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Optimal Shrinkage Estimations for the Parameters of Exponential Distribution Based on Record Values

Estimación shrinkage de los parámetros de la distribución exponencial basada en valores record

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Abstract

This paper studies shrinkage estimation after the preliminary test for the parameters of exponential distribution based on record values. The optimal value of shrinkage coefficients is also obtained based on the minimax regret criterion. The maximum likelihood, pre-test, and shrinkage estimators are compared using a simulation study. The results to estimate the scale parameter show that the optimal shrinkage estimator is better than the maximum likelihood estimator in all cases, and when the prior guess is near the true value, the pre-test estimator is better than shrinkage estimator. The results to estimate the location parameter show that the optimal shrinkage estimator is better than maximum likelihood estimator when a prior guess is close to the true value. All estimators are illustrated by a numerical example.

***Key words*:** Exponential Distribution, Minimax Regret, Record Value, Risk Function, Shrinkage Estimator.

Resumen

Este artículo estudia la estimación shrinkage posterior al test preliminar de los parámetros de la distribución exponencial basada en valores record. El valor óptimo de los coeficientes de shrinkage es obtenido también usando el criterio minimax regret. La máxima verosimilitud, pre-test, y los estimadores shrinkage son obtenidos usando estudios de simulación. Los resultados de la estimación del parámetro de escala muestran que el estimador shrinkage es major que el de máxima verosimilitud en todos los casos, y cuando el valor a priori es cercano del valor real, el estimador pre-test es major que el

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estimador shrinkage. Los resultados de estimación del parámetro de localización muestran que el estimador de shrinkage óptimo es major que el de máxima verosimilitud cuando el valor a priori es cercano al real. Todos los estimadores son ilustrados con un ejemplo numérico.

***Palabras clave*:** estimador shrinkage, distribución exponencial, minimax regret, función de riesgo, valor record.

1. Introduction

The unknown parameters in a distribution can usually be estimated by the maximum likelihood estimator (MLE) or the uniformly minimum-variance unbiased estimator (UMVUE). These estimators are solely based on the sample information. Sometimes, prior (non-sample) information about the parameters is available from previous experience or expert knowledge. To incorporate the non-sample information into the statistical procedure, a null hypothesis regarding the information is usually formulated and tested (see e.g. Bancroft 1944, Bancroft & Han 1977, Han, Rao & Ravichandran 1988). To estimate the parameter of interest, both sample information and prior information are used in shrinkage estimation when the null hypothesis is not rejected. However, if the null hypothesis is rejected, then only the sample information is used in the estimation.

The two-parameter exponential distribution has been used widely in the field of life testing and reliability theory. The shrinkage estimators of the scale parameter have been proposed by Bhattacharya & Srivastava (1974) and Pandey (1983). Chiou & Han (1989) gave a shrinkage estimation for threshold parameter. Chiou & Han (1995) proposed a pre-test estimator and a pre-test shrinkage estimator (PTSE) for the location parameter. Chiou (1990) gave an estimation of scale parameter for two exponential distributions based on censored data. The shrinkage estimation for the difference between location parameters for two exponential distribution is given by Chiou & Miao (2005). Using a LINEX loss function, the performance of the shrinkage testimators for the scale parameter of an exponential distribution is studied by Pandey (1997) and Prakash & Singh (2008).

Record values are of interest and important in many real life applications involving data relating to meteorology, sport, economics and life testing. Let $X_1, X_2, \dots$ be a sequence of independent and identically distributed random variables that has a same distribution. An observation X_j will be called an upper record value if exceeds in value all of the preceding observations, i.e., if $X_j > X_i$, for every $i < j$. The sequence of record times $T_n, n \geq 0$ is defined as follows: $T_1 = 1$ with probability 1 and, for ≥ 1, $T_n = \min\{j : X_j > X_{T_{n-1}}\}$. The sequence of upper record values is then defined by $\{X_{U(n)} = X_{T_n},\, n = 0, 1, 2, \dots\}$. For details on record values and other interesting topics related to records see Ahsanullah (1995) and Arnold, Balakrishnan & Nagaraja (1998).

In this paper, the PTSE for the parameters of exponential distribution is evaluated based on record values. The optimal value of shrinkage coefficient is obtained by using the minimax regret criterion. This paper is organized as follows: In Section 2, we will give a pre-test estimation for the scale parameter of exponential distribution based on record values. Then, the optimal value of the coefficient in PTSE for this parameter is obtained. Similarly, the optimal value of the coefficient in PTSE for the location parameter in two exponential distribution is obtained in Section 3. The PTSE and MLE are compared in Section 4 using Monte Carlo simulation. These estimators are also illustrated using a numerical example in Section 5.

2. Optimal Shrinkage Estimation for the Scale Parameter

Suppose that we observe n upper record values $X_{U(1)}, \ldots, X_{U(n)}$ from an exponential model with the following probability density function (pdf):

$$f(x) = \frac{1}{\theta} \exp(-\frac{x}{\theta}), \qquad x > 0.$$

It is well-known that $X_{U(n)}$ is a complete sufficient statistic for θ. Also, $\widehat{\theta}_{ML} = \frac{1}{n} X_{U(n)}$ is the MLE of θ, and $\frac{2n\widehat{\theta}_{ML}}{\theta}$ has a chi-square distribution with $2n$ degrees of freedom (see Arnold et al. 1998). Also, to test $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$, the likelihood ratio test reject H_0 when $\frac{2n\widehat{\theta}_{ML}}{\theta_0} < C_1$ or $\frac{2n\widehat{\theta}_{ML}}{\theta_0} > C_2$ where $C_1 = \chi^2_{2n,\alpha/2}$ and $C_2 = \chi^2_{2n,1-\alpha/2}$ where $\chi^2_{2n,\gamma}$ is the γth quantile of chi-square with $2n$ degrees of freedom.

Assume that θ_0 is a prior guess of θ. A pre-test estimator for θ based on $\widehat{\theta}_{ML}$ is given as

$$\widehat{\theta}_{pt} = \begin{cases} \theta_0 & C_1 < \frac{2X_{U(n)}}{\theta_0} < C_2 \\ \widehat{\theta}_{ML} & \text{otherwise.} \end{cases} \tag{1}$$

Zakerzadeh & Karimi (2013) showed that under the weighted square error loss $L(\theta; d) = (\frac{d}{\theta} - 1)^2$ the risk function of this estimator is

$$Risk(\widehat{\theta}_{pt}, \theta) = -\frac{A_2}{4n^2} + \frac{A_1}{n} + (\delta^2 - 2\delta)A_0 + \frac{1}{n},$$

where $A_j = \int_{C_1\delta}^{C_2\delta} w^j g(w)dw$, $j = 0, 1, 2$, and $\delta = \frac{\theta_0}{\theta}$, and $g(.)$ is the pdf of chi-square random variable with $2n$ degrees of freedom. Note that the risk function depends on α through C_1 and C_2.

The proposed pre-test estimator in (1), uses θ_0 to estimate θ when the preliminary test fails to reject the null hypothesis $H_0 : \theta = \theta_0$. Instead of using θ_0, we can use a linear combination of θ_0 and $\widehat{\theta}_{ML}$ when the preliminary test fails to reject H_0 (when $\delta \approx 1$). This gives a shrinkage estimator which assigns suitable weights to θ_0 and $\widehat{\theta}_{ML}$. The estimator is

$$\widehat{\theta}_s = \begin{cases} K\theta_0 + (1-K)\widehat{\theta}_{ML} & C_1 < \frac{2X_{U(n)}}{\theta_0} < C_2 \\ \widehat{\theta}_{ML} & \text{otherwise,} \end{cases} \tag{2}$$

where $0 \leq K \leq 1$. Note that $\widehat{\theta}_{pt}$ approaches θ_0 as $\alpha \to 0$ and it approaches $\widehat{\theta}_{ML}$ as $\alpha \to 1$; however, $\widehat{\theta}_s$ approaches $\widehat{\theta}_{pt}$ as $K \to 1$ and it approaches $\widehat{\theta}_{ML}$ as $K \to 0$.

The shrinkage coefficient, K is not defined explicitly as a function of the test statistic. The weighting function approach is intuitively appealing, but the mean square error of the resulting estimator usually cannot be derived unless the weighting function is in some simple form. Unfortunately, a different value of significance level (α) or a different value of shrinkage coefficient (K) results in a different estimator. The choice of these values depends on the decision criterion. Here, we obtain an optimal value of K for $\widehat{\theta}_s$, based on a regret function. At first we evaluate the mean square error of $\widehat{\theta}_s$.

Lemma 1. *For a fixed value of α, the mean square error of $\widehat{\theta}_s$ is a function of K and take the following form:*

$$MSE(\widehat{\theta}_s) = K^2 G_2^*(\delta) + K G_1^*(\delta) + \frac{\theta^2}{n}, \tag{3}$$

where $\delta = \frac{\theta_0}{\theta}$, $G_1^*(\delta) = \frac{\theta^2+\theta_0\theta}{n} A_1 - \frac{\theta^2}{2n^2} A_2 - 2\theta\theta_0 A_0$, and $G_2^*(\delta) = \frac{\theta^2}{4n^2} A_2 - \frac{\theta_0\theta}{n} A_1 + \theta_0^2 A_0$.

Proof. Consider $\delta = \frac{\theta_0}{\theta}$. Then

$$\begin{aligned}
E(\widehat{\theta}_s) &= E\left[\left(K\theta_0 + (1-K)\widehat{\theta}_{ML}\right) I\left(C_1\delta < \frac{2X_{U(n)}}{\theta} < C_2\delta\right)\right] \\
&+ E\left[\widehat{\theta}_{ML}\left(1 - I\left(C_1\delta < \frac{2X_{U(n)}}{\theta} < C_2\delta\right)\right)\right] \\
&= K\theta_0 A_0 + \theta - \frac{K\theta}{2n} A_1,
\end{aligned}$$

where $I(.)$ is the indicator function. Similarly the second moment of $\widehat{\theta}_s$ is

$$E(\widehat{\theta}_s^2) = K^2 {\theta_0}^2 A_0 + \theta^2(1 + \frac{1}{n}) + \frac{\theta^2}{4n^2}(K^2 - 2K) A_2 + \frac{\theta_0\theta}{n} K(1-K) A_1,$$

and the proof is completed. □

Corollary 1. *Under the weighted square error loss function $L(\hat{\theta};\theta) = (\frac{\hat{\theta}}{\theta} - 1)^2 = \frac{(\hat{\theta}-\theta)^2}{\theta^2}$, the risk function of $\widehat{\theta}_s$ is*

$$R_1(\delta, K) = K^2 G_2(\delta) + K G_1(\delta) + \frac{1}{n}, \tag{4}$$

where $G_1(\delta) = \frac{1+\delta}{n} A_1 - \frac{1}{2n^2} A_2 - 2\delta A_0$, and $G_2(\delta) = \frac{1}{4n^2} A_2 - \frac{\delta}{n} A_1 + \delta^2 A_0$.

Theorem 1. *If $G_2(\delta) \leq 0$, then the infimum value of $R_1(\delta$, w.r.t. K is the minimum value of $R_1(\delta, 0)$ and $R_1(\delta, 1)$. If $G_2(\delta) > 0$ then the infimum value of $R_1(\delta, K)$ occurs at $K_0 = \frac{-G_1(\delta)}{2G_2(\delta)}$ if $K_0 \in (0,1)$, and it occurs at $K = 0$ or $K = 1$ if $K_0 \notin (0,1)$, and therefore,*

$$\inf_K R_1(\delta, K) = \begin{cases} \min\{R_1(\delta,0), R_1(\delta,1), R_1(\delta,K_0)\} & \text{if } K_0 \in (0,1) \\ \min\{R_1(\delta,0), R_1(\delta,1)\} & \text{otherwise.} \end{cases} \tag{5}$$

***Proof*.** The proof is straight forward. □

The regret function is defined as

$$REG_1(\delta, K) = R_1(\delta, K) - \inf_K R_1(\delta, K).$$

Chiou & Han (1989) also discussion the same topic: the regret function $REG(\delta, K)$ takes a maximum value at δ_L and δ_U, for fixed α (see Figure 1). Thus the minimax regret criterion determines K^*, optimal value of K, such that

$$REG_1(\delta_L, K^*) = REG_1(\delta_U, K^*).$$

Considering $n = 5$ and $\alpha = 0.05$, the optimal K is 0.132. The plot of risk functions for $K = 0, 1, 0.132$ is shown in Figure 1. To find K^*, we consider two cases:

Case I: The value of K^* for some degrees of freedom is presented in Table 1 for $\alpha = 0.05$.

Case II: The value of K^* for some degrees of freedom is presented in Table 1 for $\alpha = 0.16$. For the AIC optimal level of significance see Inada (1984).

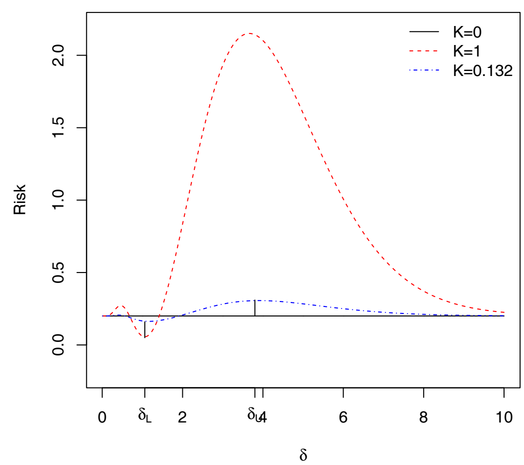

FIGURE 1: The risk function of K with $n = 5$ and $\alpha = 0.05$.

TABLE 1: Optimal value of K for some n in shrinkage estimation for the scale parameter.

	$df = 2n$												
α	4	6	8	10	12	14	16	18	20	22	24	26	28
0.05	0.059	0.089	0.113	0.132	0.146	0.158	0.168	0.176	0.184	0.190	0.195	0.200	0.205
0.16	0.079	0.114	0.139	0.158	0.172	0.184	0.194	0.203	0.210	0.216	0.222	0.227	0.231

***Note* 1.** In general, if we observe n upper record values $X_{U(1)}, \dots, X_{U(n)}$ from location-scale exponential distribution

$$f(x) = \frac{1}{\theta} \exp\{-\frac{1}{\theta}(x - \eta)\}, \qquad x > \eta, \tag{6}$$

the MLE of θ based on record values is $\widehat{\theta}_{ML} = \frac{X_{U(n)} - X_{U(1)}}{n}$, and $\frac{2n\widehat{\theta}_{ML}}{\theta}$ has a chi-square distribution with $2n-2$ degrees of freedom. In this case, we got the same result as with the special case, in absence of location parameter with a new statistic and degree of freedom.

3. Optimal Shrinkage Estimation for the Threshold Parameter

In the following discussion we always suppose that we observe n upper record values $X_{U(1)}, \dots, X_{U(n)}$ from an exponential model with the pdf in (6).

It is well-known that under the hypothesis $H_0 : \eta = \eta_0$,

$$F_0 = \frac{(n-1)(X_{U(1)} - \eta_0)}{X_{U(n)} - X_{U(1)}} \sim F_{(2,2n-2)},$$

and the likelihood ratio test rejects $H_0 : \eta = \eta_0$ against $H_1 : \eta \neq \eta_0$, when $F_0 > F_{(2,2n-2),1-\alpha}$. Therefore, by assuming a prior guess such as $\eta = \eta_0$, a pre-test estimator for η is given as

$$\widehat{\eta}_{pt} = \begin{cases} \eta_0 & 0 < F_0 < c \\ \widehat{\eta}_{ML} & \text{otherwise,} \end{cases} \tag{7}$$

where $c = F_{(2,2n-2),1-\alpha}$ and $\widehat{\eta}_{ML} = X_{U(1)}$. The properties of $\widehat{\eta}_{pt}$ depends on whether $\eta < \eta_0$ or $\eta > \eta_0$.

The pre-test estimator given in (7) uses the prior estimate η_0 when the pre-test accepts the null hypothesis. Instead of using η_0, we can use a linear combination of η_0 and $\widehat{\eta}$ when the pre-test accepts , this which gives a PTSE which assigns suitable weights to η_0 and $\widehat{\eta}$ rather than assigning weight 1 to η_0 and 0 to $\widehat{\eta}$ when the pre-test accepts. This PTSE is

$$\widehat{\eta}_s = \begin{cases} K\eta_0 + (1-K)\widehat{\eta}_{ML} & 0 < F_0 < c \\ \widehat{\eta}_{ML} & \text{otherwise.} \end{cases} \tag{8}$$

Lemma 2. *For PTSE of η, $\widehat{\eta}_s$, we have*

$$\begin{aligned} E(\widehat{\eta}_s) &= \eta + \theta + K(\eta_0 - \eta)D_1 - KD_2, \\ E(\widehat{\eta}_s^2) &= 2\theta^2 + \eta^2 + 2\theta\eta + \left[K^2(\eta_0-\eta)^2 - 2K\eta^2 + 2K\eta_0\eta\right] D_1 \\ &\quad + \left[-2K^2(\eta_0 - \eta) + 2K\eta_0 - 4K\eta\right] D_2 + (K^2 - 2K)D_3, \end{aligned}$$

where

i. $D_1 = e^{\delta}(1 - d^{(n-1)})$, $D_2 = \theta e^{\delta}\left[(-\delta+1) - (-\delta+1)d^{n-1} - cd^n\right]$, $D_3 = e^{\delta}\left[(\eta_0-\eta)^2 - (\eta_0-\eta)^2 d^{n-1} - \frac{nc^2\theta^2}{n-1}d^{n+1} - 2(\eta_0 - \eta)c\theta d^n\right] + 2\theta D_2$ *and* $d = \frac{1}{1+\frac{c}{n-1}}$, *when* $\eta \leq \eta_0$, *ii.* $D_i = \iint_H w^i f(w,t)dwdt$, $i = 0, 1, 2$, *and* $H = \{(w,t) : 0 \leq w \leq \frac{ct}{n-1} + \eta_0 - \eta, t > \frac{n-1}{c}(\eta - \eta_0)\}$, *when* $\eta > \eta_0$.

Proof. i) Let $W = X_{U(1)} - \eta$, $T = X_{U(n)} - X_{U(1)}$, $A = \left\{(w,t) : \eta_0 - \eta \leq w \leq \frac{ct}{n-1} + \eta_0 - \eta\right\}$ and A^c be the complement of A. Then

$$\begin{aligned}
E(\widehat{\eta}_s) &= E\left((K\eta_0 + (1-K)(W+\eta))I(0 < F_0 < c)\right) + E\left((W+\eta)(1 - I(0 < F_0 < c))\right)\\
&= E(W+\eta) + \iint_A \left(K\eta_0 + (1-K)(w+\eta)\right) f(w,t)dwdt - \iint_A (w+\eta) f(w,t)dwdt,\\
&= \theta + \eta + K(\eta - \eta_0)\iint_A f(w,t)dwdt - K\iint_A wf(w,t)dwdt,
\end{aligned}$$

$$\begin{aligned}
E(\widehat{\eta}_s^2) &= E\left((K\eta_0 + (1-K)(W+\eta))^2 I(0 < F_0 < c)\right) + E\left((W+\eta)^2(1 - I(0 < F_0 < c))\right)\\
&= E((W+\eta)^2) + \iint_A \left(K\eta_0 + (1-K)(w+\eta)\right)^2 f(w,t)dwdt\\
&- \iint_A (w+\eta)^2 f(w,t)dwdt,\\
&= 2\theta^2 + \eta^2 + 2\theta\eta + \left(K^2\eta_0^2 + K^2\eta^2 - 2K\eta^2 + 2K(1-K)\eta_0\eta\right)\iint_A f(w,t)dwdt\\
&+ \left(2K^2 - 4K\eta + 2K(1-K)\eta_0\right)\iint_A wf(w,t)dwdt\\
&+ (K^2 - 2K)\iint_A w^2 f(w,t)dwdt,
\end{aligned}$$

where $f(w,t)$ is the joint pdf of W and T. Note that W has an exponential distribution with parameter θ and T has a gamma distribution with parameters $n-1$ and θ. Also, they are independent random variables. Therefore, if $\eta \leq \eta_0$,

$$\begin{aligned}
\iint_A f(w,t)dwdt &= \int_0^\infty \int_{\eta_0-\eta}^{\eta_0-\eta+\frac{ct}{n-1}} \frac{1}{\theta} e^{-w/\theta} f_T(t)dwdt\\
&= e^{\frac{-(\eta_0-\eta)}{\theta}} \int_0^\infty \left(1 - e^{-\frac{ct}{(n-1)\theta}}\right) f_T(t)dt\\
&= e^{\frac{-(\eta_0-\eta)}{\theta}} \left(1 - (1 + \frac{c}{n-1})^{-(n-1)}\right) = D_1,
\end{aligned}$$

$$\begin{aligned}
\iint_A wf(w,t)dwdt &= \int_0^\infty \int_{\eta_0-\eta}^{\eta_0-\eta+\frac{ct}{n-1}} \frac{w}{\theta} e^{-w/\theta} f_T(t)\,dw\,dt\\
&= \int_0^\infty \Big[(\eta_0 - \eta)e^{-(\eta_0-\eta)/\theta} - (\eta_0 - \eta + \frac{ct}{n-1})e^{-(\eta_0-\eta+\frac{ct}{n-1})/\theta}\\
&+ \theta e^{\frac{-(\eta_0-\eta)}{\theta}} - \theta e^{-(\eta_0-\eta+\frac{ct}{n-1})/\theta}\Big] f_T(t)dt
\end{aligned}$$

$$
\begin{aligned}
&= \theta e^{\frac{-(\eta_0-\eta)}{\theta}} \Bigg[(\frac{\eta_0-\eta}{\theta}+1) - (\frac{\eta_0-\eta}{\theta}+1)(1+\frac{c}{n-1})^{-(n-1)} \\
&- c(1+\frac{c}{n-1})^{-n} \Bigg] = D_2,
\end{aligned}
$$

$$
\begin{aligned}
\iint_A w^2 f(w,t)dwdt &= \int_0^\infty \left[\int_{\eta_0-\eta}^{\eta_0-\eta+\frac{ct}{n-1}} \frac{w^2}{\theta} e^{-w/\theta} dw \right] f_T(t)dt \\
&= e^{-(\frac{\eta_0-\eta}{\theta})} \Bigg[(\eta_0-\eta)^2 - (\eta_0-\eta)^2(1+\frac{c}{n-1})^{-(n-1)} \\
&- \frac{nc^2\theta^2}{n-1}(1+\frac{c}{n-1})^{-(n+1)} - 2(\eta_0-\eta)c\theta(1+\frac{c}{n-1})^{-n} \Bigg] + 2\theta D_2 \\
&= D_3,
\end{aligned}
$$

and the proof is completed.

ii) The proof is similar to part i).

□

In this section, we study the PTSE $\widehat{\eta}_s$ following the same procedure given in Section 2. Consider the loss function $L(\eta,\hat{\eta}) = \frac{(\eta-\hat{\eta})^2}{\theta^2}$. In the following lemma, the risk of $\widehat{\eta}_s$ is calculated under this loss function and is denoted by $R_2(\delta, K)$.

Lemma 3. *i) Let $\eta \leq \eta_0$. Then*

$$
\begin{aligned}
R_2(\delta,K) &= e^{\delta}(1-d^{n-1})[2K^2+2K\delta-4K] + e^{\delta}d^n[4Kc-2K\delta c-2cK^2] \\
&+ e^{\delta}d^{n+1}\left[\frac{nc^2}{n-1}(-K^2+2K)\right\} + 2
\end{aligned}
$$

ii) Let $\eta > \eta_0$. Then

$$
\begin{aligned}
R_2(\delta,K) &= E_3[2K^2-4K+2(K^2-K)\delta] + d^{n-1}E_2[-K^2\delta^2-2K\delta-2K^2+4K] \\
&+ cd^nE_1[K^2\delta^2-2K\delta-2K^2+4K] + \frac{c^2nd^{n+1}}{n-1}E_0[-K^2+2K] + 2.
\end{aligned}
$$

where $E_i = \sum_{j=0}^{n-i} \frac{1}{j!} e^{-\frac{\delta(n-1)}{c}} (\delta(1+\frac{n-1}{c}))^j$, $i=0,1,2$ and
$E_3 = \sum_{j=0}^{n-2} \frac{1}{j!} e^{-\frac{\delta(n-1)}{c}} (\frac{\delta(n-1)}{c})^j$.

Proof. The proof is obvious. □

Using the regret function and similar to Section 2, we obtained the optimal K. We consider two cases:

Case I: The optimal value of K for some n are presented in Table 2 for $\alpha = 0.05$.

Case II: The optimal value of K for some n are presented in Table 2 for $\alpha = 0.16$, which is AIC optimal level of significance (see Inada 1984).

TABLE 2: Optimal value of K for some n in shrinkage estimation for the location parameter.

	n															
α	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0.05	0.44	0.21	0.24	0.28	0.30	0.32	0.33	0.34	0.34	0.35	0.35	0.35	0.36	0.36	0.36	0.36
0.16	0.31	0.33	0.40	0.45	0.49	0.51	0.53	0.55	0.56	0.57	0.58	0.59	0.60	0.61	0.62	0.63

4. Simulation Study

We performed a simulation study and generated record values from an exponential distribution with $\eta = 0$ and $\theta = 1$ with $n = 6, 11, 16$. The simulation was repeated 2000 times, and we obtained $\hat{\theta}_{ML}$, $\hat{\theta}_{pt}$, and $\hat{\theta}_s$, for some prior guess θ_0 of θ. We also, obtained $\hat{\eta}_{ML}$, $\hat{\eta}_{pt}$, and $\hat{\eta}_s$, for some prior guess η_0 of η. The bias and MSE of the estimators are calculated, and the results are presented in Tables 4 and 5.

From Table 4, it can be concluded that i) $\hat{\theta}_s$ is better than $\hat{\theta}_{ML}$ in all cases, ii) for prior guess (θ_0) near the true value of $\theta(= 1)$, $\hat{\theta}_{pt}$ is better than $\hat{\theta}_s$ and iii) for prior guess (θ_0) far from the true value of $\theta(= 1)$, $\hat{\theta}_s$ is better than $\hat{\theta}_{pt}$.

From Table 5, it can be concluded that i) $\hat{\eta}_{ML}$ is better than $\hat{\eta}_s$ when η_0 (prior guess) is very far from the true value of $\eta(= 0)$, otherwise $\hat{\eta}_s$ is better than $\hat{\eta}_{ML}$, and ii) $\hat{\eta}_s$ is better than $\hat{\eta}_{pt}$ for η_0 less than the true value of η, and vice versa.

TABLE 3: Bias and MSE of the estimators for the scale parameter.

		Bias			MSE		
n	θ_0	$\widehat{\theta}_{ML}$	$\widehat{\theta}_{pt}$	$\widehat{\theta}_s$	$\widehat{\theta}_{ML}$	$\widehat{\theta}_{pt}$	$\widehat{\theta}_s$
6	0.2	-0.176	-0.180	-0.177	0.167	0.172	0.167
	0.4	-0.176	-0.209	-0.175	0.167	0.194	0.161
	0.6	-0.176	-0.220	-0.187	0.167	0.164	0.163
	0.8	-0.176	-0.132	-0.167	0.167	0.082	0.147
	1.0	-0.176	0.007	-0.140	0.167	0.033	0.134
	1.2	-0.176	0.165	-0.131	0.167	0.073	0.128
	1.4	-0.176	0.307	-0.100	0.167	0.195	0.137
	1.6	-0.176	0.435	-0.095	0.167	0.374	0.136
	1.8	-0.176	0.559	-0.071	0.167	0.602	0.141
11	0.2	-0.084	-0.084	-0.084	0.094	0.094	0.094
	0.4	-0.084	-0.103	-0.088	0.094	0.111	0.095
	0.6	-0.084	-0.145	-0.094	0.094	0.121	0.093
	0.8	-0.084	-0.119	-0.096	0.094	0.069	0.085
	1.0	-0.084	0.002	-0.069	0.094	0.020	0.064
	1.2	-0.084	0.150	-0.045	0.094	0.067	0.072
	1.4	-0.084	0.272	-0.020	0.094	0.175	0.080
	1.6	-0.084	0.343	-0.018	0.094	0.320	0.088
	1.8	-0.084	0.376	-0.001	0.094	0.464	0.107
16	0.2	-0.070	-0.070	-0.070	0.061	0.061	0.061
	0.4	-0.070	-0.077	-0.071	0.061	0.075	0.068
	0.6	-0.070	-0.119	-0.076	0.061	0.094	0.068
	0.8	-0.070	-0.117	-0.072	0.061	0.055	0.055
	1.0	-0.070	0.003	-0.052	0.061	0.015	0.044
	1.2	-0.070	0.140	-0.021	0.061	0.057	0.048
	1.4	-0.070	0.242	0.003	0.061	0.156	0.056
	1.6	-0.070	0.266	0.004	0.061	0.264	0.073
	1.8	-0.070	0.237	0.005	0.061	0.329	0.086

Table 4: Bias and MSE of the estimators for the location parameter.

		Bias			MSE		
n	η_0	$\widehat{\eta}_{ML}$	$\widehat{\eta}_{pt}$	$\widehat{\eta}_s$	$\widehat{\eta}_{ML}$	$\widehat{\eta}_{pt}$	$\widehat{\eta}_s$
6	-4	1.000	-0.346	0.596	1.970	6.461	1.998
	-3	1.000	-0.760	0.463	1.970	5.925	1.802
	-2	1.000	-0.910	0.420	1.970	4.044	1.427
	-1	1.000	-0.597	0.520	1.970	1.735	1.181
	0	1.000	0.165	0.764	1.970	0.651	1.358
	1	1.000	0.701	0.933	1.970	0.912	1.644
	2	1.000	0.912	0.990	1.970	1.478	1.882
	3	1.000	0.967	0.995	1.970	1.677	1.882
	4	1.000	0.993	1.005	1.970	1.876	1.986
11	-4	1.005	0.378	0.786	1.995	4.210	2.134
	-3	1.005	-0.261	0.556	1.995	4.973	1.971
	-2	1.005	-0.777	0.368	1.995	4.243	1.688
	-1	1.005	-0.492	0.493	1.995	2.079	1.395
	0	1.005	0.184	0.726	1.995	0.729	1.272
	1	1.005	0.703	0.901	1.995	0.982	1.565
	2	1.005	0.883	0.956	1.995	1.409	1.773
	3	1.005	0.960	0.986	1.995	1.693	1.872
	4	1.005	1.029	1.040	1.995	2.049	2.144
16	-4	1.012	0.668	0.889	2.051	3.285	2.145
	-3	1.012	-0.071	0.613	2.051	4.647	2.069
	-2	1.012	-0.583	0.445	2.051	4.277	1.905
	-1	1.012	-0.487	0.469	2.051	2.129	1.406
	0	1.012	0.182	0.702	2.051	0.788	1.287
	1	1.012	0.700	0.899	2.051	0.905	1.494
	2	1.012	0.899	0.976	2.051	1.389	1.771
	3	1.012	0.930	0.952	2.051	1.608	1.764
	4	1.012	0.954	0.959	2.051	1.764	1.814

Table 5: ML, pre-test, and shrinkage estimations for the scale and location parameters.

θ_0	$\widehat{\theta}_{ML}$	$\widehat{\theta}_{pt}$	$\widehat{\theta}_s$	η_0	$\widehat{\eta}_{ML}$	$\widehat{\eta}_{pt}$	$\widehat{\eta}_s$
1	8.167	8.167	8.167	8.4	9.30	9.30	9.30
2	8.167	8.167	8.167	8.5	9.30	9.30	9.30
3	8.167	8.167	8.167	8.6	9.30	9.30	9.30
4	8.167	8.167	8.167	8.7	9.30	8.70	9.17
5	8.167	5	7.882	8.8	9.30	8.80	9.20
6	8.167	6	7.972	8.9	9.30	8.90	9.22
7	8.167	7	8.062	9.0	9.30	9.0	9.24
8	8.167	8	8.152	9.1	9.30	9.1	9.26
9	8.167	9	8.242	9.2	9.30	9.2	9.28
10	8.167	10	8.332	9.3	9.30	9.3	9.30
11	8.167	11	8.422	9.4	9.30	9.30	9.30
12	8.167	12	8.512	9.5	9.30	9.30	9.30
13	8.167	13	8.602	9.6	9.30	9.30	9.30
14	8.167	14	8.692	9.7	9.30	9.30	9.30
15	8.167	15	8.782	9.8	9.30	9.30	9.30

5. Numerical Example

The following example is based on a data set discussed by Dunsmore (1983) and Balakrishnan & Chan (1994).

A rock crushing machine is kept working as long as the size of the crushed rock is larger than the rocks crushed before. Otherwise it is reset. The data given below represent the sizes of the crushed rocks up to the third reset of the machine $9.3, 0.6, 24.4, 18.1, 6.6, 9.0, 14.3, 6.6, 13.0, 2.4, 5.6, 33.8$

The upper records are 9.3, 24.4, 33.8. It follows that the MLE's of θ and η are $\widehat{\theta}_{ML} = 8.16$ and $\widehat{\eta}_{ML} = 9.3$, respectively. Based on these records we will compute the various estimators for the two-parameter exponential model. Table 3 present the values of the pre-test and shrinkage estimators for various choices of the prior guesses of the scale parameter (θ_0) and location parameter (η_0).

It can be seen that three estimators of θ are equal, for θ_0 is less than 5, but the null hypothesis is not rejected for $\theta_0 \geq 5$, and so $\hat{\theta}_{pt}$ is equal to the prior guess and all estimators are different. Also, all estimators for η are different for $8.7 \leq \eta_0 \leq 9.3$, and otherwise the null hypothesis is rejected, so they are equal.

6. Conclusion

In some cases, there is non-sample prior information about the parameter of a population. Therefore, we can use both sample and non-sample information to estimate the parameter of interest. In this paper, we considered inference about the location-scale exponential distribution when the record values are available as sample information. For the scale parameter, we proposed a PTSE based on the MLE. It is a linear combination of the prior information and MLE when the preliminary test fails to reject the null hypothesis. We evaluated the MSE of the PTSE and obtained its optimal value based on the regret function, numerically. Simulation studies showed that the optimal PTSE is better than the MLE in all cases. Similarly, we obtained the optimal PTSE for the location parameter of exponential distribution. Simulation studies showed that this estimator is better than the MLE when the prior guess is close to the true value. In the end, all estimators are illustrated by a numerical example for some prior guess.

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