defined as

$$S_t = \sum_{i}^{N_t} X_i \tag{1}$$

The univariate CPP has many applications in various areas such as transport, ecology, radiobiology, quality control, telecommunications (see Ata & Özel 2012, Chen, Randolph & Tian-Shy 2005, Gudowska-Nowak, Lee, Nasonova, Ritter & Scholz 2007, Özel & Inal 2008, Robin 2002, Rosychuk, Huston & Prasad 2006). However, the investigation of the properties of the univariate CPP mixtures is much more complicated than the homogeneous Poisson process. The applications of the univariate CPP often run into the obstacle of numerical evaluation of the corresponding probability functions. Hence, moment characteristics of the univariate CPP play a very important role in the probability theory.

Bivariate stochastic processes have also received considerable attention in the literature, in an effort to explain phenomena in various areas of application (see Kocherlakota & Kocherlakota 1997, Özel 2011*a*, Wienke, Ripatti, Palmgren & Yashin 2010, Wienke 2011). Paired count data in time arise in a wide context including marketing (number of purchases of different products), epidemiology (incidents of different diseases in a series of districts), accident analysis (the number of accidents in a site before and after infrastructure changes), medical research (the number of seizures before and after treatment), sports (the number of goals scored by each one of the two opponent teams in soccer) and econometrics (number of voluntary and involuntary job changes). In this study we consider the following BCPP. Let $\{N_t, t \geq 0\}$ be a homogeneous Poisson process and let X_i, Y_i, $i = 1, 2, \ldots$, be independent of the process $\{N_t, t \geq 0\}$ Then the BCPP is defined as

$$\left(S_t^{(1)} = \sum_{i}^{N_t} X_i, S_t^{(2)} = \sum_{i}^{N_t} Y_i \right) \tag{2}$$

where $X_i, Y_i,\ i = 1, 2, \ldots$, are mutually independent random variables.

The CPP is studied in Özel & Inal (2008) but mainly from the evaluation of its probability function. The recursive formulas for the joint probability functions of the BCPP in (2) are derived by Hesselager (1996) and Sundt (1992). Özel & Inal (2008) defined a different kind of BCPP and obtained the joint probability function, moments and cumulants. On the other hand, non-existence of moment characteristics obstacles usage of them in probability theory itself and its applications in seismology, actuarial science, survival analysis, etc. Consequently, since relative results are sparse and case oriented, the aim of this study is to obtain the moment characteristics and covariance structures of the univariate CPP and BCPP.

The paper is organised as follows. In Section 2, moments, cumulants and some relationships are derived for the first time and special cases are obtained for the univariate CPP. In Section 3, new explicit expressions for the moments, cumulants, covariances, and correlation coefficients of the BCPP are derived. In Section 4, the results are illustrated on two real data sets. The conclusion is given in Section 5.

2. The Univariate Compound Poisson Process

2.1. Moments of the Univariate CPP

The moment generating function (mgf) makes it possible to compute general (raw) moments of $\{S_t, t \geq 0\}$. Let X_i, $i = 1, 2, \ldots$, be i.i.d. discrete random variables in (1) with the probabilities $P(X_i = j) = p_j$, $j = 0, 1, \ldots$ The common mgf of X_i, $i = 1, 2, \ldots$, is given by $M_x(u) = \sum_{j=0}^{\infty} p_j u^j = p_0 + p_1 u + p_2 u^2 + \cdots$ and the mgf of $\{S_t, t \geq 0\}$ is given by

$$\begin{aligned} M_{S_t}(u) &= \sum_{n=0}^{\infty} \exp(-\lambda t) \frac{(\lambda t)^n}{n!} [M_x(u)]^n \\ &= \exp(-\lambda t) \left[1 + \frac{\lambda t M_x(u)}{1!} + \frac{[\lambda t M_x(u)]^2}{2!} + \cdots \right] \\ &= \exp(\lambda t \, [M_x(u) - 1]) \end{aligned} \tag{3}$$

Let us assume that the random variable X takes finite values $j = 0, 1, \ldots, m$. Define the parameters $\lambda_j = \lambda p_j$, $j = 0, 1, \ldots, m$, then we have

$$M_{S_t}(u) = \exp\left[-\lambda t (1 - p_0)\right] \exp\left[\lambda_1 t \exp(u) + \ldots + \lambda_m t \exp(u^m)\right] \tag{4}$$

Thus, the rth general moment of the univariate CPP can be obtained by differentiating (4) with respect to u and substituting in $\mu'_r = E\left(S_t^r\right) = \left.\frac{d^r}{du^r} M_{S_t}(u)\right|_{u=0}$, $r = 1, 2, \ldots, n$, after some algebraic manipulations, the general moments of $\{S_t, t \geq 0\}$ are obtained as follows:

$$\begin{aligned} \mu'_1 &= (\lambda t \xi_1) \\ \mu'_2 &= (\lambda t \xi_1)^2 + (\lambda t \xi_2) \\ \mu'_3 &= (\lambda t \xi_1)^3 + 3\,(\lambda t \xi_1)\,(\lambda t \xi_2) + (\lambda t \xi_3) \\ \mu'_4 &= (\lambda t \xi_1)^4 + 6 (\lambda t \xi_1)^2\,(\lambda t \xi_2) + 4\,(\lambda t \xi_3)\,(\lambda t \xi_1) + 3 (\lambda t \xi_2)^2 + (\lambda t \xi_4) \\ \mu'_5 &= (\lambda t \xi_1)^5 + 10 (\lambda t \xi_1)^3\,(\lambda t \xi_2) + 10\,(\lambda t \xi_3)\,(\lambda t \xi_1)^2 + 15\,(\lambda t \xi_1)\,(\lambda t \xi_2)^2 \\ &\quad + 5\,(\lambda t \xi_4)\,(\lambda t \xi_1) + 10\,(\lambda t \xi_2)\,(\lambda t \xi_3) + (\lambda t \xi_5) \end{aligned} \tag{5}$$

where $\xi_r = E(x^r)$, $r = 1, 2, \ldots, n$, is the rth general moment of X_i, $i = 1, 2, \ldots$, and $\{N_t, t \geq 0\}$ is a homogeneous Poisson process with parameter $\lambda > 0$ in (1).

A recursive formula for the factorial moments of $\{S_t, t \geq 0\}$ is derived from (3). For this aim, we observe that

$$\frac{dM_{S_t}(u)}{du} = \lambda t M_{S_t}(u) \frac{dM_X(u)}{du}$$

so that applying the Leibniz differentiation rule for $r \geq 1$ we obtain

$$\begin{aligned} \mu'_r &= \lambda t \, \frac{d^{r-1}}{du^{r-1}} \left[\exp\left[\lambda t (M_x(u) - 1)\right] \frac{dM_x(u)}{du} \right] \Bigg|_{u=0} \\ &= \lambda t \sum_{k=0}^{r-1} \binom{r-1}{k} \frac{d^k M_{S_t}(u)}{du^k} \frac{d^{r-k} M_X(u)}{du^{r-k}} \Bigg|_{u=0} \end{aligned}$$

Then, the following recursive formula for the general moments of $\{S_t, t \geq 0\}$ is given by

$$\mu'_r = \lambda t \sum_{k=0}^{r-1} \binom{r-1}{k} \mu'_k \; \xi_{r-k}$$

where ξ_r, $r = 1, 2, \ldots, n$, is the rth general moment of X_i, $i = 1, 2, \ldots$, in (1).

Now consider the central moments μ_r of $\{S_t, t \geq 0\}$. The generating function $G_{S_t}(u)$ of μ_r, if the rth central moment exists, is defined by the relation

$$G_{S_t}(u) = E\left[\exp(u(S_t - \mu))\right] = \exp(-u\mu) M_{S_t}(u) \tag{6}$$

where $\mu'_r = \mu = E\left(S_t\right) = \lambda t \xi_1$. Then, rth central moment of $\{S_t, t \geq 0\}$ can be obtained by

$$\mu_r = E(S_t - \mu)^r = \frac{d^r}{du^r} G_{S_t}(0) = \frac{d^r}{du^r} \exp(-u\mu) M_{S_t}(u) \Bigg|_{u=0} \tag{7}$$

From (4) and (7), we have

$$\begin{aligned} \mu_1 &= (\mu + \lambda t \xi_1) \\ \mu_2 &= (\mu + \lambda t \xi_1)^2 + (\lambda t \xi_2) \\ \mu_3 &= (\mu + \lambda t \xi_1)^3 + 3\left(\mu + \lambda t \xi_1\right)\left(\lambda t \xi_2\right) + \left(\lambda t \xi_3\right) \\ \mu_4 &= (\mu + \lambda t \xi_1)^4 + 6(\mu + \lambda t \xi_1)^2 \left(\lambda t \xi_2\right) + 4\left(\mu + \lambda t \xi_1\right)\left(\lambda t \xi_3\right) \\ &\qquad + 3(\lambda t \xi_2)^2 + (\lambda t \xi_4) \\ \mu_5 &= (\mu + \lambda t \xi_1)^5 + 10(\mu + \lambda t \xi_1)^3 \left(\lambda t \xi_2\right) + 10(\mu + \lambda t \xi_1)^2 \left(\lambda t \xi_3\right) \\ &\quad + 15\left(\mu + \lambda t \xi_1\right)\left(\lambda t \xi_2\right)^2 + 5\left(\mu + \lambda t \xi_1\right)\left(\lambda t \xi_4\right) + 10\left(\lambda t \xi_2\right)\left(\lambda t \xi_3\right) + \left(\lambda t \xi_5\right) \end{aligned} \tag{8}$$

where ξ_r, $r = 1, 2, \ldots, n$, is the rth general moment of X_i, $i = 1, 2, \ldots$

Commonly used indices of the shape of a distribution are the moment ratios such as skewness and kurtosis. Since $\{S_t, t \geq 0\}$ has finite moments of orders up to the third, then the skewness of S_t is defined as

$$\sqrt{\beta_1} = E\left(\frac{S_t - \mu}{\sigma}\right)^3 = \frac{\mu_3}{\mu_2^{3/2}} \tag{9}$$

where σ is the standard deviation of S_t. From (9), the skewness of $\{S_t, t \geq 0\}$ is obtained using the central moments in (8) as follows:

$$\sqrt{\beta_1} = \frac{(\mu + \lambda t\xi_1)^3 + 3\,(\mu + \lambda t\xi_1)\,(\lambda t\xi_2) + (\lambda t\xi_3)}{\left[(\mu + \lambda t\xi_1)^2 + (\lambda t\xi_2)\right]^{3/2}} \tag{10}$$

Similarly, the kurtosis of $\{S_t, t \geq 0\}$ is obtained from (8) as

$$\begin{aligned}\beta_2 &= E\left[\frac{S_t - \mu}{\sigma}\right]^4 - 3 = \frac{\mu_4}{\mu_2^2} - 3 \\ &= \frac{4\,(\mu + \lambda t\xi_1)\,(\lambda t\xi_3) - 2(\mu + \lambda t\xi_1)^4 + (\lambda t\xi_4)}{\left[(\mu + \lambda t\xi_1)^2 + (\lambda t\xi_2)\right]^2}\end{aligned} \tag{11}$$

Since $M_{S_t}(u)$ is exponential form in (4), it is useful to consider the cumulants (semi invariants) κ_r, defined formally as the coefficients of the Taylor expansion of the logarithm of the characteristic function $\varphi_{S_t}(u)$ and having the cumulant generating function

$$C_{S_t}(u) = \ln \varphi_{S_t}(u) = \sum_{r=1}^{\infty} \kappa_r \frac{(iu)^r}{r!} \tag{12}$$

where i denotes the imaginary number ($i^2 = -1$) and the characteristic function of $\{S_t, t \geq 0\}$ is given by $\varphi_{S_t}(u) = \exp\left[\lambda t(\varphi_X(u) - 1)\right]$. Here, $\varphi_X(u)$ the common characteristic function of X_i, $i = 1, 2, \ldots$ Then, if X takes finitely many values $j = 0, 1, \ldots, m$, we get

$$\begin{aligned}C_{S_t}(u) &= \lambda t\left[\varphi_X(u) - 1\right] \\ &= \lambda t\left[p_0 + p_1 \exp(iu) + p_2 \exp(2iu) + \cdots + p_m \exp(miu)\right] - \lambda t \\ &= \lambda t\left[(p_0 - 1) + p_1 \exp(iu) + p_2 \exp(2iu) + \cdots + p_m \exp(miu))\right]\end{aligned} \tag{13}$$

Using Taylor series expansion, we obtain a cumulant generating function from (13) as

$$\begin{aligned}C_{S_t}(u) = \lambda t\Bigg[p_1\left(\frac{(iu)}{1!} + \cdots\right) + p_2\left(\frac{(2iu)}{1!} + \cdots\right) + \cdots \\ + p_m\left(\frac{(miu)}{1!} + \cdots\right)\Bigg]\end{aligned}$$

$$
\begin{aligned}
&= \left[\frac{(iu)}{1!} \{\lambda t(p_1 + \cdots)\} + \frac{(iu)^2}{2!} \{\lambda t(p_1 + \cdots)\} + \cdots \right. \\
&\qquad\qquad \left. + \frac{(iu)^m}{m!} \{\lambda t(p_1 + \cdots)\} \right] \\
&= \sum_{r=1}^{\infty} \left(\lambda t \sum_{j=1}^{\infty} r^j p_j \right) \frac{(iu)^r}{r!} \\
&= \sum_{r=1}^{\infty} \lambda t E(X^r) \frac{(iu)^r}{r!}
\end{aligned} \tag{14}
$$

Then, for every $r = 1, 2, \dots, n$ we have

$$\kappa_r = \lambda t \xi_r \tag{15}$$

Here, $\xi_r = E(X^r)$, $r = 1, 2, \dots n$, is the rth general moment of X_i, $i = 1, 2, \dots$ We also obtain a relationship between the general moments and the ordinary cumulants of $\{S_t, t \geq 0\}$ as follows:

$$
\begin{aligned}
\mu'_1 &= \kappa_1 \\
\mu'_2 &= \kappa_1^2 + \kappa_2 \\
\mu'_3 &= \kappa_1^3 + 3\kappa_1\kappa_2 + \kappa_3 \\
\mu'_4 &= \kappa_1^4 + 6\kappa_1^2\kappa_2 + 4\kappa_1\kappa_3 + 3\kappa_2^2 + \kappa_4 \\
\mu'_5 &= \kappa_1^5 + 10\kappa_1^3\kappa_2 + 10\kappa_1^2\kappa_3 + 15\kappa_1\kappa_2^2 + 5\kappa_1\kappa_4 + 10\kappa_2\kappa_3 + \kappa_5
\end{aligned} \tag{16}
$$

In problems with discrete random variables one often uses the factorial moments. Let $\mu_{[r]}$ be the rth factorial moment of $\{S_t, t \geq 0\}$ in (1). $\mu_{[r]}$ can be obtained by inverting the factorial moment generating function (fmgf) of $\{S_t, t \geq 0\}$

$$\mu_{[r]} = \left. \frac{d^r}{du^r} P_{S_t}(1+u) \right|_{u=0} \tag{17}$$

where fmgf of $\{S_t, t \geq 0\}$ is

$$
\begin{aligned}
P_{S_t}(1+u) = \exp[-\lambda t(1-p_0)] \exp[\lambda_1 t \exp(1+u) + \lambda_2 t \exp((1+u)^2) \\
+ \cdots + \lambda_m t \exp((1+u)^m)]
\end{aligned} \tag{18}
$$

Here, the random variable X has finite values $j = 0, 1, \dots m$. Differentiating (18) and substituting in (17), after some manipulations, we obtain the factorial moments as follows:

$$\begin{aligned}
\mu_{[1]} &= (\lambda t\xi_{[1]}) \\
\mu_{[2]} &= (\lambda t\xi_{[1]})^2 + (\lambda t\xi_{[2]}) \\
\mu_{[3]} &= (\lambda t\xi_{[1]})^3 + 3(\lambda t\xi_{[1]})(\lambda t\xi_{[2]}) + (\lambda t\xi_{[3]}) \\
\mu_{[4]} &= (\lambda t\xi_{[1]})^4 + 6(\lambda t\xi_{[1]})^2(\lambda t\xi_{[2]}) + 4(\lambda t\xi_{[3]})(\lambda t\xi_{[1]}) + 3(\lambda t\xi_{[2]})^2 + (\lambda t\xi_{[4]}) \\
\mu_{[5]} &= (\lambda t\xi_{[1]})^5 + 10(\lambda t\xi_{[1]})^3(\lambda t\xi_{[2]}) + 10(\lambda t\xi_{[3]})(\lambda t\xi_{[1]})^2 + 15(\lambda t\xi_{[1]})(\lambda t\xi_{[2]})^2 \\
&\quad + 5(\lambda t\xi_{[4]})(\lambda t\xi_{[1]}) + 10(\lambda t\xi_{[2]})(\lambda t\xi_{[3]}) + (\lambda t\xi_{[5]})
\end{aligned} \tag{19}$$

where $\xi_{[r]} = E[(X)(X-1)\cdots(X-(r-1))]$, $r = 1, 2, \ldots, n$, is the rth factorial moment of X_i, $i = 1, 2, \ldots$ If $E(X^r) < \infty$, factorial moments of $\{S_t, t \geq 0\}$ can also be calculated recursively. We observe that

$$\frac{dg_{S_t}(u)}{du} = \lambda t \exp[\lambda t(g_X(u) - 1)] \frac{dg_X(u)}{du} = \lambda t g_{S_t}(u) \frac{dg_X(u)}{du}$$

Now using the Leibniz formula for the derivatives of higher orders, we get

$$\frac{d^r g_{S_t}(u)}{du^r} = \lambda t \sum_{k=0}^{r-1} \binom{r-1}{k} \frac{d^{r-k-1} g_{S_t}(u)}{du^{r-k-1}} \frac{d^{k+1} g_X(u)}{du^{k+1}} \tag{20}$$

From (21) and the relations $\mu_{[r]} = \left.\frac{d^r g_{S_t}(u)}{du^r}\right|_{u=1}$, $\xi_r = \left.\frac{d^r g_X(u)}{du^r}\right|_{u=1}$, we have

$$\mu_{[r]} = \lambda t \sum_{k=0}^{r-1} \binom{r-1}{k} \mu_{[r-k-1]} \xi_{[k+1]} \tag{21}$$

The logarithm of the fmgf is called factorial cumulant generating function (fcgf). The coefficient of $u^r/r!$ in the Taylor expansion of this function is the rth factorial cumulant $\kappa_{[r]}$. The fcgf is given by $\ln P(1+u) = \sum_{r=1}^{\infty} \frac{\kappa_{[r]} u^r}{r!}$ where $\kappa_{[r]}$ denotes the rth factorial cumulant. Then, the factorial cumulants of $\{S_t, t \geq 0\}$ are given by

$$\kappa_{[r]} = \lambda t \xi_{[r]} \tag{22}$$

where $\xi_{[r]} = E[(X)(X-1)\ldots(X-(r-1))]$, $r = 1, 2, \ldots, n$, is the rth factorial moment of X_i, $i = 1, 2, \ldots$.

Let us point out that the factorial cumulants are related to the ordinary cumulants in the same way as the factorial moments are related to the general moments for $\{S_t, t \geq 0\}$. A relationship of the factorial cumulants with the ordinary cumulants is also obtained for $\{S_t, t \geq 0\}$ as follows:

$$\begin{aligned}
\kappa_{[1]} &= \kappa_1 \\
\kappa_{[2]} &= \kappa_2 - \kappa_1 \\
\kappa_{[3]} &= \kappa_3 - 3\kappa_2 + 2\kappa_1 \\
\kappa_{[4]} &= \kappa_4 - 6\kappa_3 + 11\kappa_2 - 6\kappa_1 \\
\kappa_{[5]} &= \kappa_5 - 10\kappa_4 + 35\kappa_3 - 50\kappa_2 + 24\kappa_1
\end{aligned} \tag{23}$$

The binomial moments, closely connected with $\mu_{[r]}$, are defined as $B_r = E\binom{S_t}{r} = \frac{1}{r!}\mu_{[r]}$. The binomial moment generating function is $B_{S_t}(u) = \sum_{j=0} B_j u^j = \sum_{j=0} \mu_{[r]} \frac{u^j}{j!} = P_{S_t}(1+u)$, so that, if $M_{S_t}(u)$ exists, then

$$B_r = \frac{1}{r!}\frac{d^r}{du^r}P_{S_t}(1+u)\bigg|_{u=0}$$

Hence, rth binomial moments of $\{S_t, t \geq 0\}$ is obtained as follows:

$$\begin{aligned}
B_1 &= \frac{(\lambda t\xi_{[1]})}{1!}, \\
B_2 &= \frac{\left[(\lambda t\xi_{[1]})^2 + (\lambda t\xi_{[2]})\right]}{2!} \\
B_3 &= \frac{\left[(\lambda t\xi_{[1]})^3 + 3(\lambda t\xi_{[1]})(\lambda t\xi_{[2]}) + (\lambda t\xi_{[3]})\right]}{3!} \\
B_4 &= \frac{\left[(\lambda t\xi_{[1]})^4 + 6(\lambda t\xi_{[1]})^2(\lambda t\xi_{[2]}) + 4(\lambda t\xi_{[3]})(\lambda t\xi_{[1]}) + 3(\lambda t\xi_{[2]})^2 + (\lambda t\xi_{[4]})\right]}{4!} \\
B_5 &= \frac{\left[(\lambda t\xi_{[1]})^5 + 10(\lambda t\xi_{[1]})^3(\lambda t\xi_{[2]}) + 10(\lambda t\xi_{[3]})(\lambda t\xi_{[1]})^2 + 15(\lambda t\xi_{[1]})(\lambda t\xi_{[2]})^2\right]}{5!} \\
&+ \frac{\left[5(\lambda t\xi_{[4]})(\lambda t\xi_{[1]}) + 10(\lambda t\xi_{[2]})(\lambda t\xi_{[3]}) + (\lambda t\xi_{[5]})\right]}{5!}
\end{aligned} \tag{24}$$

where $\xi_{[r]} = E[(X)(X-1)\dots(X-(r-1))]$, $r = 1, 2, \dots, n$, is the rth factorial moment of X_i, $i = 1, 2, \dots$.

2.2. The Covariance Structure of CPP

In this section, we derived the covariance between $\{N_t, t \geq 0\}$ and $\{S_t, t \geq 0\}$ (1) for the case that $\{N_t, t \geq 0\}$ is a homogeneous Poisson process with parameter $\lambda > 0$ and X_i, $i = 1, 2, \dots$. are discrete random variables with finite values $j = 0, 1, \dots$ The characteristic function of the random vector N_t, S_t is defined as

$$\varphi_{N_t,S_t}(u,v) = E\left[\exp(iuN_t + ivS_t)\right] = E\left[E(\exp(iuN_t + ivS_t)|\, N_t)\right] \tag{25}$$

where i is the imaginary number. (25) can be written as

$$\begin{aligned}
\varphi_{N_t,S_t}(u,v) &= E\left[E(\exp(iuN_t + iv\sum_{i=1}^{N_t} X_i\ )|\, N_t = n)\right] \\
&= E\left[\exp(iuN_t)\prod_{i=1}^{n}\exp(ivX_i)\right] \\
&= E\left[\exp(iuN_t)\left(\prod_{i=1}^{n}\varphi_X(v)\right)\right] \\
&= E\left[\exp(iuN_t)\varphi_X^n(v)\right]
\end{aligned} \tag{26}$$

where $\varphi_X(v)$ is the characteristic function of X_i, $i = 1, 2, \dots$ Since $\{N_t, t \geq 0\}$ has a homogeneous Poisson process with parameter λ, we get

$$\begin{aligned} \varphi_{N_t,S_t}(u,v) &= \exp(-\lambda t) \sum_{k=0}^{\infty} \frac{[\lambda t \varphi_X(v) \exp(iu)]^k}{k!} \\ &= \exp[\lambda t(\varphi_X(v)\exp(iu) - 1)] \end{aligned} \tag{27}$$

To derive the covariance, we have $E(N_t) = \lambda t$ and $E(S_t) = \lambda t E(X)$. In order to complete the derivation of the covariance of N_t and S_t, we need to evaluate $E(N_t S_t) = \left.\frac{\partial^2 \varphi_{N_t,S_t}(u,v)}{\partial u \partial v}\right|_{u=v=0}$. The derivative of $\varphi_{N_t,S_t}(u,v)$ with respect to u is $\frac{\partial \varphi_{N_t,S_t}(u,v)}{\partial u} = i\lambda t \varphi_X(u) \varphi_{N_t,S_t}(u,v)$ and the derivative of the latter with respect to v is

$$\begin{aligned} \frac{\partial^2 \varphi_{N_t,S_t}(u,v)}{\partial u \partial v} &= i\lambda t \varphi_X(v) \lambda t \exp(iu) \varphi'_X(v) \varphi_{N_t,S_t}(u,v) + i\lambda t \varphi'_X(v) \varphi_{N_t,S_t}(u,v) \\ &= i\lambda t \varphi_{N_t,S_t}(u,v) \varphi'_X(v) \left[\varphi_X(v) \lambda t \exp(iu) + 1\right] \end{aligned}$$

Since $\varphi_{N_t,S_t}(0,0) = \varphi_X(0) = 1$ and $\varphi'_X(0) = iE(X)$, it follows that

$$\left.\frac{\partial^2 \varphi_{N_t,S_t}(u,v)}{\partial u \partial v}\right|_{u=v=0} = i^2 \lambda t(\lambda t + 1) E(X)$$

Therefore, $E(N_t, S_t) = \lambda t(\lambda t + 1)E(X)$ and the covariance of $\{N_t, t \geq 0\}$ and $\{S_t, t \geq 0\}$ is given by

$$Cov(N_t, S_t) = \lambda t E(X) = \lambda t \xi_1 \tag{28}$$

Hence, the coefficient of correlation is

$$\rho = Corr(N_t, S_t) = \frac{Cov(N_t, S_t)}{\sqrt{Var(N_t) Var(S_t)}} = \frac{\xi_1}{\sqrt{\xi_2}} \tag{29}$$

where $Var(N_t) = \lambda t$ and $Var(S_t) = \lambda t E(X^2) = \lambda t \xi_2$.

2.3. Special Cases of the Univariate CPP

In this section we study some special cases of the univariate CPP. Expressions for various moments and cumulants are presented. The Neyman type A, B and Pólya-Aeppli are four major CPPs. The Neyman type A and B processes are defined by Neyman (1939) as 'contagious'. This definition implies that each favourable event enhances the probability of each succeding event. The Pólya-Aeppli process is derived by Getis (1974) to model the clustered point process. Note that some examples of such processes with their corresponding probability functions are discussed in Özel & Inal (2012).

Example 1. The Neyman Type A Process: Let $\{N_t, t \geq 0\}$ be a homogeneous Poisson process with parameter $\lambda > 0$ and let X_i, $i = 1, 2, \dots$ be Poisson distributed with parameter ν in (1), then $\{S_t, t \geq 0\}$ is called a Neyman type A or Poisson-Poisson process. First four moments and cumulants of the Neyman type A process are given in Table 1 .

Table 1: First four moments and cumulants of the Neyman type A process.

μ_1'	(λtv)
μ_2'	$(\lambda tv)^2 + [\lambda t(v + v^2)]$
μ_3'	$(\lambda tv)^3 + 3(\lambda tv)[\lambda t(v + v^2)] + [\lambda t(v + 3v^2 + v^3)]$
μ_4'	$(\lambda tv)^4 + 6(\lambda tv)^2[\lambda t(v + v^2)] + 4(\lambda tv)[\lambda t(v + 3v^2 + v^3)] + [\lambda t(v + 7v^2 + 6v^3 + v^4)]$
μ_1	$(2\lambda tv)$
μ_2	$(2\lambda tv)^2 + [\lambda t(v + v^2)]$
μ_3	$(2\lambda tv)^3 + 6(\lambda tv)[\lambda t(v + v^2)] + [\lambda t(v + 3v^2 + v^3)]$
μ_4	$(2\lambda tv)^4 + 6(2\lambda tv)^2[\lambda t(v + v^2)] + 4(2\lambda tv)[\lambda t(v + 3v^2 + v^3)] + [\lambda t(v + 7v^2 + 6v^3 + v^4)]$
κ_1	(λtv)
κ_2	$[\lambda t(v + v^2)]$
κ_3	$[\lambda t(v + 3v^2 + v^3)]$
κ_4	$[\lambda t(v + 7v^2 + 6v^3 + v^4)]$
$\mu_{[1]}$	(λtv)
$\mu_{[2]}$	$(\lambda tv)^2 + (\lambda tv)$
$\mu_{[3]}$	$(\lambda tv)^3 + 3(\lambda tv)^2 + (\lambda tv)$
$\mu_{[4]}$	$(\lambda tv)^4 + 6(\lambda tv)^3 + 7(\lambda tv)^2 + (\lambda tv)$
$\kappa_{[1]}$	(λtv)
$\kappa_{[2]}$	(λtv)
$\kappa_{[3]}$	(λtv)
$\kappa_{[4]}$	(λtv)
B_1	(λtv)
B_2	$[(\lambda tv)^2 + (\lambda tv)]/2!$
B_3	$[(\lambda tv)^3 + 3(\lambda tv)^2 + (\lambda tv)]/3!$
B_4	$[(\lambda tv)^4 + 6(\lambda tv)^3 + 7(\lambda tv)^2 + (\lambda tv)]/4!$

Example 2. The Neyman Type B Process: Let $\{N_t, t \geq 0\}$ be a homogeneous Poisson process with parameter $\lambda > 0$ and let X_i, $i = 1, 2, \dots$ be binomial distributed with parameters m and p in (1), then $\{S_t, t \geq 0\}$ has a Neyman type B or Poisson-binomial process. First four moments and cumulants of the Neyman type B process are presented in Table 2.

Example 3. The Pólya-Aeppli Process: Let $\{N_t, t \geq 0\}$ be a homogeneous Poisson process with parameter $\lambda > 0$ and let X_i, $i = 1, 2, \dots$ be geometric distributed random variables with parameter θ. Then, $\{S_t, t \geq 0\}$ has a Pólya-Aeppli or geometric Poisson process. First four moments and cumulants of the Pólya-Aeppli process are given in Table 3.

TABLE 2: First four moments and cumulants of the Neyman type B process.

μ_1'	(λtmp)
μ_2'	$(\lambda tmp)^2 + \lambda t[mp + m(m-1)p^2]$
μ_3'	$(\lambda tmp)^3 + 3(\lambda tmp)^2 + (\lambda tmp) + 6[\lambda tmp(m(m-1)p^2)] + [\lambda t(m(m-1)(m-2)p^3)]$
	$(\lambda tmp)^4 + 6(\lambda tmp)^2[\lambda t(mp + (m-1)p^2)] + 4(\lambda tmp)[\lambda t(mp + 3m(m-1)p^2$
	$+m(m-1)(m-2)p^3)]$
μ_4'	$(\lambda tv)^4 + 6(\lambda tv)^2[\lambda t(v+v^2)] + 4(\lambda tv)[\lambda t(v+3v^2+v^3)] + [\lambda t(v+7v^2+6v^3+v^4)]$
μ_1	$(2\lambda tmp)$
μ_2	$(2\lambda tmp)^2 + [\lambda t(mp + m(m-1)p^2)]$
μ_3	$(2\lambda tmp)^3 + 3(\lambda tmp)[\lambda t(mp + m(m-1)p^2)] + [\lambda t(mp + 3m(m-1)p^2$
	$+m(m-1)(m-2)p^3)]$
μ_4	$(2\lambda tmp)^4 + 6(2\lambda tmp)^2[\lambda t(mp + m(m-1)p^2)] + 4(2\lambda tmp)[\lambda t(mp + 3m(m-1)p^2$
	$+m(m-1)(m-2)p^3)] + 3[\lambda t(mp + m(m-1)p^2)]^2 + [\lambda t(mp + 7m(m-1)p^2$
	$+6m(m-1)(m-2)p^3 + m(m-1)(m-2)(m-3)p^4)]$
κ_1	(λtmp)
κ_2	$[\lambda t(mp + m(m-1)p^2)]$
κ_3	$[\lambda t(mp + 3m(m-1)p^2 + m(m-1)(m-2)p^3)]$
κ_4	$[\lambda t(mp + 7m(m-1)p^2 + 6m(m-1)(m-2)p^3 + m(m-1)(m-2)(m-3)(m-4)p^4)]$
$\mu_{[1]}$	(λtmp)
$\mu_{[2]}$	$(\lambda tmp)^2 + (\lambda tm(m-1)p^2)$
$\mu_{[3]}$	$(\lambda tmp)^3 + 3(\lambda tmp(mp + m(m-1)p^2))$
	$+(\lambda t(mp + 3m(m-1)p^2 + m(m-1)(m-2)p^3)$
$\mu_{[4]}$	$(\lambda tmp)^4 + 6(\lambda tmp)^3[\lambda t(mp + m(m-1)p^2)] + 4(\lambda tmp)[\lambda t(mp + 3m(m-1)p^2$
	$+m(m-1)(m-2)p^3)] + 3[\lambda t(mp + m(m-1)p^2)]^2 + [\lambda t(mp + 7m(m-1)p^2$
	$+6m(m-1)(m-2)p^3 + m(m-1)(m-2)(m-3)p^4)]$
$\kappa_{[1]}$	(λtmp)
$\kappa_{[2]}$	$[\lambda tm(m-1)p^2]$
$\kappa_{[3]}$	$[\lambda tm(m-1)(m-2)p^3]$
$\kappa_{[4]}$	$[\lambda tm(m-1)(m-2)(m-3)p^4]$
B_1	(λtmp)
B_2	$[(\lambda tmp)^2 + (\lambda tm(m-1)p^2)]/2!$
B_3	$[(\lambda tmp)^3 + 3(\lambda tmp)[\lambda tm(m-1)p^2] + [\lambda tm(m-1)(m-2)p^3]]/3!$
B_4	$[(\lambda tmp)^4 + 6(\lambda tmp)^2[\lambda tm(m-1)p^2] + 4(\lambda tmp)[\lambda tm(m-1)(m-2)p^3]$
	$+3[\lambda tm(m-1)p^2]^2 + [\lambda tm(m-1)(m-2)(m-3)p^4]]/4!$

Note that the random variable X has infinite values both the Neyman type A and the Pólya-Aeppli process. However, the moments and cumulants of these processes can be obtained using (13), (18) and (31). This is due to the probability $P(X_i = j)$ and $\lambda_j = \lambda p_j$ approach zero for $j \to \infty$.

Table 3: First four moments and cumulants of the Pólya-Aeppli process.

μ_1'	$[\lambda t(1-\theta)/\theta]$
μ_2'	$[\lambda t(1-\theta)/\theta]^2+[\lambda t(1-\theta)(2-\theta)/\theta^2]$
μ_3'	$[\lambda t(1-\theta)/\theta]^3+3[\lambda t(1-\theta)/\theta][\lambda t(1-\theta)(2-\theta)/\theta^2]+[\lambda t(1-\theta)(6+\theta(\theta-6))/\theta^3]$
μ_4'	$[\lambda t(1-\theta)/\theta]^4+6[\lambda t(1-\theta)/\theta]^2[\lambda t(1-\theta)(2-\theta)/\theta^2]+4[\lambda t(1-\theta)/\theta][\lambda t(1-\theta)$ $(6+\theta(\theta-6))/\theta^3]+3[\lambda t(1-\theta)(2-\theta)/\theta^2]^2+[\lambda t(2-\theta)(1-\theta)(12+(\theta-12)\theta)/\theta^4]$
μ_1	$[2\lambda t(1-\theta)/\theta]$
μ_2	$[2\lambda t(1-\theta)/\theta]^2+[\lambda t(1-\theta)(2-\theta)/\theta^2)]$
μ_3	$[2\lambda t(1-\theta)/\theta]^3+3[2\lambda t(1-\theta)/\theta][\lambda t(1-\theta)(2-\theta)/\theta^2]+[\lambda t(1-\theta)(6+\theta(\theta-6))/\theta^3]$
μ_4	$[2\lambda t(1-\theta)/\theta]^4+6[2\lambda t(1-\theta)/\theta]^2[\lambda t(1-\theta)(2-\theta)/\theta^2]+4[2\lambda t(1-\theta)/\theta][\lambda t(1-\theta)(6+$ $\theta(\theta-6))/\theta^3]+3[\lambda t(1-\theta)(2-\theta)/\theta^2]^2+[\lambda t(2-\theta)(1-\theta)(12+(\theta-12)\theta)/\theta^4]$
κ_1	$[\lambda t(1-\theta)/\theta]$
κ_2	$[\lambda t(2-\theta)(1-\theta)/\theta^2]$
κ_3	$[\lambda t(1-\theta)(6+\theta(\theta-6))/\theta^3]$
κ_4	$[\lambda t(1-\theta)(2-\theta)(12+\theta(\theta-12))/\theta^4]$
$\mu_{[1]}$	$[\lambda t(1-\theta)/\theta]$
$\mu_{[2]}$	$[\lambda t(1-\theta)/\theta]^2+[\lambda t(2-\theta)(1-\theta)/\theta^2]$
$\mu_{[3]}$	$[\lambda t(1-\theta)/\theta]^3+3[\lambda t(2-\theta)(1-\theta)/\theta^2][\lambda t(1-\theta)/\theta]+[\lambda t(1-\theta)(6+(\theta-6)\theta)/\theta^3]$
$\mu_{[4]}$	$[\lambda t(1-\theta)/\theta]^4+6[\lambda t(1-\theta)/\theta]^2[\lambda t(2-\theta)(1-\theta)/\theta^2]+4[\lambda t(1-\theta)/\theta)(\lambda t(1-\theta)$ $(6+(\theta-6)\theta)/\theta^3]+3[\lambda t(2-\theta)(1-\theta)/\theta^2]^2+[\lambda t(2-\theta)(1-\theta)(12+(\theta-12)\theta)/\theta^4]$
$\kappa_{[1]}$	$[\lambda t(1-\theta)/\theta]$
$\kappa_{[2]}$	$2[\lambda t(1-\theta)/\theta]^2$
$\kappa_{[3]}$	$6[\lambda t(1-\theta)/\theta]^3$
$\kappa_{[4]}$	$24[\lambda t(1-\theta)/\theta]^4$
B_1	$[\lambda t(1-\theta)/\theta]$
B_2	$[(\lambda t(1-\theta)/\theta)^2+2(\lambda t(1-\theta)/\theta)^2]/2!$
B_3	$[(\lambda t(1-\theta)/\theta)^3+6(\lambda t(1-\theta)/\theta)(\lambda t(1-\theta)/\theta)^2+6(\lambda t(1-\theta)/\theta)^3]/3!$
B_4	$[(\lambda t(1-\theta)/\theta)^4+12(\lambda t(1-\theta)/\theta)^2(\lambda t(1-\theta)/\theta)^2+24(\lambda t(1-\theta)/\theta)(\lambda t(1-\theta)/\theta)^3$ $+12(\lambda t(1-\theta)/\theta)^2+24(\lambda t(1-\theta)/\theta)^4]/4!$

3. The Bivariate Compound Poisson Process

In this section, we turn now to the consideration of factorial moments, cumulants, and the coefficient of correlation for the BCPP. Let $\{N_t, t\geq 0\}$ be a homogeneous Poisson process with parameter $\lambda>0$ and let X_i, Y_i, $i=1,2,\ldots$, be mutually i.i.d. discrete random variables taking finite values with the probabilities $P(X_i=j)=p_j$, $j=0,1,\ldots m$ and $P(Y_i=k)=q_k$, $k=0,1,\ldots \ell$ in (2). We start by finding factorial moments $\mu_{[r,s]}$ for $r=1,2,\ldots$, $s=1,2,\ldots$ For this purpose, we first compute the joint probability generating function (pgf) $S_t^{(1)}$ and $S_t^{(2)}$ as follows

$$\begin{aligned} g_{S_t^{(1)},S_t^{(2)}}(u_1,u_2) &= \sum_{s_1=0}^{\infty}\sum_{s_2=0}^{\infty} P\left(\sum_i^{N_t} X_i=s_1, \sum_i^{N_t} Y_i=s_2\right) u_1^{s_1}u_2^{s_2} \\ &= \sum_{s_1=0}^{\infty}\sum_{s_2=0}^{\infty}\sum_{n=0}^{\infty} P\left(\sum_i^{n} X_i=s_1, \sum_i^{n} Y_i=s_2\right) P_{N_t}(n) u_1^{s_1}u_2^{s_2} \end{aligned}$$

where $P_{N_t}(n)=P(N_t=n)$, $n=0,1,\ldots$, is the probability function of the homogeneous Poisson process. Since X_i, Y_i, $i=1,2,\ldots$ are i.i.d. random variables, we get

$$\begin{aligned} g_{S_t^{(1)},S_t^{(2)}}(u_1,u_2) &= p_{N_t}(0) + p_{N_t}(1) \\ &\qquad \sum_{s_1=0}^{\infty}\sum_{s_2=0}^{\infty} P(X_1=s_1)P(Y_1=s_2)u_1^{s_1}u_2^{s_2} + \cdots \\ &= p_{N_t}(0) + p_{N_t}(1)g_{X_1}(u_1)g_{Y_1}(u_2) \\ &\qquad + p_{N_t}(2)g_{X_1+X_2}(u_1)g_{Y_1+Y_2}(u_2) + \cdots \\ &= p_{N_t}(0) + p_{N_t}(1)g_X(u_1)g_Y(u_2) \\ &\qquad + p_{N_t}(2)[g_X(u_1)]^2[g_Y(u_2)]^2 + \cdots \end{aligned} \tag{30}$$

where $g_X(u_1)$, $g_Y(u_2)$ are the common pgfs of X_i, Y_i, $i = 1, 2, \ldots$, respectively. Using (30), it is more convenient to deal with

$$\begin{aligned} g_{S_t^{(1)},S_t^{(2)}}(u_1,u_2) &= g_{N_t}[g_X(u_1)g_Y(u_2)] \\ &= \exp[\lambda t[g_X(u_1)g_Y(u_2) - 1]] \\ &= \exp(-\lambda t)\exp[\lambda t(p_0q_0 + p_0q_1u_2 + \cdots + p_0q_lu_2^l + p_1q_0u_1 \\ &\quad + p_1q_1u_1u_2 + \cdots + p_1q_lu_1u_2^l + p_mq_0u_1^m + \cdots + p_mq_lu_1^mu_2^l)] \end{aligned} \tag{31}$$

The joint pgf in (31) can be differentiated any number of times with respect to r and s and evaluated at $(0, 0)$ yielding

$$\mu_{[r,s]} = \left.\frac{\partial^{r+s} g_{S_t^{(1)},S_t^{(2)}}(u_1,u_2)}{\partial u_1^r \partial u_2^s}\right|_{u_1=u_2=1} \tag{32}$$

Differentiating (31) and substituting in (32), after some algebraic manipulations, the factorial moments of $S_t^{(1)}$ and $S_t^{(2)}$ are given by

$$\begin{aligned} \mu_{[1,1]} &= (\lambda t\xi_{[1]})(\lambda t\varsigma_{[1]}) + (\lambda t\xi_{[1]}\varsigma_{[1]}) \\ \mu_{[2,1]} &= (\lambda t\xi_{[1]})^2(\lambda t\varsigma_{[1]}) + (\lambda t\xi_{[1]})(\lambda t\xi_{[1]}\varsigma_{[1]}) + (\lambda t\xi_{[2]})(\lambda t\varsigma_{[1]}) + (\lambda t\xi_{[2]}\varsigma_{[1]}) \\ \mu_{[2,2]} &= (\lambda t\xi_{[1]})^2(\lambda t\varsigma_{[1]})^2 + (\lambda t\xi_{[1]})(\lambda t\varsigma_{[1]})(\lambda t\xi_{[1]}\varsigma_{[1]}) + (\lambda t\xi_{[2]})(\lambda t\varsigma_{[1]})^2 \\ &\quad + (\lambda t\varsigma_{[1]})(\lambda t\xi_{[2]}\varsigma_{[1]}) + (\lambda t\xi_{[1]})^2(\lambda t\varsigma_{[2]}) + (\lambda t\xi_{[1]})(\lambda t\xi_{[1]}\varsigma_{[2]}) \\ &\quad + (\lambda t\xi_{[1]}\varsigma_{[1]})^2 + (\lambda t\xi_{[2]})(\lambda t\varsigma_{[2]}) + (\lambda t\xi_{[2]}\varsigma_{[2]}) \\ \mu_{[2,3]} &= (\lambda t\xi_{[1]})^2(\lambda t\varsigma_{[1]})^3 + (\lambda t\xi_{[1]})(\lambda t\varsigma_{[1]})^2(\lambda t\xi_{[1]}\varsigma_{[1]}) + (\lambda t\xi_{[2]})(\lambda t\varsigma_{[1]})^3 \\ &\quad + (\lambda t\varsigma_{[1]})^2(\lambda t\xi_{[2]}\varsigma_{[1]}) + (\lambda t\varsigma_{[1]})(\lambda t\xi_{[1]}\varsigma_{[1]})^2 + (\lambda t\xi_{[1]})(\lambda t\varsigma_{[1]})(\lambda t\xi_{[1]}\varsigma_{[2]}) \\ &\quad + (\lambda t\xi_{[1]})^2(\lambda t\varsigma_{[2]})(\lambda t\varsigma_{[1]}) + (\lambda t\xi_{[2]})(\lambda t\varsigma_{[2]})(\lambda t\varsigma_{[1]}) + (\lambda t\varsigma_{[1]})(\lambda t\xi_{[2]}\varsigma_{[2]}) \\ &\quad + (\lambda t\xi_{[2]})(\lambda t\xi_{[1]}\varsigma_{[1]})(\lambda t\varsigma_{[1]}) + (\lambda t\varsigma_{[2]})(\lambda t\xi_{[2]}\varsigma_{[1]}) + (\lambda t\xi_{[1]}\varsigma_{[1]})(\lambda t\xi_{[1]}\varsigma_{[2]}) \\ &\quad + (\lambda t\xi_{[1]}\varsigma_{[3]})(\lambda t\varsigma_{[1]}) + (\lambda t\varsigma_{[3]})(\lambda t\xi_{[1]})^2 + (\lambda t\xi_{[2]})(\lambda t\varsigma_{[3]}) + (\lambda t\xi_{[2]}\varsigma_{[3]}) \end{aligned} \tag{33}$$

where $\xi_{[r]} = E[X(X-1)\ldots(X-(r-1))]$, $r = 1, 2, \ldots$, is the rth factorial moment of X_i, $i = 1, 2, \ldots$ and where $\varsigma_{[s]} = E[Y(Y-1)\ldots(Y-(s-1))]$, $s = 1, 2, \ldots$, is the sth factorial moment of Y_i, $i = 1, 2, \ldots$ in (2). Note that $\mu_{[r,s]} = \mu_{[s,r]}$ for $r = 1, 2, \ldots$, $s = 1, 2, \ldots$

Similar to univariate CPP, let X_i, Y_i, $i = 1, 2, \ldots$, have finite values with the probabilities $P(X_i = j) = p_j$, $j = 0, 1, \ldots m$ and $P(Y_i = k) = q_k$, $k = 0, 1, \ldots \ell$. (31) and (34) can be used when $P(X_i = j) = p_j$ and $P(Y_i = k) = q_k$ approach to zero for $j, k \to \infty$.

The joint cumulant generating function of $S_t^{(1)}$ and $S_t^{(2)}$ is given by

$$\begin{aligned}\kappa(u_1, u_2) = -\lambda t + \lambda t[(p_0 q_0 + \cdots + p_0 q_r \exp(u_2^r)) + (p_1 q_0 \exp(u_1) + \cdots \\ + p_1 q_r \exp(u_1) \exp(u_2^r)) + (p_m q_0 \exp(u_1^m) + \cdots \\ + p_m q_r \exp(u_1^m) \exp(u_2^r))]\end{aligned} \tag{34}$$

From (35) we have

$$\kappa_{r,s} = \lambda t(\xi_r \varsigma_s), \ r = 1, 2, \ldots, s = 1, 2, \ldots \tag{35}$$

where $\xi_r = E(X^r)$, $r = 1, 2, \ldots$, and $\varsigma_s = E(Y^s)$, $s = 1, 2, \ldots$, are expected values of X_i and Y_i, $i = 1, 2, \ldots$, respectively.

The covariance of $S_t^{(1)}$ and $S_t^{(2)}$ is obtained using (34)

$$\begin{aligned}Cov\left(S_t^{(1)}, S_t^{(2)}\right) &= E\left(S_t^{(1)} S_t^{(2)}\right) - E\left(S_t^{(1)}\right) E\left(S_t^{(2)}\right) \\ &= \lambda t(\lambda t + 1)\xi_1 \varsigma_1 - (\lambda t \xi_1)(\lambda t \varsigma_1) \\ &= \lambda t \xi_1 \varsigma_1\end{aligned} \tag{36}$$

Then, the coefficient of correlation for $S_t^{(1)}$ and $S_t^{(2)}$ is given by

$$\begin{aligned}\rho = Corr\left(S_t^{(1)}, S_t^{(2)}\right) &= \frac{Cov\left(S_t^{(1)}, S_t^{(2)}\right)}{\sqrt{Var\left(S_t^{(1)}\right) Var\left(S_t^{(2)}\right)}} = \frac{\lambda t \xi_1 \varsigma_1}{\sqrt{[\lambda t E\left(X^2\right)]\left[\lambda t E\left(Y^2\right)\right]}} \\ &= \frac{\xi_1 \varsigma_1}{\sqrt{\xi_2 \varsigma_2}}\end{aligned} \tag{37}$$

4. Numerical Examples

To illustrate the usage of the univariate CPP and BCPP, we present two data sets. The first data is taken from Meintanis (1997) and Özel & Inal (2010). It corresponds to the number of traffic accidents and fatalities recorded on Sundays of each month over the period 1997-2004 in the region of Groningen. In this study the same data is used to show applicability of the univariate CPP the with following random variables: N_t is the number of Sunday accidents which occurs in Groningen between years 1997-2004; X_i, $i = 1, 2, \ldots$, are the number of fatalities the ith type of accident; $S_t = \sum_i^{N_t} X_i$ is the total number of fatalities in the time interval $(0, t]$.

The homogeneous Poisson process provide an adequate fit to the number of Sunday accidents ($p-value < 0.01$, $\chi^2 = 2.94$) for $\lambda = 9.84$ (in month). The independency of X_i, $i = 1, 2, \ldots$, and $\{N_t, t \geq 0\}$ is shown using the $Spearman's\ \rho$ test ($Spearman's\ \rho = 0.084$; $p = 0.432$). Then, we have to decide the best distribution of X_i, $i = 1, 2, \ldots$ among the Poisson, binomial and geometric distributions for the number of fatalities. For this aim, a goodness of fit test can be performed to choose the correct distribution (Agresti 2002). However, one can take into consideration the number of values of X_i, $i = 1, 2, \ldots$ If X_i, $i = 1, 2, \ldots$ have finite values, the binomial distribution can be used. Similarly, geometric or Poisson distribution can be more suitable when X_i, $i = 1, 2, \ldots$ have infinite values. The goodness of fit test is applied to decide the best distribution. It is found seen that the Poisson distribution with parameter $\nu = 0.53$ ($p-value < 0.001$, $\chi^2 = 0.20$), the binomial distribution with parameters $m = 4$, $p = 0.12$ ($p-value < 0.01$, $\chi^2 = 1.52$) and the geometric distribution with parameter $\theta = 0.62$ ($p-value < 0.001$, $\chi^2 = 0.06$) fit the data. Then it can be said that $\{S_t, t \geq 0\}$ has a Pólya-Aeppli process. Note that the goodness-of-fit are applied sequentially without taking into account the dependence amongst these tests, which of course influences the overall size of the test, i.e., when we test all hypothesis each at level α, the computation of the overall level becomes more complicated.

The moments and cumulants for the Pólya-Aeppli process are computed from Table 3 for the parameters $\lambda = 9.84$; $\theta = 0.62$ and several values of t. The results are presented in Table 4. Then, the values of the skewness, kurtosis, $Cov(N_t, S_t)$ and $Corr(N_t, S_t)$ are computed for the Pólya-Aeppli process and the results are given in Table 5.

Table 4: The moments and cumulants of the Pólya-Aeppli process for the traffic accidents in Groningen.

t	μ'_1	μ'_2	μ'_3	μ'_4	$\mu_{[1]}$	$\mu_{[2]}$	$\mu_{[3]}$	$\mu_{[4]}$
0.5	3.02	15.81	109.04	922.47	3.02	12.79	67.66	396.66
1	6.03	49.80	613.16	7885.24	6.03	43.77	366.71	3352.23
2	12.06	172.34	3246.47	64054.05	12.06	160.28	2317.11	35671.34
3	18.09	367.62	9216.11	241607.41	18.09	349.53	7167.36	154264.24
4	24.12	635.66	19838.25	645397.44	24.12	611.53	16233.64	448189.00
	μ_1	μ_2	μ_3	μ_4	$\kappa_{[1]}$	$\kappa_{[2]}$	$\kappa_{[3]}$	$\kappa_{[4]}$
0.5	6.03	43.08	340.80	3513.44	3.02	18.19	164.52	1984.45
1	12.06	164.95	2458.88	41475.28	6.03	72.75	1316.17	31751.14
2	24.12	620.87	16855.10	487496.27	12.06	290.98	10529.37	508018.17
3	36.19	1367.78	53718.02	2201987.32	18.09	654.71	35536.61	2571841.99
4	48.25	2405.66	123577.01	6556890.97	24.12	1163.92	84234.93	8128290.74
	κ_1	κ_2	κ_3	κ_4	B_1	B_2	B_3	B_4
0.5	3.02	6.71	20.90	86.33	3.02	6.39	11.28	16.53
1	6.03	19.45	41.80	172.67	6.03	21.88	61.12	139.68
2	12.06	38.91	83.61	345.33	12.06	80.14	386.18	1486.31
3	18.09	58.36	125.41	518.00	18.09	174.77	1194.56	6427.68
4	24.12	77.82	167.21	690.66	24.12	305.77	2705.61	18674.54

TABLE 5: The skewness, kurtosis, covariance, and the coefficient of correlation of the Pólya-Aeppli process for the traffic accidents in Groningen.

t	$\sqrt{\beta_1}$	β_2	$Cov(N_t, S_t)$	$Corr(N_t, S_t)$
0.5	0.002131	-1.107261	3.015484	0.52475
1	0.000274	-1.475558	6.030968	
2	0.000035	-1.735354	12.061935	
3	0.000010	-1.822979	18.092903	
4	0.000004	-1.867004	24.123871	

A second data set comes from earthquakes in Turkey which is given by Özel (2011a) and Özel (2011b). The mainshocks with surface wave magnitudes $M_s \geq 5.0$ that occurred in Turkey between 1900 and 2009, their foreshock and aftershock sequences are considered. For the construction of a model to explain the total number of foreshocks and aftershocks with the BCPP in (2), the following random variables are defined: N_t is the number of mainshocks that occurred in Turkey between 1903 and 2009; X_i, $i = 1, 2, \ldots$ are the number of foreshocks of ith mainshock; Y_i, $i = 1, 2, \ldots$ are the number of aftershocks of the ith mainshock; and $\left(S_t^{(1)} = \sum_i^{N_t} X_i, S_t^{(2)} = \sum_i^{N_t} Y_i\right)$ is the total number of foreshocks and aftershocks for the mainshocks. The goodness of fit test is performed to compare the observed frequency distribution to the theoretical Poisson distribution. Chi-square value ($\chi^2 = 0.051$ with $df = 9$, $p - value = 0.525$) indicates that $\{N_t, t \geq 0\}$ fits the Poisson process with parameter $\lambda = 1.037$ (in years) at the level of 0.05. Spearman's ρ test verifies the absence of correlation between N_t and X_i, $i = 1, 2, \ldots$ (Spearman's $\rho = 0.071$; $p = 0.412$). No correlation is also found between N_t and Y_i, $i = 1, 2, \ldots$ (Spearman's $\rho = 0.034$; $p = 0.589$). Similarly, it is shown that there is no statistically significant dependence between X_i and Y_i, $i = 1, 2, \ldots$ (Spearman's $\rho = 0.048$; $p = 0.493$). As discussed by Özel (2011b), if the occurrence of foreshock sequences is assumed to be independent of the occurrence of mainshocks, then the distribution of foreshocks can be treated as a binomial distribution. The goodness-of-fit test for the binomial distribution provided an adequate fit with a p-value of 0.999 and chi-squared value $\chi^2 = 0.003$ with 34 degrees of freedom. This means that the binomial distribution with parameters ($m = 35$, $p = 0.15$) fits the probability function of X_i, $i = 1, 2, \ldots$ It is pointed out that the number of aftershocks of a mainshock has a geometric distribution (Christophersen & Smith 2000). After obtaining the frequency distribution of aftershocks and the goodness-of-fit test ($\chi^2 = 1.587$ with $df = 35$), it is seen that Y_i, $i = 1, 2, \ldots$ have a geometric distribution with parameter $\theta = 0.175$. Then, we can write $\left(S_t^{(1)} = \sum_i^{N_t} X_i, S_t^{(2)} = \sum_i^{N_t} Y_i\right)$ and suggest that $(S_t^{(1)}, S_t^{(2)})$ has a BCPP. So, the joint factorial moments and cumulants are calculated from (33) and (35) for the parameters $\lambda = 1.037$; $\theta = 0.175$; ($m = 35$, $p = 0.15$) and several values of t. Then, $Cov\left(S_t^{(1)}, S_t^{(2)}\right)$ and $Corr\left(S_t^{(1)}, S_t^{(2)}\right)$ are computed from (34) and (36) and the results are presented in Table 6.

TABLE 6: The moments, cumulants, covariance, and coefficient of correlation of the BCPP for the earthquakes in Turkey.

t	$\mu_{[1,1]}$	$\mu_{[2,1]}$	$\mu_{[2,2]}$	$\mu_{[2,3]}$	$\kappa_{1,1}$	$\kappa_{2,1}$	$\kappa_{2,2}$	$\kappa_{2,3}$	$Cov(N_t, S_t)$	$Corr(N_t, S_t)$
0.5	19.49	152.43	546.09	26543.35	12.83	78.28	816.35	3422.87	12.83	0.6237
1	52.28	551.27	713.82	128434.33	25.67	156.56	1632.71	6845.74	25.67	
2	157.79	2522.87	11344.95	1118387.43	51.33	313.12	3265.42	13691.49	51.33	
2.5	230.51	4313.00	27684.65	2578433.75	64.16	391.40	4081.77	17114.36	64.16	
3	316.54	6784.23	57395.00	5330414.38	77.00	469.68	4898.13	20537.23	77.00	

5. Conclusion

In this paper, the moments, cumulants, skewness, kurtosis and covariance of the univariate CPP are derived. Some special cases of the univariate CPP are provided and a numerical example based on the traffic accidents in Groningen is given. Then, BCPP is defined and some important probabilistic characteristics such as moments, cumulants, covariances, and the coefficient of correlation for the BCPP are obtained.

Earthquake is an unavoidable natural disaster for Turkey. Application to the earthquake data in Turkey is presented to illustrate the usage of the BCPP and its properties. Earthquakes could be regarded as discrete events, representing some real but not well-known tectonic process. Following that scheme and keeping in mind the highly random characteristics of all earthquake parameters, it is quite natural to consider a sequence of earthquakes as a stochastic process. The stochastic modeling of the earthquake occurrence has proved very useful in earthquake prediction studies, in understanding the nature of the earthquake phenomena, and in assessing seismicity and seismic hazard. Existing approaches in the research of seismic hazard assessment are generally based on the homogeneous Poisson process. However, new studies have been done using CPP and BCPP and give more information than homogeneous Poisson process. For this reason, the factorial moments and cumulants of BCPP, which are obtained in this study, can be a good tool to understand earthquake behaviour.

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