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Estimation a Stress-Strength Model for $P(Y_{r:n_1} < X_{k:n_2})$ Using the Lindley Distribution

En estimación del estrés fuerza modelo en la caja $P(Y_{r:n_1} < X_{k:n_2})$ de distribución Lindley

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Abstract

The problem of estimation reliability in a multicomponent stress-strength model, when the system consists of k components have strength each component experiencing a random stress, is considered in this paper. The reliability of such a system is obtained when strength and stress variables are given by Lindley distribution. The system is regarded as alive only if at least r out of k $(r < k)$ strength exceeds the stress. The multicomponent reliability of the system is given by $R_{r,k}$. The maximum likelihood estimator (MLE), uniformly minimum variance unbiased estimator (UMVUE) and Bayes estimator of $R_{r,k}$ are obtained. A simulation study is performed to compare the different estimators of $R_{r,k}$. Real data is used as a practical application of the proposed model.

***Key words*:** Bayes Estimator, Lindley Distribution, Maximum Likelihood Estimator, Order Statistics, Stress-Strength Model, Uniformly Minimum Variance Unbiased Estimator.

Resumen

El problema de la fiabilidad de estimación en el modelo de estrés-fuerza multicomponente, cuando el sistema consta de componentes k tiene fuerza, cada componente experimentando un estrés al azar se considera en este documento. Se obtiene la fiabilidad de estos sistemas cuando las variables de fuerza y tensión están dadas por la distribución Lindley. El sistema es considerado como vivo solo si al menos r de $k(r < k)$ fuerzas superan el estrés. La fiabilidad de varios componentes del sistema viene dado por $R_{r,k}$. El estimador de máxima verosimilitud (MLE), se obtienen estimadores insesgados

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de varianza uniformemente mínima (UMVUE) y el estimador de Bayes $R_{r,k}$. Se realizó un estudio de simulación para comparar los diferentes estimadores de $R_{r,k}$. Se utilizaron datos reales como aplicación práctica para el modelo propuesto.

***Palabras clave*:** distribución de Lindley, estadísticas de orden, estimador de Bayes, estimador insesgado de varianza uniformemente mínima, estimador insesgado de varianza mínima, modelo de estrés-fuerza.

1. Introduction

The Lindley distribution originally developed by Lindley (1958, 1965) in the context of Bayesian statistics, is a counter example of fiducial statistics. The Lindley distribution has the following probability density function (p.d.f)

$$f(x;\theta) = \frac{\theta^2}{(\theta+1)}(1+x)e^{-\theta x} \qquad x>0, \qquad \theta>0, \tag{1}$$

The corresponding cumulative distribution function (c.d.f) is

$$F(x;\theta) = 1-(1+\frac{\theta}{1+\theta}x)e^{-\theta x} \qquad x>0, \qquad \theta>0, \tag{2}$$

and the corresponding survival function is

$$S(x;\theta) = (1+\frac{\theta}{1+\theta}x)e^{-\theta x} \qquad x>0, \qquad \theta>0. \tag{3}$$

Ghitany & Atieh (2008) studied the mathematical and statistical properties of the Lindley distribution. They have shown that this distribution is better a model than the well-known exponential distribution in some particular cases. Al-Mutairi, Ghitany & Kundu (2013) investigated the stress- strength model using the Lindley distribution and in this paper we will investigate the multicomponent stress-strength model of the Lindley distribution. Also Figure 1 shows that the Lindley distribution for different values of θ is positively skewed. Many authors have discussed the Lindley distribution as a model of lifetime data such as Krishna & Kumar (2011), Singh, Singh & Singh (2008) and Singh, Gupta & Sharma (2014), and Al-Mutairi et al. (2013) studied stress-strength model. Also the inverse Lindley distribution discussed as stress-strength model has been studied by Sharma, Singh, Singh & Agiwal (2014, 2015).

The reliability parameter $R = P(Y < X)$ is referred to as a stress-strength model, which is used in quality control, engineering statistics, and other fields. In a reliability context, the stress-strength model describes the life of a component that has a random strength variable X and is subjected to random variable stress Y. The system fails if and only if the stress is greater than strength at any time. The estimation of a stress-strength model when X and Y are random variables having a specified distribution has been discussed by many authors including Birnbaum (1956), Basu (1964), Downtown (1973), Tong (1974, 1977), Beg (1980),

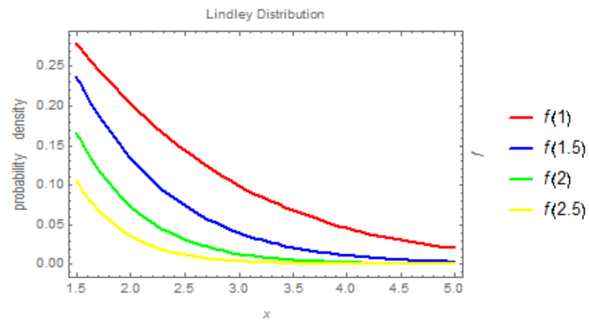

Figure 1: Lindley distribution.

Iwase (1987), McCool (1991). Recently, Ali, Pal & Woo (2012), Wong (2012), Shahsanaei & Daneshkhah (2013), Hussian (2013), Al-Mutairi et al. (2013), Ghitany, Al-Mutairi & Aboukhamseen (2015) and Najarzadegan, Babaii, Rezaei & Nadarajah (2016). Also, the estimation of reliability in a multicomponent stress-strength model $R_{r,k} = P(\text{at least } r \text{ of } X_1, \dots, X_k \text{ exceed } Y)$ has been discussed by many authors including from Bhattacharyya & Johnson (1974), and Pandey, Uddin & Ferdous (1992). Some of the recent work on the multicomponent stress strength has been undertaken by Eryilmaz (2008), Pakdaman & Ahmadi (2013), Rao & Kantan (2010), Kizilaslan & Nadar (2015), Dey, Mazucheli & Anis (2017), Rao (2012), and Rao, Muhammad & Osama (2016).

In this paper, the system of reliability $R_{r,k} = P(Y_{r:n_1} < X_{k:n_2})$ in the Lindley distribution case is derived. Special cases of $R_{r,k}$ can be found in Section 2. The maximum likelihood estimator (MLE) of $R_{r,k}$, the uniform minimum variance unbiased estimator (UMVUE) of $R_{r,k}$, and the Bayes estimator of $R_{r,k}$ are obtained in Section 3. In Section 4, a simulation study is performed to compare the estimators of the reliability system. In Section 5, real data is used as a practical application of the proposed procedure. Finally, we conclude in Section 6.

2. System of Reliability

Let X and Y be two random variables as part of the Lindley distribution with parameters q and p, respectively. Suppose $X_1, \dots, X_{n_2}$ and $Y_1, \dots, Y_{n_1}$ are two independent samples from X and Y, respectively. The strength and the stress are assumed to be independent. Based on these assumptions, we find the system of reliability to be

$$R_{r,k} = P(Y_{r:n_1} < X_{k:n_2}) = \int_0^{\infty} F_{Y_{r:n_1}}(x) f_{X_{k:n_2}}(x)\, dx, \tag{4}$$

where, $F_{Y_{r:n_1}}(x)$ and $f_{X:n_2}(x)$ are the rth cumulative density function and kth probability density function of $Y_{r:n_1}$ and $X_{k:n_2}$ respectively. And,

$$f_{X_{k:n_2}}(x) = k \begin{pmatrix} n_2 \\ k \end{pmatrix} F^{k-1}(x)(1 - F(x)]^{n_2-k} f(x), \tag{5}$$

$$F_{Y_{r:n_1}}(x) = \sum_{j=r}^{n_1} \binom{n_1}{j} F^j(x)[1-F(x)]^{n_1-j}. \tag{6}$$

From (5) and (6) in (4), we obtain

$$\begin{aligned} R_{r,k} = {} & k\frac{q^2}{1+q}\binom{n_2}{k}\sum_{j=r}^{n_1}\binom{n_1}{j}\int_0^\infty (1+x)e^{-qx}\left[1-\left(1+\frac{p}{1+p}x\right)e^{-px}\right]^j \\ & \times\left[\left(1+\frac{p}{1+p}x\right)e^{-px}\right]^{n_1-j}\left[1-\left(1+\frac{q}{1+q}x\right)e^{-qx}\right]^{k-1} \\ & \times\left[\left(1+\frac{q}{1+q}x\right)e^{-qx}\right]^{n_2-k} dx \end{aligned}$$

Based on some calculations and binomial theory, we obtain

$$\begin{aligned} R_{r,k} = {} & \frac{kq^2}{1+q}\binom{n_2}{k}\sum_{j=r}^{n_1}\sum_{l_1=0}^{n_1-j}\sum_{l_2=0}^{n_2-k}\sum_{l_3=0}^{k-1}\sum_{l_4=0}^{k-l_3-1}\sum_{l_5=0}^{j}\sum_{l_6=0}^{j-l_5}(-1)^{k+j-l_3-l_5-1}\binom{n_1}{j} \\ & \times\binom{n_1-j}{l_1}\binom{n_2-k}{l_2}\binom{k-1}{l_3}\binom{k-l_3-1}{l_4}\binom{j}{l_5} \\ & \times\binom{j-l_5}{l_6}\left(\frac{q}{1+q}\right)^{n_2-l_2-l_3-l_4-1}\left(\frac{p}{1+p}\right)^{n_1-l_1-l_5-l_6} \\ & \times\left(\int_0^\infty (1+x)x^{n_1+n_2-l_1-l_2-l_3-l_4-l_5-l_6-1}e^{-x(q(n_2-l_3)+p(n_1-l_5))}\,dx\right) \end{aligned}$$

$$\begin{aligned} R_{r,k} = {} & \frac{kq^2}{1+q}\binom{n_2}{k}\sum_{j=r}^{n_1}\sum_{l_1=0}^{n_1-j}\sum_{l_2=0}^{n_2-k}\sum_{l_3=0}^{k-1}\sum_{l_4=0}^{k-l_3-1}\sum_{l_5=0}^{j}\sum_{l_6=0}^{j-l_5}(-1)^{k+j-l_3-l_5-1}\binom{n_1}{j} \\ & \times\binom{n_1-j}{l_1}\binom{n_2-k}{l_2}\binom{k-1}{l_3}\binom{k-l_3-1}{l_4}\binom{j}{l_5}\binom{j-l_5}{l_6} \\ & \times\left(\frac{q}{1+q}\right)^{n_2-l_2-l_3-l_4-1}\left(\frac{p}{1+p}\right)^{n_1-l_1-l_5-l_6} \\ & \times\left(\frac{\Gamma(n_1+n_2-l_1-l_2-l_3-l_4-l_5-l_6)}{(q(n_2-l_3)+p(n_1-l_5))^{(n_1+n_2-l_1-l_2-l_3-l_4-l_5-l_6)}}\right) \\ & \times\left(\frac{n_1+n_2-l_1-l_2-l_3-l_4-l_5-l_6}{(q(n_2-l_3)+p(n_1-l_5))}+1\right), \end{aligned} \tag{7}$$

where $\Gamma(.)$ is a gamma function.

We now present some special cases of $R_{r,k}$ with a different arrangement of the components.

1. For $r = n_1$ and $k = 1$, the minimum strength component is subjected to the maximum stress component. In this case, the probability $R_{n_1,1}$ is the reliability of a series system with an n_2 component

$$R_{n_1,1} = \frac{n_2 q^2}{(1+q)} \sum_{l_2=0}^{n_2-1} \sum_{l_5=0}^{n_1} \sum_{l_6=0}^{n_1-l_5} (-)^{n_1-l_5} \binom{n_2-1}{l_2} \binom{n_1}{l_5} \binom{n_1-l_5}{l_6} \\ \times \left(\frac{q}{1+q}\right)^{n_2-l_2-1} \left(\frac{p}{1+p}\right)^{n_1-l_5-l_6} \\ \times \left(\frac{\Gamma(n_1+n_2-l_2-l_5-l_6)}{(qn_2+p(n_1-l_5))^{(n_1+n_2-l_2-l_5-l_6)}}\right) \left(\frac{n_1+n_2-l_2-l_5-l_6}{(qn_2+p(n_1-l_5))}+1\right). \tag{8}$$

2. For $r = n_1$ and $k = n_2$, the maximum strength component is subjected to the maximum stress component. Then, R_{n_1,n_2} is the reliability of a parallel system with an n_2 component

$$R_{n_1,n_2} = \frac{n_2 q^2}{1+q} \sum_{l_3=0}^{n_2-1} \sum_{l_4=0}^{n_2-l_3-1} \sum_{l_5=0}^{n_1} \sum_{l_6=0}^{n_1-l_5} (-1)^{n_2+n_1-l_3-l_5-1} \binom{n_2-1}{l_3} \\ \times \binom{n_2-l_3-1}{l_4} \binom{n_1}{l_5} \binom{n_1-l_5}{l_6} \\ \times \left(\frac{q}{1+q}\right)^{n_2-l_3-l_4-1} \left(\frac{p}{1+p}\right)^{n_1-l_5-l_6} \\ \times \left(\frac{\Gamma(n_1+n_2-l_3-l_4-l_5-l_6)}{(q(n_2-l_3)+p(n_1-l_5))^{(n_1+n_2-l_3-l_4-l_5-l_6)}}\right) \\ \times \left(\frac{n_1+n_2-l_3-l_4-l_5-l_6}{(q(n_2-l_3)+p(n_1-l_5))}+1\right). \tag{9}$$

3. For $r = 1$ and $k = 1$, the minimum strength component is subjected to the minimum stress component. Then,

$$R_{1,1} = \frac{n_2 n_1 q^2}{1+q} \sum_{j=1}^{n_1} \sum_{l_1=0}^{n_1-1} \sum_{l_2=0}^{n_2-1} \sum_{l_5=0}^{j} \sum_{l_6=0}^{j-l_5} (-1)^{j-l_5} \binom{n_1}{j} \binom{n_1-j}{l_1} \\ \times \binom{n_2-1}{l_2} \binom{j}{l_5} \binom{j-l_5}{l_6} \left(\frac{q}{1+q}\right)^{n_2-l_1-l_2-1} \\ \times \left(\frac{p}{1+p}\right)^{n_1-l_1-l_5-l_6} \left(\frac{\Gamma(n_1+n_2-l_1-l_2-l_5-l_6)}{(qn_2+p(n_1-l_5))^{(n_1+n_2-l_1-l_2-l_5-l_6)}}\right) \\ \times \left(\frac{n_1+n_2-l_1-l_2-l_5-l_6}{(qn_2+p(n_1-l_5))}+1\right). \tag{10}$$

4. For $r = n_1$ and $k = k$, the kth strength order component is subjected to the maximum stress component.

$$
\begin{aligned}
R_{n_1,k} = & \frac{kq^2}{1+q}\begin{pmatrix} n_2 \\ k \end{pmatrix}\sum_{l_2=0}^{n_2-k}\sum_{l_3=0}^{k-1}\sum_{l_4=0}^{k-l_3-1}\sum_{l_5=0}^{n_1}\sum_{l_6=0}^{n_1-l_5}\begin{pmatrix} n_2-k \\ l_2 \end{pmatrix} \\
& \times \begin{pmatrix} k-1 \\ l_3 \end{pmatrix}\begin{pmatrix} k-l_3-l_4-1 \\ l_4 \end{pmatrix}\begin{pmatrix} n_1 \\ l_5 \end{pmatrix}\begin{pmatrix} n_1-l_5 \\ l_6 \end{pmatrix} \\
& \times \left(\frac{q}{1+q}\right)^{n_2-l_2-l_3-l_4-1}\left(\frac{p}{1+p}\right)^{n_1-l_1-l_5-l_6} \\
& \times \left(\frac{\Gamma(n_1+n_2-l_1-l_2-l_3-l_4-l_5-l_6)}{(q(n_2-l_3)+p(n_1-l_5))^{(n_1+n_2-l_1-l_2-l_3-l_4-l_5-l_6)}}\right) \\
& \times \left(\frac{n_1+n_2-l_1-l_2-l_3-l_4-l_5-l_6}{(q(n_2-l_3)+p(n_1-l_5))}+1\right).
\end{aligned} \tag{11}
$$

3. Different Estimators of $R_{r,k}$

In this section the three estimation methods for $R_{r,k}$ that were applied were the maximum likelihood estimator (MLE), the uniformly minimum variance unbiased estimator (UMVUE), and the Bayes estimator of $R_{r,k}$ using the Lindley approximation.

3.1. Maximum Likelihood Estimator

Let $X_1, \dots, X_{n_2}$ be a random sample of the strengths of the n_2 systems that are distributed as Lindley random variables with parameter q and $Y_1, \dots, Y_{n_1}$. Let these be a random samples of stresses of n_1 systems that are distributed as Lindley random variables with the parameter p. Then the log likelihood function of the observed samples is

$$
\begin{aligned}
\log L(p,q) = & \, n_1[\log p^2 - \log(1+p)] + \sum_{i=1}^{n_1}\log(1+y_i) - p\sum_{i=1}^{n_1} y_i \\
& + n_2[\log q^2 - \log(1+q)] + \sum_{j=1}^{n_2}\log(1+x_j) - q\sum_{j=1}^{n_2} x_j
\end{aligned}
$$

Ghitany & Atieh (2008) showed that the maximum estimator of p and q, denoted by $\hat{p}$ and $\hat{q}$, are

$$
\hat{q}_{MLE} = \frac{(1-\bar{X}) + \sqrt{(\bar{X}-1)^2 + 8\bar{X}}}{2\bar{X}} \tag{12}
$$

$$
\hat{p}_{MLE} = \frac{(1-\bar{Y}) + \sqrt{(\bar{Y}-1)^2 + 8\bar{Y}}}{2\bar{Y}}. \tag{13}
$$

By using the invariance property of the maximum likelihood estimator the maximum estimator of $R_{r,k}$ can be obtain. This is denoted by $\hat{R}^{MLE}_{r,k}$, replacing p and q in equation (7) by their maximum estimators. Hence $\hat{R}^{MLE}_{r,k}$ is given by

$$\begin{aligned}\hat{R}^{MLE}_{R,K} = & \frac{k\hat{q}^2_{MLE}}{1+\hat{q}_{MLE}} \begin{pmatrix} n_2 \\ k \end{pmatrix} \sum_{j=r}^{n_1} \sum_{l_1=0}^{n_1-j} \sum_{l_2=0}^{n_2-k} \sum_{l_3=0}^{k-1} \sum_{l_4=0}^{k-l_3-1} \sum_{l_5=0}^{j} \sum_{l_6=0}^{j-l_5} (-1)^{k+j-l_3-l_5-1} \begin{pmatrix} n_1 \\ j \end{pmatrix} \\ & \times \begin{pmatrix} n_1-j \\ l_1 \end{pmatrix} \begin{pmatrix} n_2-k \\ l_2 \end{pmatrix} \begin{pmatrix} k-1 \\ l_3 \end{pmatrix} \begin{pmatrix} k-l_3-1 \\ l_4 \end{pmatrix} \begin{pmatrix} j \\ l_5 \end{pmatrix} \\ & \times \begin{pmatrix} j-l_5 \\ l_6 \end{pmatrix} \left(\frac{\hat{q}_{MLE}}{1+\hat{q}_{MLE}}\right)^{n_2-l_2-l_3-l_4-1} \left(\frac{\hat{p}_{MLE}}{1+\hat{p}_{MLE}}\right)^{n_1-l_1-l_5-l_6} \\ & \times \left(\frac{\Gamma(n_1+n_2-l_1-l_2-l_3-l_4-l_5-l_6)}{(\hat{q}_{MLE}(n_2-l_3)+\hat{p}_{MLE}(n_1-l_5))^{(n_1+n_2-l_1-l_2-l_3-l_4-l_5-l_6)}}\right) \\ & \times \left(\frac{n_1+n_2-l_1-l_2-l_3-l_4-l_5-l_6}{(\hat{q}_{MLE}(n_2-l_3)+\hat{p}_{MLE}(n_1-l_5))}+1\right)\end{aligned} \tag{14}$$

Where $\hat{p}_{MLE}$ and $\hat{q}_{MLE}$ are defined in equation (12) and (13) respectively.

Now, we find the asymptotic distribution of $\hat{R}^{MLE}_{r,k}$ because it is difficult to find the explicit distribution. To find the asymptotic distribution and the confidence interval of $R_{r,k}$, we use the algorithm below.

Algorithm:

1. Find the asymptotic variance of $\hat{p}_{MLE}$ and $\hat{q}_{MLE}$ as follows

$$var(\hat{p}_{MLE}) = E[-\frac{\partial^2 LogL(p,q)}{\partial p^2}] = \frac{2n_1}{p^2} - \frac{n_1}{(1+p)^2},$$

$$var(\hat{q}_{MLE}) = E[-\frac{\partial^2 LogL(p,q)}{\partial q^2}] = \frac{2n_2}{q^2} - \frac{n_2}{(1+q)^2}$$

2. Find the asymptotic variance of $\hat{R}^{MLE}_{r,k}$as is presented by Rao (1973)

$$var(\hat{R}^{MLE}_{r,k}) = var(\hat{p}_{MLE})(\frac{\partial R_{r,k}}{\partial p})^2 + var(\hat{q}_{MLE})(\frac{\partial R_{r,k}}{\partial q})^2$$

3. As $n_1 \to \infty$ and $n_2 \to \infty$ then $\frac{\hat{R}^{MLE}_{r,k} - R_{r,k}}{\sqrt{var(\hat{R}^{MLE}_{r,l}}} \xrightarrow{D} N(0,1)$. Hence, an asymptotic $100(1-\alpha)\%$ confidence interval for $R_{r,k}$ could be written as $\hat{R}^{MLE}_{r,k} \mp Z_{\frac{\alpha}{2}} \sqrt{Var(\hat{R}^{MLE}_{r,k})}$ where $Z_{\frac{\alpha}{2}}$is the upper $\frac{\alpha}{2}$-quantile of standard normal distribution.

3.2. Uniformly minimum variance unbiased estimator (UMVUE) of $R_{r,k}$

To find the UMVUE of $R_{r,k}$ which is denoted by $\hat{R}^{U}_{r,k}$, we need to prove the following theorem.

Theorem 1. *If $X_1, \dots, X_n$ is a random sample from the Lindley distribution with parameter θ, then the probability density function of $z = X_1 + \cdots + X_n$ is*

$$g(z;n,\theta) = \sum_{k=0}^{n} \binom{n}{k} \frac{\theta^{2n}}{(1+\theta)^n \Gamma(2n-k)} z^{2n-k-1} e^{-\theta z}, z > 0, \theta > 0$$

***Proof*.** See Al- Mutairi et al (2013). □

Now, let $X_1, \dots, X_{n_2}$ be a random sample of the strengths of n_2 systems that are distributed as Lindley random variables with parameter q, let $Y_1, \dots, Y_{n_1}$ be a random sample of stresses of n_1 systems that are distributed as Lindley random variables with parameter p. Also, let $U = \sum_{i=1}^{n_2} X_i$ and $V = \sum_{j=1}^{n_1} Y_j$ be complete and sufficient statistics for p and q, respectively. Hence $\hat{R}^U_{r,k}$ can obtained as

$$\hat{R}^U_{r,k} = E[\phi(X_{1:n_2}, Y_{2:n_1})\,|U = u, V = v]$$

where, $\phi(X_{1:n_2}, Y_{1:n_2}) = \begin{cases} 1 & Y_{1:n_1} < X_{1:n_2} \\ 0 & Y_{1:n_1} > X_{1:n_2} \end{cases}$

Therefore,

$$\hat{R}^U_{r,k} = \int_0^{\min(u,v)} \int_y^u f_{1:n_2}(X_{1:n_2} = x\,|X_{i::n_2} = u) f_{1:n_1}(Y_{1:n_1} = y\,|Y_{j:n_1} = v] dxdy$$

$$= \frac{1}{ij} \int_0^{\min(u,v)} \int_y^u (1 - \frac{F(x)}{F(u)})^{i-1} (1 - \frac{F(y)}{F(v)})^{j-1} \frac{f(x)}{F(u)} \frac{f(y)}{F(v)} dxdy$$

where, $1 < i < n_2$ and $1 < j < n_1$. Using binomial theorem we get

$$\begin{aligned} \hat{R}^U_{r,k} = \frac{1}{ij} \sum_{m_1=0}^{i-1} \sum_{m_2=0}^{j-1} \binom{i-1}{m_1} \binom{j-1}{m_2} \\ \times \int_0^{\min(u,v)} \int_y^u (\frac{F(x)}{F(u)})^{i-m_1-1} (\frac{F(y)}{F[v)})^{j-m_2-1} \frac{f(x)}{F(u)} \frac{f(y)}{F(v)} dxdy \end{aligned} \quad (15)$$

where,

$$f(y) = \frac{p^2}{1+p}(1+y)e^{-py} \qquad y > 0, \qquad p > 0$$

$$f(x) = \frac{q^2}{1+q}(1+x)e^{-qx} \qquad x > 0, \qquad q > 0,$$

$$F(x) = 1 - (1 + \frac{py}{1+p})e^{-py} \qquad y > 0, \qquad p > 0,$$

$$F(x) = 1 - (1 + \frac{qx}{1+q})e^{-qx} \qquad x > 0, \qquad q > 0,$$

$$F(u) = \sum_{k=0}^{n_1} \sum_{s=2n_2-k}^{\infty} \binom{n_2}{k} \frac{q^k}{s!(1+q)^{n_2}} (\frac{u}{q})^s e^{-\frac{u}{q}},$$

$$F(v) = \sum_{k_1=0}^{n_1} \sum_{s_1=2n_1-k}^{\infty} \left(\begin{array}{c} n_1 \\ k_1 \end{array} \right) \frac{p^{k_1}}{s_1!(1+p)^{n_1}} (\frac{v}{p})^{s_1} e^{-\frac{v}{p}}$$

When calculating the integral equation (15), when $u \leq v$ and $u > v$, we get $\hat{R}^U_{r,k}$.

3.3. Bayes estimator of$R_{r,k}$

To find the Bayesian estimators of unknown parameters p, q, and the stress-strength reliability model $R_{r,k}$, which is denoted by $\hat{R}^B_{r,k}$, we consider a non-informative and an informative gamma prior for unknown parameters p and q (see Jeffrey 1961). Let $X_1, \dots, X_{n_2}$ be a random sample of the strengths of n_2 systems that are distributed as Lindley random variables with parameter q, and let $Y_1, \dots, Y_{n_1}$ be a random sample of the stresses of n_1 systems that are distributed as Lindley random variables with parameter p. We assume p and q have gamma prior distributions of the following forms

$$\pi(p) = \frac{b_1^{a_1}}{\Gamma(a_1)} p^{a_1-1} e^{-pb_1}, \qquad p > 0, \qquad a_1 > 0, \qquad b_1 > 0 \tag{16}$$

and,

$$\pi(q) = \frac{b_2^{a_2}}{\Gamma(a_2)} q^{a_2-1} e^{-qb_2}, \qquad q > 0, \qquad a_2 > 0, \qquad b_2 > 0 \tag{17}$$

where, a_1, a_2, b_1, and b_2 are known.

The joint posterior distribution of p and q is defined by

$$\pi(p, q \mid data) = kL(p, q \mid data)\pi(p)\pi(q)$$

where, $k = 1/\int_0^\infty \int_0^\infty L(p, q \mid data)\pi(p)\pi(q)\ dpdq$, and

$$L(p, q \mid data) = \frac{p^{2n_1} q^{2n_2}}{(1+p)^{n_1}(1+q)^{n_2}} \prod_{i=1}^{n_2}(1+x_i) \prod_{j=1}^{n_1}(1+y_j) e^{-p\sum_{j=1}^{n_1} y_j - q\sum_{i=1}^{n_2} x_i}.$$

The Bayes estimator of any parametric function $R_{r,k}$ under square error loss function (SELF) can be written as

$$\hat{R}^B_{r,k} = \int_0^\infty \int_0^\infty R_{r,k}\pi(p, q\,|data)dpdq \tag{18}$$

We have no closed form for $\hat{R}^B_{r,k}$, hence numerical computations are needed.

3.3.1. Lindley Approximation

Lindley (1980) proposed an approximation technique to find the Bayes estimators of stress-strength parameters p, q and $R_{r,k}$ under the squared error loss function, which are given by $\theta^*_{l,SELF} = \hat{\theta}_{l,MLE} + \hat{\rho}_l\hat{\sigma}_{ll} + 0.5(\hat{L}_{lll}\hat{\sigma}_{ll}) = \theta_{l,MLE}$, $l = 1, 2$ and,

$$\begin{aligned}\hat{R}^B_{r,k} = R^*_{SELF} = \hat{R_{r,k}} + \frac{1}{2}[\hat{\sigma}_{11}(\hat{R}_{11} + 2\hat{R}_1\hat{\rho}_1) + \hat{\sigma}_{22}(\hat{R}_{22} + 2\hat{R}_2\hat{\rho}_2)] \\ + \frac{1}{2}[\hat{L}_{111}\hat{R}_1\hat{\sigma}^2_{11} + \hat{L}_{222}\hat{R}_2\hat{\sigma}^2_{22}]\end{aligned} \tag{19}$$

where, $\hat{\theta}_{1,MLE} = \hat{p}_{MLE}$, $\hat{\theta}_{2,MLE} = \hat{q}_{MLE}$,

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = [I(\underset{-}{\theta})]^{-1} = \begin{pmatrix} \frac{-2n_1}{\theta_1^2} + \frac{n_1}{(1+\theta_1)^2} & 0 \\ 0 & \frac{-2n_2}{\theta_2^2} + \frac{n_2}{(1+\theta_2)^2} \end{pmatrix}^{-1}$$

$I(\underset{-}{\theta})$ is the asymptotic expected Fisher information matrix, $\underset{-}{\theta} = (\theta_1, \theta_2)$,$\theta_1 = p$, $\theta_2 = q$, $\rho_l = \frac{\partial Log\pi(\theta_1,\theta_2)}{\partial\theta_l}$, $\pi(\theta_1, \theta_2) = \pi_j(\theta_1, \theta_2)$ is the joint prior wing a methodology given by Jeffrey (1961), which ca be presented by the following formula

$$\pi_j(\theta_1, \theta_2) = \frac{\sqrt{(\theta_1^2 + 4\theta_1 + 2)(\theta_2^2 + 4\theta_2 + 2)}}{\theta_1\theta_2(1 + \theta_1)(1 + \theta_2)}, \tag{20}$$

or $\pi(\theta_1, \theta_2) = \pi_g(\theta_1, \theta_2)$ is the joint prior when θ_1 and θ_2 have prior gamma distribution as in equation (16) and (17), respectively, which is then given as

$$\pi_g(\theta_1, \theta_2) \propto \theta_1^{a_1-1}\theta_2^{a_2-1}e^{-(\theta_1 b_1 + \theta_2 b_2)}, \tag{21}$$

$$\begin{aligned}L_{lll} = \frac{\partial^3 LogL(\theta_1, \theta_2)}{\partial\theta_l^3}, L_{111} = \frac{4n_1}{\theta_1^3} - \frac{2n_1}{(1 + \theta_1)^3}, \\ L_{222} = \frac{4n_2}{\theta_2^3} - \frac{2n_2}{(1 + \theta_2)^3}, L_{112} = L_{122} = 0,\end{aligned}$$

Let,

$$\begin{aligned}R_{r,k} =& k\begin{pmatrix} n_2 \\ k \end{pmatrix}\sum_{j=r}^{n_1}\sum_{l_1=0}^{n_1-j}\sum_{l_2=0}^{n_2-k}\sum_{l_3=0}^{k-1}\sum_{l_4=0}^{k-l_3-1}\sum_{l_5=0}^{j}\sum_{l_6=0}^{j-l_5}(-1)^{k+j-l_3-l_5-1}\begin{pmatrix} n_1 \\ j \end{pmatrix} \\ &\times\begin{pmatrix} n_1 - j \\ l_1 \end{pmatrix}\begin{pmatrix} n_2 - k \\ l_2 \end{pmatrix}\begin{pmatrix} k - 1 \\ l_3 \end{pmatrix}\begin{pmatrix} k - l_3 - 1 \\ l_4 \end{pmatrix} \\ &\times\begin{pmatrix} j \\ l_5 \end{pmatrix}\begin{pmatrix} j - l_5 \\ l_6 \end{pmatrix}ABCDE\end{aligned}$$

where:

$$A = \frac{q^2}{(1 + q)}, B = (\frac{p}{1 + p})^{n_1-l_1-l_5-l_6}, C = (\frac{q}{1+})^{n_2-l_2-l_3-l_4-1}$$

$$D = (\frac{\Gamma(n_1 + n_2 - l_1 - l_2 - l_3 - l_4 - l_5 - l_6)}{(q(n_2 - l_3) + p(n_1 - l_5))^{(n_1+n_2-l_1-l_2-l_3-l_4-l_5-l_6}}),$$

$$E = (\frac{n_1 + n_2 - l_1 - l_2 - l_3 - l_4 - l_5 - l_6}{(q(n_2 - l_3) + p(n_1 - l_5))} + 1), R_l = \frac{\partial R_{r,k}}{\partial \theta_l}.$$

Where,

$$\frac{\partial R_{r,k}}{\partial \theta_1} = \binom{n_2}{k} \sum_{j=r}^{n_1} \sum_{l_1=0}^{n_2-k} \sum_{l_2=0}^{n_1-j} \sum_{l_3=0}^{k-1} \sum_{l_4=0}^{k-l_3-1} \sum_{l_5=0}^{j} \sum_{l_6=0}^{j-l_5} k(-1)^{k+j-l_3-l_5-1} \binom{n_1}{j}$$
$$\times \binom{n_2-k}{l_1} \binom{n_1-j}{l_2} \binom{k-1}{l_3} \binom{k-l_3-1}{l_4} \binom{j}{l_5}$$
$$\times \binom{j-l_5}{l_6} \left(BCDE\frac{dA}{d\theta_1} + ACDE\frac{dB}{d\theta_1} + ABCE\frac{\partial D}{\partial \theta_1} + ABCD\frac{\partial D}{\partial \theta_1} \right)$$

$$\frac{\partial R_{r,k}}{\partial \theta_2} = \binom{n_2}{k} \sum_{j=r}^{n_1} \sum_{l_1=0}^{n_2-k} \sum_{l_2=0}^{n_1-j} \sum_{l_3=0}^{k-1} \sum_{l_4=0}^{k-l_3-1} \sum_{l_5=0}^{j} \sum_{l_6=0}^{j-l_5} k(-1)^{k+j-l_3-l_5-1} \binom{n_1}{j}$$
$$\times \binom{n_2-k}{l_1} \binom{n_1-j}{l_2} \binom{k-1}{l_3} \binom{k-l_3-1}{l_4} \binom{j}{l_5}$$
$$\times \binom{j-l_5}{l_6} \left(ABDE\frac{dC}{d\theta_2} + ABCE\frac{\partial D}{\partial \theta_2} + ABCD\frac{\partial D}{\partial \theta_2} \right)$$

Similarly we can calculate $R_{ll} = \frac{\partial^2 R}{\partial \theta_l^2}$. Note that all terms and derivatives written with a hat are calculated by replacing θ_1, θ_2, and R with their maximum likelihood estimators.

4. Simulation Study

In this section we perform a simulation study to:

1. Study the behavior of $\hat{R}_{r,k}^{MLE}$ by using different sample sizes. The average bias and average mean square error are computed. Also the average confidence length of the simulated 95% confidence intervals are computed.

2. We Lindley's approximation to compute $\hat{R}_{r,k}^{B}$. Also average bias and average mean square error are computed.

3. Compare the performance of $\hat{R}_{r,k}^{MLE}$ and $\hat{R}_{r,k}^{B}$.

4.1. Simulation Study to Investigate the Behavior of $\hat{R}_{r,k}^{MLE}$

To study the behavior of $\hat{R}_{r,k}^{MLE}$, we use the following steps

1. Compute the true value of $R_{r,k}$ with given parameters $(p,q) = (3,1.5)$, (2.5,1.5), (2,1.5), (1.5,1.5), (1.5,2), (1.5,2.5),(1.5,3) and the sample sizes $n_1 = n_2 = 5$, 10, 30.
2. For the given sample sizes and given parameters in (1) generate random samples from the Lindley distribution
3. Compute$\hat{p}_{MLE}$,$\hat{q}_{MLE}$ and $\hat{R}_{r,k}^{MLE}$.
4. Repeat (1) and (2) $N = 10^4$ times.
5. Compute Bias and mean square error (MSE).

***Note* 1.** To avoid the difficulty of computations, we take $r = 1$, $k = 3$ and perform the study for $R_{1,3}$.

From Table 1, we can observe that the bias decreases as p decreases and q becomes fixed; it also decreases as q increases and p becomes fixed. Also, MSE decreases as the sample sizes increases.

Table 1: The average bias and average mean square error for different sample sizes for $R_{1,3}$.

(p,q)	Sample Sizes					
	$n_1 = n_2$ =5		$n_1 = n_2$ =10		$n_1 = n_2 = 30$	
	Bias	MSE	Bias	MSE	Bias	MSE
(3,1.5)	−0.0016	0.0001	−0.0006	0.0000	0.0003	0.0000
(2.5,1.5)	−0.0029	0.0004	−0.0005	0.0000	0.0004	0.0000
(2,1.5)	−0.0029	0.0004	−0.0036	0.0001	−0.0008	0.0002
(1.5,1.5)	0	0	0	0	0	0
(1.5,2)	0.0099	0.0049	0.0152	0.0023	0.0016	0
(1.5,2.5)	0.02522	0.0031	0.0025	0.0006	−0.0027	0.0002
(1.5,3)	0.0233	0.0027	0.0056	0.0003	−0.0051	0.0007

From Table 2, we can observe that the average confidence length decreases as p decreases and q becomes fixed; it also decreases as q increases and p becomes fixed.

Table 2: Average confidence length of the simulated 95% confidence intervals of $R_{1,3}$.

(p,q)	Sample Sizes		
	$n_1 = n_2$ =5	$n_1 = n_2$ =10	$n_1 = n_2$ =30
(3,1.5)	0.3096	0.1502	0.0161
(2.5,1.5)	0.1108	0.1799	0.0233
(2,1.5)	0.0271	0.0963	0.0118
(1.5,1.5)	0.3174	0.0535	0.0069
(1.5,2)	0.0056	0.1153	0.0110
(1.5,2.5)	0.0038	0.064	0.0260
(1.5,3)	0.0022	0.0345	0.0256

4.2. Study the behavior of $\hat{R}^{B}_{r,k}$

To study the behavior of $\hat{R}^{B}_{r,k}$ we use the following algorithm

1. Equation (19) is used to compute $\hat{R}^{B}_{r,k}$ when the joint prior $\pi(\theta_1, \theta_2) = \pi_j(\theta_1, \theta_2)$
2. Repeat (1) $N = 10^4$ times.
3. Compute bias and mean square error (MSE).

***Note* 2.** Note: we use the sample sizes $n_1 = n_2 = 5$, 10.

From Table 3, we can observe that the MSE decreases as the sample size increases. Also it decreases as q increases and p becomes fixed, and increases as p decreases and q becomes fixed. Also, from Table 1 and Table 3 we can observe that the MSE of $\hat{R}^{MLE}_{r,k}$ is less than $\hat{R}^{B}_{r,k}$.

Table 3: The average bias and average mean square error for different sample sizes for $R_{1,3}$(Bayes estimator).

(p, q)	Sample Sizes			
	$n_1 = n_2 = 5$		$n_1 = n_2 = 10$	
	Bias	MSE	Bias	MSE
(3,1.5)	−0.0205	0.0021	−0.0026	0.0000
(2.5,1.5)	−0.0304	0.0046	−0.0035	0.0001
(2,1.5)	−0.0323	0.0052	−0.0003	0.0000
(1.5,1.5)	−0.0041	0.0000	−0.0127	0.0016
(1.5,2)	−0.0411	0.0084	0.0000	0.0000
(1.5,2.5)	−0.0077	0.0003	0.0000	0.0000
(1.5,3)	−0.0050	0.0001	−0.0443	0.0196

5. Data Analysis

To decide whether the proposed model in the previous section can be used in practice, we consider two real data sets reported by Lawless (1982) and Proschan (1963). The first data set is obtained from Lawless (1982) and it represents the number of revolutions before the failure of 23 ball bearings in life tests, which are as follows:

Data Set I: 17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.80, 51.84, 51.96, 54.12, 55.56, 67.80, 68.44, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12,105.84, 127.92, 128.04, 173.40.

The second data set is obtained from Proschan (1963) and represents the times between the successive failures of 15 air conditioning (AC) units in a Boeing 720 airplane, which are as follows:

Data Set II:12, 21, 26, 27, 29, 29, 48, 57, 59, 70, 74, 153, 326, 386, 502.

To estimate the stress strength model using the above data sets, we use the following steps:

1. Check the validity of the Lindley distribution for given data sets by using the Kolmogrov-Smirnov (K-S) test.
2. Find the maximum likelihood estimators for p and q.
3. Compute the maximum likelihood estimator of $R_{r,k}$ and asymptotic confidence interval.

Table 4 and Figures 2 and 3 show the result of goodness of fit and the maximum estimators of $R_{r,k}$, p, and q.

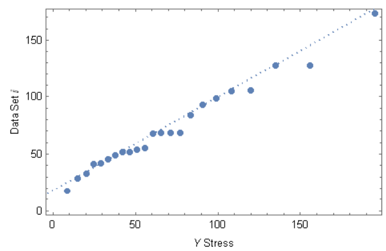

FIGURE 2: Fitted Lindley distribution for data set I.

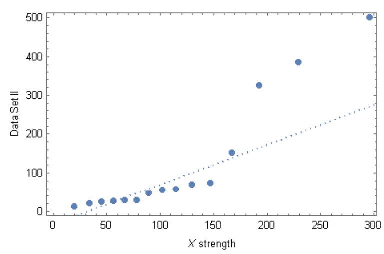

FIGURE 3: Fitted Lindley distribution for data set II.

TABLE 4: The model fitting summary for both the data sets.

Data set	$K-S$	$p-value$
Data set (1)	0.1928	0.318
Data set (2)	0.102	0.698

TABLE 5: Data analysis results.

Parameters	(r,k)	$(\hat{p}_{MLE},\hat{q}_{MLE})$	$\hat{R}_{r,k}^{MLE}$	$95\%ACI$
(p,q)	(1,3)	(0.0273,0.0163)	0.215	(0.0384,0.3915)

6. Conclusions

In this paper, we have considered the problem of estimation reliability in a multicomponent stress-strength model $R_{r,k} = P[Y_{r:n_1} < X_{k:n_2}]$ for which the stress and strength variables are given by a Lindley distribution. The three estimation methods of $R_{r,k}$ applied were the maximum likelihood, the uniformly minimum variance unbiased, and the Bayes estimators. By simulation we made a comparison between the maximum likelihood and Bayes estimators. In both estimators the mean square error decreases as sample sizes increases. Also, the maximum likelihood estimator has a mean square error that is less than the Bayes estimator, as can be seen in Table 1 and Table 3. Real data was used as a practical application of the proposed model. Finally we recommend that the Lindley distribution is used as the multicomonent stress-strength model.

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